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U-ACTION ON PERTURBED HEEGAARD FLOER HOMOLOGY

Zhongtao Wu

This paper has two purposes. First, as a continuation of [**27**], we apply a similar method to compute the perturbed HF^+ for some special classes of fibered three-manifolds in the second highest spin^c-structures, including the mapping tori of Dehn twists along a single non-separating curve and along a transverse pair of curves. Second, we establish an adjunction inequality for the perturbed Heegaard Floer homology, which indicates a potential connection between the U-action on the homology group and the Thurston norm of a three-manifold. As an application, we find the U-action on the perturbed HF^+ of the above classes of fibered three-manifolds is trivial.

1. Introduction

Instanton Floer homology [**4**], Seiberg–Witten Floer homology [**12**], embedded contact homology [**7**] and a few other versions of Floer homology are siblings of Heegaard Floer homology, all of which are extremely useful invariants in their own rights. In spite of their very different origins, it is largely believed that all versions of Floer homology should be isomorphic in a proper sense. As a first step toward the conjecture, Taubes established the equivalence between Seiberg–Witten Floer cohomology and embedded contact homology [**26**], and, more recently, with Lee, the equivalence between Seiberg–Witten Floer cohomology and periodic Floer homology [**13**].

The Floer homology of a fibered three-manifold is particularly important, for it is the meeting point of various different versions of Floer homology. A significant number of computations of this nature have been carried out in, for example, [**3, 5, 9, 25**], and their results all agree. Similar computations can be done for perturbed Floer homology, in which the areas of flow-lines are kept track of, and the *Novikov ring* Λ is used as the coefficient ring. (See Definition 2.1 below for the definition of the Novikov ring.)

Following [**27**], where the perturbed Heegaard Floer homology is calculated for the product three-manifolds $\Sigma_g \times S^1$, we aim to apply a similar method to compute the perturbed HF^+ for some special classes of fibered three-manifolds. More precisely, viewing each fibered three-manifold Y as a mapping torus $\Sigma_g \times [0,1]/(x,1) \sim (\phi(x),0)$, denoted by $M(\phi)$, for some orientation-preserving diffeomorphism ϕ of Σ_g, we study the cases where ϕ can be decomposed as products of Dehn twists along a single non-separating curve, or along a transverse pair of curves.

To state the results, recall that the homology group $H_2(M(\phi);\mathbb{Z})$ of the mapping torus $M(\phi)$ can be identified with $\mathbb{Z}\oplus\ker(1-\phi_*)$ where ϕ_* denotes the action of ϕ on $H_1(\Sigma_g,\mathbb{Z})$. For a fixed integer k, let $S_k \subset \mathrm{Spin}^c(M(\phi))$ denote the collection of spin^c-structures satisfying the following two requirements:

(1) $\langle c_1(\mathfrak{s}), [\Sigma_g]\rangle = 2k$.
(2) $\langle c_1(\mathfrak{s}_k), [T]\rangle = 0$, for all classes $[T]$ coming from $H_1(\Sigma_g)$.

According to the adjunction inequality for Heegaard Floer homology [**17**], $HF^+(M(\phi);\mathfrak{s}) = 0$ unless $\mathfrak{s}$ satisfy the conditions above and $|k| \le g-1$. We shall focus on the computation of the perturbed homology group HF^+ in S_{g-2} with a generic perturbation ω, denoted by the notation $HF^+(M(\phi), g-2;\omega)$. Let $g > 2$ so that S_{g-2} consists of entirely non-torsion spin^c-structures; we have the following main theorem.

Theorem 1.1. *Assume $g > 2$.*

(1) *Let $M(t_\gamma^n)$ denote the mapping torus of multiple Dehn twists along a non-separating curve γ, and let ω be a generic perturbation. Then*
$$HF^+(M(t_\gamma^n), g-2;\omega) = (\Lambda[U]/U)^{2g-2}.$$
(2) *Let $M(t_\gamma^m t_\delta^n)$ denote the mapping torus of multiple Dehn twists along a transverse pair of curves γ and δ, and let ω be a generic perturbation. Then*
$$HF^+(M(t_\gamma^m t_\delta^n), g-2;\omega) = \begin{cases} (\Lambda[U]/U)^{2g-2+|mn|} & \text{if } mn < 0\ , \\ (\Lambda[U]/U)^{2g-4+|mn|} & \text{if } mn > 0\ . \end{cases}$$
(3) *Let $M(t_\gamma^{m_1} t_\delta^{n_1} t_\gamma^{m_2})$ denote the mapping torus of multiple Dehn twists along a transverse pair of curves γ and δ, where $m_1, m_2, n_1 > 0$; and let ω be a generic perturbation. Then*
$$HF^+(M(t_\gamma^{m_1} t_\delta^{n_1} t_\gamma^{m_2}), g-2;\omega) = (\Lambda[U]/U)^{2g-4+(m_1+m_2)n_1}.$$
(4) *Let $M(t_\gamma^{m_1} t_\delta^{n_1} \cdots t_\gamma^{m_k} t_\delta^{n_k})$ denote the mapping torus of multiple Dehn twists along a transverse pair of curves γ and δ, where $m_i \cdot n_j < 0$; and let ω be a generic perturbation. Then*
$$HF^+(M(t_\gamma^{m_1} t_\delta^{n_1} \cdots t_\gamma^{m_k} t_\delta^{n_k}), g-2;\omega) = (\Lambda[U]/U)^{|L|},$$
where L denotes the Lefschetz number of the monodromy.

In [**2**], Cotton–Clay computes the perturbed symplectic Floer homology for all area-preserving surface diffeomorphisms, which provides a lower bound on the number of *fixed points* of symplectomorphisms in given mapping classes. Note that Theorem 1.1 agrees with his results. We shall also compare with [**6**], in which computations of the perturbed Heegaard Floer homology are carried out for the mapping torus of a periodic diffeomorphism. Fink shows that the rank of the homology in second highest Spin^c structures S_{g-2} is exactly the *Lefschetz number* of the corresponding monodromy ϕ.

The unperturbed counterpart of the problem is considered in [**9**]. By presenting M_ϕ as zero-surgery on some knot K in a three-manifold, Jabuka and Mark is able to use the relationship between the knot Floer homology of K and the Floer homology of surgeries on K to determine the Heegaard Floer homology of certain mapping tori $M(\phi)$, mostly overlapping with the cases considered here. However, some extra difficulties arise as the higher differentials of certain spectral sequences is non-vanishing when one attempts to adapt their method in the perturbed case. Hence, we take an alternative approach based on certain *special Heegaard Diagrams*, which will be explained in the next two sections. In the end, we find the homology group in our perturbed case is actually simpler, whose rank is, more or less, just the Euler characteristic of the corresponding homology group in the unperturbed case.

In order to determine $HF^+(M(\phi), g-2; \omega)$ as a $\Lambda[U]$-module, we could cite the result from Lekili [**14**] which readily implies the triviality of the U-action. Alternatively, we establish a more general adjunction inequality here that may be of independent interests in other occasions. The following statement can be seen as an analogy, as well as a generalization, of Theorem 7.1 of [**17**].

Theorem 1.2 (U-action Adjunction Inequality). *Let Z be a connected, embedded two-manifold that represents a non-trivial homology class in an oriented three-manifold Y, and let ω be a generic perturbation. If $\mathfrak{s}$ is a* Spin^c *structure for which $U^j \cdot HF^+(Y, \mathfrak{s}; \omega) \neq 0$, then*

$$|\langle c_1(\mathfrak{s}), [Z] \rangle| \leq 2g(Z) - 2j - 2.$$

In fact, the same conclusion holds for a perturbation ω as long as $\omega(Z) \neq 0$.

We immediately obtain, by taking $j = g$ in the above theorem:

Corollary 1.3. *If a three-manifold Y contains a homologically non-trivial, embedded two-manifold of genus g, then $U^g \cdot HF^+(Y; \omega) = 0$.*

In particular, the U-action applies trivially on $HF^+(M(\phi); \omega)$, provided we can find a homologically non-trivial torus inside the mapping torus $M(\phi)$. It turns out that every diffeomorphism considered in Theorem 1.1 fixes certain essential curve in Σ_g, thus generates the desired homologically non-trivial torus.

