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Four-ball genus bounds and a refinement of the Ozsváth-Szabó tau invariant

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Based on work of Rasmussen [Ras03], we construct a concordance invariant associated to the knot Floer complex, and exhibit examples in which this invariant gives arbitrarily better bounds on the 4-ball genus than the Ozsváth-Szabó τ invariant.

1. Introduction

The 4-*ball genus* of a knot $K \subset S^3$ is

$$g_4(K) = \min\{g(\Sigma) \mid \Sigma \text{ smoothly embedded in } B^4 \text{ with } \partial\Sigma = K\},$$

where $g(\Sigma)$ denotes the genus of the surface Σ. The 4-ball genus gives a lower bound on the unknotting number of a knot (that is, the minimal number of crossing changes needed to obtain the unknot). We say knots K_1 and K_2 are *concordant* if $g_4(K_1\# -K_2) = 0$, where $-K_2$ denotes the reverse of the mirror image of K_2.

In [OS03c], Ozsváth-Szabó defined a concordance invariant, τ, that gives a lower bound for the 4-ball genus of a knot. This invariant is sharp on torus knots, giving a new proof of the Milnor conjecture, originally proved by Kronheimer-Mrowka using gauge theory [KM93]

The knot Floer homology package [OS04a, Ras03] associates to a knot K a $\mathbb{Z}\oplus\mathbb{Z}$-filtered chain complex over the ring $\mathbb{F}[U, U^{-1}]$, where $\mathbb{F}$ denotes the field of two elements and U is a formal variable. We denote this complex $CFK^\infty(K)$. The invariant τ depends only on a single $\mathbb{Z}$-filtration, and forgets the module structure. By studying the module structure together with the full $\mathbb{Z}\oplus\mathbb{Z}$-filtration, we obtain a concordance invariant, ν^+, which gives a better bound on the 4-ball genus than τ, in the sense that

$$\tau(K) \leq \nu^+(K) \leq g_4(K). \tag{1.1}$$

Moreover, the gap between τ and ν^+ can be made arbitrarily large.

Theorem 1. *For any positive integer p, there exists a knot K with $\tau(K) \geq 0$ and*

$$\tau(K) + p \leq \nu^+(K) = g_4(K).$$

Remark 1.1. The invariant ν^+ is closely related to the sequence of local h invariants of Rasmussen [Ras03, Section 7], which Rasmussen uses to give bounds on the 4-ball genus; indeed, ν^+ corresponds to the first place in the sequence where a zero appears.

In Proposition 3.7, we also show that the gap between ν^+ and the knot signature can be made arbitrarily large.

In the case of alternating knots (or, more generally, quasi-alternating knots), the invariant ν^+ is completely determined by the signature of the knot.

Theorem 2. *Let $K \subset S^3$ be a quasi-alternating knot. Then,*

$$\nu^+(K) = \begin{cases} 0 & \text{if } \sigma(K) \geq 0, \\ -\frac{\sigma(K)}{2} & \text{if } \sigma(K) < 0. \end{cases}$$

We also have the following result when K is strongly quasipositive. See [Hed10] for background on strongly quasipositive knots.

Proposition 3. *If K is strongly quasipositive, then*

$$\nu^+(K) = \tau(K) = g_4(K) = g(K).$$

Proof. [Hed10, Theorem 1.2] states that $\tau(K) = g_4(K) = g(K)$ if and only if K is strongly quasipositive. Since $\tau(K) \leq \nu^+(K) \leq g_4(K)$, the result follows. □

Organization. In Section 2, we define the invariant ν^+ and prove various properties. In Section 3, we construct an infinite family of knots in order to prove Theorem 1. Throughout, we work over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$.

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2. The invariant ν^+

Heegaard Floer homology, introduced by Ozsváth and Szabó [OS04b], is an invariant for closed oriented Spin^c 3–manifolds $(Y, \mathfrak{s})$, taking the form of a collection of related homology groups: $\widehat{HF}(Y, \mathfrak{s})$, $HF^{\pm}(Y, \mathfrak{s})$, and $HF^{\infty}(Y, \mathfrak{s})$. There is a U–action on the Heegaard Floer homology groups $HF^{\pm}$ and HF^{∞}. When $\mathfrak{s}$ is torsion, there is an absolute Maslov $\mathbb{Q}$–grading on the Heegaard Floer homology groups. The U–action decreases the grading by 2.

For a rational homology 3–sphere Y with a Spin^c structure $\mathfrak{s}$, $HF^+(Y, \mathfrak{s})$ can be decomposed as the direct sum of two groups: the first group is the image of $HF^{\infty}(Y, \mathfrak{s}) \cong \mathbb{F}[U, U^{-1}]$ in $HF^+(Y, \mathfrak{s})$, which is isomorphic to $\mathcal{T}^+ = \mathbb{F}[U, U^{-1}]/U\mathbb{F}[U]$, and its minimal absolute $\mathbb{Q}$–grading is an invariant of $(Y, \mathfrak{s})$, denoted by $d(Y, \mathfrak{s})$, the *correction term* [OS03a]; the second group is the quotient modulo the above image and is denoted by $HF_{\mathrm{red}}(Y, \mathfrak{s})$. Altogether, we have

$$HF^+(Y, \mathfrak{s}) = \mathcal{T}^+ \oplus HF_{\mathrm{red}}(Y, \mathfrak{s}).$$

We briefly recall the large N surgery formula of [OS04a, Theorem 4.4]. We use the notation of [NW15]. Let $CFK^{\infty}(K)$ denote the knot Floer complex of K, which takes the form of a $\mathbb{Z} \oplus \mathbb{Z}$-filtered, $\mathbb{Z}$-graded chain complex over $\mathbb{F}[U, U^{-1}]$. The U-action lowers each filtration by one. We will be particularly interested in the quotient complexes

$$A_k^+ = C\{\max\{i, j-k\} \geq 0\} \quad \text{and} \quad B^+ = C\{i \geq 0\}$$

where i and j refer to the two filtrations. The complex B^+ is isomorphic to $CF^+(S^3)$. There is a map

$$v_k^+ : A_k^+ \to B^+$$

defined by projection. One can also define a map

$$h_k^+ : A_k^+ \to B^+$$

defined by projection to $C\{j \geq k\}$, followed by shifting to $C\{j \geq 0\}$ via the U-action, and concluding with a chain homotopy equivalence between $C\{j \geq 0\}$ and $C\{i \geq 0\}$. These maps correspond to the maps induced on HF^+ by the two handle cobordism from $S^3_N(K)$ to S^3 [OS04a, Theorem 4.4].

