Prove all the following statements.

1. (Kühnel Ch.2 Q.4) A regular curve between two points $p, q$ in $\mathbb{R}^n$ with minimal length is necessarily the line segment from $p$ to $q$. Hint: Consider the Schwarz inequality $\langle X, Y \rangle \leq \|X\| \cdot \|Y\|$ for the tangent vector and the difference vector $q - p$.

2. (Kühnel Ch.2 Q.1) The curvature and the torsion of a Frenet curve $c(t)$ in $\mathbb{R}^3$ are given by the formulas
   \[
   \kappa(t) = \frac{\|\dot{c} \times \ddot{c}\|}{\|\dot{c}\|^3} \quad \text{and} \quad \tau(t) = \frac{\text{Det}(\dot{c}, \ddot{c}, \dddot{c})}{\|\dot{c} \times \ddot{c}\|^2}
   \]
   for an arbitrary parametrization. For a plane curve we have
   \[
   \kappa(t) = \frac{\text{Det}(\dot{c}, \dddot{c})}{\|\dot{c}\|^3}.
   \]

3. (Kühnel Ch.2 Q.9-11) Let a plane curve be given in polar coordinates $(r, \varphi)$ by $r = r(\varphi)$. Using the notation $r' = \frac{dr}{d\varphi}$, the arc length in the interval $[\varphi_1, \varphi_2]$ can be calculated as
   \[
   s = \int_{\varphi_1}^{\varphi_2} \sqrt{r'^2 + r^2} d\varphi,
   \]
   and the curvature is given by
   \[
   \kappa(\varphi) = \frac{2r'^2 - rr'' + r^2}{(r'^2 + r^2)^{3/2}}.
   \]
   Calculate the curvature of the curve given by $r(\varphi) = a\varphi$ ($a$ is a constant), the so-called Archimedean spiral, see Figure 2.12. Moreover, show that (i) The length of the curve given in polar coordinates by $r(t) = \exp(t), \varphi(t) = at$ with a constant $a$ (the logarithmic spiral) in the interval $(-\infty, t]$ is proportional to the radius $r(t)$, see Figure 2.12. (ii) The position vector of the logarithmic spiral has a constant angle with the tangent vector.
4. (Kühnel Ch.2 Q.6) If a circle of unit radius is rolled along a line (without friction), then a fixed point on that circle has its trajectory the so-called cycloid, see Figure 2.11. Find the arc-length parametrization \( c(s) : [0, L] \to \mathbb{R}^2 \) for one complete arch of the cycloid. Calculate its curvature \( \kappa(s) \).

![Figure 2.11. Cycloid](image)

5. (Kühnel Ch.2 Q.3) Let \( c(s) \) be a regular curve parametrized by arc length. If \( \kappa(s) \neq 0 \) for all \( s \), then the evolute of \( c \) is defined to be the curve

\[
\gamma(s) := c(s) + \frac{1}{\kappa(s)} e_2(s)
\]

where \( \{e_1(s), e_2(s)\} \) is the Frenet frame of \( c(s) \) and \( \kappa(s) \) is the (signed) curvature. Show that \( \gamma \) is regular precisely where \( \kappa' 
eq 0 \), and that the tangent to \( \gamma \) at the point \( s = s_0 \) intersects the curve \( c \) at \( s = s_0 \) perpendicularly. Moreover, prove that the evolute of a cycloid (see Problem 4 above) is also a cycloid.

6. (Kühnel Ch.2 Q.7) Find the unique (up to rigid motions) arc-length parametrized plane curve \( c(s) : (0, \infty) \to \mathbb{R}^2 \) whose curvature is given by \( \kappa(s) = s^{-1/2} \).

**Suggested Exercises**

1. (Kühnel Ch.2 Q.2) At every point \( p \) of a regular plane curve \( c \) with \( c''(p) \neq 0 \) (or, equivalently, \( \kappa(p) \neq 0 \)) there is a parabola which has a point of third order contact with the curve at \( p \). The point of contact is the vertex of the parabola if and only if \( \kappa'(p) = 0 \).

2. (Kühnel Ch.2 Q.5) If all tangent vectors to the curve \( c(t) = (3t, 3t^2, 2t^3) \) are drawn from the origin, then their endpoints are on the surface of a circular cone with axis be the line \( x - z = y = 0 \).

3. (Kühnel Ch.2 Q.8) The Frenet two-frame of a plane curve with given curvature function \( \kappa(s) \) can be described by the exponential series for the matrix

\[
\begin{pmatrix}
0 & \int_0^s \kappa(t)dt \\
-\int_0^s \kappa(t)dt & 0
\end{pmatrix}
\]

It follows that

\[
\begin{pmatrix}
e_1(s) \\
e_2(s)
\end{pmatrix} = \sum_{i=0}^{\infty} \frac{1}{i!} \left(\begin{pmatrix}0 & \int_0^s \kappa \\
\int_0^s \kappa & 0
\end{pmatrix} \right)^i.
\]