Our paper is organized as follows. In Section 2, we collect some preliminary results on perturbed Heegaard Floer homology. We also review the construction of a special Heegaard diagram, which will be used throughout the paper. In Section 3, we extract and reformulate a standard argument from [**27**], and use it as a principal tool in determining the rank of the perturbed Heegaard Floer homology of various mapping tori. In Section 4, we establish the U-action adjunction inequality as a formal consequence of Heegaard–Floer cobordism invariants. This, along with the computations in the preceding section, leads to Theorem 1.1.

2. Preliminaries

2.1. Perturbed Heegaard Floer homology. Let $(\Sigma, \alpha, \beta, z)$ be a pointed Heegaard diagram of a three-manifold Y. The Heegaard Floer chain complex $CF^+(Y)$ is freely generated by $[x, i]$ where x is an intersection point of Lagrangian tori $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ and $i \in \mathbb{Z}_{\geq 0}$, and the differential is given by

$$\partial^+[x,i] = \sum_y \left(\sum_{\{\phi \in \pi_2(x,y) | n_z(\phi) \leq i\}} \#\widehat{\mathcal{M}}(\phi)[y, i - n_z(\phi)] \right).$$

The above definition only makes sense under certain admissibility conditions so that the sum on the right-hand side of the differential is finite. However, there is a variant of Heegaard Floer homology where Novikov rings and perturbations by closed two-forms are introduced without any admissibility condition, called the perturbed Heegaard Floer homology. See [**11**] for a more detailed account.

Definition 2.1. The Novikov ring Λ is the ring whose elements are formal power series of the form $\sum_{r \in \mathbb{R}} a_r T^r$ with $a_r \in \mathbb{Z}_2$ such that $\#\{a_r | a_r \neq 0, r < N\} < \infty$ for any $N \in \mathbb{R}$. In fact, Λ is a field.

Define a perturbed chain complex which is freely generated over Λ by $[x, i]$ as before, and whose differential is given by

$$\partial^+[x,i] = \sum_y \left(\sum_{\{\phi \in \pi_2(x,y) | n_z(\phi) \leq i\}} \#\widehat{\mathcal{M}}(\phi) T^{\mathcal{A}(\phi)} \cdot [y, i - n_z(\phi)] \right),$$

where $\mathcal{A}(\phi)$ denotes the area pre-assigned to the domain $\mathcal{D}(\phi)$ by $\mathcal{A}$. If ϕ_1 and ϕ_2 are two topological discs that connect an intersection point x to y, then their difference is a periodic domain $\mathcal{P}$; and there is a unique two-form $\eta \in H^2(Y; \mathbb{R})$ satisfying the equality $\mathcal{A}(\phi_1) - \mathcal{A}(\phi_2) = \eta([\mathcal{P}])$ for all choices of ϕ_1 and ϕ_2. We denote $HF^+(Y; \eta)$ for the homology of this chain complex. We remark that although the differential depends on the choice of a representative of the class η, the isomorphism class of the homology group $HF^+(Y; \eta)$ is determined by $\ker(\eta) \cap H_2(Y; \mathbb{Z})$.

Recall that a two-form ω is said to be generic if $\ker(\omega) \cap H_2(Y;\mathbb{Z}) = 0$, or equivalently, $\omega(\mathcal{P}) \neq 0$ for any integral periodic domain $\mathcal{P}$. For a generic form, $HF^+(Y,\omega)$ is defined without any admissibility conditions on the Heegaard diagram.

Perturbed Heegaard Floer homology shares many common properties with the unperturbed homology. In particular, we will need the following characterization for the Euler characteristic of HF^+ [**17**].

Lemma 2.2. *For a non-torsion Spin^c structure $\mathfrak{s}$, $HF^+(Y,\mathfrak{s};\eta)$ is finitely generated, and the Euler characteristic*

$$\chi(HF^+(Y,\mathfrak{s};\eta)) = \chi(HF^+(Y,\mathfrak{s})) = \pm\tau_t(Y,\mathfrak{s}),$$

where τ_t is Turaev's torsion function, with respect to the component t of $H^2(Y;\mathbb{R}) - 0$ containing $c_1(\mathfrak{s})$.

Recall that the Heegaard Floer chain complex can be equipped with a $\mathbb{Z}/2\mathbb{Z}$-grading, and $\chi(HF^+(Y,\mathfrak{s}))$ is simply $\mathrm{rank}HF^+(Y,\mathfrak{s})_{\mathrm{even}} - \mathrm{rank}HF^+(Y,\mathfrak{s})_{\mathrm{odd}}$. Different ways of assigning the $\mathbb{Z}/2\mathbb{Z}$-grading account for the sign ambiguity in the statement. Turaev's torsion function, derived from certain complicated group rings over CW-complex, is often rather hard to compute. For fibered three manifolds, the situation is much simplified by the following remarkable identity [**8, 24**].

Lemma 2.3. *If we denote $\tau_t(M(\phi),k)$ for the sum of all Turaev's torsion functions over the set of the spin^c-structures S_k, then*

$$\tau_t(M(\phi),k) = L(S^{g-1-k}\phi),$$

where the latter is the Lefschetz number of the induced function of ϕ over the symmetric product $S^{g-1-k}\Sigma_g$.

In particular when $k = g-2$,

$$\tau_t(M(\phi), g-2) = L(\phi).$$

Let us remind the reader that the Lefschetz number of a continuous map $\phi : M \longrightarrow M$ is defined by

$$L(\phi) := \sum_i (-1)^i \mathrm{Tr}(\phi_* : H_i(M) \to H_i(M)).$$

2.2. A special Heegaard diagram. In order to compute the homology for general fibered three manifolds, we need to use certain special Heegaard diagram, first introduced by Ozsváth and Szabó in studying contact invariant [**20,** Section 3]. Figure 1 is the special Heegaard Diagram for $\Sigma_g \times S^1$. It consists of two twice punctured $4g$-gons and a standard identification on their edges, representing two genus g surfaces with opposite orientations that glued together through the pairs of holes that produces a genus $2g+1$ surface. In the text below, we shall refer to the top $4g$-gon in Figure 1 as

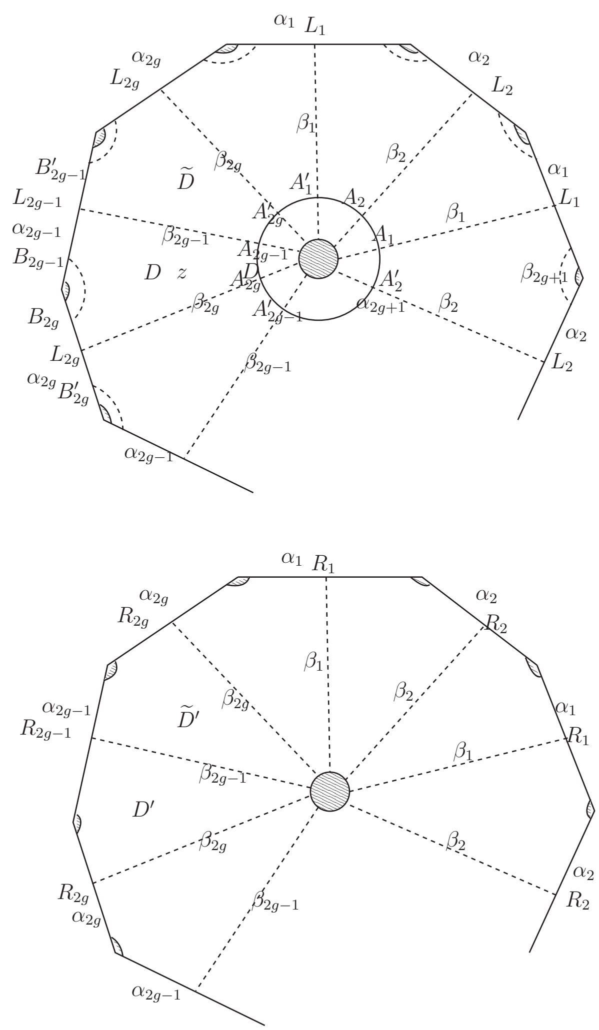

Figure 1. The special Heegaard Diagram of $\Sigma_g \times S^1$. It consists of two twice punctured $4g$-gons and a standard identification on their edges. Here, the top polygon, which shall be also referred to as the "left" one has the usual counterclockwise, while the bottom polygon, which shall be also referred to as the "right" one has the other orientation. They represent two genus g surfaces, glued together through the pairs of holes that produces a genus $2g+1$ surface.

the "left" one and the bottom $4g$-gon as the "right" one for the sake of consistency with [**20**]. All the α's and β's curves are drawn along with their intersection points marked. We list some of the important properties of this special Heegaard diagram:

- each $\alpha_i \cap \beta_i$ twice, denoted by L_i and R_i respectively, $1 \leq i \leq 2g$;
- $\alpha_i \cap \beta_j = \emptyset$, when $i \neq j$, $1 \leq i, j \leq 2g$;
- $\alpha_{2g+1} \cap \beta_i$ twice, denoted by A_i and A'_i, respectively, $1 \leq i \leq 2g$;
- $\alpha_i \cap \beta_{2g+1}$ twice, denoted by B_i and B'_i, respectively, $1 \leq i \leq 2g$.

