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On rational sliceness of Miyazaki's fibered, −amphicheiral knots

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Abstract

We prove that fibered, −amphicheiral knots with irreducible Alexander polynomials are rationally slice. This contrasts with the result of Miyazaki that $(2n, 1)$-cables of these knots are not ribbon. We also show that the concordance invariants ν^+ and Υ from Heegaard Floer homology vanish for a class of knots that includes rationally slice knots. In particular, the ν^+- and Υ-invariants vanish for these cable knots.

1. *Introduction*

Recall that a knot $K \subset S^3$ is called *slice* if it bounds an embedded disk D in D^4, and it is called *ribbon* if it bounds an immersed disk in S^3 with only ribbon singularities. (The present paper considers only the smooth category unless otherwise specified.) One easily sees that every ribbon knot is a slice knot. An outstanding open problem, posed by Fox and known as the *slice-ribbon conjecture*, asks if the converse is true. As an attempt to approach the problem, Casson and Gordon introduced the notion of *homotopy ribbon knots* in [**6**]. A knot K is homotopy ribbon if it bounds an embedded disk D in a homotopy 4-ball V so that the inclusion induced map

$$\pi_1(S^3 \smallsetminus K) \to \pi_1(V \smallsetminus D)$$

is surjective. Since every ribbon knot is homotopy ribbon, the slice-ribbon problem can be divided into two parts, namely whether every slice knot is homotopy ribbon, and whether every homotopy ribbon knot is ribbon [**27**, Problem 4.22].

In [**6**, Theorem 5.1], it is proved that a fibered knot is homotopy ribbon if and only if the monodromy of its fiber extends over the handlebody bounded by the fiber. Hinging on this theorem, several non-homotopy ribbon knots have been constructed (for example, see [**1**, **2**, **4**, **14**, **33**, **34**]). Most of these examples are algebraically slice, and detecting their non-sliceness is an interesting problem.

Specifically, Miyazaki [**34**] showed that the connected sum of iterated torus knots

$$T_{2,3;2,13} \# T_{2,15} \# -T_{2,3;2,15} \# -T_{2,13}$$

is algebraically slice but not homotopy ribbon [**34**, Example 1]. Its algebraic sliceness is proved by Livingston and Melvin [**31**]. Around two decades later, using Casson–Gordon invariants, Hedden, Kirk and Livingston [**17**] showed that $T_{2,3;2,13} \# T_{2,15} \# -T_{2,3;2,15} \# -T_{2,13}$ is not even topologically slice. Our paper will center around another collection of algebraically slice, non-homotopy ribbon fibered knots considered by Miyazaki in [**34**, Example 2]. Their construction is based on the following specific family of knots that we will refer to as *Miyazaki knots* for the rest of the paper.

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DEFINITION 1. A *Miyazaki knot* K is a fibered, −amphicheiral knot with irreducible Alexander polynomial $\Delta_K(t)$.

The precise definition of a −amphicheiral knot is given in Definition 3. We remark that the figure-eight knot 4_1 is a Miyazaki knot. We give more examples of satellite Miyazaki knots in Section 4.

EXAMPLE 1 [**34**, Example 2]. For any Miyazaki knot K, its $(2n, 1)$-cable $K_{2n,1}$ is algebraically slice, but not homotopy ribbon for any $n \neq 0$.

Note that if any $K_{2n,1}$ is slice, then this knot will be a counterexample to the slice-ribbon conjecture. In view of the slice-ribbon conjecture, one asks the following open question. The special case of the (2,1)-cable of 4_1 is asked by Kawauchi in [**25**].

QUESTION 1 (Kawauchi). Is $K_{2n,1}$ slice when K is Miyazaki? In particular, is the (2,1)-cable of 4_1 a slice knot?

By the slice-ribbon conjecture, for any Miyazaki knot K, it is expected that its $(2n, 1)$-cables $K_{2n,1}$ are not slice. In contrast, our first main result, Theorem 1.1, shows that the $(2n, 1)$ cables of any Miyazaki knot K are *rationally* slice.

To state and motivate our results, we first introduce some terminology, and discuss previously known results.

DEFINITION 2 (Rationally slice knots). We say a knot $K \subset S^3$ is *rationally slice* if there exists an embedded disk D in a rational homology 4-ball V such that $\partial(V, D) = (S^3, K)$. In this situation, the knot K is called *strongly rationally slice* if, in addition, the following inclusion induced map is an isomorphism

$$H_1(S^3 \smallsetminus K; \mathbb{Z})/\text{torsion} \longrightarrow H_1(V \smallsetminus D; \mathbb{Z})/\text{torsion}.$$

A standard Thom-Pontrjagin argument of [**29**] can be easily adapted to show that any strongly rationally slice knot is algebraically slice.

EXAMPLE 2. It is known that 4_1 is rationally slice (for example, see [**7**, Theorem 4.16]), but 4_1 is *not* strongly rationally slice since 4_1 is not algebraically slice.

We remark that our notion of rational sliceness is different from the one used in [**25**, **26**], but coincides with the one used in recent literatures including [**7**, **11**, **36**]. (In [**25**, **26**], rationally slice knots and strongly rationally slice knots in our sense are called *weakly rationally slice* and *rationally slice*, respectively.)

In his unpublished manuscript [**25**], Kawauchi showed that the (2,1)-cable of 4_1 is strongly rationally slice. In [**26**, Corollary 1.2], Kawauchi showed a more general statement that the $(2n, 1)$-cables of any strongly −amphicheiral knot are strongly rationally slice. (See the precise definition of a strongly −amphicheiral knot in Definition 3.) We remark that Hartley [**16**] found examples of −amphicheiral knots that are not strongly −amphicheiral, so there is a subtle difference between the two notions of −amphicheirality and strongly −amphicheirality.

It was implicit in the proof of [**26**, Corollary 1.2] that strongly −amphicheiral knots are rationally slice. (For the reader's convenience, we include in Lemma 3.1 an adaptation of Kawauchi's arguments that prove this fact.) It is unknown whether any general −amphicheiral knot is rationally slice or not. Our first main result is that any Miyazaki knot is rationally slice, and its $(2n, 1)$ cables are strongly rationally slice.

THEOREM 1.1. *Let K be a Miyazaki knot. Then K is rationally slice, and its $(2n, 1)$-cables $K_{2n,1}$ are strongly rationally slice.*

In fact, we prove a stronger statement that K is $\mathbb{Z}[\frac{1}{2}]$-slice and $K_{2n,1}$ are strongly $\mathbb{Z}[\frac{1}{2}]$-slice where $\mathbb{Z}[\frac{1}{2}]$ is the subring of $\mathbb{Q}$ generated by $\frac{1}{2}$. Since it is straightforward to generalize Definition 2 to (strong) R-sliceness for any subring R of $\mathbb{Q}$, we do not spell out the precise definition.