Similarly, one can consider the subquotient complexes

$$\widehat{A}_k = C\{\max\{i, j-k\} = 0\} \quad \text{and} \quad \widehat{B} = C\{i = 0\} \cong \widehat{CF}(S^3)$$

and the maps

$$\widehat{v}_k : \widehat{A}_k \to \widehat{B} \quad \text{and} \quad \widehat{h}_k : \widehat{A}_k \to \widehat{B}.$$

The invariant τ is defined in [OS03c] to be

$$\tau(K) = \min\{k \in \mathbb{Z} \mid \iota_k \text{ induces a nontrivial map on homology}\},$$

where $\iota_k : C\{i = 0, j \leq k\} \to \widehat{CF}(S^3)$ denotes inclusion. A slightly stronger concordance invariant, ν, is defined in [OS11, Definition 9.1] to be

$$\nu(K) = \min\{k \in \mathbb{Z} \,|\, \widehat{v}_k : \widehat{A}_k \to \widehat{CF}(S^3) \text{ induces a nontrivial map in homology}\}.$$

The invariant $\nu(K)$ gives a lower bound for $g_4(K)$ and is equal to either $\tau(K)$ or $\tau(K) + 1$; in particular, in many cases ν gives a better 4-ball genus than τ.

We can further refine these bounds by considering maps on CF^+ rather than $\widehat{CF}$.

Definition 2.1. Define $\nu^+(K)$ by

$$\nu^+(K) = \min\{k \in \mathbb{Z} \,|\, v_k^+ : A_k^+ \to CF^+(S^3), \;\; v_k^+(1) = 1\}.$$

Here, 1 denotes the lowest graded generator of the subgroup $\mathcal{T}^+$ in the homology of the complex, and we abuse our notations by identifying A_k^+ and $CF^+(S^3)$ with their homologies.

According to [NW15], the definition of $\nu^+(K)$ is equivalent to the smallest k such that $V_k = 0$, where V_k is the U-exponent of v_k^+ at sufficiently high gradings. We can define H_k similarly in terms of h_k^+. By [NW14, Equation (13)] and [HLZ15, Lemma 2.5], the V_k's and H_k's satisfy

$$H_k = V_{-k} \tag{2.1}$$

$$H_k = V_k + k \tag{2.2}$$

$$V_k - 1 \leq V_{k+1} \leq V_k \tag{2.3}$$

and are related to the correction terms in the surgery formula [NW15, Proposition 1.6]:

Proposition 2.2. *Suppose $p, q > 0$, and fix $0 \le i \le p-1$. Then*

$$d(S^3_{p/q}(K), i) = d(L(p,q), i) - 2\max\{V_{\lfloor \frac{i}{q} \rfloor}, H_{\lfloor \frac{i-p}{q} \rfloor}\}. \tag{2.4}$$

We have the following properties for ν^+.

Proposition 2.3. *The invariant ν^+ satisfies:*

1) *ν^+ is a smooth concordance invariant.*
2) *$\nu^+(K) \ge 0$, and the equality holds if and only if $V_0 = 0$.*
3) *$\nu^+(K) \ge \nu(K) \ge \tau(K)$.*

Proof. To see 1, note that V's are determined by the d-invariants of the surgered manifolds $S^3_n(K)$ [NW15, Proposition 1.6], and the d-invariants are concordance invariants. To see 2, note that $V_{-1} > H_{-1} = V_1 \ge 0$ by Equations (2.1) and (2.2). To see 3, chase the commutative diagram

$$\begin{tikzcd} \widehat{A}_k \arrow[r, "j_A"] \arrow[d, "\widehat{v}_k"'] & A^+_k \arrow[d, "v^+_k"'] \\ \widehat{B} \arrow[r, "j_B"] & B^+. \end{tikzcd}$$

□

The ν^+ invariant can be computed explicitly for quasi-alternating knots, a generalization of alternating knots introduced in [MO08]. In fact, Theorem 2 states that ν^+ is completely determined by the signature of the knot, just as the τ invariant:

$$\nu^+(K) = \begin{cases} 0 & \text{if } \sigma(K) \ge 0, \\ -\frac{\sigma(K)}{2} & \text{if } \sigma(K) < 0. \end{cases}$$

Proof of Theorem 2. Let K be quasi-alternating. By [OS03b, Corollary 1.5] and [MO08, Theorem 2], $d(S^3_1(K)) = 0$ when $\sigma(K) \ge 0$. This proves that $\nu^+(K) = 0$ when $\sigma(K) \ge 0$. On the other hand, the proof of Theorem 1.4 of [OS03b], together with [MO08, Theorem 2], implies that for any $s > 0$,

$$H_{\le s+\frac{\sigma}{2}-2}(A^+_s) \cong HF^+_{\le s+\frac{\sigma}{2}-2}(S^3).$$

In particular, if we let $s = -\sigma/2$ when $\sigma(K) < 0$, then

$$H_{\le -2}(A^+_s) \cong HF^+_{\le -2}(S^3) \cong 0.$$

Here, the gradings of the homology of both sides are inherited from the grading on $CFK^\infty(K)$. Thus, the element $1 \in \mathcal{T}^+ \subset H_*(A_s^+)$ has grading $-2V_s$. In light of the vanishing of the homology group $H_{\leq -2}(A_s^+)$, we must have $V_s = 0$. So

$$\nu^+(K) \leq s = -\sigma(K)/2$$

from the definition. We also know that

$$\nu^+(K) \geq \tau(K) = -\sigma(K)/2$$

for a quasi-alternating knot K. Hence, $\nu^+(K) = -\sigma(K)/2$. □

Next, we show that ν^+ also give a lower bound for the four-ball genus of a knot.

