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BOUNDARY TREATMENTS FOR MULTILEVEL METHODS ON UNSTRUCTURED MESHES*

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Abstract. In applying multilevel iterative methods on unstructured meshes, the grid hierarchy can allow general coarse grids whose boundaries may be nonmatching to the boundary of the fine grid. In this case, the standard coarse-to-fine grid transfer operators with linear interpolants are not accurate enough at Neumann boundaries so special care is needed to correctly handle different types of boundary conditions. We propose two effective ways to adapt the standard coarse-to-fine interpolations to correctly implement boundary conditions for two-dimensional polygonal domains, and we provide some numerical examples of multilevel Schwarz methods (and multigrid methods) which show that these methods are as efficient as in the structured case. In addition, we prove that the proposed interpolants possess the local optimal L^2-approximation and H^1-stability, which are essential in the convergence analysis of the multilevel Schwarz methods. Using these results, we give a condition number bound for two-level Schwarz methods.

Key words. iterative methods, domain decomposition, unstructured meshes

AMS subject classifications. 65F10, 65N30, 65N55

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1. Introduction.

Unstructured grids have become popular in scientific computing because they can be easily adapted to complex geometries and sharp gradients in the solution [3, 12, 17]. However, in order to compete with structured meshes which can exploit the regularity of the mesh, there is a need to develop efficient solvers on unstructured meshes including good multilevel algorithms such as domain decomposition or multigrid methods. Since no natural coarse grids exist as in structured meshes, practical multilevel domain decomposition and multigrid algorithms must allow coarser grids which are nonquasi uniform with boundaries and interior elements which are not necessarily matching to that of the fine mesh. The traditional solvers need to be modified so that their efficiency will not be adversely affected by this lack of structure and to ensure that a proper sequence of coarse subspaces exists for the domain decomposition or multigrid methods.

Providing a coarse grid hierarchy for multilevel methods poses some difficulties when using unstructured meshes, and several different approaches have been developed recently (see, for instance, [14, 15, 18, 19]). One technique generates a coarse grid hierarchy by using independent grids created by some grid generator (for example, the one which produced the original grid). Another approach uses agglomeration techniques to create a coarse space hierarchy. Still another method uses a graph approach by forming maximal independent sets (MIS) of the boundaries and interiors

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of the mesh and then retriangulating the resulting vertex set. The advantage of using a MIS approach is that the grids are node-nested, and thus efficient methods can be used to create the interpolation and restriction operators needed to transfer information from one level to the other. A disadvantage, however, is that for complicated geometries, particularly in three dimensions, special care must be taken to ensure that the coarse grids which are produced are valid and preserve the important geometric features of the fine domain.

Using MIS coarsening to generate a coarse grid hierarchy, it was shown in [7] that for domain decomposition methods for elliptic problems on unstructured meshes, the same optimal convergence rate can be achieved as in the structured case provided that the coarse grid domain covers the Neumann boundary part of the fine grid domain, but no such requirement is needed for homogeneous Dirichlet boundary conditions. This was demonstrated numerically with problems on the unit square by physically extending the coarse grid domain beyond the Neumann boundaries and using linear interpolation.

In this paper, we will extend this idea to include interpolants with nonzero extensions which do not require the coarse grid domain be modified to cover the Neumann boundary part of the fine grid domain, and we will provide some analysis on a crucial step in the convergence analysis of two-level Schwarz methods on unstructured meshes using such coarse-to-fine interpolants. We will follow the general framework for convergence analyses applicable to unstructured meshes in [7, 8, 9], which can be viewed as a natural extension of the one formulated by Xu [23] for structured meshes. Some preliminary results can be found in [5].

This paper is arranged as follows. The considered elliptic problem is introduced in section 2, and the coarse-to-fine grid transfer operators along with several particular interpolants are defined in section 3. In section 4, we provide some numerical results on multilevel Schwarz (cf. [2, 24]) and multigrid methods using the coarse-to-fine grid transfer operators proposed in section 3. Previous numerical results on multilevel Schwarz methods on structured grids can be found in [21, 24]. The results we present here, however, appear to be the first on multilevel Schwarz on unstructured grids.

Section 5 gives an optimal condition number bound for the two-level additive Schwarz method. The optimal L^2-approximation and H^1-stability properties of the interpolants, which are essential in the convergence analysis of the multilevel Schwarz methods, are shown in the appendix. As multilevel additive methods need some more technical tools, for example, stability of the inverse of the coarse-to-fine interpolant (cf. Chan–Zou [9]), a full multilevel convergence theory is beyond the scope of this paper. Though it is not clear to us whether the two-level convergence results can be extended to the multilevel case, we emphasize that the proposed interpolants are applicable to the general multilevel case and the numerical results in section 4 show that, in practice, optimal convergence for the multilevel case can be achieved using these interpolants. We summarize with some conclusions in section 6.

2. The elliptic problem. Let us consider the elliptic boundary value problem

$$
\begin{aligned}
-\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) + b\,u &= f \ \text{ in } \Omega,\\
u &= 0 \text{ on } \Gamma_D,\\
\sum_{i,j=1}^{2} a_{ij}\frac{\partial u}{\partial x_j}\gamma_i &= 0 \text{ on } \Gamma_N,
\end{aligned}
$$

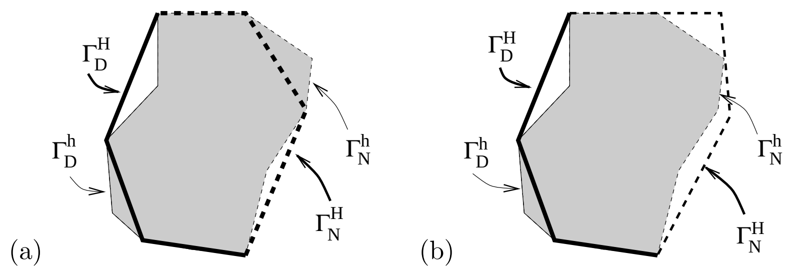

FIG. 1. *Zero extension interpolants:* (a) $\mathcal{I}_h^0$*: with unmodified coarse boundaries;* (b) $\mathcal{I}_h^1$*: with modified coarse boundaries to cover the parts where Neumann conditions exist (dashed lines). Thick lines represent coarse grid boundaries.*

where $(a_{ij}(x))$ is symmetric and uniformly positive definite and $b(x) \geq 0$ in Ω. Ω is a polygonal domain, and Γ_D and Γ_N are two curves consisting of piecewise straight lines with $\bar{\Gamma}_D \cup \bar{\Gamma}_N = \partial\Omega$. $\gamma = (\gamma_1, \gamma_2)$ is the unit outward normal to $\partial\Omega$.