We give some remarks on the proof of Theorem 1.1 to clarify our input. Kawauchi showed in [**24**] that hyperbolic $-$amphicheiral knots are strongly $-$amphicheiral knots. Hence, Theorem 1.1 for hyperbolic Miyazaki knots follows from [**26**, Corollary 1.2]. By Thurston's uniformization theorem for Haken manifolds [**38**], every knot is either hyperbolic, a torus knot, or a satellite knot. Since torus knots are not $-$amphicheiral, we only need to consider satellite knots. In order to prove Theorem 1.1, we give a nice description of satellite Miyazaki knots in Lemma 3.2; namely, we show that a companion of any satellite Miyazaki knot is also a Miyazaki knot with smaller genus, and a pattern of any satellite Miyazaki knot is unknotted and fibered. (We give infinitely many examples of satellite Miyazaki knots in Section 4.)

Recall from Example 1 that the $(2n, 1)$-cables of any Miyazaki knot are not ribbon. Hinging on results in [**34**], we show that any non-trivial linear combination of the $(2n, 1)$-cables of any fixed Miyazaki knot are not ribbon.

THEOREM 1.2. *For any Miyazaki knot K, any non-trivial linear combination of the $(2n, 1)$-cables of K is not ribbon.*

REMARK 1. It is interesting to compare Theorem 1.2 with the result of Abe and Tagami [**1**, Lemma 3.1] based on the work of Baker [**2**] that any non-trivial linear combination of tight, prime fibered knots is not ribbon.

We can relate Theorem 1.2 with the rational knot concordance group. Let $\mathcal{C}$ and $\mathcal{C}_{\mathbb{Q}}$ be the knot concordance group and the rational knot concordance group. That is, $\mathcal{C}_{\mathbb{Q}}$ is the set of rational concordance classes of knots in rational homology 3-spheres (for an excellent survey of the structure of $\mathcal{C}_{\mathbb{Q}}$, see [**7**]). There is a natural, inclusion induced map $\mathcal{C} \to \mathcal{C}_{\mathbb{Q}}$ from the smooth concordance group to the rational concordance group. It is natural to study the structures of the kernel and the cokernel of the map $\mathcal{C} \to \mathcal{C}_{\mathbb{Q}}$ which measure the subtle difference between knot concordance and rational knot concordance. Up to now, it is known that $\mathrm{Ker}(\mathcal{C} \to \mathcal{C}_{\mathbb{Q}})$ contains a $\mathbb{Z}_2^\infty$-subgroup and $\mathrm{Coker}(\mathcal{C} \to \mathcal{C}_{\mathbb{Q}})$ contains a $\mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty$ subgroup (see [**7**, Theorem 1.4]). Hence, finding a $\mathbb{Z}^\infty$-subgroup in $\mathrm{Ker}(\mathcal{C} \to \mathcal{C}_{\mathbb{Q}})$ is an intriguing open problem. By Theorem 1.2, an affirmative answer to this problem is a consequence of the slice-ribbon conjecture.

COROLLARY 1.3. *Assume that the slice-ribbon conjecture is true. If K is a Miyazaki knot, then the set $\{K_{2n,1}\}_{n=1}^\infty$ generates a $\mathbb{Z}^\infty$-subgroup of $\mathrm{Ker}(\mathcal{C} \to \mathcal{C}_{\mathbb{Q}})$.*

REMARK 2. Let $\mathcal{C}^{\mathrm{top}}$ and $\mathcal{C}_{\mathbb{Q}}^{\mathrm{top}}$ be the topological knot concordance group and the topological rational knot concordance group, respectively. The same proof works for the following analogous statement. For any Miyazaki knot K, if the topologically slice-homotopy ribbon conjecture is true, then any non-trivial linear combination of knots in $\{K_{2n,1}\}_{n=1}^\infty$ is not slice, and hence $\{K_{2n,1}\}_{n=1}^\infty$ generates a $\mathbb{Z}^\infty$-subgroup of $\mathrm{Ker}(\mathcal{C}^{\mathrm{top}} \to \mathcal{C}_{\mathbb{Q}}^{\mathrm{top}})$ for any Miyazaki knot K. For a survey and the state of the art for the topologically slice-homotopy ribbon conjecture, see [**10**].

One may attempt to prove the non-sliceness of cables of Miyazaki knots. This leads us to look at their concordance invariants coming from Heegaard Floer theory. We will be primarily working with the ν^+-invariant [**23**], as it gives the strongest obstruction to sliceness among several closely related invariants, including the invariants τ, ε, and Υ. For a detailed description and comparison of concordance invariants from Heegaard Floer theory, we refer the reader to Hom's survey article [**22**].

In [**39**], the second author gave a cabling formula for the ν^+-invariants, and the formula implies that $\nu^+(K) = 0$ if and only if $\nu^+(K_{2n,1}) = 0$ for all n. Hence, if we can find a Miyazaki knot K with non-vanishing ν^+, then the knots $K_{2n,1}$ would be non-slice since they also have non-vanishing ν^+, giving a negative answer to the aforementioned question due to Kawauchi. This observation motivated the study of the ν^+-invariant of Miyazaki knots. Combined with Theorem 1.1, our second main result, Theorem 1.4, establishes that the ν^+-invariant vanishes for any Miyazaki knot.

Theorem 1.4. *Let K be a rationally slice knot. Then*

(i) $\nu^+(K) = \nu^+(-K) = 0$*;*
(ii) $\nu^+(P(K)\# - P(U)) = \nu^+(-P(K)\#P(U)) = 0$ *for all patterns P;*
(iii) $\nu^+(P(K)) = \nu^+(P(U))$ *for all patterns P.*

In particular, the ν^+-invariant vanishes for any Miyazaki knot and any strongly $-$amphicheiral knot.

As far as the authors know, it is unknown whether the ν^+-invariant vanishes for $-$amphicheiral knots, or more generally, knots of finite concordance order. (In contrast, τ, ε, and Υ of these knots vanish for elementary reasons.) Theorem 1.4 indicates that such examples with non-vanishing ν^+ must be rare because of the vanishing results for strongly $-$amphicheiral knots. We remark that this is related to the botany problem of knot Floer homology [**18**] since there is a bifiltered chain complex C such that $\nu^+(C) \neq 0$, but $\nu^+(C \otimes C) = 0$, (for example, see [**21**, Figure 3]).

Actually, in the proof of Theorem 1.4, we show that the same conclusions hold under a weaker assumption that K is rationally 0-bipolar. For the definition of a rationally 0-bipolar knot, see Definition 4. Since there is a non-slice knot (for example, 4_1) which is rationally 0-bipolar, Theorem 1.4 establishes that the collection of the ν^+-invariants of all the satellites of a knot cannot detect its sliceness. This result is inspired by recent work of Cha and the first named author [**8**] where an analogous statement for the Rasmussen s-invariant is proved.

We finish the introduction by posing two questions for future research, whose answers are now known to be true for strongly $-$amphicheiral knots by the work of this paper.