Proposition 2.4. $\nu^+(K) \leq g_4(K)$

Proof. This follows from [Ras03, Corollary 7.4]. The function $h_k(K)$ in [Ras03] is the same as V_k in [NW15]. □

Remark 2.5. [Ras03, Corollary 7.4] states that $g_4(K) \geq V_k + k$ for all $k \leq g_4(K)$, so one might wonder if other V_k's can give stronger 4-ball genus bounds. However, since $V_k - 1 \leq V_{k+1} \leq V_k$, it follows that ν^+ is the best 4-ball genus bound obtainable from the sequence of V_k's.

3. Four-ball genus bound

In this section, we exhibit some examples of knots whose ν^+ invariant is arbitrarily better than the corresponding τ invariant. Hence, the ν^+ invariant indeed gives us significantly improved four-ball genus bound for some particular knots. We will show that for any integer $n \geq 2$, there exists a knot K with $\tau(K) \geq 0$ and

$$\tau(K) + n = \nu^+(K) = g_4(K).$$

Let $K_{p,q}$ denote the (p,q)-cable of K, where p denotes the longitudinal winding. Without loss of generality, we will assume throughout that $p > 0$. Let $T_{p,q}$ denote the (p,q)-torus knot (that is, the (p,q)-cable of the unknot), and $T_{p,q;m,n}$ the (m,n)-cable of $T_{p,q}$. We begin with a single example of a knot for which ν^+ gives a better 4-ball genus bound than τ.

Proposition 3.1. *Let K be the knot $T_{2,9}\# - T_{2,3;2,5}$. We have*

$$\tau(K) = 0, \qquad \nu(K) = 1, \qquad \text{and} \qquad \nu^+(K) = 2.$$

Proof. The torus knot $T_{2,9}$ is an L-space knot, as is $T_{2,3;2,5}$ [Hed09, Theorem 1.10], so their knot Floer complexes are completely determined by their Alexander polynomials [OS05, Theorem 1.2] (cf. [Hom14b, Remark 6.6]). We have that

$$\Delta_{T_{2,9}}(t) = t^8 - t^7 + t^6 - t^5 + t^4 - t^3 + t^2 - t + 1$$

and

$$\begin{aligned}\Delta_{T_{2,3;2,5}}(t) &= \Delta_{T_{2,3}}(t^2) \cdot \Delta_{T_{2,5}}(t)\\ &= t^8 - t^7 + t^4 - t + 1.\end{aligned}$$

Furthermore, we have that $CFK^\infty(-K) \cong CFK^\infty(K)^*$ [OS04a, Section 3.5], where $CFK^\infty(K)^*$ denotes the dual of $CFK^\infty(K)$. Thus, $CFK^\infty(-T_{2,3;2,5})$ is generated over $\mathbb{F}[U, U^{-1}]$ by

$$[y_0, 0, -4], \quad [y_1, -1, -4], \quad [y_1, -1, -1], \quad [y_3, -3, -1], \quad [y_4, -4, 0],$$

where we write $[y, i, j]$ to denote that the generator y has filtration level (i, j). The differential is given by

$$\begin{aligned}\partial y_0 &= y_1\\ \partial y_2 &= y_1 + y_3\\ \partial y_4 &= y_3.\end{aligned}$$

The complex $CFK^\infty(T_{2,9})$ is generated by

$$\begin{gathered}[x_0, 0, 4], \quad [x_1, 1, 4], \quad [x_2, 1, 3], \quad [x_3, 2, 3], \quad [x_4, 2, 2],\\ [x_5, 3, 2], \quad [x_6, 3, 1], \quad [x_7, 4, 1], \quad [x_8, 4, 0].\end{gathered}$$

The differential is given by

$$\begin{aligned}\partial x_1 &= x_0 + x_2\\ \partial x_3 &= x_2 + x_4\\ \partial x_5 &= x_4 + x_6\\ \partial x_7 &= x_6 + x_8.\end{aligned}$$

The complexes $CFK^\infty(-T_{2,3;2,5})$ and $CFK^\infty(T_{2,9})$ are depicted in Figures 1 and 2, respectively. (More precisely, CFK^∞ consists of the complexes pictured tensored with $\mathbb{F}[U, U^{-1}]$, where U lowers i and j each by 1.) In particular, we see that $\tau(-T_{2,3;2,5}) = -4$ since y_0 generates the vertical homology, and that $\tau(T_{2,9}) = 4$ since x_0 generates the vertical homology. Since τ is additive under connected sum, it follows that

$$\tau(-T_{2,3;2,5} \# T_{2,9}) = 0,$$

as desired.

