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A CABLING FORMULA FOR THE ν^+ INVARIANT

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Abstract. We prove a cabling formula for the concordance invariant ν^+, defined by the author and Hom in a previous work. This gives rise to a simple and effective four-ball genus bound for many cable knots.

1. Introduction

The invariant ν^+, or equivalently ν^-, is a concordance invariant defined by the author and Hom [4], and by Ozsváth-Stipsicz-Szabó [10] based on Rasmussen's local h invariant [16]. It gives a lower bound on the four-ball genus of knots and can get arbitrarily better bounds than those from the Ozsváth-Szabó τ invariant. In this paper, we prove a cabling formula for ν^+. The main result is:

Theorem 1.1. *For $p, q > 0$ and the cable knot $K_{p,q}$, we have*

$$\nu^+(K_{p,q}) = p\nu^+(K) + \frac{(p-1)(q-1)}{2}$$

when $q \geq (2\nu^+(K) - 1)p - 1$.

As an application of the cabling formula, one can use ν^+ to bound the four-ball genus of cable knots; in certain special cases, ν^+ determines the four-ball genus precisely.

Corollary 1.2. *Suppose K is a knot such that $\nu^+(K) = g_4(K) = n$. Then*

$$\nu^+(K_{p,q}) = g_4(K_{p,q}) = pn + \frac{(p-1)(q-1)}{2}$$

for $q \geq (2n-1)p - 1$.

Take $K = T_{2,5}\#2T_{2,3}\# - T_{2,3;2,5}$, for example. It is known that $g_4(K) = \nu^+(K) = 2$ [4, Corollary 3.2]. Using Corollary 1.2, we can determine the four-ball genus of any cable knots $K_{p,q}$ when $q \geq 3p - 1$. This generalizes [4, Proposition 3.5].

From a different perspective, this article is partly motivated by a question of Akio Kawauchi asking whether "the $(2,1)$-cable of a strongly negative-amphicheiral knot (in particular, the figure-eight knot) is slice". A knot K in the three-sphere S^3 is *strongly negative-amphicheiral* if there is an orientation-reversing involution τ on S^3 such that $\tau(K) = K$ and the fixed point set $\mathrm{Fix}(\tau) = S^0 \subset K$. In [5], Kawauchi proved that the $(2,1)$-cable of a strongly negative-amphicheiral knot

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is rationally slice. On the other hand, Miyazaki showed that the $(2,1)$-cable of a strongly negative-amphicheiral knot with an irreducible Alexander polynomial (e.g., the figure-eight knot) is not ribbon [8]. Therefore, if the $(2,1)$-cable of any one of such knots is slice, this gives a counterexample to the *slice-ribbon conjecture*.

As ν^+ of the figure-eight knot is 0, its $(2,1)$-cable has $\nu^+ = 0$ by Theorem 1.1; our ν^+ invariant does not obstruct the sliceness of the $(2,1)$-cable of the figure-eight knot. On the contrary, if $\nu^+(K) > 0$ holds for some strongly negative-amphicheiral knot K, then $\nu^+(K_{2,1}) > 0$ by Proposition 5.2; in particular, it will imply that $K_{2,1}$ is not a slice knot. This leads us to ask the following question.

Question 1.3. Does $\nu^+(K) = 0$ hold for all strongly negative-amphicheiral knots K?

Unlike ν^+, it is fairly straightforward to see that the Ozsváth-Szabó τ invariant of an amphicheiral knot is always 0. Concerning the behavior of τ invariant under knot cabling, the question was well studied [2], [15], [17], culminating in Hom's explicit formula in [3]. In contrast to the rather explicit computational approach used in these papers, our method of study is based on a special relationship between ν^+ and surgery of knots, and thus avoids the potential difficulty associated to the computation of the knot Floer complex $CFK^\infty(K_{p,q})$.

In order to carry out our proposed method, we need to compute the correction terms on both sides of the reducible surgery (4), which we describe in Section 3. The most technical part of the argument is to identify the projection map of the Spin^c structures in the reducible surgery, and this is discussed in Section 4. The proof of the main theorem follows in Section 5.