Question 2. Are $-$amphicheiral knots rationally slice? More generally, are knots of finite concordance order rationally slice?

Question 3. Does the ν^+-invariant vanish for $-$amphicheiral knots? More generally, does the ν^+-invariant vanish for knots of finite concordance order?

The remainder of this paper is organized as follows. In Section 2, we discuss the amphicheirality, or more generally, the symmetry of knots and links by recalling several results of Hartley [**15**, **16**]. We will relate the symmetry of a satellite knot with the symmetry of its companion and pattern, and also relate the symmetry of a knot in a solid torus with the symmetry of an associated 2-component link. These relationships will be crucial to the understanding of satellite Miyazaki knots. In Section 3, we show that every Miyazaki knot is either hyperbolic or a satellite knot with a Miyazaki companion with smaller genus. Subsequently, we apply an inductive argument to prove the main result, Theorem 1.1, on rational sliceness of Miyazaki knots. We also prove Theorem 1.2 in Section 3. We then exhibit an infinite family of satellite

Miyazaki knots in Section 4, and make a digression to observe that all known examples are strongly $-$amphicheiral. Finally, in Section 5, we take a different point of view and use Heegaard Floer homology to study Miyazaki knots, proving Theorem 1.4. The discussion in this section constitutes a rather independent unit of the paper.

2. *Preliminaries on $-$amphicheiral knots*

In this section, we survey results on $-$amphicheiral knots that will be a key ingredient for proving Theorem 1.1. First, we define $-$amphicheiral knots and strongly $-$amphicheiral knots.

Definition 3. A knot K in S^3 is *$-$amphicheiral* if there exists an orientation reversing homeomorphism $f\colon (S^3,K)\to(S^3,K)$ such that $f(K)$ is K with the reversed orientation. A knot K in S^3 is *strongly $-$amphicheiral* if we can choose f to be an involution.

Let $L=L_1\sqcup\cdots\sqcup L_n$ be a link in S^3. More generally, we say that (S^3,L) *has symmetry* $(\alpha,\varepsilon_1,\ldots,\varepsilon_n)$ if there exists a self-homeomorphism f of S^3 of class α that restricts to a self-homeomorphism of each component L_i of class ε_i for each i. Here, α takes the value ±1 or $\iota_\pm$, which stands for orientation preserving/reversing homeomorphisms or involutions of S^3, respectively; and $\varepsilon_i=\pm1$ depending on whether $f|_{L_i}$ preserves or reverses orientation of L_i. In particular, a knot is (S^3,K) is $-$amphicheiral if it has symmetry $(-1,-1)$, and it is strongly $-$amphicheiral if it has symmetry $(\iota_-,-1)$.

For the purpose of this article, we will primarily focus on $-$amphicheiral satellite knots. Recall that a *pattern* P is a knot in $S^1\times D^2$. For any knot J, let $P(J)$ denote a satellite knot K with pattern $(S^1\times D^2,P)$ and companion J. For a pattern $P\subset S^1\times D^2$, let *the associated link* of P be a link $(S^3,\mu_P\sqcup P(U))$ where μ_P denotes a meridian of the ambient solid torus $S^1\times D^2$. In particular, the winding number of P is the linking number $\mathrm{lk}(\mu_P,P(U))$.

It turns out that the symmetry of a satellite knot $P(J)$ is almost completely determined by the symmetries of P and J, as follows. Following the notation of Hartley [**16**], we say a pattern $P\subset S^1\times D^2$ has symmetry $([\alpha,\varepsilon_1],\varepsilon_2)$ if there exists a self-homeomorphism of the solid torus of class α that maps the longitude class $[\lambda]$ to $\varepsilon_1[\lambda]$ and restricts to a self-homeomorphism of K of the class ε_2. As before, α takes the value ±1 or $\iota_\pm$ that stands for orientation preserving/reversing homeomorphisms or involutions of the solid torus $S^1\times D^2$, respectively; and ε_1, ε_2 take the value ±1.

We are now in a position to state [**16**, Theorem 4.1(1)], which relates the symmetry of a satellite knot $P(J)$ with the symmetry of the companion J and the pattern P.

Theorem 2.1 [**16**, Theorem 4.1(1)]. *Suppose P is a pattern and J is a non-trivial prime knot such that neither J nor its mirror image is a companion of $P(U)$. Let $\alpha=\pm1$ or $\iota_\pm$ and $\varepsilon=\pm1$. Then $(S^3,P(J))$ has symmetry (α,ε) if and only if (S^3,J) has symmetry (α,ε_1) and $(S^1\times D^2,P)$ has symmetry $([\alpha,\varepsilon_1],\varepsilon)$ for some $\varepsilon_1=\pm1$.*

A pattern $P\subset S^1\times D^2$ is called an *unknotted pattern* if $P(U)$ is an unknot. The mild technical condition of Theorem 2.1 is satisfied for all unknotted patterns. We sketch the proof of Theorem 2.1 for the reader's convenience. We deform a given homeomorphism f of $(S^3,P(J))$ of symmetry (α,ε) to an isotopic homeomorphism f' which fixes J and a tubular neighborhood V of it. Suppose f' maps $[\lambda]$ to $\varepsilon_1[\lambda]$, then the induced self-homeomorphism $f'|_V$ has the symmetry $([\alpha,\varepsilon_1],\varepsilon)$ on $(V,K)\cong(S^1\times D^2,P)$. Since f' also fixes J which is in the same class of the longitude λ, the self-homeomorphism f' realizes the symmetry (α,ε_1) for (S^3,J). The converse of Theorem 2.1 can be proved in a similar manner, and we refer to Hartley's original manuscript [**16**].

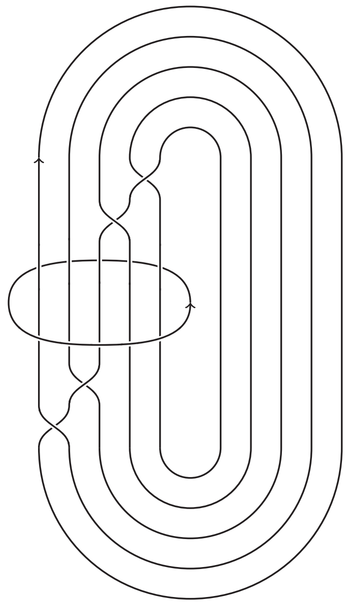

FIGURE 1. *The associated link of a pattern P where P is the closure of a braid of the form $\sigma_4\sigma_3^{-1}\sigma_2\sigma_1^{-1}$. It is an unknotted pattern and has symmetry* $([\iota_-, -1], -1)$.