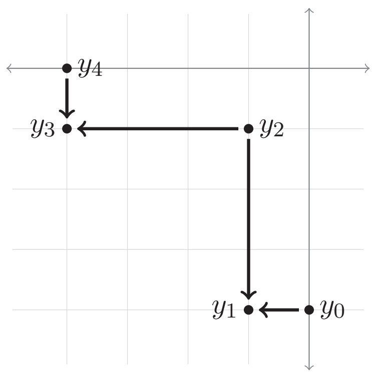

Figure 1: $CFK^\infty(-T_{2,3;2,5})$

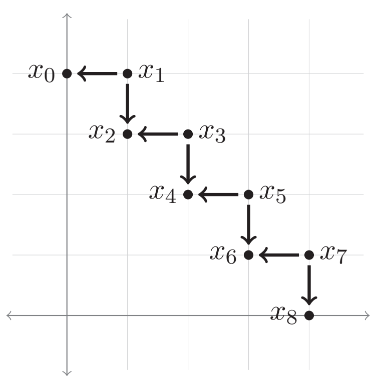

Figure 2: $CFK^\infty(T_{2,9})$

The knot Floer complex satisfies a Künneth formula [OS04a, Theorem 7.1]:

$$CFK^\infty(K_1\#K_2) \cong CFK^\infty(K_1) \otimes_{\mathbb{F}[U,U^{-1}]} CFK^\infty(K_2).$$

In particular, we may compute $CFK^\infty(T_{2,9}\# - T_{2,3;2,5})$ as the tensor product of $CFK^\infty(T_{2,9})$ and $CFK^\infty(-T_{2,3;2,5})$, where

$$[x,i,j]\otimes[y,k,\ell] = [xy, i+k, j+\ell].$$

The generators, filtration levels, and differentials in the tensor product are listed below.

$$\begin{aligned}
\partial[x_0y_0,0,0] &= x_0y_1\\
\partial[x_1y_0,1,0] &= x_1y_1 + x_0y_0 + x_2y_0\\
\partial[x_2y_0,1,-1] &= x_2y_1\\
\partial[x_3y_0,2,-1] &= x_3y_1 + x_2y_0 + x_4y_0\\
\partial[x_4y_0,2,-2] &= x_4y_1\\
\partial[x_5y_0,3,-2] &= x_5y_1 + x_4y_0 + x_6y_0\\
\partial[x_6y_0,3,-3] &= x_6y_1\\
\partial[x_7y_0,4,-3] &= x_7y_1 + x_6y_0 + x_8y_0\\
\partial[x_8y_0,4,-4] &= x_8y_1\\
\partial[x_0y_1,-1,0] &= 0\\
\partial[x_1y_1,0,0] &= x_0y_1 + x_2y_1\\
\partial[x_2y_1,0,-1] &= 0\\
\partial[x_3y_1,1,-1] &= x_2y_1 + x_4y_1\\
\partial[x_4y_1,1,-2] &= 0\\
\partial[x_5y_1,2,-2] &= x_4y_1 + x_6y_1\\
\partial[x_6y_1,2,-3] &= 0\\
\partial[x_7y_1,3,-3] &= x_6y_1 + x_8y_1\\
\partial[x_8y_1,3,-4] &= 0\\
\partial[x_0y_2,-1,3] &= x_0y_1 + x_0y_3\\
\partial[x_1y_2,0,3] &= x_1y_1 + x_1y_3 + x_0y_2 + x_2y_2\\
\partial[x_2y_2,0,2] &= x_2y_1 + x_2y_3\\
\partial[x_3y_2,1,2] &= x_3y_1 + x_3y_3 + x_2y_2 + x_4y_2\\
\partial[x_4y_2,1,1] &= x_4y_1 + x_4y_3
\end{aligned}$$

$$
\begin{aligned}
\partial[x_5y_2, 2, 1] &= x_5y_1 + x_5y_3 + x_4y_2 + x_6y_2 \\
\partial[x_6y_2, 2, 0] &= x_6y_1 + x_6y_3 \\
\partial[x_7y_2, 3, 0] &= x_7y_1 + x_7y_3 + x_6y_2 + x_8y_2 \\
\partial[x_8y_2, 3, -1] &= x_8y_1 + x_8y_3 \\
\partial[x_0y_3, -4, 3] &= 0 \\
\partial[x_1y_3, -3, 3] &= x_0y_3 + x_2y_3 \\
\partial[x_2y_3, -3, 2] &= 0 \\
\partial[x_3y_3, -2, 2] &= x_2y_3 + x_4y_3 \\
\partial[x_4y_3, -2, 1] &= 0 \\
\partial[x_5y_3, -1, 1] &= x_4y_3 + x_6y_3 \\
\partial[x_6y_3, -1, 0] &= 0 \\
\partial[x_7y_3, 0, 0] &= x_6y_3 + x_8y_3 \\
\partial[x_8y_3, 0, -1] &= 0 \\
\partial[x_0y_4, -4, 4] &= x_0y_3 \\
\partial[x_1y_4, -3, 4] &= x_1y_3 + x_0y_4 + x_2y_4 \\
\partial[x_2y_4, -3, 3] &= x_2y_3 \\
\partial[x_3y_4, -2, 3] &= x_3y_3 + x_2y_4 + x_4y_4 \\
\partial[x_4y_4, -2, 2] &= x_4y_3 \\
\partial[x_5y_4, -1, 2] &= x_5y_3 + x_4y_4 + x_6y_4 \\
\partial[x_6y_4, -1, 1] &= x_6y_3 \\
\partial[x_7y_4, 0, 1] &= x_7y_3 + x_6y_4 + x_8y_4 \\
\partial[x_8y_4, 0, 0] &= x_8y_3
\end{aligned}
$$

We perform the following change of basis on $CFK^\infty(T_{2,9}\# -T_{2,3;2,5})$. In the linear combinations below, we have ordered the terms so that the first basis element has the greatest filtration and thus determines the filtration level of the linear combination.