2. The invariant ν^+

In this section, we review the definition and properties of the ν^+ invariant from [4] and relevant backgrounds in Heegaard Floer theory. Heegaard Floer homology is a collection of invariants for closed three-manifolds Y in the form of homology theories $HF^\infty(Y)$, $HF^+(Y)$, $HF^-(Y)$, $\widehat{HF}(Y)$ and $HF_{\mathrm{red}}(Y)$. In Ozsváth-Szabó [13] and Rasmussen [16], a closely related invariant is defined for null-homologous knots $K \subset Y$, taking the form of an induced filtration on the Heegaard Floer complex of Y. In particular, let $CFK^\infty(K)$ denote the knot Floer complex of $K \in S^3$, which is $\mathbb{Z} \oplus \mathbb{Z}$-filtered over $\mathbb{F}[U, U^{-1}]$ and that the action of U decreases each filtration by one. Consider the quotient complexes

$$A_k^+ = C\{\max\{i, j-k\} \geq 0\} \quad \text{and} \quad B^+ = C\{i \geq 0\}$$

where i and j refer to the two filtrations. The complex B^+ is isomorphic to $CF^+(S^3)$. Associated to each k, there is a graded, module map

$$v_k^+ : A_k^+ \to B^+$$

defined by projection and another map

$$h_k^+ : A_k^+ \to B^+$$

defined by projection to $C\{j \geq k\}$, followed by shifting to $C\{j \geq 0\}$ via the U-action, and concluding with a chain homotopy equivalence between $C\{j \geq 0\}$ and $C\{i \geq 0\}$. Finally, the ν^+ invariant is defined as

$$\nu^+(K) := \min\{k \in \mathbb{Z} \,|\, v_k^+ : A_k^+ \to CF^+(S^3),\ \ v_k^+(1) = 1\}. \tag{1}$$

Here, 1 denotes the lowest graded generator of the non-torsion class in the homology of the complex, and we abuse our notation by identifying A_k^+ and $CF^+(S^3)$ with their homologies.

Recall that in the large N surgery, v_k^+ corresponds to the maps induced on HF^+ by the two handle cobordism from $S^3_N(K)$ to S^3 [13, Theorem 4.4]. This allows one to extract four-ball genus bound from functorial properties of the cobordism map. We list below some additional properties of ν^+, all of which can be found in [4].

(a) ν^+ is a smooth concordance invariant, taking non-negative integer values.
(b) $\tau(K) \leq \nu^+(K) \leq g_4(K)$. (See [12] for the definition of τ.)
(c) For a quasi-alternating knot K, $\nu^+(K) = \begin{cases} 0 & \text{if } \sigma(K) \geq 0, \\ -\frac{\sigma(K)}{2} & \text{if } \sigma(K) < 0. \end{cases}$
(d) For a strongly quasi-positive knot K,

$$\nu^+(K) = \tau(K) = g_4(K) = g(K).$$

For a rational homology three-sphere Y with a Spin^c structure $\mathfrak{s}$, $HF^+(Y,\mathfrak{s})$ is the direct sum of two groups: the first group is the image of $HF^\infty(Y,\mathfrak{s}) \cong \mathbb{F}[U,U^{-1}]$ in $HF^+(Y,\mathfrak{s})$, which is isomorphic to $\mathcal{T}^+ = \mathbb{F}[U,U^{-1}]/U\mathbb{F}[U]$, and its minimal absolute $\mathbb{Q}$–grading is an invariant of $(Y,\mathfrak{s})$, denoted by $d(Y,\mathfrak{s})$, the *correction term* [11]; the second group is the quotient modulo the above image and is denoted by $HF_{\text{red}}(Y,\mathfrak{s})$. Altogether, we have

$$HF^+(Y,\mathfrak{s}) = \mathcal{T}^+ \oplus HF_{\text{red}}(Y,\mathfrak{s}).$$

Using this splitting, we can associate for each integer k and the knot K a non-negative integer $V_k(K)$ that equals the U-exponent of the map v_k^+ restricted to[1] $\mathcal{T}^+ \in A_k^+$. This sequence of $\{V_k\}$ is non-increasing, i.e., $V_k \geq V_{k+1}$, and stabilizes at 0 for large k. Observe that the minimum k for which $V_k = 0$ is the same as $\nu^+(K)$ defined in (1). This enables us to reinterpret the ν^+ invariant in the following more concise way:

$$\nu^+(K) = \min\{k \in \mathbb{Z} \mid V_k = 0\}. \tag{2}$$

In addition, the sequence $\{V_k\}$ completely determines the correction terms of manifolds obtained from knot surgery. This can be seen from the surgery formula [9, Proposition 1.6],

$$d(S^3_{p/q}(K), i) = d(L(p,q), i) - 2\max\{V_{\lfloor \frac{i}{q} \rfloor}, V_{\lfloor \frac{p+q-1-i}{q} \rfloor}\}, \tag{3}$$

for $p, q > 0$ and $0 \leq i \leq p-1$. We will explain this formula in greater detail in Section 4.

We conclude this section by mentioning that an invariant equivalent to ν^+, denoted ν^- by Ozsváth and Szabó, was formulated in terms of the chain complex CFK^- in [10]. That invariant played an important role to establish a four-ball genus bound for a one-parameter concordance invariant $\Upsilon_K(t)$ defined in the same reference. For the purpose of this paper, we will not elaborate on that definition.

3. Reducible surgery on cable knots

Recall that the (p,q)-cable of a knot K, denoted $K_{p,q}$, is a knot supported on the boundary of a tubular neighborhood of K with slope p/q with respect to the

[1] Again, we abuse the notation by identifying A_k^+ with its homology.

standard framing of this torus. A well-known fact in low-dimensional topology states that the pq-surgery on $K_{p,q}$ results in a reducible three-manifold.

Proposition 3.1.

$$S^3_{pq}(K_{p,q}) \cong S^3_{q/p}(K)\#L(p,q). \tag{4}$$

The above homeomorphism is exhibited in many references (cf. [1], [2]). For self-containedness, we include a proof of Proposition 3.1 below. Not only is this reducible surgery a key ingredient of establishing our main result Theorem 1.1, the geometric description of the homeomorphism is also crucial for justifying Lemma 4.1. Our exposition will follow [2].

Proof of Proposition 3.1. Denote $N(K)$ the tubular neighborhood of K and $E(K) = S^3 - N(K)$ its complement, and let $T(K)$ be the boundary torus of $N(K)$. The cable $K_{p,q}$ is embedded in $T(K)$ as a curve of slope p/q. Consider the tubular neighborhood $N(K_{p,q})$ of the cable. The solid torus $N(K_{p,q})$ intersects $T(K)$ at an annular neighborhood $A = N(K_{p,q}) \cap T(K)$, and the boundary of this annulus consists of two parallel copies of $K_{p,q}$, denoted by λ and λ', each of which have linking number pq with $K_{p,q}$. Therefore, the surgery slope of coefficient pq is given by λ (or equivalently, λ'), and the pq-surgery on $K_{p,q}$ is performed by gluing a solid torus to the knot complement $E(K_{p,q})$ in such a way that the meridian is identified with a curve isotopic to λ.

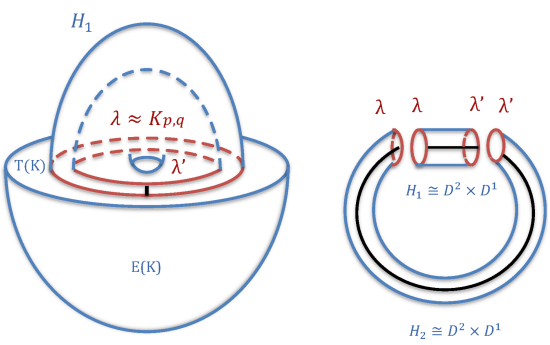

FIGURE 1. Attach the two-handle H_1 to $E(K)$ along $K_{p,q}$. The two disk-ends of H_1 are identified with the corresponding disk-ends of the other two-handle H_2 that is attached to $N(K)$.