EXAMPLE 3. As an application of Theorem 2.1, we prove that the satellite knot $K = P(J)$ is strongly −amphicheiral, where J is the figure-eight knot and P is the pattern whose associated link $\mu_P \sqcup P(U)$ is depicted in Figure 1. In [**30**, p. 371], Livingston showed that the link has symmetry $([\iota_-, -1], -1)$. Since (S^3, J) has symmetry $(\iota_-, -1)$, $K = P(J)$ is strongly −amphicheiral by Theorem 2.1. We can generalize this construction by considering an infinite family of the 3-braids of the form $\prod_{i=0}^{2n} \sigma_{2n-i}^{(-1)^i}$ since the same argument works (compare [**30**, Figure 13]).

In [**16**, Theorem 4.1(3)], Hartley relates the symmetry of a pattern $(S^1 \times D^2, P)$ with the symmetry of the associated link $(S^3, \mu_P \sqcup P(U))$ of P. The most relevant to our purpose is the following special case.

LEMMA 2.2 [**16**, Theorem 4.1(3)]. *Suppose P is a pattern and $\varepsilon_i = \pm 1$ for $i = 0, 1, 2$. Then $(S^1 \times D^2, P)$ has symmetry $([\varepsilon_0, \varepsilon_1], \varepsilon_2)$ if and only if its associated link $(S^3, \mu_P \sqcup P(U))$ has symmetry $(\varepsilon_0, \varepsilon_0\varepsilon_1, \varepsilon_2)$. Similarly, $(S^1 \times D^2, P)$ has symmetry $([\iota_-, \varepsilon_1], \varepsilon_2)$ if and only if $(S^3, \mu_P \sqcup P(U))$ has symmetry $(\iota_-, -\varepsilon_1, \varepsilon_2)$*

The following lemma gives a simple criterion on symmetries of a pattern $(S^1 \times D^2, P)$ with non-trivial winding number.

LEMMA 2.3. *Suppose P is a pattern with non-zero winding number. If $(S^1 \times D^2, P)$ has symmetry $([-1, \varepsilon_1], \varepsilon_2)$, then $\varepsilon_1\varepsilon_2 = 1$.*

Proof. By Lemma 2.2, $(S^1 \times D^2, P)$ has symmetry $([-1, \varepsilon_1], \varepsilon_2)$ if and only if its associated link $(S^3, \mu_P \sqcup P(U))$ has symmetry $(-1, -\varepsilon_1, \varepsilon_2)$. To obtain a contradiction, assume that $\varepsilon_1\varepsilon_2 = -1$. Since $(S^3, \mu_P \sqcup P(U))$ has symmetry $(-1, -\varepsilon_1, \varepsilon_2)$,

$$\operatorname{lk}(\mu_P, P(U)) = -\operatorname{lk}(-\epsilon_1\mu_P, \epsilon_2 P(U)) = -\operatorname{lk}(\mu_P, P(U)),$$

and hence $\mathrm{lk}(\mu_P, P(U)) = 0$. This is a contradiction to our assumption that the winding number of P is non-zero since the winding number of P is equal to $\mathrm{lk}(\mu_P, P(U)) = 0$. □

3. *Miyazaki knots are rationally slice*

The goal of this section is to prove Theorem 1.1. We begin with a lemma that is built on work of Kawauchi [**24**, **26**]. Recall that $\mathbb{Z}[\frac{1}{2}]$ is the subring of $\mathbb{Q}$ generated by $\frac{1}{2}$.

LEMMA 3.1. *Suppose that K is a knot in S^3.*

(i) *If K is $-$amphicheiral and hyperbolic, then K is strongly $-$amphicheiral.*
(ii) *If K is strongly $-$amphicheiral, then K is $\mathbb{Z}[\frac{1}{2}]$-slice, and hence K is rationally slice.*
(iii) *If K is strongly $-$ amphicheiral, then $Q(K)$ is strongly $\mathbb{Z}[\frac{1}{2}]$-slice, and hence strongly rationally slice for any pattern Q such that $Q(U)$ is slice and the winding number of Q is even.*

Proof. (i) and (iii) are special cases of [**24**, Lemma 1; **24**, Theorem 1.1], respectively. We sketch below a proof of (ii), which is adapted from [**26**, Lemma 2.3].

Suppose K is strongly $-$amphicheiral. That is, there is an orientation reversing involution $\tau\colon S^3 \to S^3$ such that $\tau(K) = K$ and $\mathrm{Fix}(\tau) = S^0 \subset K$. Let M be the 0-surgery manifold of S^3 along K. We first observe that τ naturally extends to an orientation reversing, free involution $\tau_M\colon M \to M$. Choose an oriented longitude λ of K and an oriented meridian μ of K. Choose a τ-invariant neighborhood $\nu(K)$ of K. Since $\tau(K) = K$ and $\mathrm{Fix}(\tau) = S^0 \subset K$, $\tau|_{\partial\nu(K)}$ is an orientation reversing free involution such that $\tau|_{\partial\nu(K)}$ sends the homology classes $[\lambda]$ and $[\mu]$ to $-[\lambda]$ and $[\mu]$. Therefore, $\tau|_{\partial\nu(K)}(\lambda)$ and $\tau|_{\partial\nu(K)}(\mu)$ are isotopic to $-\lambda$ and μ, respectively. Note that $M = S^3 \smallsetminus \nu(K) \cup_i S^1 \times D^2$ where the gluing map $i\colon \partial\nu(K) \to S^1 \times D^2$ is given by $i(\lambda) = \partial D^2$ and $i(\mu) = S^1$. Since $\tau|_{\partial\nu(K)}(\lambda) = -\lambda$ and $\tau|_{\partial\nu(K)}(\mu) = \mu$, $\tau|_{S^3\smallsetminus\nu(K)}$ extends to an orientation reversing, free involution $\tau_M\colon M \to M$.

Let M_τ be the quotient space $M/{\sim}$ where $x \sim \tau_M(x)$ for all $x \in M$. Since τ is orientation reversing, M_τ is a non-orientable 3-manifold. Since τ_M is an orientation reversing involution, the quotient map $\pi\colon M \to M_\tau$ is the orientation double cover. Let W be the twisted I-bundle over M_τ. That is, W is the mapping cylinder of $\pi\colon M \to M_\tau$ and $\partial W = M$. Note that M_τ is a deformation retract of W and hence $H_*(M_\tau;\mathbb{Z}) \cong H_*(W;\mathbb{Z})$. In the proof of [**26**, Lemma 2.3], it is proved that $H_*(W;\mathbb{Q}) = H_*(S^1;\mathbb{Q})$ but we will prove a stronger statement that $H_*(W;\mathbb{Z}[\frac{1}{2}]) \cong H_*(S^1;\mathbb{Z}[\frac{1}{2}])$. Since M_τ is a connected and non-orientable 3-manifold,

$$H_i(W;\mathbb{Z}) \cong H_i(M_\tau;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ 0 & \text{if } i = 3, 4. \end{cases}$$

In [**26**, Lemma 2.3], it is proved that $H_1(W, M;\mathbb{Z}) = \mathbb{Z}_2$ and $H_1(W;\mathbb{Q}) = \mathbb{Q}$. As M is the 0-surgery manifold of S^3 along K, $H_1(M;\mathbb{Z}) = \mathbb{Z}$. From the homology long exact sequence of a pair (W, M), we have an exact sequence

$$\mathbb{Z} \longrightarrow H_1(W;\mathbb{Z}) \longrightarrow \mathbb{Z}_2 \longrightarrow 0.$$