$$
\begin{aligned}
z_0 &= x_0y_0 \\
z_1 &= x_0y_1 \\
z_2 &= x_0y_2 + x_1y_3 + x_3y_3 + x_4y_4 \\
z_3 &= x_1y_2 \\
z_4 &= x_2y_2 + x_3y_3 + x_1y_1 + x_4y_4 \\
z_5 &= x_3y_2 + x_5y_4 + x_1y_0
\end{aligned}
$$

$$
\begin{aligned}
z_6 &= x_4y_2 + x_5y_3 + x_3y_1 + x_6y_4 + x_2y_0 \\
z_7 &= x_5y_2 + x_7y_4 + x_3y_0 \\
z_8 &= x_6y_2 + x_7y_3 + x_5y_1 + x_4y_0 \\
z_9 &= x_7y_2 \\
z_{10} &= x_8y_2 + x_7y_1 + x_4y_0 + x_5y_1 \\
z_{11} &= x_8y_3 \\
z_{12} &= x_8y_4 \\
w_0^i &= x_{2i+1}y_4 && i = 0,1,2,3 \\
w_1^i &= x_{2i}y_4 && i = 0,1,2,3 \\
w_2^i &= x_{2i}y_3 && i = 0,1,2,3 \\
w_3^i &= x_{2i+1}y_3 + x_{2i+2}y_4 && i = 0,1,2,3 \\
w_0^{i+4} &= x_{2i+1}y_0 && i = 0,1,2,3 \\
w_1^{i+4} &= x_{2i+1}y_1 + x_{2i}y_0 && i = 0,1,2,3 \\
w_2^{i+4} &= x_{2i+2}y_1 && i = 0,1,2,3 \\
w_3^{i+4} &= x_{2i+2}y_0 && i = 0,1,2,3.
\end{aligned}
$$

See Figure 3.

Notice that the basis elements $\{z_i\}_{i=0}^{12}$ generate a direct summand C of $CFK^\infty(T_{2,9}\# -T_{2,3;2,5})$. See Figure 4. Since the total homology of this summand is non-zero, this summand determines both ν and ν^+. We write $\widehat{A}_s$ and A_s^+ to refer to the associated subquotient complexes of C.

The vertical homology of C is generated by z_0. The generator z_0 in $C\{i=0\}$ is not the image of any cycle in $\widehat{A}_0$. On the other hand, z_0 is non-zero in $H_*(\widehat{A}_1)$. Hence $\nu(T_{2,9}\# -T_{2,3;2,5}) = 1$.

The cycle z_6 generates $H_*(C)$. Moreover, the cycle Uz_6 is non-zero in $H_*(A_1^+)$; see Figure 5. The cycle Uz_6 is a boundary in A_2^+ as in Figure 6, while the cycle z_6 is non-zero in $H_*(A_2^+)$. It follows that $\nu^+(T_{2,9}\# - T_{2,3;2,5}) = 2$, as desired. □

Corollary 3.2. *Let* $K = T_{2,5}\#2T_{2,3}\# -T_{2,3;2,5}$. *Then*

$$\tau(K) = 0, \quad \nu(K) = 1, \quad \text{and} \quad \nu^+(K) = 2.$$

Proof. By [HKL16, Theorem B.1],

$$CFK^\infty(T_{2,5}\#2T_{2,3}) \cong CFK^\infty(T_{2,9}) \oplus A,$$

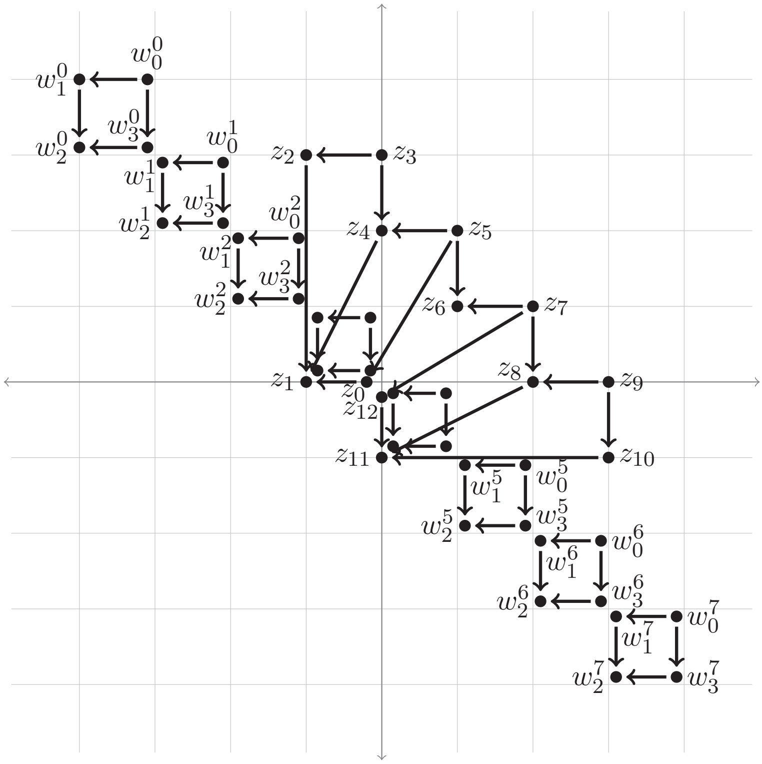

Figure 3: $CFK^\infty(T_{2,9}\# - T_{2,3;2,5})$ after a change of basis

where A is acyclic (i.e., its total homology vanishes). Since acyclic summands do not affect τ, ν, and ν^+, the result follows. $\square$

Lemma 3.3. *Let* $K = T_{2,5}\#2T_{2,3}\# - T_{2,3;2,5}$. *Then* $g_4(K) = 2$.

Proof. When $p, q > 0$, the genus of $T_{p,q}$ is equal to $\frac{(p-1)(q-1)}{2}$. We can construct a genus 4 Seifert surface F for $-T_{2,3;2,5} = (-T_{2,3})_{-2,5}$ by taking two parallel copies of the genus one Seifert surface for $-T_{2,3}$ and connecting them with 5 half-twisted bands. The knot $-T_{2,3}\#T_{-2,5}$ sits on F. To see this, consider one copy of the Seifert surface for $-T_{2,3}$ together with the half-twisted bands and a small neighborhood of a segment connecting the ends of the bands.