Alternatively, one can think of the above gluing as attaching a pair of two-handles H_1, H_2 to $E(K_{p,q})$. See Figure 1. Since the exterior of $K_{p,q}$ is homeomorphic to

$$E(K_{p,q}) = E(K) \cup_{T(K)-A} N(K),$$

its pq-surgery can be decomposed as

$$S^3_{pq}(K_{p,q}) = [E(K) \cup H_1] \cup [N(K) \cup H_2].$$

As the two-handles are attached along essential curves on $T(K)$ (isotopic to λ), $E(K) \cup H_1$ and $N(K) \cup H_2$ end up having a common boundary homeomorphic to S^2. This proves that $S^3_{pq}(K_{p,q})$ is a reducible manifold.

To further identify the two pieces of the reducible manifold, note that the attaching curve is isotopic to λ, which has slope p/q on $T(K)$. It follows that

$$N(K) \cup H_2 \cong L(p,q) - D^3.$$

From the perspective of $E(K)$, the curve λ has slope q/p. Thus, the other piece is

$$E(K) \cup H_1 \cong S^3_{q/p}(K) - D^3.$$

This completes the proof. □

4. Spin^c structures in reducible surgery

Let us take a closer look at the surgery formula (3), in which there is an implicit identification of Spin^c structures

$$\sigma : \mathbb{Z}/p\mathbb{Z} \to \mathrm{Spin}^c(S^3_{p/q}(K)). \tag{5}$$

For simplicity, we use an integer $0 \le i \le p-1$ to denote the Spin^c structure $\sigma([i])$, when $[i] \in \mathbb{Z}/p\mathbb{Z}$ is the congruence class of i modulo p. The identification can be made explicit by the procedure in Ozsváth and Szabó [14, Section 4,7]. In particular, it is independent of the knot K on which the surgery is applied[2] and it is affine,

$$\sigma[i+1] - \sigma[i] = [K'] \in H_1(S^3_{p/q}(K)) \cong \mathrm{Spin}^c(S^3_{p/q}(K)),$$

where K' is the dual knot of the surgery on K, and Spin^c structures are affinely identified with the first homology. Moreover, the conjugation map J on Spin^c structures can be expressed as

$$J(\sigma([i])) = \sigma([p+q-1-i]) \tag{6}$$

(cf. [7, Lemma 2.2]). We will use these identifications throughout this paper.

In [11, Proposition 4.8], Ozsváth and Szabó made an identification of Spin^c structures on lens spaces through their standard genus 1 Heegaard diagram, which coincide with the above identification through surgery (on the unknot). They also proved the following recursive formula for the correction terms of lens spaces:

$$d(L(p,q),i) = \frac{(2i+1-p-q)^2 - pq}{4pq} - d(L(q,r),j) \tag{7}$$

for positive integers $p > q$ and $0 \le i < p+q$, where r and j are the reductions modulo q of p and i, respectively. Substituting in $q = 1$, one sees:

$$d(L(p,1),i) = \frac{(2i-p)^2 - p}{4p}. \tag{8}$$

For a reducible manifold $Y = Y_1 \# Y_2$, there are projections from Spin$^c(Y)$ to the Spin^c structures of the two factors Spin$^c(Y_1)$ and Spin$^c(Y_2)$. Particularly, this applies to the case of the reducible surgery $S^3_{pq}(K_{p,q}) \cong S^3_{q/p}(K) \# L(p,q)$. In terms of the canonical identification (5), we write $\phi_1 : \mathbb{Z}/pq\mathbb{Z} \to \mathbb{Z}/q\mathbb{Z}$ and $\phi_2 : \mathbb{Z}/pq\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ for the two projections. These two maps are independent of the knot K, which we determine explicitly in the next lemma.

[2]Thus, formula (3) may be interpreted as comparing the correction terms of the "same" Spin^c structures of surgery on different knots.

With the above notation and identifications of Spin^c structures understood, we apply the surgery formula (3) to both sides of the reducible manifold $S^3_{pq}(K_{p,q}) \cong S^3_{q/p}(K)\#L(p,q)$ and deduce

$$d(L(pq,1),i) - 2V_i(K_{p,q}) = d(L(q,p),\phi_1(i)) + d(L(p,q),\phi_2(i)) - 2\max\{V_{\lfloor \frac{\phi_1(i)}{p} \rfloor}(K), V_{\lfloor \frac{p+q-1-\phi_1(i)}{p} \rfloor}(K)\} \tag{9}$$

for all $i \leq \frac{pq}{2}$. Here we used the fact that $V_i \geq V_{pq-i}$ when $i \leq \frac{pq}{2}$, as $\{V_k\}$ is a non-increasing sequence. When K is the unknot, (9) simplifies to

$$d(L(pq,1),i) - 2V_i(T_{p,q}) = d(L(q,p),\phi_1(i)) + d(L(p,q),\phi_2(i)) \tag{10}$$

as all V_i's are 0 for the unknot.