Since $H_1(W;\mathbb{Q}) = \mathbb{Q}$, the exact sequence gives $H_1(W;\mathbb{Z}) = \mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}_2$. Since $\mathbb{Z}[\frac{1}{2}]$ is a torsion free, abelian group, $\mathbb{Z}[\frac{1}{2}]$ is a flat $\mathbb{Z}$-module. Hence, $H_*(-;\mathbb{Z}[\frac{1}{2}]) \cong H_*(-;\mathbb{Z}) \otimes \mathbb{Z}[\frac{1}{2}]$ by the universal coefficient theorem. It follows that

$$H_i(W;\mathbb{Z}[\tfrac{1}{2}]) \cong H_*(W;\mathbb{Z}) \otimes \mathbb{Z}[\tfrac{1}{2}] = \begin{cases} \mathbb{Z}[\frac{1}{2}] & \text{if } i = 0, 1 \\ 0 & \text{if } i = 3, 4. \end{cases}$$

Now, we prove that $H_2(W;\mathbb{Z}[\frac{1}{2}]) = 0$. Let p be an odd prime. From the universal coefficient theorem,

$$H_i(W;\mathbb{Z}_p) = \begin{cases} \mathbb{Z}_p & \text{if } i = 0, 1 \\ 0 & \text{if } i = 3, 4. \end{cases}$$

As mentioned above, $H_*(W;\mathbb{Q}) \cong H_*(S^1;\mathbb{Q})$, so the Euler characteristic $\chi(W)$ is zero. Therefore,

$$0 = \chi(W) = \sum_{i=0}^{4} (-1)^i \dim_{\mathbb{Z}_p} H_i(W;\mathbb{Z}_p) = \dim_{\mathbb{Z}_p} H_2(W;\mathbb{Z}_p)$$

which implies that $H_2(W;\mathbb{Z}_p) = 0$ for any odd prime p. Recall that $H_2(W;\mathbb{Q}) \cong H_2(S^1;\mathbb{Q}) = 0$. This implies that $H_2(W;\mathbb{Z})$ is a torsion abelian group. For an odd prime p, by the universal coefficient theorem, $0 = H_2(W;\mathbb{Z}_p) \cong H_2(W;\mathbb{Z}) \otimes \mathbb{Z}_p$. From this, we see that the order of $H_2(W;\mathbb{Z})$ is a power of 2 and $H_2(W;\mathbb{Z}[\frac{1}{2}]) = H_2(W;\mathbb{Z}) \otimes \mathbb{Z}[\frac{1}{2}] = 0$.

In summary, we have observed that if K is a strongly −amphicheiral knot in S^3, then the 0-surgery manifold M bounds a 4-manifold W such that $H_*(W;\mathbb{Z}[\frac{1}{2}]) \cong H_*(S^1;\mathbb{Z}[\frac{1}{2}])$. It is well known that the existence of such a W is equivalent to the condition that K is $\mathbb{Z}[\frac{1}{2}]$-slice (see [**12**, Proposition 1.5]). From the universal coefficient theorem, it is easy to see that K is also rationally slice. This completes the proof of (ii). □

Our next lemma concerns satellite Miyazaki knots.

Lemma 3.2. *If a satellite knot $K = P(J)$ is Miyazaki, then P is an unknotted, fibered pattern, J is Miyazaki, and $g(J) < g(K)$.*

Assuming Lemma 3.2, we prove Theorem 1.1.

Proof of Theorem 1.1. Suppose that K is a non-trivial Miyazaki knot. In other words, K is fibered, −amphicheiral with $\Delta_K(t)$ irreducible. We show a slightly general statement that K is rationally slice, and $Q(K)$ is strongly rationally slice for any pattern Q such that $Q(U)$ is slice and the winding number of Q is even. (The original conclusion follows as a special case that Q is the $(2n, 1)$-cable pattern.) Throughout the proof, Q denotes an arbitrary pattern such that $Q(U)$ is slice and the winding number of Q is even.

We first observe that Theorem 1.1 holds for any hyperbolic Miyazaki knot. Suppose that the Miyazaki knot K is hyperbolic. In this case, by Lemma 3.1(i), K is strongly −amphicheiral. It follows from (ii) and (iii) of Lemma 3.1 that K is rationally slice, and $Q(K)$ is strongly rationally slice.

Recall the following facts mentioned in the introduction.

(1) Every knot is either a hyperbolic knot, a torus knot, or a satellite knot.
(2) Non-trivial torus knots are not −amphicheiral.

Hence, it remains to prove Theorem 1.1 for satellite Miyazaki knots.

We now use an induction on $g(K)$. Suppose that $g(K) = 1$. By the classification of genus 1 fibered knots (for example, [**5**, Proposition 5.14]), K is either the trefoil knot or the figure-eight knot. Since K is −amphicheiral, K is the figure-eight knot. Since the figure-eight knot is hyperbolic, we already have shown that Theorem 1.1 holds for this case.

Suppose that K is a satellite knot such that $g(K) > 1$. As an induction hypothesis, we have that if K' is Miyazaki and $g(K') < g(K)$, then K' is $\mathbb{Z}[\frac{1}{2}]$-slice, and $Q'(K')$ is strongly rationally slice for any pattern Q' such that $Q'(U)$ is slice, and the winding number of Q'

is even. If $K = P(J)$, then P is an unknotted pattern, and J is Miyazaki knot such that $g(J) < g(K)$ by Lemma 3.2.

By the induction hypothesis, we have that the knot J is $\mathbb{Z}[\frac{1}{2}]$-slice, that is, J bounds a slice disk Δ in a $\mathbb{Z}[\frac{1}{2}]$-homology 4-ball V. The subsequent construction of a slice disk for K in V is standard: Since $P \subset S^1 \times D^2$ is an unknotted pattern, regarding $P \subset \partial(D^2 \times D^2)$, P bounds a disk $D \subset D^2 \times D^2$. As the tubular neighborhood $\nu(\Delta) \cong \Delta \times D^2$ of Δ in V is diffeomorphic to $D^2 \times D^2$, the image of the disk $D \subset D^2 \times D^2 \cong \nu(\Delta) \subset V$ under the above diffeomorphism is then the desired slice disk for $K = P(J) \subset V$. This shows that K is $\mathbb{Z}[\frac{1}{2}]$-slice.

It remains to show that $Q(K)$ is strongly rationally slice for any pattern Q such that $Q(U)$ is slice, and the winding number of Q is even. Fix such a pattern Q. Now we can regard $Q(K) = Q(P(J))$ as a satellite knot with companion J and pattern $Q \circ P$. Here, the pattern $Q \circ P$ is characterized by the property that $Q \circ P(L)$ is isotopic to $Q(P(L))$ for any knot L.