Recall that $S_k \subset \mathrm{Spin}^c(M(\phi))$ is the set of spin^c-structures satisfying the following two conditions

(1) $\langle c_1(\mathfrak{s}), [\Sigma_g] \rangle = 2k$.
(2) $\langle c_1(\mathfrak{s}_k), [T] \rangle = 0$ for all classes $[T]$ coming from $H_1(\Sigma_g)$.

We can find the generators of S_k in this Heegaard diagram.

- For $k \geq g$, S_k is empty.
- For $k = g-1$, S_{g-1} consists of a pair of generators: $(A_{2g}, B_{2g}, L_1, L_2, \ldots, L_{2g-1})$ and $(A_{2g-1}, B_{2g-1}, L_1, \ldots, L_{2g-2}, L_{2g})$.
- For $k = g-2$, S_{g-2} consists of $(2g-1)$ pairs of generators:
  $a_1 := (A_{2g}, B_{2g}, R_1, L_2, L_3, \ldots, L_{2g-1})$,
  $a_2 := (A_{2g}, B_{2g}, L_1, R_2, L_3, \ldots, L_{2g-1})$
  $\ldots$
  $a_{2g-2} = (A_{2g}, B_{2g}, L_1, L_2, \ldots, R_{2g-2}, L_{2g-1})$
  and
  $b_1 := (A_{2g-1}, B_{2g-1}, R_1, L_2, L_3, \ldots, L_{2g-2}, L_{2g})$,
  $b_2 := (A_{2g-1}, B_{2g-1}, L_1, R_2, L_3, \ldots, L_{2g-2}, L_{2g})$
  $\ldots$
  $b_{2g-2} = A_{2g-1}, B_{2g-1}, L_1, L_2, \ldots, R_{2g-2}, L_{2g})$
  and
  $a_0 := (A_{2g}, B_{2g}, L_1, L_2, \ldots, R_{2g-1})$,
  $b_0 := (A_{2g-1}, B_{2g-1}, L_1, L_2, \ldots, R_{2g})$.

  Here, a_0 and b_0 are distinguished from the other generators by the fact that there is a disk D' connecting them without passing the basepoint z. We call them *fake generators*. The remaining $(2g-2)$ pairs, on the other hand, are called *essential generators*. By making a choice of the $\mathbb{Z}/2\mathbb{Z}$-grading so that $a_i \in CF^+(Y)_{\mathrm{odd}}$ and $b_i \in CF^+(Y)_{\mathrm{even}}$, we can resolve the sign ambiguity in Lemma 2.2:

$$\chi(HF^+Y, \mathfrak{s}; \eta)) = \chi(HF^+Y, \mathfrak{s})) = \tau_t(Y, \mathfrak{s}).$$

- When $0 < k < g-1$, S_k consists of $\binom{2g-1}{g-1-k}$ pairs of generators: simply replace $(g-1-k)$ of L_i by R_i in the coordinates of the generators of S_{g-1}. Among them, $\binom{2g-2}{g-2-k}$ pairs are fake and $\binom{2g-2}{g-1-k}$ pairs are essential.

We claim that the above is a complete list of all generators in S_k. Again, recall the following Chern class formula [**17,** Section 7.1]:

$$\langle c_1(\mathfrak{s}_z(x), [\mathcal{P}]\rangle = \chi(\mathcal{P}) - 2n_z(\mathcal{P}) + 2\sum_{x_i \in x} n_{x_i}(\mathcal{P}),$$

where $\mathcal{P}$ is a domain whose boundary is a sum of α and β curves and x is a generator of the Heegaard Floer homology. We check $\langle c_1(\mathfrak{s}_z(a_i)), [\Sigma_g]\rangle = 2(g-2)$.

Clearly, the periodic domain $\mathcal{P}$ in the formula corresponding to the homology representative $[\Sigma_g]$ is represented by the the union of all hexagons lying in the left-hand-side polygon between α_{2g+1} and β_{2g+1}, which is itself a genus-g surface with two punctures; thus, the Euler measure $\chi(\mathcal{P}) = -2g$. It is also easy to see that

$$n_z(\mathcal{P}) = 1, n_{A_{2g}}(\mathcal{P}) = n_{B_{2g}}(\mathcal{P}) = \frac{1}{2}, n_{L_i}(\mathcal{P}) = 1, n_{R_i}(\mathcal{P}) = 0.$$

Plugging into the Chern class formula, we obtain

$$\langle c_1(\mathfrak{s}_z(a_i)), [\Sigma_g]\rangle = -2g - 2 + 2\left(\frac{1}{2} + \frac{1}{2} + 2g - 2\right) = 2g - 4$$

as desired.

Indeed, it is a very similar calculation using the Chern class formula to show that $\langle c_1(\mathfrak{s}_z(a_i)), [T]\rangle = 0$ for all a_i's. Here, each class $[T]$ is represented by some embedded torus in the three-manifold, as well as by unions of hexagons in the Heegaard diagram. In particular, the unions $D \cup D' \cup D_1 \cup D_1'$ and $D \cup D' \cup D_2 \cup D_2'$ in Figure 1 are examples of such periodic domains. By applying the Chern class formula on these two periodic domains, we can further see that every essential generator in S_k must contain intersection points $(A_{2g}, B_{2g}, L_{2g-1})$ or $(A_{2g-1}, B_{2g-1}, L_{2g})$, while every fake generator must contain intersection points $(A_{2g}, B_{2g}, R_{2g-1})$ or $(A_{2g-1}, B_{2g-1}, R_{2g})$. This fact enabled us to simplify the enumeration of generators of $\Sigma_g \times S^1$ by a great deal, and we would like to point out that the same simplification remains valid for all three-manifolds considered in this paper. (Although it is definitely not true for an arbitrary three-manifold Y with $b_1(Y) = 1$.)

In general, the special Heegaard diagram for an arbitrary mapping torus is obtained in a similar manner. The α and β curves inside the left-hand-side $4g$-gon are always the same as those inside $\Sigma_g \times S^1$, which we would refer later as a standard diagram. Inside the right-hand-side $4g$-gon, whereas the α's curves remain unaltered, the β's curves twist according to ϕ. Therefore, it is only necessary to exhibit the right-hand-side $4g$-gon of the Heegaard diagram, as it encodes essentially all the information of the manifold.

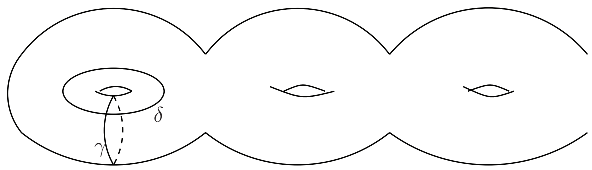

Figure 2. The standard position of a transverse pair of curves is represented by γ and δ.