Take the boundary sum of F with the genus two Seifert surface for $T_{2,5}$ and with two copies of the genus one Seifert surface for $T_{2,3}$ to obtain a surface F'. The surface F' is a genus 8 Seifert surface for K. The genus

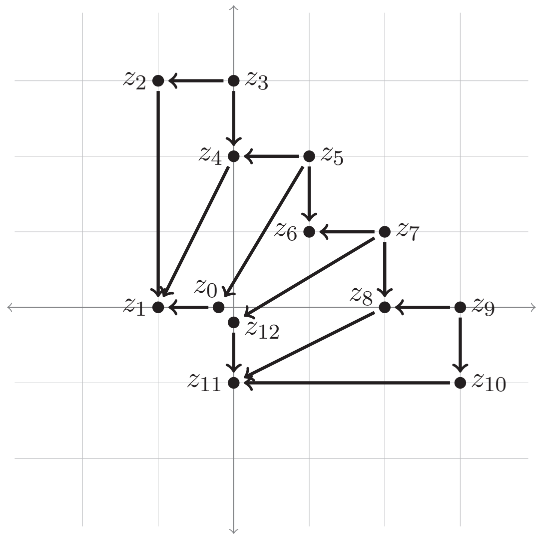

Figure 4: The relevant summand of $CFK^\infty(T_{2,9}\# - T_{2,3;2,5})$

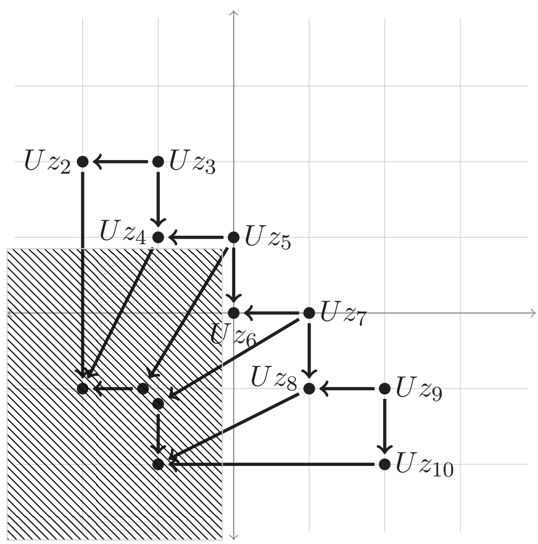

Figure 5: The generators $\{Uz_i\}$ in A_1^+

6 slice knot $J = -T_{2,3}\#T_{-2,5}\#T_{2,3}\#T_{2,5}$ sits on this surface. Performing

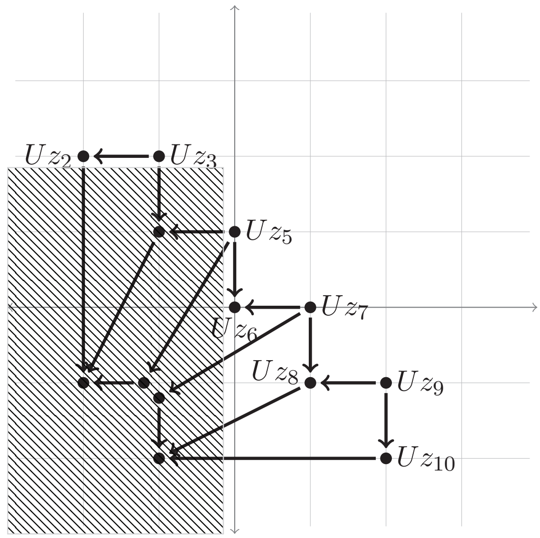

Figure 6: The generators $\{Uz_i\}$ in A_2^+

surgery along J on F' in B^4 yields a genus two slice surface for K. Since $\nu^+(K) = 2$ and $\nu^+(K) \leq g_4(K)$, it follows that $g_4(K) = 2$. □

In order to prove the main theorem, we will consider certain cables of the knot $K = T_{2,5}\#2T_{2,3}\# - T_{2,3;2,5}$. We first compute τ of these cables.

Lemma 3.4. *Let K be the knot $T_{2,5}\#2T_{2,3}\# - T_{2,3;2,5}$. Then*

$$\tau(K_{p,3p-1}) = \frac{3p(p-1)}{2}.$$

Proof. Recall from [Hom14a, Definition 3.4] that the invariant $\varepsilon(K)$ is defined to be -1 if $\tau(K) < \nu(K)$. The equality then follows from [Hom14a, Theorem 1], which states that if $\varepsilon(K) = -1$, then

$$\tau(K_{p,q}) = p\tau(K) + \frac{(p-1)(q+1)}{2}.$$
□

Proposition 3.5. *Let K be the knot $T_{2,5}\#2T_{2,3}\# - T_{2,3;2,5}$. Then*

$$\nu^+(K_{p,3p-1}) = g_4(K_{p,3p-1}) = \frac{p(3p-1)}{2} + 1.$$

Proof. Let $p, q > 0$. For an arbitrary knot J and its cable $J_{p,q}$, there is a reducible surgery

$$S^3_{pq}(J_{p,q}) \cong S^3_{q/p}(J)\#L(p,q).$$

We apply the surgery formula (2.4) for the above knot surgery when J is the unknot. Note that $\max\{V_i, H_{i-pq}\} = V_i$ when $0 \le i \le \frac{pq}{2}$ since $V_i = H_{-i}$ and $H_{i-1} \le H_i$. Thus, we have

$$d(L(pq,1),i) - 2V_i(T_{p,q}) = d(L(q,p),\phi_1(i)) + d(L(p,q),\phi_2(i)) \tag{3.1}$$

for all $0 \le i \le \frac{pq}{2}$.