Since P is an unknotted pattern, $Q \circ P(U)$ is isotopic to $Q(U)$, and hence slice. Note that the winding number of $Q \circ P$ is even since it is a multiple of the winding number of Q which is also even. Therefore, by the induction hypothesis, $Q(K) = Q \circ P(J)$ is strongly $\mathbb{Z}[\frac{1}{2}]$-slice. This completes the proof. □

The proof of Lemma 3.2 is the most technical part of the paper. In the course of the proof, we will make use of results on −amphicheiral, satellite knots that we discussed earlier in Section 2, and the following criterion for fibered, satellite knots (see, for example, [**19**, Theorem 1]).

LEMMA 3.3 (Criterion for fibered, satellite knots). *A satellite knot $K = P(J)$ is fibered if and only if both the companion knot J and the pattern P are fibered.*

We will also need the following lemma.

LEMMA 3.4. *If K is a fibered knot with irreducible Alexander polynomial, then K is a prime knot. In particular, Miyazaki knots are prime.*

Proof. Assume to the contrary that $K = K_1 \# K_2$, the connected sum of two non-trivial knots K_1 and K_2. Since K is fibered, K_1 and K_2 are also fibered by Lemma 3.3. (Connected sum is a satellite operation.) As the genus of a fibered knot is half the degree of its Alexander polynomial, we have $\deg \Delta_{K_i}(t) = 2g(K_i) > 0$ for $i = 1, 2$. This implies that the Alexander polynomial $\Delta_K(t) = \Delta_{K_1}(t)\Delta_{K_2}(t)$ is the product of two non-constant polynomials, thus contradicting the irreducibility of the Alexander polynomial $\Delta_K(t)$. Hence K must be prime. □

Now we prove Lemma 3.2.

Proof of Lemma 3.2. Let K be a satellite, Miyazaki knot. That is, K is satellite, fibered, and −amphicheiral with $\Delta_K(t)$ irreducible. By Lemma 3.4, K is prime. Since K is a satellite knot, we can write $K = P(J)$ such that J is a non-trivial knot and the pattern P is not isotopic to the core of the solid torus. Since K is fibered, Lemma 3.3 implies that $(S^1 \times D^2, P)$ and J are fibered.

The winding number w of P must be non-zero, according to the proof of [**19**, Theorem 1]. It follows that $\Delta_J(t^w) \neq 1$. We have a cabling formula $\Delta_K(t) = \Delta_{P(U)}(t)\Delta_J(t^w)$ where w is the winding number. Since P and J are fibered, $\deg \Delta_{P(U)}(t) = 2g(P)$ and $\deg \Delta_J(t) = 2g(J)$. From the irreducibility of $\Delta_K(t)$ and $\Delta_J(t^w) \neq 1$, we conclude that $\Delta_{P(U)}(t) = 1$ and hence $P(U)$ is the unknot.

After a possible simultaneous change of the orientations of P and K, we can assume that the winding number w is positive. By [**19**, Corollary 1], if P is a winding number 1 unknotted pattern such that $(S^1 \times D^2, P)$ is fibered, then P is isotopic to the core of $S^1 \times D^2$. Therefore, $|w| \geqslant 2$. From the cabling formula and fiberedness, $g(K) = |w|g(J) > g(J)$. Since $\Delta_K(t)$ is irreducible, $\Delta_J(t^w) = \Delta_K(t)$ implies that $\Delta_J(t)$ is also irreducible.

It remains to prove that J is −amphicheiral. For this purpose, we apply Theorem 2.1. We first check that J and P satisfies the hypothesis of Theorem 2.1. In the beginning of the proof, we proved that a fibered knot with irreducible Alexander polynomial is prime. Therefore, J is also prime. Recall that we are assuming J is non-trivial. We proved that $P(U)$ is the unknot. Therefore, neither J nor its mirror image is a companion of $P(U)$.

Since K is −amphicheiral, it has symmetry $(-1, -1)$. Theorem 2.1 then implies that (S^3, J) has symmetry $(-1, \varepsilon_1)$ and $(S^1 \times D^2, P)$ has symmetry $([-1, \varepsilon_1], -1)$ for some $\varepsilon_1 = \pm 1$. Since P is a pattern with non-zero winding number, Lemma 2.3 gives that $\varepsilon_1 = -1$. This shows that (S^3, J) has symmetry $(-1, -1)$ and hence J is −amphicheiral. This completes the proof. □

We finish this section by proving Theorem 1.2.

Proof of Theorem 1.2. Let K be a Miyazaki knot. We show that any non-trivial combination of its $(2n, 1)$-cables $K_{2n,1}$ is not ribbon as a consequence of [**34**, Theorems 8.5.1 and 8.6].

Suppose that $\#_{n=1}^{m} a_n K_{2n,1}$ is homotopy ribbon and $a_k \neq 0$ for some k. Define $\mathcal{K}$ to be the class of all iterated cables of prime fibered knots J such that there is no non-trivial Laurent polynomial $f(t)f(t^{-1})$ that divides $\Delta_J(t)$. By Lemma 3.4 and the definition of Miyazaki knots, K is prime and fibered. It follows that $K_{2n,1}$ is in $\mathcal{K}$ for any n.

Miyazaki proved in [**34**, Theorem 8.6] that if $J_1, \ldots, J_m$ are knots in $\mathcal{K}$ such that $J_1 \# \cdots \# J_m$ is homotopy ribbon, then $m = 2l$ and J_{2i-1} is isotopic to $-J_{2i}$ for all $i = 1, \ldots, l$ after a possible relabeling of indices. Note that $g(K_{2n,1}) = 2ng(K)$ for any n. Hence, $K_{2i,1}$ is not isotopic to $\pm K_{2j,1}$ if $i \neq j$. Therefore, the aforementioned result of Miyazaki [**34**, Theorem 8.6] implies that a_k is even and $K_{2k,1}$ is −amphicheiral. Thus, $K_{2n,1} \# K_{2n,1}$ is ribbon since it is isotopic to $K_{2n,1} \# - K_{2n,1}$.

In [**34**, Theorem 8.5.1] it is shown that if J is a non-trivial fibered knot and $\#_{i=1}^{m} J_{p,q}$ is homotopy ribbon, then so is J. Therefore, K is homotopy ribbon. On the other hand, the Alexander polynomial of K is irreducible since K is Miyazaki. It follows that K is not algebraically slice, and hence is not homotopy ribbon. This is a contradiction and completes the proof. □

4. *Examples of Miyazaki knots*

Our next proposition exhibits an infinite family of satellite Miyazaki knots. In particular, we will see that the knot given in Example 3 is Miyazaki. To the best of the authors' knowledge, there has been no construction of a satellite Miyazaki knot before.