Let $\mathcal{T}^h$ be a given fine triangulation of the domain Ω with triangular elements, and let V^h be the piecewise linear finite element space defined on $\mathcal{T}^h$ with functions vanishing at the nodal points lying in the Dirichlet boundary part Γ_D. Suppose $\mathcal{T}^H$ is a coarse triangulation of the domain Ω with its elements forming a polygonal domain Ω^H. With unstructured meshes, the MIS coarsening strategy for generating a coarse grid hierarchy may produce coarse grid domains whose boundaries do not match that of the fine domain. Note then that Ω^H is allowed to be nonnested and nonmatching with Ω, so in general we have $\Omega^H \neq \Omega$ (see Fig. 1). Moreover, we do not require the coarse grid $\mathcal{T}^H$ to have anything to do with the fine grid $\mathcal{T}^h$, i.e., none of the nodes of $\mathcal{T}^H$ need be nodes of $\mathcal{T}^h$, but only that it is shape regular. No assumption on quasi uniformity is made on the grids $\mathcal{T}^h$ and $\mathcal{T}^H$. Let V^H be the piecewise linear finite element space corresponding to the coarse grid triangulation $\mathcal{T}^H$ and the boundary condition in V^H be defined as follows: each boundary node $x_i^H \in \partial\Omega^H$ in $\mathcal{T}^H$ is assigned the same boundary condition type (Dirichlet or Neumann) as the closest fine boundary node to x_i^H. By changing boundary conditions for a few coarse boundary nodes, if needed, the coarse boundary nodes can be arranged in such a way that two neighbors of each Neumann (resp., Dirichlet) node are also of Neumann (resp., Dirichlet) type with only two Neumann (resp., Dirichlet) nodes near two junctions between Γ_D and Γ_N to have one Dirichlet and one Neumann node as its two neighbors.

It is intuitively obvious that for the coarse grid, $\mathcal{T}^H$, to assist in accelerating the convergence of iterative methods on the fine grid, $\mathcal{T}^h$, it cannot be allowed to be too small compared with the fine grid. Therefore, we always assume that Ω^H covers a significant part of Ω. More accurately, we assume that there exists a positive constant C such that for any point $x \in \partial\Omega$, we have

$$\text{dist}(x,\ \tau^H) \leq C\, d(\tau^H),$$

where τ^H is the closest element in $\mathcal{T}^H$ to x and $d(\tau^H)$ the diameter of τ^H.

Two-level Schwarz methods. Here and below, the subscript which is a domain or subdomain stresses that the integral involved is done over the related domain or subdomain, e.g., A_Ω and A_{Ω^k} below.

The finite element approximation to the original elliptic problem can be formulated as follows. Find $u \in V^h$ such that

$$A_\Omega(u,v) = (f,v) \quad \forall\, v \in V^h, \tag{2.1}$$

where the bilinear form $A_\Omega(\cdot\,,\cdot)$ is defined by

$$A_\Omega(u,v) = \int_\Omega \left(\sum_{i,j=1}^{2} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + b\, uv \right) dx.$$

To construct the two-level additive Schwarz method for solving the finite element system (2.1), we first decompose the domain Ω into p nonoverlapping subdomains $\tilde{\Omega}^k$ $(1 \le k \le p)$, then extend each $\tilde{\Omega}^k$ to a larger one Ω^k such that the distance between $\partial\Omega^k$ and $\partial\tilde{\Omega}^k$ is bounded from below by $\delta_k > 0$. We assume that $\partial\Omega^k$ does not cut through any element $\tau^h \in \mathcal{T}^h$.

Corresponding to each subdomain Ω^k, we define a subspace V^k of V^h by

$$V^k = \{v \in V^h;\ \ v = 0 \ \text{ on } \ \Omega \setminus \Omega^k\}.$$

We now introduce a fine-grid operator A on V^h and a coarse-grid operator A_H on V^H by

$$(Au,v)_\Omega = A_\Omega(u,v) \quad \forall\, u,v \in V^h, \quad (A_H u,v)_{\Omega^H} = A_{\Omega^H}(u,v) \quad \forall\, u,v \in V^H$$

and a local operator A_k on each subspace V^k by

$$(A_k u,v)_{\Omega^k} = A_{\Omega^k}(u,v)\ \ \forall\, u,v \in V^k.$$

Let $f_h \in V^h$ be the L^2 projection of f; then the finite element system (2.1) is equivalent to the equation

$$Au = f_h,$$

which can be solved by the preconditioned CG method with the two-level additive Schwarz preconditioner M. We next construct M. To this aim, we need a "prolongation" operator $\mathcal{I}_k$ from each subspace V^k to V^h and $\mathcal{I}_h$ from V^H to V^h, respectively. Then the two-level additive Schwarz preconditioner M can be formulated as

$$M = \mathcal{I}_h A_H^{-1} Q_h + \sum_{k=1}^{p} \mathcal{I}_k A_k^{-1} Q_k \tag{2.2}$$

with $Q_h : V^h \to V^H$ and $Q_k : V^h \to V^k$ being the adjoints of $\mathcal{I}_h$ and $\mathcal{I}_k$, respectively (cf. [9]).