3. Calculations for fibered three manifolds

Standard classification results in surfaces imply that any simple non-separating curve can be mapped to the standard position γ, and that any pair of transverse curves can be mapped to γ and δ in Figure 2, by a suitable surface automorphism. Hence, for simplicity, we always assume the curves to lie in the standard position in the forthcoming discussions. We are going to compute the rank of $HF^+(M(\phi), g-2; \omega)$ for various mapping tori by a method based on ideas from [**27**]. A few simplification is made in the argument although, and it is reformulated in a form most suitable for its subsequent applications.

Throughout the section, g is implicitly assumed to be greater than 2.

3.1. A standard argument. Recall from the proceeding section that there are $2g-2$ pairs of essential generators $a_i \xrightarrow{D} b_i$ in S_{g-2} with a holomorphic disk D connecting them; and there is a single pair of fake generators $a_0 \xleftarrow{D'} b_0$ with a holomorphic disk D' connecting them. Also note that both the topological disks D and D' can be represented by some holomorphic disks ϕ, so that the algebraic number of holomorphic disk in the corresponding moduli space of disks in the homology class of ϕ is given by $\#\widehat{\mathcal{M}}(\phi) = \pm 1$ (See [**23,** Section 9]). Arguments below will show that a_0 and b_0 do not survive in the homology, hence justifying the name "fake generators" that we have called them.

In general, let us denote:

$CF^{\text{ess}}_{\text{odd}}$:= Vector space generated by all essential generators supported in odd grading.
(generated by all a_i's, $1 \le i \le g-2$ in $CF(\Sigma_g \times S^1, g-2)$).

$CF^{\text{ess}}_{\text{even}}$:= Vector space generated by all essential generators supported in even grading.
(generated by all b_i's, $1 \le i \le g-2$ in $CF(\Sigma_g \times S^1, g-2)$).

$CF^{\text{fake}}_{\text{odd}}$:= Vector space generated by all fake generator supported in odd grading.

$$\begin{tikzcd}
CF^{ess}_{odd} \arrow[d, "D", "n_z=1"'] & \cdots & CF^{ess}_{odd}\cdot U^{k} \arrow[d, "D"] & CF^{fake}_{odd} & \cdots & CF^{fake}_{odd}\cdot U^{-k} \\
CF^{ess}_{even}\cdot U^{-1} & & CF^{ess}_{even}\cdot U^{-k-1} & CF^{fake}_{even} \arrow[u, "D'"] & & CF^{fake}_{even}\cdot U^{-k} \arrow[u, "D'", "n_z=0"']
\end{tikzcd}$$

Figure 3. The chain complex of $CF^+(Y)$.

(generated by a_0 in $CF(\Sigma_g \times S^1, g-2)$).

$CF^{\text{fake}}_{\text{even}} :=$ Vector space generated by all fake generators supported in even grading
(generated by b_0 in $CF(\Sigma_g \times S^1, g-2)$).

$CF^+_{\text{odd}} := (CF^{\text{ess}}_{\text{odd}}) \oplus CF^{\text{fake}}_{\text{odd}}) \cdot (1 \oplus U^{-1} \oplus U^{-2} + \cdots).$

$CF^+_{\text{even}} := (CF^{\text{ess}}_{\text{even}}) \oplus CF^{\text{fake}}_{\text{even}}) \cdot (1 \oplus U^{-1} \oplus U^{-2} + \cdots).$

We summarize these information of the chain complex $CF^+(Y)$ in Figure 3.

It contains all the generators of $CF^+(Y, g-2)$, though the boundary map ∂ of this chain complex is apparently incomplete as here represented. We can get around this difficulty by cleverly choosing a generic form ω in light of the fact that $HF^+(Y, g-2; \omega)$ is an invariant for generic perturbation ω. To this end, choose a generic two form ω such that $\omega(D) = \omega(D') \ll \omega(\text{other regions})$. Then the above complex would be the E_1 page of the spectral sequence if there were an area filtration on the Heegaard diagram. Unfortunately, such an area filtration does not exist due to non-admissiblity of the Heegaard diagram. Nevertheless, this idea can still carry through by other means and is made precise by the following technical lemma, which enables us to compute $HF^+(Y, g-2; \omega)$ without any further knowledge on the chain complex, provided that certain condition on Euler characteristic is satisfied.

Lemma 3.1. *Suppose the generators and a partial information of the boundary map ∂ of a chain complex $CF^+(Y)$ are reflected as in Figure 3. If we know, in addition, that $\chi(HF^+(Y)) = -rank\, CF^{\text{ess}}_{\text{odd}}$, then*

$$rank\, HF^+_{\text{even}}(Y; \omega) = 0,$$
$$rank\, HF^+_{\text{odd}}(Y; \omega) = -\chi(HF^+(Y))$$

for the generic perturbation ω.

Proof. As mentioned above, it suffices to prove the lemma for a generic two-form ω with $\omega(D) = \omega(D') \ll \omega$ (other regions). Suppose x represents a non-zero class in HF^+_{odd}, we will show:

(1) $x \notin CF^{\text{ess}}_{\text{odd}} \cdot (U^{-1} \oplus U^{-2} + \cdots)$.
(2) there is an element $x' \in CF^{\text{ess}}_{\text{odd}} \cdot (1 \oplus U^{-1} \oplus U^{-2} + \cdots)$ such that $[x] = [x'] \in HF^+_{\text{odd}}$.

If we can prove these two claims, then each class in HF^+_{odd} would uniquely correspond to an element in the subspace $CF^{\text{ess}}_{\text{odd}}$; so $\text{rank}HF^+_{\text{odd}} \leq \text{rank}CF^{\text{ess}}_{\text{odd}}$. Since we also have $\text{rank}HF^+_{\text{odd}} - \text{rank}HF^+_{\text{even}} = \text{rank}CF^{\text{ess}}_{\text{odd}}$, the desired equalities follow immediately.

To prove (1), note that every element of $CF^{\text{ess}}_{\text{odd}} \cdot (U^{-1} \oplus U^{-2} + \cdots)$ can be written as $x = \sum a_i U^{-j} k_{ij}$, where $k_{ij} \in \Lambda$ and $a_i \in CF^{\text{ess}}_{\text{odd}}$. Suppose $k_{i_1 j_1}$ is one of the coefficients with the lowest order term in T. Then

$$\partial x = b_{i_1} U^{-(j_1-1)} \cdot (k_{i_1 j_1} T^{\omega(D)} + \text{higher order terms in } T) + \cdots \ .$$

Hence $\partial x \neq 0$, if $x \neq 0$; so x is not a cycle.

To prove (2), we first compute the determinant of the ∂-matrix from $CF^{\text{fake}}_{\text{even}}$ to $CF^{\text{fake}}_{\text{odd}}$. There is a unique lowest order term $T^{N \cdot \omega(D')}$ coming from the holomorphic disk D' in diagonal entries, where N is the number of generators and thus also the size of the matrix ($N = 1$ in the case of $\Sigma_g \times S^1$ that corresponds to the unique pair of generators a_0, b_0 and the holomorphic disk D' that connects them). Consequently, the determinant is nonzero. As this ∂-matrix has entries in the Novikov ring Λ, which is itself a field, it follows that $\det \neq 0$ is equivalent to the invertibility of the matrix; so the boundary map ∂ is surjective.

We would like to extend the above argument to the differential from the larger space $CF^{\text{fake}}_{\text{even}} \cdot (1 \oplus U^{-1} \oplus \cdots \oplus U^{-k})$ to $CF^{\text{fake}}_{\text{odd}} \cdot (1 \oplus U^{-1} \oplus \cdots + \oplus U^{-k})$. Suppose $x_1, x_2, \ldots, x_N$ and $y_1, y_2, \cdots, y_N$ are the generators of $CF^{\text{fake}}_{\text{even}}$ and $CF^{\text{fake}}_{\text{odd}}$ respectively, and there is a holomorphic disk D' connecting x_i and y_i for each i. Then $CF^{\text{fake}}_{\text{even}} \cdot (1 \oplus U^{-1} \oplus \cdots \oplus U^{-k})$ (resp. $CF^{\text{fake}}_{\text{odd}} \cdot (1 \oplus U^{-1} \oplus \cdots \oplus U^{-k})$) can be viewed as an Λ-vector space generated by a basis of $N(k+1)$ elements $[x_i, j]$ (resp. $[y_i, j]$), where $1 \leq i \leq N$ and $0 \leq j \leq k$. We can construct an associated ∂-matrix with entries in Λ of size $N(k+1)$ according to the following rule: if $\partial[x_{i_1}, j_1] = c^{i_2 j_2}_{i_1 j_1}[y_{i_2}, j_2] + \cdots$, then record $c^{i_2 j_2}_{i_1 j_1}$ in the entry of the matrix that corresponds to the row for $[x_{i_1}, j_1]$ and the column for $[y_{i_2}, j_2]$. Complicated as this matrix appears to be, we claim that it has nonzero determinant and thus invertible. The key observation is that there is a unique lowest order term $T^{\omega(D')}$ in each diagonal entry of the matrix c^{ij}_{ij}. This corresponds to the fact that there is a holomorphic disk D' connecting $[x_i, j]$ and $[y_i, j]$ for each i. Therefore, there is a unique lowest order term $T^{N(k+1)\omega(D')}$ in the expression of the determinant, and consequently the determinant must not be zero. It follows that this matrix is surjective.