Here, we identify the Spin^c structure of a rational homology sphere by an integer i as in [NW15], and $\phi_1(i)$ and $\phi_2(i)$ are the projection of the Spin^c structure to the two factors of the reducible manifold. In particular, we can identify $\phi_1(i)$ with some integers between 0 and $q-1$ and $\phi_2(i)$ with some integers between 0 and $p-1$. The maps ϕ_1 and ϕ_2 are independent of the knot J, and in principle, can be determined from an explicit geometric description of the reducible surgery (cf [Hed09]). For the purpose of our argument below, we do not need it.

Similarly, apply (2.4) for an arbitrary knot J. We have

$$\begin{aligned} &d(L(pq,1),i) - 2V_i(J_{p,q}) \\ &= d(L(q,p),\phi_1(i)) - 2\max\left\{V_{\lfloor \frac{\phi_1(i)}{p} \rfloor}(J), H_{\lfloor \frac{\phi_1(i)-q}{p} \rfloor}(J)\right\} \\ &\quad + d(L(p,q),\phi_2(i)). \end{aligned}$$

for all $i \le \frac{pq}{2}$.

Compared with Equation (3.1) and using the fact $V_i(T_{p,q}) \ge 0$, we deduce that for all $i \le \frac{pq}{2}$,

$$\begin{aligned} V_i(J_{p,q}) &= V_i(T_{p,q}) + \max\left\{V_{\lfloor \frac{\phi_1(i)}{p} \rfloor}(J), H_{\lfloor \frac{\phi_1(i)-q}{p} \rfloor}(J)\right\} \\ &\ge \max\left\{V_{\lfloor \frac{\phi_1(i)}{p} \rfloor}(J), H_{\lfloor \frac{\phi_1(i)-q}{p} \rfloor}(J)\right\} \end{aligned}$$

From now on, let us specialize to the case when $K = T_{2,5}\#2T_{2,3}\#-T_{2,3;2,5}$ and $q = 3p-1$. We claim that

$$\max\left\{V_{\lfloor \frac{\phi_1(i)}{p} \rfloor}(K), H_{\lfloor \frac{\phi_1(i)-q}{p} \rfloor}(K)\right\} > 0.$$

To see this, note that $V_0(K), V_1(K) > 0$ as $\nu^+(K) = 2$. When $0 \le \phi_1(i) < 2p$, $V_{\lfloor \frac{\phi_1(i)}{p} \rfloor}(K) > 0$. Otherwise, $2p \le \phi_1(i) < q = 3p-1$, and then $H_{\lfloor \frac{\phi_1(i)-q}{p} \rfloor}(K) > 0$ since $H_{-k} = V_k$ and $V_0(K), V_1(K) > 0$.

Hence, $V_i(K_{p,q}) > 0$ for all $i \leq \frac{pq}{2}$. This implies that

$$\nu^+(K_{p,3p-1}) \geq \frac{p(3p-1)}{2} + 1.$$

On the other hand,

$$g_4(K_{p,q}) \leq pg_4(K) + \frac{(p-1)(q-1)}{2},$$

since one can construct a slice surface for $K_{p,q}$ from p parallel copies of a slice surface for K together with $(p-1)q$ half-twisted bands. By Lemma 3.3, $g_4(K) = 2$, so when $q = 3p-1$, the right-hand side of the above inequality is $\frac{p(3p-1)}{2} + 1$. Hence

$$\frac{p(3p-1)}{2} + 1 \leq \nu^+(K_{p,3p-1}) \leq g_4(K_{p,3p-1}) \leq \frac{p(3p-1)}{2} + 1,$$

so $\nu^+(K_{p,3p-1}) = g_4(K_{p,3p-1}) = \frac{p(3p-1)}{2} + 1$. □

Note that $\nu^+(K_{p,3p-1}) - \tau(K_{p,3p-1}) = p+1$ for $K = T_{2,5}\#2T_{2,3}\# -T_{2,3;2,5}$. This proves Theorem 1.

A similar argument shows that ν^+ gives a sharp four-ball genus bound for certain other cable knots as well.

Proposition 3.6. *Let K be a knot with $\nu^+(K) = g_4(K) = n$, then*

$$\nu^+(K_{p,(2n-1)p-1}) = g_4(K_{p,(2n-1)p-1}) = \frac{p((2n-1)p-1)}{2} + 1.$$

Proof. Let $q = (2n-1)p - 1$. In the proof of Proposition 3.5, we showed

$$V_i(K_{p,q}) \geq \max\left\{V_{\lfloor \frac{\phi_1(i)}{p} \rfloor}(K), H_{\lfloor \frac{\phi_1(i)-q}{p} \rfloor}(K)\right\}$$

for all $0 \leq i \leq \frac{pq}{2}$. We claim that

$$\max\left\{V_{\lfloor \frac{\phi_1(i)}{p} \rfloor}(K), H_{\lfloor \frac{\phi_1(i)-q}{p} \rfloor}(K)\right\} > 0.$$

To see this, note that $V_i(K) > 0$ for all $i < n$. When $0 \leq \phi_1(i) < np$, $V_{\lfloor \frac{\phi_1(i)}{p} \rfloor}(K) > 0$. Otherwise, $np \leq \phi_1(i) < q = (2n-1)p - 1$, and then

$H_{\lfloor \frac{\phi_1(i)-q}{p} \rfloor}(K) > 0$. Hence, $V_i(K_{p,q}) > 0$ for all $i \leq \frac{pq}{2}$. This implies that

$$\nu^+(K_{p,q}) \geq \frac{pq}{2} + 1 = \frac{p((2n-1)p-1)}{2} + 1.$$

On the other hand,

$$\begin{aligned} g_4(K_{p,q}) &\leq pg_4(K) + \frac{(p-1)(q-1)}{2} \\ &= pn + \frac{(p-1)((2n-1)p-2)}{2} \\ &= \frac{p((2n-1)p-1)}{2} + 1. \end{aligned}$$