Since $V^k \subset V^h$, $1 \le k \le p$, the natural injection $\mathcal{I}_k$ can be taken as the prolongation operator from V^k to V^h. The coarse grid space V^H, however, is not generally a subspace of V^h as the coarse elements are often not the unions of some fine elements in the unstructured grid, even if $\Omega^H = \Omega$. It was shown in [7] that unstructured grid methods were as efficient as those for structured grids. However, in addition to the nonnestedness of the coarse grid space induced by the unstructured grid, when the coarse grid boundary $\partial\Omega^H$ does not match with the original boundary $\partial\Omega$, the coarse space V^H will not be a subspace of the fine space V^h. We focus on this case and define a general interpolant $\mathcal{I}_h$ in section 3 which can be used for the prolongation operator from V^H to V^h.

3. Coarse-to-fine interpolations. To construct a coarse-to-fine transfer operator, one may easily come up with the standard nodal value interpolant associated with the fine space V^h. But notice that this interpolant is well defined only for those fine nodes also lying in the coarse domain $\bar{\Omega}^H$ and is meaningless for those fine nodes lying outside $\bar{\Omega}^H$. A simple and natural way to remove this barrier is to assign those fine node values by zero. This is indeed a reasonable and efficient thing to do when the assignment is done along the coarse boundary part of Dirichlet type (which is also near the fine boundary part of Dirichlet type). We shall denote this interpolant as the coarse-to-fine interpolant, $\mathcal{I}_h^0$.

$\mathcal{I}_h^0$: Zero extension with unmodified coarse boundaries. Where coarse grid boundary conditions are of Dirichlet type, the standard nodal value interpolants with zero extensions can be accurate enough for interpolating fine grid values outside the coarse grid domain Ω^H (cf. Fig. 1(a)); we refer to [6, 7] for the theoretical and numerical justifications of $\mathcal{I}_h^0$.

Although the interpolant $\mathcal{I}_h^0$ is appropriate to use at Dirichlet boundaries, it is not accurate enough to use at Neumann boundaries, or not accurate at all sometimes; see the numerical results in [7] and section 4. To achieve better efficiency, we should modify this intergrid operator to account for the Neumann condition. We now propose two general ways to treat such boundaries:

1. Modify the coarse grid domain to cover any fine grid boundaries of Neumann type.
2. Increase the accuracy of the interpolants by accounting for the Neumann condition for those fine nodes in $\Omega\backslash\Omega^H$.

The first approach is motivated by the fact that standard nodal value interpolants can still be used with efficiency where the coarse grid covers the Neumann boundary part of the fine grid. This was first proposed and justified in [7]. We shall denote this operator as the coarse-to-fine interpolant, $\mathcal{I}_h^1$.

$\mathcal{I}_h^1$: Zero extension with modified coarse boundaries. Modify the original coarse grid domain Ω^H to make it appropriately larger so that it covers the Neumann boundary part of the fine grid domain (see Fig. 1(b)). Let us still denote the modified coarse grid domain by Ω^H. Then for all $v^H \in V^H$, the interpolant $\mathcal{I}_h^1$ is defined as

$$\mathcal{I}_h^1 v^H(x_j^h) = \begin{cases} v^H(x_j^h) & \text{for } x_j^h \in \Omega \cap \bar{\Omega}^H, \\ 0 & \text{for } x_j^h \in \Omega \setminus \bar{\Omega}^H. \end{cases}$$

This is a natural extension of v^H by zero outside the Dirichlet boundary part of the coarse grid domain. Similar zero extensions were used in Kornhuber–Yserentant [16] to embed an arbitrarily complicated domain into a square or cube in constructing multilevel methods on nested and quasi-uniform meshes for second-order elliptic problems with purely Dirichlet boundary conditions.

Although the coarse-to-fine operator $\mathcal{I}_h^1$ works well for mixed boundary conditions, one has to modify the original coarse grid so that it covers the Neumann boundary part of the find grid domain. This can be difficult to do for very complicated domains. To avoid modifying the original coarse grid, we now consider standard finite element interpolants which are modified only near Neumann boundaries. To do so, we first introduce some notation. Let τ_{lr}^H be any coarse boundary element in $\mathcal{T}^H$ which is made up of the three vertices x_l^H, x_r^H, x_i^H and which has an edge on the boundary $\partial\Omega^H$, denoted by $x_l^H x_r^H$. We use $\Omega(x_l^H, x_r^H)$ to denote the union of all fine elements, if any, which has a nonempty intersection with the unbounded domain formed by the edge $x_l^H x_r^H$ and two outward normal lines to $x_l^H x_r^H$ at two vertices x_l^H, x_r^H (cf.

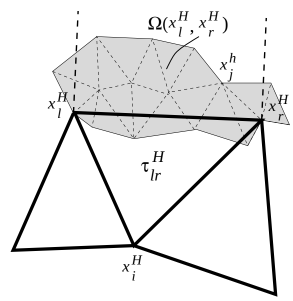

FIG. 2. *Shaded region,* $\Omega(x_l^H, x_r^H)$*, shows the fine grid part which is not completely covered by the coarse grid domain.*

Fig. 2). By including a few more fine elements in some $\Omega(x_l^H, x_r^H)$, if necessary, we may assume that the fine grid part $(\Omega \setminus \Omega^H)$ is included in the union of all $\Omega(x_l^H, x_r^H)$. Moreover, we assume

(H1) diam $\Omega(x_l^H, x_r^H) \le \mu_0$ diam τ_{lr}^H,

which implies the measure of $\Omega(x_l^H, x_r^H)$ is bounded by the measure of τ_{lr}^H:

$$|\Omega(x_l^H, x_r^H)| \le \mu\, |\tau_{lr}^H|,$$

where μ_0 and μ are two positive constants independent of H and h. Without any difficulty, the constant μ_0, and so μ, can be allowed in our subsequent results to depend on the two nodes x_l^H, x_r^H. In this case, μ_0 and μ will enter all the related bounds naturally.