Note that $c_{i_1j_1}^{i_2j_2}$ vanishes whenever $j_2 > j_1$. This implies that the image of the differential from $CF_{\text{even}}^{\text{fake}} \cdot (1 \oplus U^{-1} \oplus \cdots \oplus U^{-k})$ to $CF_{\text{odd}}^{\text{fake}} \cdot (1 \oplus U^{-1} \oplus \cdots)$ lands entirely inside the subspace $CF_{\text{odd}}^{\text{fake}} \cdot (1 \oplus U^{-1} \oplus \cdots \oplus U^{-k})$. Hence, for any $b \in CF_{\text{odd}}^{\text{fake}} \cdot (1 \oplus U^{-1} \oplus \cdots \oplus U^{-k})$, using the surjection proved in the last paragraph, we can always find $a \in CF_{\text{even}}^{\text{fake}} \cdot (1 \oplus U^{-1} \oplus \cdots \oplus U^{-k})$ such that the projection of ∂a in $CF_{\text{odd}}^{\text{fake}} \cdot (1 \oplus U^{-1} \oplus \cdots)$ is b. Choose a large enough k, and let this b be the projection of x (the same x that appears at the beginning of the proof) in $CF_{\text{odd}}^{\text{fake}} \cdot (1 \oplus U^{-1} \oplus \cdots)$; also let $\partial a = y \in CF_{\text{odd}}^{+}$, so $[y] = 0 \in HF_{\text{odd}}^{+}$. Let $x' = x - y$, then x' projects to 0 in $CF_{\text{odd}}^{\text{fake}} \cdot (1 \oplus U^{-1} \oplus \cdots)$. Hence $x' \in CF_{\text{odd}}^{\text{ess}} \cdot (1 \oplus U^{-1} \oplus \cdots)$ as desired. □

We remark that the preceding argument is applicable to any three-manifold as long as the conditions of the assumption are met. In particular, it holds for $Y = \Sigma_g \times S^1$

$$\chi(HF^+(Y, g-2)) = -\text{rank} CF_{\text{odd}}^{\text{ess}} = 2 - 2g,$$

from which the computation of $HF^+(Y, g-2; \omega)$ in [**27**] follows. For the remaining section, we would apply this method to determine the rank of the perturbed Heegaard Floer homology for various other mapping tori, and would refer it as the "standard" argument.

3.2. Multiple Dehn twists along a non-separating curve $\phi = t_\gamma^n$.

Assume that the monodromy $\phi = t_\gamma^n$; the right-hand side of the special Heegaard diagram of $M(t_\gamma^n)$ looks like Figure 4.

We proceed to enumerate all the generators in the set of the Spin^c structures S_{g-2} in the Heegaard diagram. Observe that apart from n intersection points between α_2 and β_1, Dehn twists along γ introduce does not introduce any new intersection points; and a routine calculation using the Chern class formula finds no other additional generator than the $2g-1$ pairs that initially existed, among which $2g-2$ pairs are essential.

Apply Lemma 2.2 and 2.3: $\chi(HF^+(M(t_\gamma^n), \mathfrak{s}_{g-2}; \omega)) = L(t_\gamma^n) = 2 - 2g$. The condition $\chi(HF^+(M(t_\gamma^n), \mathfrak{s}_{g-2}) = 2 - 2g = -\text{rank} CF_{\text{odd}}^{\text{ess}}$ is satisfied, so we can apply the standard argument and obtain the following.

Proposition 3.2. $HF^+(M(t_\gamma^n), g-2; \omega) = \Lambda_{\text{odd}}^{2g-2}$.

3.3. Multiple Dehn twists along a transverse pair of curves $\phi = t_\gamma^m t_\delta^n$.

Assume the monodromy $\phi = t_\gamma^m t_\delta^n$. There are two cases: either $m \cdot n < 0$ or $m \cdot n > 0$.

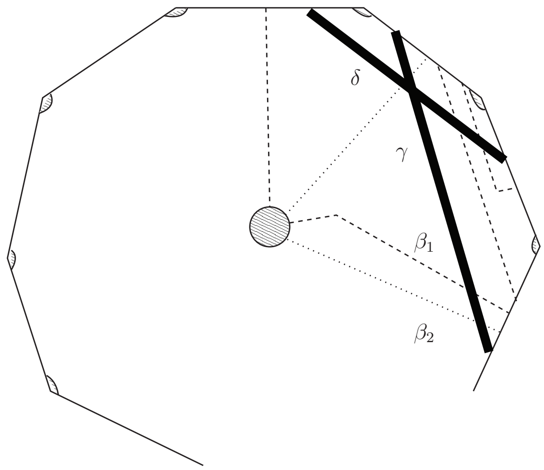

Figure 4. The Heegaard diagram for $M(t_\gamma^n)$, when $n = 2$. The pair of non-separating curves γ and δ in standard positions are exhibited in thick lines.

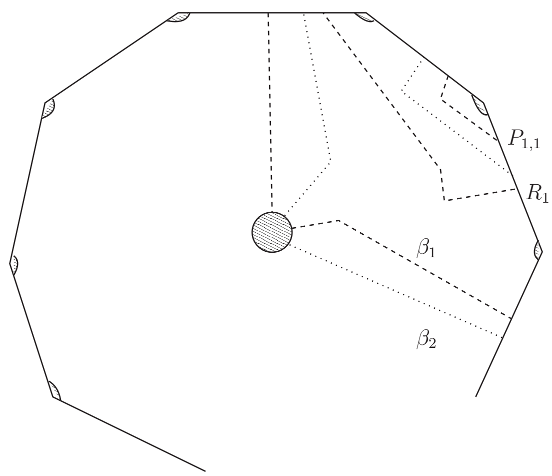

Figure 5. The Heegaard diagram for $M(t_\gamma^m t_\delta^n)$, when $m = 1, n = -1$. Here, β_1 is represented by the dashed curve, while β_2 is represented by the dotted curve.

Consider the case $m \cdot n < 0$ first. We have the Heegaard Diagram in Figure 5.