For the rest of the section, assume $Y = S^3_{pq}(K_{p,q})$, $Y_1 = S^3_{q/p}(K)$ and $Y_2 = L(p,q)$, and denote by $K' \subset Y_1 = S^3_{q/p}(K)$ and $K'_{p,q} \subset Y = S^3_{pq}(K_{p,q})$ the dual knots of the surgeries on K and $K_{p,q}$, respectively.

Lemma 4.1. *The projection maps ϕ_1 and ϕ_2 of the* Spin^c *structures are given by:*

$$\phi_1(i) = i - \frac{(p-1)(q-1)}{2} \pmod q;$$

$$\phi_2(i) = i - \frac{(p-1)(q-1)}{2} \pmod p.$$

Proof. Since the projection maps are affine, we assume

$$\phi_1(i) = a_1 \cdot i + b_1 \pmod q,$$

$$\phi_2(i) = a_2 \cdot i + b_2 \pmod p.$$

Note that the maps ϕ_1, ϕ_2 are generally not homomorphisms. Nevertheless, we claim that $\phi_1(i+1) - \phi_1(i) = 1 \pmod q$. Under Ozsváth-Szabó's canonical identification $\sigma : \mathbb{Z}/pq\mathbb{Z} \to \text{Spin}^c(Y)$, we have $\sigma[i+1] - \sigma[i] = [K'_{p,q}] \in H_1(Y) \cong \mathbb{Z}/pq\mathbb{Z}$. Thus, it amounts to showing that $\phi_1[K'_{p,q}] = [K']$ under the projection of Y into the first factor Y_1 and the identification (5).

This can be seen from the geometric description of the reducible surgery in the last section: The dual knot $K'_{p,q}$, which is isotopic to the closed black curve on the right of Figure 1, projects to an arc in $E(K) \cup H_1$ on the left of Figure 1. This arc is closed up in Y_1 by connecting it to a simple arc in $D^3 = Y_1 - (E(K) \cup H_1)$. Since the curve intersects λ once, it must represent[3] $[K'] \in H_1(Y_1)$. Hence

$$1 = \phi_1(i+1) - \phi_1(i) = (a_1 \cdot (i+1) + b_1) - (a_1 \cdot i + b_1) \pmod q$$

from which we see $a_1 = 1$. A similar argument proves $a_2 = 1$.

To determine b_1, note that the projection ϕ_1 commutes with the conjugation J as operations on Spin^c structures. After substituting the equation $\phi_1(i) = i + b_1 \pmod q$ into $\phi_1 \circ J = J \circ \phi_1$ and applying (6), we get

$$(pq - i) + b_1 = p + q - 1 - (i + b_1) \pmod q.$$

So

$$b_1 = \begin{cases} -\frac{(p-1)(q-1)}{2} & \text{if } q \text{ is odd,} \\ -\frac{(p-1)(q-1)}{2} \text{ or } -\frac{(p-1)(q-1)}{2} + \frac{q}{2} & \text{if } q \text{ is even,} \end{cases}$$

[3]To be accurate, this is true up to a proper choice of orientation of K'.

where the identity is understood modulo q as before. Similar arguments also imply:

$$b_2 = \begin{cases} -\frac{(p-1)(q-1)}{2} & \text{if } p \text{ is odd,} \\ -\frac{(p-1)(q-1)}{2} \text{ or } -\frac{(p-1)(q-1)}{2} + \frac{p}{2} & \text{if } p \text{ is even.} \end{cases}$$

We argue that $b_1 = b_2 = -\frac{(p-1)(q-1)}{2}$. This is evidently true when both p and q are odd integers. When p is even and q is odd, we want to exclude the possibility $b_1 = -\frac{(p-1)(q-1)}{2}$ and $b_2 = -\frac{(p-1)(q-1)}{2} + \frac{p}{2}$ using the method of proof by contradiction. A similar argument will address the case for which p is odd and q is even, and thus completes the proof.