We remark that (H1) restricts the size of the fine grid part near the edge $x_l^H x_r^H$ but outside the coarse grid domain Ω^H; that is, each local fine grid part $\Omega(x_l^H, x_r^H)$ is not allowed to be too large compared to its nearest coarse element τ_{lr}^H. This is a reasonable requirement in applications.

Then the standard nodal value interpolant associated with the fine space V^h can be generalized outward to each local fine grid part $\Omega(x_l^H, x_r^H)$ using three given linear functions θ_1, θ_2, and θ_3, which are defined in $\bar\Omega \cup \bar\Omega^H$ but bounded in $\Omega(x_l^H, x_r^H) \cup \tau_{lr}^H$ and satisfy

$$\theta_1(x) + \theta_2(x) + \theta_3(x) = 1, \quad \forall\, x \in \bar\Omega \cup \bar\Omega^H. \tag{3.1}$$

Note that the functions θ_1, θ_2, and θ_3 above are not necessarily nonnegative, and although they are element τ_{lr}^H-related, we will not use any index to specify this relation in order to simplify the notation. Then for any coarse function $v^H \in V^H$, we define an operator Θ_h by

$$\Theta_h v^H(x) = \theta_1(x) v^H(x_l^H) + \theta_2(x) v^H(x_r^H) + \theta_3(x) v^H(x_i^H), \quad \forall\, x \in \Omega(x_l^H, x_r^H) \cup \tau_{lr}^H$$

and assume that

(H2) $\Theta_h v^H = v^H$ on the edge $x_l^H x_r^H$,

which means $\Theta_h v^H$ is indeed an extension of v^H. For convenience, later on we will always regard $\Theta_h v^H$ as a function defined also outside $\Omega(x_l^H, x_r^H) \cup \tau_{lr}^H$ by extending it naturally.

With the above notation, we can introduce the general coarse-to-fine interpolant $\mathcal{I}_h$.

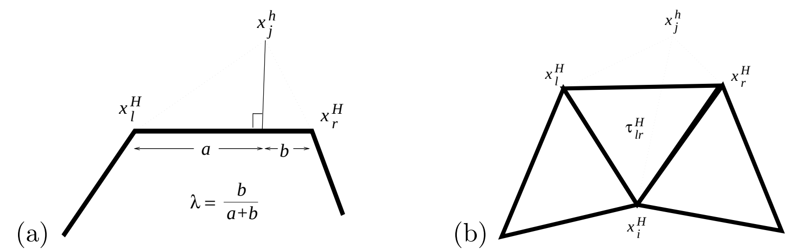

FIG. 3. *More accurate interpolants:* (a) $\mathcal{I}_h^2$*: Fine nodal values outside the coarse domain are interpolated with coarse nodal values on the nearest coarse grid edge;* (b) $\mathcal{I}_h^3$*: Fine nodal values outside the coarse domain are interpolated with nodal values on the nearest coarse element* τ_{lr}^H*. Thick lines represent coarse grid boundaries or elements, and dotted lines show the coarse nodes used to interpolate the fine nodal value at* x_j^h*.*

DEFINITION 3.1. *For any coarse function* v^H *in* V^H*, its image under the coarse-to-fine interpolant* $\mathcal{I}_h$ *is specified as follows:*

(C1) *For any fine node* x_j^h *in* $\bar{\Omega} \cap \bar{\Omega}^H$*,*

$$\mathcal{I}_h v^H(x_j^h) = v^H(x_j^h).$$

(C2) *For any fine node* x_j^h *in* $\Omega(x_l^H, x_r^H) \setminus \bar{\Omega}^H$ *with both* x_l^H *and* x_r^H *of Neumann nodes,*

$$\mathcal{I}_h v^H(x_j^h) = \Theta_h v^H(x_j^h).$$

(C3) *For any fine node* x_j^h *in* $\Omega(x_l^H, x_r^H) \setminus \bar{\Omega}^H$ *with both* x_l^H *and* x_r^H *of Dirichlet nodes,*

$$\mathcal{I}_h v^H(x_j^h) = 0.$$

(C4) *For any fine node* x_j^h *in* $\Omega(x_l^H, x_r^H) \setminus \bar{\Omega}^H$ *with one of* x_l^H *and* x_r^H *being the Neumann node and one being the Dirichlet node,*

$$\mathcal{I}_h v^H(x_j^h) = 0 \quad if \quad x_j^h \quad \text{is a fine boundary node of Dirichlet type,}$$
$$\mathcal{I}_h v^H(x_j^h) = \Theta_h v^H(x_j^h) \quad \text{otherwise.}$$

The following are two concrete examples of interpolants which satisfy the above definition and assumptions. We give only the corresponding forms of Θ_h's required in the definition.

$\mathcal{I}_h^2$: Nearest edge interpolation. Define the interpolant at x_j^h by using the nodes of the coarse boundary edge closest to x_j^h (see Fig. 3):

$$\mathcal{I}_h^2 v^H(x_j^h) = \lambda(x_j^h) v^H(x_l^H) + (1 - \lambda(x_j^h)) v^H(x_r^H),$$

where x_l^H and x_r^H are the nodes of the coarse boundary edge closest to x_j^h and λ is the ratio of the lengths of two segments of the edge $x_l^H x_r^H$ cut off by the normal line passing through x_j^h to the edge (see Fig. 3). This kind of interpolation was used by Bank–Xu [1] in their construction of a hierarchical basis on an unstructured mesh.