Denote the $|mn|$ extra intersection between α_1 and β_1 by $P_{i,j}$, where $1 \leq i \leq |m|$ and $1 \leq j \leq |n|$. There are $(2g - 1 + |mn|)$ pairs of generators in S_{g-2}, among which $(2g-2+|mn|)$ pairs are essential:

$$\begin{gathered}(A_{2g}, B_{2g}, R_1, L_2, \ldots, L_{2g-1}),\\ (A_{2g}, B_{2g}, L_1, R_2, \ldots, L_{2g-1})\\ \cdots\\ (A_{2g}, B_{2g}, L_1, L_2, \ldots, R_{2g-1})\end{gathered}$$

and

$$\begin{gathered}(A_{2g-1}, B_{2g-1}, R_1, L_2, \ldots, L_{2g-2}, L_{2g}),\\ (A_{2g-1}, B_{2g-1}, L_1, R_2, \ldots, L_{2g-2}, L_{2g})\\ \cdots\\ (A_{2g-1}, B_{2g-1}, L_1, L_2, \ldots, R_{2g}),\end{gathered}$$

and

$$\begin{gathered}(A_{2g}, B_{2g}, P_{i,j}, L_2, \ldots, L_{2g-1}),\\ (A_{2g-1}, B_{2g-1}, P_{i,j}, L_2, \ldots, L_{2g}).\end{gathered}$$

To compute the Lefschetz number of $L(t_\gamma^m t_\delta^n)$, note that both t_γ and t_δ act trivially on $H_0(\Sigma_g)$, $H_2(\Sigma_g)$, and a $(2g-2)$-dimensional subspace of $H_1(\Sigma_g)$. While on the two-dimensional subspace spanned by the Poincare duals of γ and δ, they act by $\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & \\ -1 & 1 \end{pmatrix}$, respectively. Then,

$$\operatorname{Tr}\left(\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}^m \begin{pmatrix} 1 & \\ -1 & 1 \end{pmatrix}^n\right) = \operatorname{Tr}\left(\begin{pmatrix} 1 & m \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \\ -n & 1 \end{pmatrix}\right) = 2 - mn$$

and the Lefschetz number is

$$\begin{aligned} L(\phi) &= \sum_{i=0}^{2} (-1)^i \operatorname{Tr}(\phi_* : H_i(M) \to H_i(M)) \\ &= 1 - ((2g-2) + (2-mn)) + 1 \\ &= 2 - 2g + mn. \end{aligned}$$

The condition $\chi(HF^+(M(t_\gamma^n), \mathfrak{s}_{g-2}) = 2 - 2g + mn = -\mathrm{rank} CF^{\mathrm{ess}}_{\mathrm{odd}}$ is satisfied, so we can apply the standard argument and obtain the following.

Proposition 3.3. $HF^+(M(t_\gamma^m t_\delta^n), g-2; \omega) = \Lambda_{\mathrm{odd}}^{2g-2+|mn|}$, $m \cdot n < 0$.

Let us proceed to the case $m \cdot n > 0$. By symmetry, it suffices to consider $m, n > 0$.

We have the following Heegaard diagram (Figure 6), that can be subsequently simplified to Figure 7 by an isotopy on β_1. Note that the intersections R_1 and $P_{m,n}$ disappear in the new diagram. In this Heegaard diagram, there

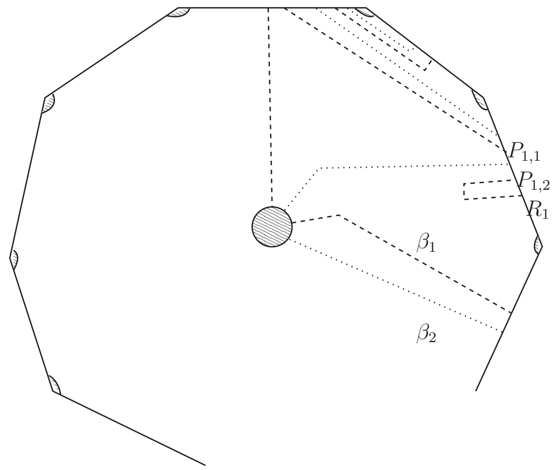

Figure 6. The Heegaard diagram for $M(t_\gamma^m t_\delta^n)$, when $m = 1, n = 2$. β_1 is represented by the dashed curve, while β_2 is represented by the dotted curve.

are $(2g-3+mn)$ pairs of generators in S_{g-2}, among which $2g-4+mn$ pairs are essential:

$$(A_{2g}, B_{2g}, L_1, R_2, \ldots, L_{2g-1})$$
$$\cdots$$
$$(A_{2g}, B_{2g}, L_1, L_2, \ldots, R_{2g-1})$$

and

$$(A_{2g-1}, B_{2g-1}, L_1, R_2, \ldots, L_{2g})$$
$$\cdots$$
$$(A_{2g-1}, B_{2g-1}, L_1, L_2, \ldots, R_{2g})$$

and

$$(A_{2g}, B_{2g}, P_{i,j}, L_2, \ldots, L_{2g-1})$$
$$(A_{2g-1}, B_{2g-1}, P_{i,j}, L_2, \ldots, L_{2g})$$

where $(i,j) \neq (m,n)$.

As alluded to earlier, there are multiple Spin^c structures in the set S_{g-2}. In fact, the spin^c-structures are naturally identified with the second cohomology group $H^2(M(t_\gamma^m t_\delta^n), \mathbb{Z}) = \mathbb{Z}^{2g-1} \oplus \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$. Applying the Chern class formula, we find

$$(A_{2g}, B_{2g}, P_{i,j}, L_2, \ldots, L_{2g-1})$$
$$\cdots$$
$$(A_{2g-1}, B_{2g-1}, P_{i,j}, L_2, \ldots, L_{2g})$$

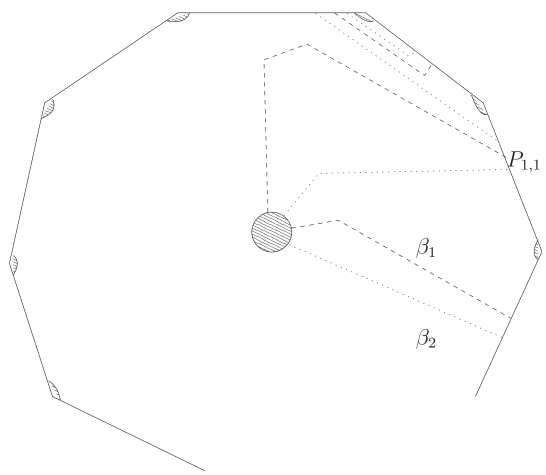

Figure 7. The simplified Heegaard diagram after isotopying β_1. Note that the intersections R_1 and $P_{1,2}$ disappear in this new diagram.

with $(i,j) \neq (m,n)$ lying on $mn-1$ different spin^c-structures, denoted by $\mathfrak{s}_{i,j}$ respectively, while all the remaining generators

$$\begin{gathered}(A_{2g}, B_{2g}, L_1, R_2, \dots, L_{2g-1})\\ \cdots \\ (A_{2g}, B_{2g}, L_1, L_2, \dots, R_{2g-1}),\\ (A_{2g-1}, B_{2g-1}, L_1, R_2, \dots, L_{2g})\\ \cdots \\ (A_{2g-1}, B_{2g-1}, L_1, L_2, \dots, R_{2g})\end{gathered}$$

lying on another distinguished spin^c-structure, that we denote by $\mathfrak{s}_{m,n}$.

For each spin^c-structure $\mathfrak{s}_{i,j}$, $(i,j) \neq (m,n)$, there are exactly two generators $(A_{2g}, B_{2g}, P_{i,j}, L_2, \dots, L_{2g-1})$, $(A_{2g-1}, B_{2g-1}, P_{i,j}, L_2, \dots, L_{2g})$ which are connected by a holomorphic disk D with $n_z \neq 0$. The argument from [**27,** Section 3] for three-torus can be adapted here to show

$$HF^+(M(t_\gamma^m t_\delta^n), \mathfrak{s}_{i,j}; \omega) = \Lambda,$$

all supported in even gradings.

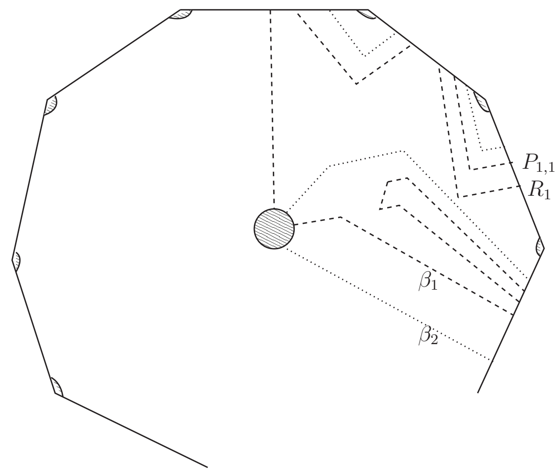

Figure 8. The Heegaard diagram of $M(t_\gamma t_\delta t_\gamma)$. An isotopy on β_1 can be carried out to cancel the pairs of intersection points R_1 and $P_{1,1}$.