We derive a contradiction by comparing the correction terms computed in two ways. From equations (7) and (8), we have

$$\begin{aligned} d(L(pq,1), j + \frac{(p-1)(q-1)}{2}) &= \frac{(2j+1-p-q)^2 - pq}{4pq} \\ &= d(L(q,p),j) + d(L(p,q),j) \end{aligned}$$

for $0 \le j < p+q$.

On the other hand, it follows from (10) that

$$d(L(pq,1), j + \frac{(p-1)(q-1)}{2}) = d(L(q,p),j) + d(L(p,q), j + \frac{p}{2})$$

for $0 \le j < p+q-1$. Here, we used the fact that $V_i(T_{p,q}) = 0$ for all $i > \frac{(p-1)(q-1)}{2}$ (since $g_4(T_{p,q}) = \frac{(p-1)(q-1)}{2}$) and the assumptions $\phi_1(i) = i - \frac{(p-1)(q-1)}{2}$ and $\phi_2(i) = i - \frac{(p-1)(q-1)}{2} + \frac{p}{2}$. Comparing the above two identities, we obtain

$$d(L(p,q),j) = d(L(p,q), j + \frac{p}{2}). \tag{11}$$

Recall from Lee-Lipshitz [6, Corollary 5.2] that correction terms of lens spaces also satisfy the identity

$$d(L(p,q), j+q) - d(L(p,q),j) = \frac{p-1-2j}{p}$$

for $0 \le j < p$. It follows

$$d(L(p,q), j + \frac{p}{2} + q) - d(L(p,q), j + \frac{p}{2}) = \frac{p-1-2(j+\frac{p}{2})}{p} = \frac{-1-2j}{p}.$$

Yet, according to (11),

$$\begin{aligned} &d(L(p,q), j + \frac{p}{2} + q) - d(L(p,q), j + \frac{p}{2}) \\ &\quad = d(L(p,q), j+q) - d(L(p,q),j) = \frac{p-1-2j}{p}. \end{aligned}$$

We reach a contradiction! This completes the proof of the lemma. □

As a quick check of Lemma 4.1, let us look at the surgery $S^3_{15}(T_{3,5}) \cong L(5,3)\#L(3,5)$. The correction terms of the three lens spaces with Spin^c structure i are computed using (7) and summarized in Table 1.

Meanwhile, we compute the projection functions $\phi_1(i)$, $\phi_2(i)$, according to the formula in Lemma 4.1, and $V_i(T_{3,5})$ from the well-known knot Floer complex CFK^∞ of torus knots. The results are summarized in Table 2. We can then verify identity (10) using the values of correction terms provided in Table 1.

TABLE 1. Table of correction terms for lens spaces

lens space \ i	0	1	2	3	4	5	6	$\cdots$
$L(15,1)$	$7/2$	$77/30$	$53/30$	$11/10$	$17/30$	$1/6$	$-1/10$	$\cdots$
$L(5,3)$	$2/5$	0	$2/5$	$-2/5$	$-2/5$	$2/5$	$2/5$	$\cdots$
$L(3,5)$	$1/6$	$1/6$	$-1/2$	$1/6$	$1/6$	$-1/2$	$1/6$	$\cdots$

In particular, note that at $i = \frac{(p-1)(q-1)}{2} = 4$, there is the column $\phi_1 = \phi_2 = V = 0$, as expected from Lemma 4.1, and there is the identity

$$d(L(15,1),4) - 2V_4 = 17/30 = 2/5 + 1/6 = d(L(5,3),0) + d(L(3,5),0)$$

that we can read off, in accordance with equation (10).

TABLE 2. Projections ϕ_1, ϕ_2 are given by Lemma 4.1, and the V_i's are computed directly from the knot Floer complex $CFK^\infty(T_{3,5})$

Function \ i	0	1	2	3	4	5	6	$\cdots$
ϕ_1	1	2	3	4	0	1	2	$\cdots$
ϕ_2	2	0	1	2	0	1	2	$\cdots$
V	2	1	1	1	0	0	0	$\cdots$

5. PROOF OF THE CABLING FORMULA

In this section, we prove Theorem 1.1. First, note the following relationship between the sequences $V_i(K_{p,q})$ and $V_i(K)$ if we compare (9) and (10).