Proposition 4.1. *Suppose J is the figure-eight knot, and P_n is the closure of a $(2n+1)$-braid of the form $\prod_{i=0}^{2n} \sigma_{2n-i}^{(-1)^i}$. Then the satellite knot $P_n(J)$ is Miyazaki for any positive integer n.*

Proof. Let $K = P_n(J)$ for some fixed n. Recall from Definition 1, we need to prove that K is fibered, −amphicheiral, and $\Delta_K(t)$ is irreducible.

A braid β is called *homogeneous* if each standard braid generator σ_i appears at least once in β and the exponent on σ_i has the same sign in each appearance in the braid word β (that is, if σ_i appears, then σ_i^{-1} does not appear). A theorem of Stallings [**37**, Theorem 2] says that $(S^1 \times D^2, \widehat{\beta})$ is fibered for any homogeneous braid β. Since P_n is the closure of a homogeneous braid $\prod_{i=0}^{2n} \sigma_{2n-i}^{(-1)^i}$, $(S^1 \times D^2, P_n)$ is fibered. As $J = 4_1$ is fibered, we see that K is also fibered from Lemma 3.3.

As we remarked in Example 3, $(S^1 \times D^2, P_n)$ has symmetry $([\iota_-, -1]. -1)$. Hence K has symmetry $(\iota_-, -1)$ by Theorem 2.1, that is, it is strongly $-$amphicheiral. It is straightforward to see that P_n is an unknotted pattern by explicitly drawing a diagram of $P_n(U)$. Next, we apply the cabling formula of the Alexander polynomial $\Delta_{P_n(J)}(t) = \Delta_{P_n}(t)\Delta_J(t^{2n+1})$ (the winding number is $2n+1$), so $\Delta_{P_n(J)}(t) = t^{2(2n+1)} - 3t^{2n+1} + 1$. By Lemma 4.2, $\Delta_{P_n(J)}(t)$ is irreducible. This completes the proof. $\square$

LEMMA 4.2. *For any positive integer n, $f_n(t) = t^{2(2n+1)} - 3t^{2n+1} + 1$ is irreducible.*

The proof of this lemma is purely algebraic in nature and was imparted to the authors by Jiu Kang Yu. The following standard fact about irreducible polynomial is needed, which can be found in [**28**, Theorem 9.1, p. 297].

LEMMA 4.3. *Let $\mathbb{F}$ be a field and m an odd integer. Let $a \in \mathbb{F}$, $a \neq 0$. Assume that for all prime numbers p such that $p \mid m$ we have $a \notin \mathbb{F}^p$. Then $x^m - a$ is irreducible in $\mathbb{F}[x]$.*

Proof of Lemma 4.2. Recall that $f_n(t)$ is irreducible over $\mathbb{Q}$ if and only if $\mathbb{Q}[t]/\langle f_n(t)\rangle$ is a degree $4n+2$ field extension of $\mathbb{Q}$. Let $u = (3+\sqrt{5})/2$ which is a root of the equation $x^2 - 3x + 1 = 0$. Then there is an isomorphism $\mathbb{Q}[t]/\langle f_n(t)\rangle \cong \mathbb{Q}(\sqrt{5})[x]/\langle x^{2n+1} - u\rangle$. Since $\mathbb{Q}(\sqrt{5})$ is a degree 2 field extension of $\mathbb{Q}$, $\mathbb{Q}(\sqrt{5})[x]/\langle x^{2n+1} - u\rangle$ is a degree $4n+2$ field extension of $\mathbb{Q}$ if and only if it is a degree $2n+1$ field extension of $\mathbb{Q}(\sqrt{5})$. The latter is equivalent to $x^{2n+1} - u$ is irreducible over $\mathbb{Q}(\sqrt{5})$. In short, $f_n(t)$ is irreducible over $\mathbb{Q}$ if and only if $x^{2n+1} - u$ is irreducible over $\mathbb{Q}(\sqrt{5})$.

We show that $x^{2n+1} - u$ is irreducible over $\mathbb{Q}(\sqrt{5})$. Since $[(3+\sqrt{5})/2] \cdot [(3-\sqrt{5})/2] = 1$, u is a unit of the ring of integers $\mathbb{Z}[(1+\sqrt{5})/2]$. It is well known that the unit group of $\mathbb{Z}[(1+\sqrt{5})/2]$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}_2$, and consists of the elements of the form $\pm v^k$, where $v = (1+\sqrt{5})/2$. In particular, $u = v^2$. The irreducibility of $x^{2n+1} - u$ over $\mathbb{Q}(\sqrt{5})$ then follows from Lemma 4.3. $\square$

Note that the Miyazaki knots constructed in Proposition 4.1 are strongly $-$amphicheiral. In general, remember that Miyazaki knots are $-$amphicheiral, and hyperbolic $-$amphicheiral knots are strongly $-$amphicheiral. We ask the following question.

QUESTION 4. Are Miyazaki knots always strongly $-$amphicheiral?

From the discussion in the previous section, we see that all Miyazaki knots can be obtained from hyperbolic ones via iterated satellite operations. This inspires us to look for an inductive approach, and we establish the following result along this direction.

PROPOSITION 4.4. *Suppose $K = P(J)$ is a Miyazaki knot with a hyperbolic companion J and a pattern of winding number* 3. *Then K is strongly $-$amphicheiral.*

Proof. From Lemma 3.2, we see that J is also Miyazaki and P is a fibered unknotted pattern. Lemma 3.1 then implies that J is strongly $-$amphicheiral. In light of Theorem 2.1, it suffices to prove that $(S^1 \times D^2, P)$ has symmetry $([\iota_-, -1], -1)$.

Note that P is a fibered pattern of winding number 3, and there are only three such patterns up to isotopy in $S^1 \times D^2$, corresponding to closures of the conjugacy classes of the 3-braids $\sigma_1\sigma_2$, $\sigma_1^{-1}\sigma_2^{-1}$ and $\sigma_1^{-1}\sigma_2$ [**32**]. The first two possibilities for P give cable knots $J_{3,1}$ and $J_{3,-1}$, respectively. We will prove that these knots are not Miyazaki, and thus these two cases do not occur. This leaves the third pattern as the only possibility. For this case, we observe that the closure of the braid $\sigma_1^{-1}\sigma_2$ has indeed the symmetry $([\iota_-, -1], -1)$ (see the explanation in Example 3), and this finishes the proof.

We prove a more general statement that any (p, q)-cable of a knot is not Miyazaki. Suppose that $K = J_{p,q}$ is a Miyazaki knot. That is, K is a −amphicheiral, fibered knot whose Alexander polynomial is irreducible. We can assume that $p > 1$ by considering $-K$ otherwise. Then $K\#K$ is ribbon since K is −amphicheiral. As in the proof of Theorem 1.2, K is homotopy ribbon by [**34**, Theorem 8.5.1]. Since the Alexander polynomial of K is not irreducible, K is not algebraically slice, and hence not homotopy ribbon. This is a contradiction and completes the proof. □

5. ν^+-*Invariants of satellites do not detect slice knots*

The notion of rationally 0-bipolar knots is defined in [**9**, Definition 2.3] as a rational version of 0-bipolar knots introduced in [**13**]. For the reader's convenience, we recall the definition here.