To determine the homology in the spin^c-structure $\mathfrak{s}_{m,n}$, note that its Euler characteristic is:

$$\begin{aligned}\chi(HF^+_{\mathfrak{s}_{m,n}}) &= \tau_t(2g-2) - \sum_{(i,j)\neq(m,n)} \chi(HF^+_{\mathfrak{s}_{i,j}}) \\ &= 2-2g+mn-(mn-1) \\ &= 3-2g.\end{aligned}$$

There are also exactly $2g-3$ pairs of essential generators, so we can apply the standard argument and conclude

$$HF^+(M(t_\gamma^m t_\delta^n), \mathfrak{s}_{m,n}; \omega) = \Lambda^{2g-3}.$$

In summary, we have:

Proposition 3.4. $HF^+(M(t_\gamma^m t_\delta^n), g-2; \omega) = \Lambda^{mn-1}_{\text{even}} \oplus \Lambda^{2g-3}_{\text{odd}}$, $m \cdot n > 0$.

3.4. Multiple Dehn twists along a transverse pair of curves $\phi = t_\gamma^{m_1} t_\delta^{n_1} t_\gamma^{m_2}$. The manifolds considered here have the form $M(t_\gamma^{m_1} t_\delta^{n_1} t_\gamma^{m_2})$, where $m_1, m_2, n_1 > 0$. The Heegaard diagram is drawn for the case $m_1 = n_1 = m_2 = 1$ in Figure 8, which can be simplified by an isotopy on β_1 to remove the intersections R_1 and $P_{1,1}$ (Figure 9). In general, there will be $2g-4+(m_1+m_2)n_1$ pairs of essential generators in a simplified Heegaard diamgram of $M(t_\gamma^{m_1} t_\delta^{n_1} t_\gamma^{m_2})$. (We spare the labour of including the diagram here, for it is not more illuminating but far more difficult to perceive.)

As $H^2(M(t_\gamma^{m_1} t_\delta^{n_1} t_\gamma^{m_2}), \mathbb{Z}) = \mathbb{Z}^{2g-1} \oplus \mathbb{Z}/(m_1+m_2)\mathbb{Z} \oplus \mathbb{Z}/n_1\mathbb{Z}$, we have $(m_1+m_2)n_1$ different spin^c-structures in S_{g-2}, denoted by $\mathfrak{s}_{i,j}$. After a

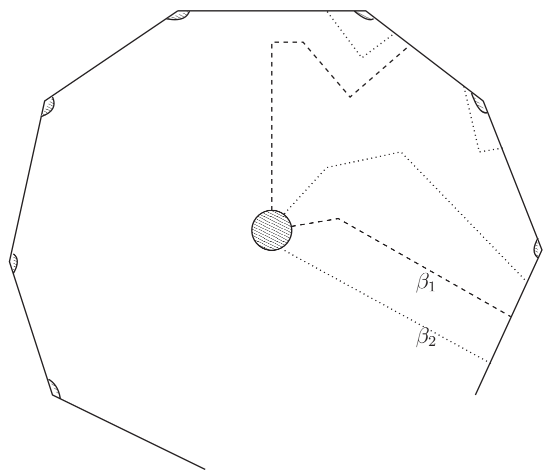

Figure 9. The simplified Heegaard diagram of $M(t_\gamma t_\delta t_\gamma)$. An isotopy on β_1 has been carried out to cancel the pairs of intersection points R_1 and $P_{1,1}$.

tedious, yet elementary, calculation using the Chern class formula, we can identify exactly a single pair of essential generators for each $\mathfrak{s}_{i,j}$ for $(i,j) \neq (m_1+m_2, n_1)$, and $2g-3$ pairs of essential generators for the remaining distinguished spin^c-structure $\mathfrak{s}_{m_1+m_2,n_1}$, much like the situation in the previous section.

Hence, for all $(i,j) \neq (m_1+m_2, n_1)$,

$$HF^+(M(t_\gamma^{m_1} t_\delta^{n_1} t_\gamma^{m_2}), \mathfrak{s}_{i,j}; \omega) = \Lambda.$$

all supported in even gradings.

The Lefschetz number of this monodromy is $2-2g+(m_1+m_2)n_1$. Thus:

$$\begin{aligned} \chi(HF^+_{\mathfrak{s}_{m_1+m_2,n_1}}) &= \tau_t(2g-2) - \sum_{(i,j)\neq(m_1+m_2,n_1)} \chi(HF^+_{\mathfrak{s}_{i,j}}) \\ &= 2-2g+(m_1+m_2)n_1 - ((m_1+m_2)n_1 - 1) \\ &= 3-2g. \end{aligned}$$

The standard argument applies once more and shows

$$HF^+(M(t_\gamma^{m_1} t_\delta^{n_1} t_\gamma^{m_2}), \mathfrak{s}_{m_1+m_2,n_1}; \omega) = \Lambda^{2g-3}.$$

Putting all the spin^c-structures together, we conclude:

Proposition 3.5. $HF^+(M(t_\gamma^{m_1} t_\delta^{n_1} t_\gamma^{m_2}), g-2; \omega) = \Lambda_{\text{even}}^{(m_1+m_2)n_1-1} \oplus \Lambda_{\text{odd}}^{2g-3}$.

3.5. Multiple Dehn twists along a transverse pair of curves $\phi = t_\gamma^{m_1} t_\delta^{n_1} \cdots t_\gamma^{m_k} t_\delta^{n_k}$. Lastly, we consider the manifolds of the form

$M(t_\gamma^{m_1}t_\delta^{n_1}\cdots t_\gamma^{m_k}t_\delta^{n_k})$, where $m_i \cdot n_j < 0$. In other words, they are the mapping tori of Dehn twists along γ and δ with alternating signs.

Let M denote the matrix

$$M := \begin{pmatrix} 1 & m_1 \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \\ -n_1 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & m_k \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \\ -n_k & 1 \end{pmatrix}.$$

Then the Lefschetz number is $4 - 2g - \mathrm{Tr}(M)$.

On the other hand, if we denote by M' the matrix

$$M' := \begin{pmatrix} 1 & |m_1| \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \\ |n_1| & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & |m_k| \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \\ |n_k| & 1 \end{pmatrix}, \tag{1}$$

a direct counting reveals a total number of $2g - 4 + \mathrm{Tr}(M')$ pairs of essential generators in the corresponding special Heegaard diagram. (Refer back to $M(t_\gamma^m t_\delta^n)$ as a special example.)

We claim $\mathrm{Tr}(M) = \mathrm{Tr}(M')$ in our case. This is trivial when $m_i > 0$, or equivalently, $n_j < 0$. In the case $m_i < 0$ and $n_j > 0$, apply induction on k to show that the diagonal entries of M are sum of monomials of even degrees, and hence, equal to the corresponding entries of M'.

From this, we see that the total number of essential generators is the minus of the Lefschetz number, which we denote by L. Hence, the standard argument implies

Proposition 3.6. *$HF^+(M(t_\gamma^{m_1}t_\delta^{n_1}\cdots t_\gamma^{m_k}t_\delta^{n_k}), g-2; \omega) = \Lambda^{|L|}_{\mathrm{odd}}, m_i \cdot n_j < 0$ where $|L| = 2g - 4 + \mathrm{Tr}(M')$, and M' is the matrix defined in (1).*

4. Adjunction inequalities

Having discussed the motivation and applications of the U-action adjunction inequality in the introduction, we are devoted to the proof of Theorem 1.2 in this section. The argument below is due primarily to Yanki Lekili.

Let us first recall the adjunction inequality by Ozsváth and Szabó [**17**].

Theorem 4.1. [**17, Theorem 7.1**] *Let $Z \subset Y$ be a connected, embedded two-manifold of genus $g(Z) > 0$ in an oriented three-manifold with $b_1(Y) > 0$. If $\mathfrak{s}$ is a Spin^c structure for which $HF^+(Y, \mathfrak{s}) \neq 0$, then*

$$|\langle c_1(\mathfrak{s}), [Z]\rangle| \leq 2g(Z) - 2.$$

While Ozsváth and Szabó proved Theorem 4.1 by constructing a particular Heegaard diagram whose generators all lie in the Spin^c structures that satisfy the adjunction inequality, we establish Theorem 1.2 in a more indirect way. Our approach depends on certain formal properties of cobordism in Heegaard Floer homology [**11, 22**].