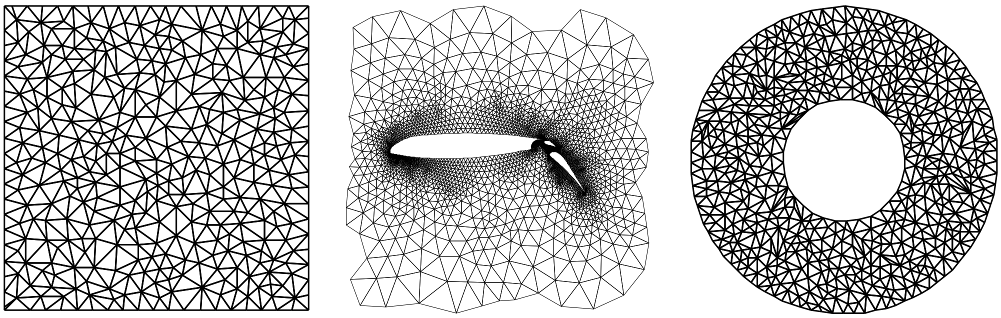

FIG. 4. *Some fine grids: an unstructured square with* 385 *nodes (left), NASA airfoil with* 4253 *nodes (center), and an annulus with* 610 *nodes (right).*

$\mathcal{I}_h^3$: Nearest element interpolation. Define the nonzero extension by using barycentric functions (see Fig. 3):

$$\mathcal{I}_h^3 v^H(x_j^h) = \lambda_l(x_j^h)v^H(x_l^H) + \lambda_r(x_j^h)v^H(x_r^H) + \lambda_i(x_j^h)v^H(x_i^H),$$

where $\lambda_l, \lambda_r, \lambda_i$ are three barycentric coordinate functions (also known as area or volume coordinates) corresponding to τ_{lr}^H.

Remark 3.1. Note that the functions λ_l, λ_r, and λ_i used in the definition of $\mathcal{I}_h^3$ satisfy $\lambda_l, \lambda_r, \lambda_i \geq 0$ for $x_j^h \in \tau_{lr}^H$, but not so for $x_j^h \notin \tau_{lr}^H$. In the case as shown in Fig. 3(b), we have $x_j^h \notin \tau_{lr}^H$, $\lambda_l(x) \geq 0, \lambda_r(x) \geq 0$, but $\lambda_i(x) \leq 0$ and $\lambda_l(x) + \lambda_r(x) + \lambda_i(x) = 1$. By (H1), we always have

$$|\lambda_l(x)| \leq \mu_1, \quad |\lambda_r(x)| \leq \mu_1, \quad \text{and} \quad |\lambda_i(x)| \leq \mu_1 \quad \forall\, x \in \Omega(x_l^H, x_r^H) \cup \tau_{lr}^H,$$

where μ_1 is a constant independent of h and H but depending only on the constant μ in (H1).

4. Numerical results. In this section, we provide some numerical results of domain decomposition and multigrid methods on unstructured meshes for elliptic problems on various fine grid domains (see Fig. 4). The well-known NASA airfoil mesh was provided by T. Barth and D. Jesperson of NASA Ames, and a fine, unstructured square and annulus were generated using Barth's two-dimensional Delaunay triangulator. All numerical experiments were performed using the Portable, Extensible Toolkit for Scientific Computation (PETSc) [13] running on a Sun SPARC 20. Piecewise linear finite elements were used for the discretizations and the resulting linear system was solved using either multilevel overlapping Schwarz or V-cycle multigrid as a preconditioner with full GMRES as an outer accelerator.

Our approach to generating a coarse grid hierarchy is to find a maximal independent set of the boundaries and the interior of the fine grid of the mesh and then retriangulate the resulting set of vertices (other coarsening algorithms can be used here). This process is then repeated recursively for the desired number of levels. An example coarse grid hierarchy of the airfoil mesh retriangulated with Cavendish's algorithm [4] is shown in Fig. 5, where G^2 refers to the first coarsening of the fine grid, G^1 is the coarsening of G^2, and G^0 is the coarsening of the G^1.

We shall present numerical results for Schwarz solvers and multigrid methods. For partitioning, all the domains (except the coarsest) were partitioned using the recursive spectral bisection method [20] with exact solves for both the subdomain problems