Lemma 5.1. *Given $p, q > 0$ and $i \leq \frac{pq}{2}$, the sequence of non-negative integers $V_i(K_{p,q})$ and $V_i(K)$ satisfy the relation*

$$V_i(K_{p,q}) = V_i(T_{p,q}) + \max\{V_{\lfloor \frac{\phi_1(i)}{p} \rfloor}(K), V_{\lfloor \frac{p+q-1-\phi_1(i)}{p} \rfloor}(K)\}. \tag{12}$$

Here, $\phi_1(i) = i - \frac{(p-1)(q-1)}{2}$ as above.

In order to evaluate $\nu^+(K_{p,q})$ from equation (2), it is enough to determine the minimum i such that $V_i(K_{p,q}) = 0$. Since $V_i(T_{p,q}) > 0$ when $i < \frac{(p-1)(q-1)}{2}$, we only need to consider $i \geq \frac{(p-1)(q-1)}{2}$ by (12).

Proof of Theorem 1.1. When $q \geq (2\nu^+(K)-1)p+1$, we have $p\nu^+(K) + \frac{(p-1)(q-1)}{2} \leq \frac{pq}{2}$, so the condition $i \leq \frac{pq}{2}$ in Lemma 5.1 is satisfied for all i in the range $\frac{(p-1)(q-1)}{2} \leq i \leq p\nu^+(K) + \frac{(p-1)(q-1)}{2}$. Equation (12) simplifies to

$$V_i(K_{p,q}) = V_{\lfloor \frac{\phi_1(i)}{p} \rfloor}(K) \tag{13}$$

as $V_i(T_{p,q}) = 0$ and $\lfloor \frac{\phi_1(i)}{p} \rfloor \leq \lfloor \frac{p+q-1-\phi_1(i)}{p} \rfloor$ for $\phi_1(i) = i - \frac{(p-1)(q-1)}{2}$. As $V_i(K) = 0$ if and only if $i \geq \nu^+(K)$, it is easy to see from (13) and Lemma 4.1 that the minimum i such that $V_i(K_{p,q}) = 0$ is $p\nu^+(K) + \frac{(p-1)(q-1)}{2}$. Hence, $\nu^+(K_{p,q}) = p\nu^+(K) + \frac{(p-1)(q-1)}{2}$. □

When $q < (2\nu^+(K)-1)p+1$, the cabling formula for $\nu^+(K_{p,q})$ is still unknown. Nevertheless, the preceding argument gives the following lower bound.

Proposition 5.2. *For $p, q > 0$ and the cable knot $K_{p,q}$,*

$$\nu^+(K_{p,q}) > \frac{pq}{2}$$

when $q < (2\nu^+(K)-1)p+1$ and $\nu^+(K) > 0$.

Proof. It is enough to show that $V_{\frac{pq}{2}}(K_{p,q}) > 0$. Remember that the minimum i such that $V_i(K_{p,q}) = 0$ is $p\nu^+(K) + \frac{(p-1)(q-1)}{2}$, according to the proof of Theorem 1.1. The statement follows from the monotonicity of $\{V_k\}$ and the inequality $\frac{pq}{2} < p\nu^+(K) + \frac{(p-1)(q-1)}{2}$. □

Theorem 1.1 is useful when one aims to determine the four-ball genus of some cable knots (e.g. Corollary 1.2).

Proof of Corollary 1.2. The four-ball genus of the cable knot is bounded above by

$$g_4(K_{p,q}) \leq pg_4(K) + \frac{(p-1)(q-1)}{2}$$

because one can construct a slice surface for $K_{p,q}$ from p parallel copies of a slice surface for K together with $(p-1)q$ half-twisted bands. Using Theorem 1.1, we see that when $\nu^+(K) = g_4(K) = n$,

$$pn + \frac{(p-1)(q-1)}{2} = \nu^+(K_{p,q}) \leq g_4(K_{p,q}) \leq pn + \frac{(p-1)(q-1)}{2},$$

from which Corollary 1.2 follows immediately. □

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