Definition 4 [**9**, Definition 2.3]. A knot K in S^3 is *rationally* 0*-bipolar* if there exist compact smooth 4-manifolds V_+ and V_- and smoothly embedded disks $D_\pm$ in $V_\pm$ such that

(i) $\partial(V_\pm, D_\pm) = (S^3, K)$;
(ii) $H_1(V_\pm; \mathbb{Q}) = 0$;
(iii) $V_\pm$ is $\pm$-definite. That is, $b_2^\pm(V_\pm) = b_2(V_\pm)$;
(iv) $[D_\pm, \partial D_\pm] = 0 \in H_2(V_\pm, S^3; \mathbb{Q})$.

We will use the following facts about rationally 0-bipolar knots.

(B1) Any rationally slice knot is rationally 0-bipolar. (Compare Definitions 2 and 4.)
(B2) A pattern $Q \subset S^1 \times D^2$ is called a *slice pattern* if $Q(U)$ is slice. If K is a rationally 0-bipolar knot, then $Q(K)$ is rationally 0-bipolar for any slice pattern Q (compare [**9**, Theorem 2.6(6)]).
(B3) $V_0(K) = V_0(-K) = 0$ if K is rationally 0-bipolar [**9**, Theorem 2.7].

We remark that (B2) and (B3) are mild generalizations of [**13**, Propositions 3.3 and 1.2]. The slice disk for $Q(K)$ in (B2) is constructed in the same way as in the proof of Theorem 1.1. (B3) was originally stated in terms of the correction terms $d(S^3_1(K)) = d(S^3_1(-K)) = 0$, and we refer the reader to [**23**] for the equivalence of these two identities. The invariant $V_0(K)$, and more generally, the sequence $\{V_k(K)\}$ of a knot K will be defined shortly.

Next, we sketch the construction of the ν^+-invariant and give some relevant background of Heegaard Floer homology. For a knot $K \subset S^3$, the Heegaard Floer knot complex $CFK^\infty(K)$ is a doubly filtered complex with a U-action that lowers each filtration index by one. Define the quotient complexes $A_k^+ = C\{\max\{i, j-k\} \geqslant 0\}$ and $B^+ = C\{i \geqslant 0\}$, where i and j refer to the two filtrations. Associated to each k, there is a graded, module map

$$v_k^+ \colon A_k^+ \to B^+.$$

Define $V_k(K)$ be the U-exponent of v_k^+ at sufficiently high gradings. This sequence of $\{V_k(K)\}$ is non-increasing, that is, $V_k(K) \geqslant V_{k+1}(K)$, and stabilizes at 0 for large k. We define $\nu^+(K)$ to be the minimum k for which $V_k(K) = 0$. We list some properties of the ν^+-invariant below.

(N1) It is a concordance invariant, taking non-negative integer values

$$\nu^+(K) = \min\{k \in \mathbb{Z} \mid V_k = 0\} \geqslant 0.$$

(N2) It is bounded above by the 4-ball genus

$$\nu^+(K) \leqslant g_4(K).$$

(N3) It bounds Ozsváth–Stipsicz–Szabó's one-parameter family of concordance invariants [35].

$$|\Upsilon_K(t)| \leqslant t \max(\nu^+(K), \nu^+(-K)).$$

Following the same argument of [8, Theorem B], we give a proof of Theorem 1.4. Note that ν^+ is not a concordance homomorphism, so part (2) of the theorem does not follow from part (1) immediately.

Proof of Theorem 1.4. As we mentioned in the introduction, we prove Theorem 1.4 under a weaker assumption that K is rationally 0-bipolar.

(i) This follows from (B3) by the definition of the ν^+-invariant.

(ii) Fix a pattern P and consider a pattern $Q = P\# - P(U)$. That is, $(S^1 \times D^2, Q)$ is the connected sum of pairs $(S^1 \times D^2, P)$ and $(S^3, -P(U))$. Note that Q is a slice pattern since $Q(U) = P(U)\# - P(U)$. Hence by (B2), $Q(K) = P(K)\# - P(U)$ is rationally 0-bipolar since K is rationally 0-bipolar. From (B3) and (N1), we conclude that

$$\nu^+(P(K)\# - P(U)) = \nu^+(-P(K)\#P(U)) = 0.$$

(iii) It is known that ν^+ is sub-additive under connected sum [3, Theorem 1.4]. From (ii) and the concordance invariance of ν^+, we have an inequality which holds for any P:

$$\nu^+(P(U)) = \nu^+(P(K)\# - P(K)\#P(U)) \leqslant \nu^+(P(K)) + \nu^+(-P(K)\#P(U)) = \nu^+(P(K)).$$

The proof of $\nu^+(P(K)) \leqslant \nu^+(P(U))$ is similar. □

Using (N3), we show that the Ozsváth-Stipsicz-Szabó Υ invariant of satellites does not detect slice knots either.

Corollary 5.1. *If K is a rationally 0-bipolar knot, then*

$$\Upsilon_{P(K)} = \Upsilon_{P(U)}$$

for all patterns P.

Proof. Since Υ is a concordance homomorphism [35, Corollary 1.12], we have

$$\Upsilon_{P(K)} - \Upsilon_{P(U)} = \Upsilon_{P(K)\#-P(U)} = 0,$$

where the second equality follows from (N3) and Theorem 1.4(1). □

We finish our discussion with a question that is motivated by [8] and this paper. From [8, Theorem 3.1; 20, Proposition 5.1], we know that the following conditions are equivalent.

(1) Two knots K and K' are ε-equivalent. That is, $\varepsilon(K\# - K') = 0$.
(2) $\tau(P(K)) = \tau(P(K'))$ for any pattern P.
(3) $\varepsilon(P(K)\# - P(K')) = 0$ for any pattern P.
(4) $\varepsilon(P(K)) = \varepsilon(P(K'))$ for any pattern P.

In particular, any two rationally 0-bipolar knots K and K' are ε-equivalent [8]. These two knots satisfy, in addition, $\Upsilon_{P(K)} = \Upsilon_{P(K')}$ from Corollary 5.1. In general, we ask:

Question 5. Is $\Upsilon_{P(K)\#-P(K')} = 0$ for all ε-equivalent knots K and K' and patterns P?

Suppose that there are ε-equivalent knots K, K', and a pattern P such that $\Upsilon_{P(K)\#-P(K')} \neq 0$. Then the knot $J = P(K)\# - P(K')$ will give the first example of a knot with $\varepsilon(J) = 0$ and $\Upsilon_J \neq 0$. Previously, only a doubly-filtered complex C was known to satisfy $\varepsilon(C) = 0$ and $\Upsilon_C \neq 0$ [**35**, Figure 6], but it is unclear whether such a complex can be realized as the knot Floer complex of a knot.

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