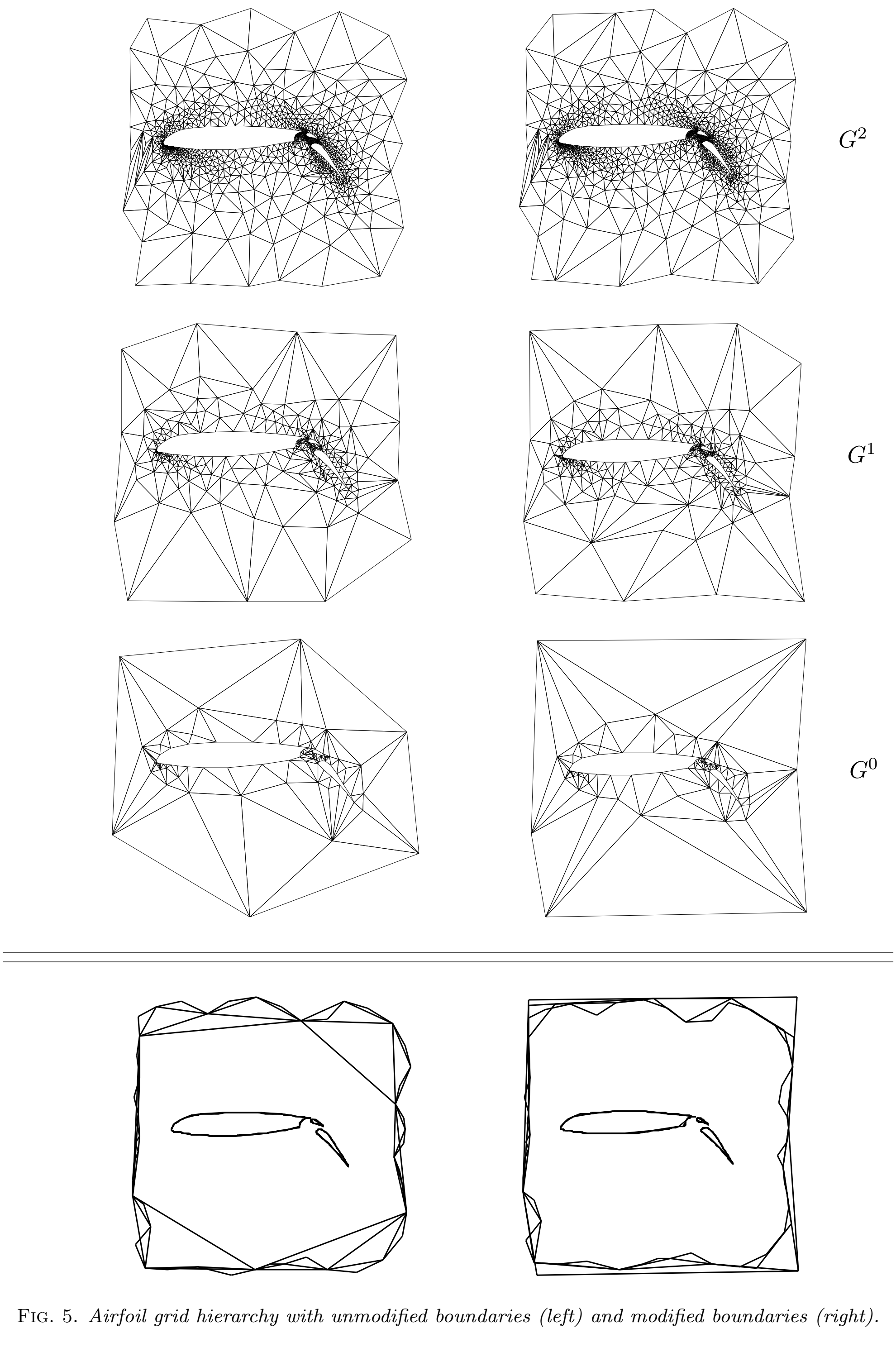

Fig. 5. *Airfoil grid hierarchy with unmodified boundaries (left) and modified boundaries (right).*

Table 1

Additive multilevel Schwarz iterations for the Poisson problem on a unit square grid. All grids (except coarsest) were partitioned using RSB. Shown is the number of GMRES iterations to convergence.

Dirichlet boundary conditions

# of levels	# of nodes	# of subdomains	# overlap elements 0	1	2
1	6409	256	84	63	50
	1522	64	45	36	27
	385	16	26	19	16
2	1522 385	64 1	19	16	16
	385 102	16 1	19	15	15
	102 29	4 1	17	15	15
3	6409 1522 385	256 64 1	28	24	25
	1522 385 102	64 16 1	32	25	26
	385 102 29	16 4 1	31	26	26
4	6409 1522 385 102	256 64 16 1	43	37	37
	1522 385 102 29	64 16 4 1	42	37	37

and the coarse grid problem. To generate overlapping subdomains, we first partition the domain into nonoverlapping subdomains and then extend each subdomain by some number of elements.

In all the experiments, the initial iterate is set to be zero and the iteration is stopped when the discrete norm of the residual is reduced by a factor of 10^{-5}.

For our first experiment, we use additive Schwarz to solve the Poisson problem on a unit square with homogeneous Dirichlet boundary conditions. Because the fine domain is so simple and Dirichlet boundary conditions are given, nonmatching boundaries are not an issue here and no special interpolants are used. We provide these results simply for completeness, as multilevel Schwarz results on unstructured grids have not been previously found in the literature to the authors' knowledge. Table 1 shows the number of GMRES iterations to convergence with varying fine grid problem and varying number of levels.

Providing a coarse grid improved convergence, and without it the method is not scalable to the case with a large number of subproblems. Interesting things to notice are that for a fixed number of levels, multilevel Schwarz is mesh-size independent, but that the number of iterations increases with the number of levels for a fixed problem size. This had also been previously observed for structured meshes using a multilevel diagonal scaling method in [21] and is due to the additive nature of the method. Also,

TABLE 2

*Additive multilevel Schwarz iterations for the elliptic problem with mildly varying coefficients on the airfoil grid (G^3) with 4253 unknowns. All grids (except coarsest) were partitioned using RSB with one element overlap. Shown is the number of GMRES iterations to convergence. * indicates identical results since no coarse grid was used.*

Dirichlet boundary conditions

# of levels	Grids	# of subdomains	Special interpolant used			
			$\mathcal{I}_h^0$	$\mathcal{I}_h^1$	$\mathcal{I}_h^2$	$\mathcal{I}_h^3$
1	G^3	32	23	*	*	*
2	G^3 G^2	32 1	15	15	15	16
3	G^3 G^2 G^1	32 8 1	23	23	23	25
4	G^3 G^2 G^1 G^0	32 8 2 1	32	33	33	35

Mixed Dirichlet/Neumann boundary conditions

# of levels	Grids	# of subdomains	Special interpolant used			
			$\mathcal{I}_h^0$	$\mathcal{I}_h^1$	$\mathcal{I}_h^2$	$\mathcal{I}_h^3$
1	G^3	32	51	*	*	*
2	G^3 G^2	32 1	43	14	15	16
3	G^3 G^2 G^1	32 8 1	53	21	23	23
4	G^3 G^2 G^1 G^0	32 8 2 1	61	27	29	30

increasing the amount of overlap improved convergence, but in practice, a one-element overlap was sufficient.

In our second experiment, we solve a mildly varying coefficient problem on the airfoil,

$$\frac{\partial}{\partial x}\left((1+xy)\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left((\sin(3y))\frac{\partial u}{\partial y}\right)=(4xy+2)\sin(3y)+9x^2\cos(6y),$$

with either a purely Dirichlet boundary condition or a mixed boundary condition: Dirichlet for $x \leq 0.2$ and homogeneous Neumann for $x > 0.2$. For this problem, the nonhomogeneous Dirichlet condition is $u = 2 + x^2 \sin(3y)$. Table 2 shows the number of GMRES iterations to convergence using additive multilevel Schwarz with the different boundary treatments.

We see the slow increase in iteration number as we increase the number of levels used. More important, we see the deterioration in the method when Neumann conditions are not properly handled.