So $\nu^+(K_{p,(2n-1)p-1}) = g_4(K_{p,(2n-1)p-1}) = \frac{p((2n-1)p-1)}{2} + 1$. $\square$

We conclude by showing that the knot signature cannot detect the four-ball genus of the knots used in Theorem 1. Recall that

$$\frac{1}{2}|\sigma(K)| \leq g_4(K).$$

Proposition 3.7. *Let $K = T_{2,5}\#2T_{2,3}\# - T_{2,3;2,5}$. Then for $p > 0$,*

$$\frac{1}{2}|\sigma(K_{p,3p-1})| + 2p - 2 \leq g_4(K_{p,3p-1}).$$

Proof. We have that $\sigma(T_{2,q}) = 1 - q$. By [Shi71, Theorem 9],

$$\sigma(K_{p,q}) = \begin{cases} \sigma(T_{p,q}) & \text{if } p \text{ is even} \\ \sigma(K) + \sigma(T_{p,q}) & \text{if } p \text{ is odd.} \end{cases}$$

Thus, $\sigma(T_{2,3;2,5}) = -4$ and since signature is additive under connected sum,

$$\begin{aligned} \sigma(T_{2,5}\#2T_{2,3}\# - T_{2,3;2,5}) &= -4 + 2(-2) - (-4) \\ &= -4. \end{aligned}$$

We showed in Lemma 3.3 that $g_4(K) = 2$, so for K, the signature is indeed strong enough to detect the four-ball genus. However, we will now show that it is not strong enough to detect the four-ball genus of $K_{p,3p-1}$. We have that

$$\begin{aligned} |\sigma(K_{p,3p-1})| &\leq |\sigma(K)| + |\sigma(T_{p,3p-1})| \\ &\leq 4 + (p-1)(3p-2) = 3p^2 - 5p + 6, \end{aligned}$$

where the second inequality follows from the fact that when $p, q > 0$,

$$|\sigma(T_{p,q})| \leq 2g_4(T_{p,q}) = (p-1)(q-1).$$

On the other hand,

$$2g_4(K_{p,3p-1}) = 3p^2 - p + 2,$$

so

$$|\sigma(K_{p,3p-1})| + 4p - 4 \leq 2g_4(K_{p,3p-1}).$$

□

Recall from Proposition 3.5 that $g_4(K_{p,3p-1}) = \nu^+(K_{p,3p-1})$. A consequence of Proposition 3.7 is that the gap between $\frac{1}{2}\sigma$ and ν^+ can be made arbitrarily large.

References

[Hed09] Matthew Hedden, *On knot Floer homology and cabling II*, Int. Math. Res. Not. IMRN (2009), no. 12, 2248–2274.

[Hed10] Matthew Hedden, *Notions of positivity and the Ozsváth-Szabó concordance invariant*, J. Knot Theory Ramifications, **19** (2010), no. 5, 617–629.

[HKL16] Matthew Hedden, Se-Goo Kim, and Charles Livingston, *Topologically slice knots of smooth concordance order two*, J. Differential Geom., **102** (2016), no. 3, 353–393.

[HLZ15] Jennifer Hom, Tye Lidman, and Nicholas Zufelt, *Reducible surgeries and Heegaard Floer homology*, Math. Res. Lett., **22** (2015), no. 3, 763–788.

[Hom14a] Jennifer Hom, *Bordered Heegaard Floer homology and the tau-invariant of cable knots*, J. Topol., **7** (2014), no. 2, 287–326.

[Hom14b] Jennifer Hom, *The knot Floer complex and the smooth concordance group*, Comment. Math. Helv., **89** (2014), no. 3, 537–570.

[KM93] P. B. Kronheimer and T. S. Mrowka, *Gauge theory for embedded surfaces. I*, Topology, **32** (1993), no. 4, 773–826. 1241873 (94k:57048)

[MO08] Ciprian Manolescu and Peter Ozsváth, *On the Khovanov and knot Floer homologies of quasi-alternating links*, Proceedings of Gökova Geometry-Topology Conference 2007, Gökova Geometry/Topology Conference (GGT), Gökova, 2008, pp. 60–81.

[NW15] Yi Ni and Zhongtao Wu, *Cosmetic surgeries on knots in S^3*, J. Reine Angew. Math., **706** (2015), 1–17.

[NW14] Y. Ni and Z. Wu, *Heegaard Floer correction terms and rational genus bounds*, Adv. Math., **267** (2014), 360–380.

[OS03a] Peter Ozsváth and Zoltán Szabó, *Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary*, Adv. Math., **173** (2003), no. 2, 179–261.

[OS03b] Peter Ozsváth and Zoltán Szabó, *Heegaard Floer homology and alternating knots*, Geom. Topol., **7** (2003), 225–254.

[OS03c] Peter Ozsváth and Zoltán Szabó, *Knot Floer homology and the four-ball genus*, Geom. Topol., **7** (2003), 615–639.

[OS04a] Peter Ozsváth and Zoltán Szabó, *Holomorphic disks and knot invariants*, Adv. Math., **186** (2004), no. 1, 58–116.

[OS04b] Peter Ozsváth and Zoltán Szabó, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. of Math. (2), **159** (2004), no. 3, 1027–1158.

[OS05] Peter Ozsváth and Zoltán Szabó, *On knot Floer homology and lens space surgeries*, Topology, **44** (2005), no. 6, 1281–1300.

[OS11] Peter S. Ozsváth and Zoltán Szabó, *Knot Floer homology and rational surgeries*, Algebr. Geom. Topol., **11** (2011), no. 1, 1–68.

[Ras03] Jacob Rasmussen, *Floer homology and knot complements*, Ph.D. thesis, Harvard University, 2003.

[Shi71] Yaichi Shinohara, *On the signature of knots and links*, Trans. Amer. Math. Soc., **156** (1971), 273–285.

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