Let W be an oriented, smooth, connected, four-dimensional cobordism with $\partial W = -Y_1 \cup Y_2$. Fix a Spin^c structure $\mathfrak{s} \in \mathrm{Spin}^c(W)$, and let $\mathfrak{t}_i$ denote its restriction to Y_i. We also fix a cohomology class $\omega \in H^2(W;\mathbb{R})$, and denote its restriction to Y_i by ω_i . Then, there is a cobordism map

$$F^+_{W,\mathfrak{s};\omega} : HF^+(Y_1, \mathfrak{t}_1; \omega_1) \longrightarrow HF^+(Y_2, \mathfrak{t}_2; \omega_2),$$

which is a smooth oriented four-manifold invariant. These maps satisfy a composition law.

Lemma 4.2. [**22, Composition Law**] *If W_1 is a cobordism from Y_1 to Y_2 and W_2 is a cobordism from Y_2 to Y_3, and we equip W_1 and W_2 with* Spin^c *structures $\mathfrak{s}_1$ and $\mathfrak{s}_2$, respectively, whose restrictions agree over Y_2. Let $W = W_1 \#_{Y_2} W_2$. Then for any $\omega \in H^2(W;\mathbb{R})$, we have*

$$F^+_{W_2,\mathfrak{s}_2;\omega|_{W_2}} \circ F^+_{W_1,\mathfrak{s}_1;\omega|_{W_1}} = \sum_{\{\mathfrak{s}\in \mathrm{Spin}^c(W) | \mathfrak{s}|_{W_i} = \mathfrak{s}_i\}} F^+_{W,\mathfrak{s};\omega}.$$

Another necessary ingredient of our proof for Theorem 1.2 is the Heegaard Floer homology of product manifolds $\Sigma_g \times S^1$.

Lemma 4.3. [**11, Theorem 9.4**] *Let η be a two-form perturbed in the S^1-direction of $\Sigma_g \times S^1$, i.e., the cohomology class $\eta \in H^2(Y,\mathbb{R})$ evaluates non-zero on the fiber Σ_g, where $g \geq 2$. Then there is an identification of $\mathbb{Z}[U]$-modules*

$$HF^+(\Sigma_g \times S^1, k; \eta) \cong X(g,d),$$

where $d = g - 1 - |k|$, and

$$X(g,d) = \bigoplus_{i=0}^{d} \Lambda^{2g-i} H^1(\Sigma_g) \otimes_{\mathbb{Z}} (\Lambda[U]/U^{d-i+1}).$$

Note that Lemma 4.3 verifies our desired adjunction inequality for the product manifold $\Sigma_g \times S^1$. It may be also helpful to compare the Lemma with both Proposition 4.5 of [**27**], in which a quite different answer is reached for a generic perturbation; and with Theorem 9.3 of [**19**], in which a very similar result is obtained for the unperturbed Heegaard Floer homology in non-torsion Spin^c structures $k \neq 0$ — simply replace Λ by $\mathbb{Z}$ in the above statement. The result of the torsion Spin^c structure $k = 0$ of the unperturbed case is quite differental though, see [**10,** Theorem 1.1].

Proof of Theorem 1.2. Take $W = Y \times [0,1]$. Let $Z \subset Y$ be a connected, embedded two-manifold of genus g in Y, and let N be the boundary of the tubular neighborhood of Z in W. Clearly, $Z \cdot Z = 0$, so N is diffeomorphic to $\Sigma_g \times S^1$. By fixing a path joining Y to Z, and taking a regular neighborhood, we break the cobordism apart into a piece W_1 from Y to $Y \# N$, and then another piece W_2 from $Y \# N$ to Y.

Suppose $\mathfrak{s}$ is a Spin^c structure on Y. It can be extended uniquely to a Spin^c structure on W, denoted by $\mathfrak{s}$ as well, as $H^2(Y \times [0,1]) \to H^2(Y)$ is an isomorphism. Let $\mathfrak{s}_i$ be the restriction of $\mathfrak{s}$ on W_i, respectively. There is actually a unique way of extending $\mathfrak{s}_i$ to a Spin^c structure $\mathfrak{s} \in \mathrm{Spin}^c(W)$, for the extension of the Spin^c structure $\mathfrak{s}_i|Y$ from Y to W is unique. Hence, the composition law of cobordism implies that

$$F^+_{W_2,\mathfrak{s}_2;\omega|_{W_2}} \circ F^+_{W_1,\mathfrak{s}_1;\omega|_{W_1}} = F^+_{W,\mathfrak{s};\omega},$$

where $\omega \in H^2(W;\mathbb{R})$ is a generic two-form in the sense that $\mathrm{Ker}(\omega) \cap H^2(W;\mathbb{Z}) = \{0\}$.

The cobordism map $F^+_{W,s;\omega}$ is an identity from $HF^+(Y,s;\omega)$ to itself, since W is a product cobordism. Hence, the cobordism map

$$F^+_{W_1,\mathfrak{s}_1;\omega|_{W_1}} : HF^+(Y,\mathfrak{s};\omega|_Y) \longrightarrow HF^+(Y\#N,\mathfrak{s}|_{Y\#N};\omega|_{Y\#N}),$$

is injective. Note that $\omega|_Y$ is a generic form on Y, which we denote by ω as well; and $\omega|_N$ is the image of ω under successive restrictions $H^2(W) \to H^2(\Sigma_g \times D^2) \to H^2(N = \Sigma_g \times S^1)$, corresponding to $\eta = PD([S^1])$ in N. Thus, we can rewrite the cobordism map as

$$F^+_{W_1,\mathfrak{s}_1;\omega|_{W_1}} : HF^+(Y,\mathfrak{s};\omega) \longrightarrow HF^+(Y,\mathfrak{s};\omega) \otimes HF^+(\Sigma_g \times S^1,\mathfrak{s};\eta).$$

Suppose $\mathfrak{s}$ is a Spin^c structure for which $U^j \cdot HF^+(Y,s;\omega) \neq 0$. Then $F^+_{W_1,\mathfrak{s}_1;\omega|_{W_1}}(U^j \cdot HF^+(Y,s;\omega)) \neq 0$ for the map is injective. As $F^+_{W_1,\mathfrak{s}_1;\omega|_{W_1}}$ is U-equivariant, we have $U^j \cdot HF^+(Y,\mathfrak{s};\omega) \otimes HF^+(\Sigma_g \times S^1,\mathfrak{s};\eta) \neq 0$. In particular, multiplying U^j on the second factor shows that $U^j \cdot HF^+(\Sigma_g \times S^1,\mathfrak{s};\eta) \neq 0$. When $g \geq 2$, we obtain $|\langle c_1(\mathfrak{s}),[Z]\rangle| \leq 2g(Z) - 2j - 2$ from Lemma 4.3. When $g = 0$ or 1, the corresponding homology groups are $HF^+(S^2 \times S^1;\eta) = 0$ and $HF^+(\mathbb{T}^3;\eta) = \Lambda[U]/U$; and the adjunction inequality also holds in these cases. $\square$

Remark 4.4. When $j \leq g - 1$, the adjunction inequality (Theorem 1.2) holds for the unperturbed Heegaard Floer homology by the same argument. However, it is unclear to the author how this can be generalized to torsion Spin^c structures (corresponding to $j = g$ and Corollary 1.3) in the unperturbed case.

Corollary 1.3 follows readily from the adjunction inequality. In particular, when specializing to the case $g = 0$, we point out that the converse is also true.

Theorem 4.5. [**15**] *A three-manifold Y contains a homologically non-trivial, embedded sphere if and only if $HF^+(Y;\omega) = 0$.*

Theorem 4.5 follows essentially from [**15,** Theorem 3.6]. In light of Corollary 1.3 and Theorem 4.5, we would like to ask: is the converse also true for higher-genus cases $g > 1$? More generally, is there any special relationship between the U-action and Thurston norms?

As a consequence of Corollary 1.3 and the results in the previous section, we obtain Theorem 1.1.

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Department of Mathematics
Caltech, MC 253-37
1200 E California Blvd
Pasadena, CA 91125
E-mail address: zhongtao@caltech.edu

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