In Table 3, we show results for the same problem solved using a hybrid multiplicative-additive Schwarz (multiplicative between levels but additive among subdomains on the same level). As in the additive case, deterioration of the method occurs when mixed boundary conditions are present. However, we can achieve optimal conver-

TABLE 3

*Hybrid multiplicative-additive multilevel Schwarz iterations for the elliptic problem with mildly varying coefficients on the airfoil grid (G^3) with 4253 unknowns. All grids (except coarsest) were partitioned using RSB with one element overlap. Shown is the number of GMRES iterations to convergence. * indicates identical results since no coarse grid was used.*

Dirichlet boundary conditions

# of levels	Grids	# of subdomains	Special interpolant used			
			$\mathcal{I}_h^0$	$\mathcal{I}_h^1$	$\mathcal{I}_h^2$	$\mathcal{I}_h^3$
1	G^3	32	23	*	*	*
2	G^3 G^2	32 1	13	13	13	14
3	G^3 G^2 G^1	32 8 1	13	13	13	14
4	G^3 G^2 G^1 G^0	32 8 2 1	13	13	13	14

Mixed Dirichlet/Neumann boundary conditions

# of levels	Grids	# of subdomains	Special interpolant used			
			$\mathcal{I}_h^0$	$\mathcal{I}_h^1$	$\mathcal{I}_h^2$	$\mathcal{I}_h^3$
1	G^3	32	51	*	*	*
2	G^3 G^2	32 1	36	13	13	14
3	G^3 G^2 G^1	32 8 1	36	13	13	14
4	G^3 G^2 G^1 G^0	32 8 2 1	36	13	13	14

gence rates, even with a varying number of levels with the hybrid method. Still further improvement can be obtained when using a multiplicative method (both on the subdomains and between levels), and the method behaves much like multigrid (see Tables 4 and 5). In fact, this is nothing more than multigrid but with a block smoother. A V-cycle multigrid method with pointwise Gauss–Seidel smoothing and two pre- and two postsmoothings per level was used to produce the results in Table 5.

Table 6 shows some multigrid results for the Poisson equation on an annulus. The forcing function is set to be one and both kinds of boundary conditions were tested. A V-cycle multigrid method with pointwise Gauss–Seidel smoothing and two pre- and two postsmoothings per level was used. When mixed boundary conditions are present, the deterioration is less pronounced in the multigrid method, but it still exists. It is interesting to note that in our previous multigrid experiments on a quasi-uniform annulus (see [5]), the observed deterioration in the method was much more dramatic than those observed here with the unstructured annulus. We believe that this was due to some extremely poor element aspect ratios on the fine grid in the quasi-uniform case, compounding the effect of the poor approximation on Neumann boundaries.

5. Two-level convergence theory. Here, we try to set up a framework for convergence theory. An important ingredient in the convergence proof for the overlapping

Table 4

*Multiplicative multilevel Schwarz iterations for the elliptic problem with mildly varying coefficients on the airfoil grid (G^3) with 4253 unknowns. All grids (except coarsest) were partitioned using RSB with one element overlap. Shown is the number of GMRES iterations to convergence. * indicates identical results since no coarse grid was used.*

Dirichlet boundary conditions

# of levels	Grids	# of subdomains	Special interpolant used			
			$\mathcal{I}_h^0$	$\mathcal{I}_h^1$	$\mathcal{I}_h^2$	$\mathcal{I}_h^3$
1	G^3	32	9	*	*	*
2	G^3 G^2	32 1	4	4	4	4
3	G^3 G^2 G^1	32 8 1	4	4	4	4
4	G^3 G^2 G^1 G^0	32 8 2 1	4	4	4	4

Mixed Dirichlet/Neumann boundary conditions

# of levels	Grids	# of subdomains	Special interpolant used			
			$\mathcal{I}_h^0$	$\mathcal{I}_h^1$	$\mathcal{I}_h^2$	$\mathcal{I}_h^3$
1	G^3	32	23	*	*	*
2	G^3 G^2	32 1	5	4	4	4
3	G^3 G^2 G^1	32 8 1	5	4	4	4
4	G^3 G^2 G^1 G^0	32 8 2 1	5	4	4	4

multilevel domain decomposition and multigrid methods is the requirement that the coarse-to-fine grid transfer operator possesses the local optimal L^2-approximation and local H^1-stability properties [7, 8, 9]. The locality of these properties is essential to the effectiveness of these methods on highly nonquasi-uniform unstructured meshes.

We need to introduce some more notation (see section 2): for $\tau^h \in \mathcal{T}^h$ and $\tau^H \in \mathcal{T}^H$,

$$N(\tau^H) = \text{union of coarse elements adjacent to } \tau^H,$$
$$B_k = \cup_{\tau^H \cap \Omega^k \neq \emptyset} \tau^H, \quad h_k = \max_{\tau^h \subset \Omega^k} h_\tau,$$
$$S_k = \cup_{\tau^H \subset B_k} N(\tau^H), \quad H_k = \max_{\tau^H \subset B_k} H_\tau.$$

Note that B_k is the union of all coarse elements having nonempty intersection with the subdomain Ω^k. We allow each Ω^k to be of quite different size and of quite different shape from other subdomains, but we make the following reasonable assumptions:

(A1) Any point $x \in \Omega$ belongs to at most q_0 subdomains of $\{\Omega^k\}_{k=1}^p$ with $q_0 > 0$ an integer.

(A2) $h_k \lesssim H_k$, and $\text{card}\{\tau^H \in \mathcal{T}^H;\ \ \tau^H \subset B_k\} \le n_0$ for $1 \le k \le p$ with $n_0 > 0$ an integer.

TABLE 5
Multigrid iterations for the elliptic problem with mildly varying coefficients on the airfoil. Shown is the number of GMRES iterations to convergence.

Dirichlet boundary conditions

# of fine grid nodes	MG levels	# of coarse grid nodes	Special interpolant used			
			$\mathcal{I}_h^0$	$\mathcal{I}_h^1$	$\mathcal{I}_h^2$	$\mathcal{I}_h^3$
4253	2	1170	4	4	4	4
	3	340	4	4	4	4
	4	101	4	4	4	4

Mixed Dirichlet/Neumann boundary conditions

# of fine grid nodes	MG levels	# of coarse grid nodes	Special interpolant used			
			$\mathcal{I}_h^0$	$\mathcal{I}_h^1$	$\mathcal{I}_h^2$	$\mathcal{I}_h^3$
4253	2	1170	6	5	4	4
	3	340	6	4	5	5
	4	101	7	5	5	5

TABLE 6
Multigrid iterations for the Poisson problem on an annulus. The exit condition was decreased to 10^{-6} from 10^{-5}. Shown is the number of GMRES iterations to convergence.

Dirichlet boundary conditions

# of fine grid nodes	MG levels	# of coarse grid nodes	Special interpolant used			
			$\mathcal{I}_h^0$	$\mathcal{I}_h^1$	$\mathcal{I}_h^2$	$\mathcal{I}_h^3$
2430	2	610	4	4	4	4
	3	160	4	4	4	4
	4	47	4	4	4	4

Mixed Dirichlet/Neumann boundary conditions

# of fine grid nodes	MG levels	# of coarse grid nodes	Special interpolant used			
			$\mathcal{I}_h^0$	$\mathcal{I}_h^1$	$\mathcal{I}_h^2$	$\mathcal{I}_h^3$
2430	2	610	6	5	4	4
	3	160	7	5	4	4
	4	47	7	5	4	4

(A3) Any point $x \in \Omega^H$ belongs to at most q_0 subdomains of $\{S_k\}_{k=1}^p$.

The following theorem gives the bound of the condition number $\kappa(MA)$ for the two-level additive Schwarz method (2.2) of section 2.

THEOREM 5.1. *Under the assumptions* (A1)–(A3), *we have*

$$\kappa(MA) \lesssim \max_{1\le k\le p} \frac{H_k^2}{\delta_k^2}.$$

Theorem 5.1 indicates that an optimal condition number may be expected if the local overlap δ_k is proportional to the local subdomain size H_k.

To prove Theorem 5.1, it is essential for any $u \in V^h$ to find a partition $u = \mathcal{I}_h u_H + \sum_{k=1}^p u_k$ with $u_k \in V^k$ $(1 \le k \le p)$ and $u_H \in V^H$ such that $\mathcal{I}_h u_H$ is bounded by u in both L^2-norm and H^1-norm and preserves the local optimal L^2-norm error approximation to u. This can be done quite routinely by using Lemma A.2 (see the appendix) and the standard partition $\{\theta_i\}_{i=1}^p$ of unity for Ω corresponding to the subdomains $\{\Omega^k\}_{i=1}^p$. We refer to [7, 8, 9] for the details.

6. Conclusions. When using general unstructured meshes, the coarse grid domain may not necessarily match that of the fine grid. For the parts of the fine grid domain which are not contained in the coarse domain, special treatments must be done to handle different boundary conditions. The transfer operators using linear interpolation with a zero extension are the most natural to implement and are effective for problems with Dirichlet boundary conditions.

For problems where Neumann boundary conditions exist, however, zero extension is no longer appropriate and special interpolants should be sought. Our numerical results show the significance of the assumption that when standard interpolations with zero extension are used, the coarse grid must cover the Neumann boundaries of the fine grid problem; otherwise deterioration of the methods occurs. The deterioration is most significant when using additive multilevel methods but can still be seen for the multiplicative methods. When coupled with highly stretched elements, the deterioration can be very significant, even for multiplicative methods.

Although modifying the coarse grid domains to ensure that this assumption is satisfied is effective, this approach can be problematic to implement for particularly complicated domains or can sometimes generate coarse grid domains which deviate significantly from the fine domain.

An alternative is to modify the interpolants so that nonzero extensions are used on those fine grid boundaries which have Neumann conditions and which are not contained within the coarse grid domain. Since we are using the multilevel methods only as preconditioners, the extension need not be particularly accurate; we used either constant extension with the nearest boundary nodal value or extension using the barycentric functions of the nearest coarse grid element, neither of which is difficult to implement.

Appendix. We now prove the lemma which implies the stability and approximation of the coarse space V^H to the fine space V^h under the coarse-to-fine grid transfer operator $\mathcal{I}_h$ and which immediately gives rise to the convergence and condition number bounds for the two-level additive Schwarz methods (cf. section 5). As multilevel additive methods need some more technical tools, for example, stability of the inverse of the coarse-to-fine interpolant and construction of a "good" partition of a fine function over the subspaces of all grid levels (cf. Chan–Zou [9]), we do not yet know whether a similar convergence result can be extended to the multilevel case.

Appendix A. Stability and approximation properties of the interpolation operator. Purely for our theoretical analysis, we now introduce a triangulation $\tilde{\mathcal{T}}^H$. Extend $\mathcal{T}^H$ to a larger but still shape-regular triangulation $\tilde{\mathcal{T}}^H$, the corresponding domain denoted by $\tilde{\Omega}^H$, such that the Neumann boundary of Ω^H is contained in $\tilde{\Omega}^H$ but the Dirichlet boundary remains the same. Let $\tilde{V}^H$ be the corresponding piecewise linear finite element space on $\tilde{\mathcal{T}}^H$ with completely homogeneous Dirichlet boundary condition. Then we have

$$V^H = \tilde{V}^H|_{\bar{\Omega}^H}.$$

We then have the following local optimal L^2-approximation and H^1-stability for the operator $\mathcal{I}_h$ on the coarse space V^H.

LEMMA A.1. *Let $\mathcal{I}_h$ be any interpolation operator defined in Definition* 3.1*, and let v^H be any coarse function in V^H. If we extend v^H onto $\tilde{V}^H$ in any way, still* *denoted by* v^H*, then for any* $\tau^H \in \mathcal{T}^H$*, we have*

$$(l1) \quad \sum_{\substack{\tau^h \cap \tau^H \neq \emptyset \\ \tau^h \subset \bar{\Omega}^H}} \|v^H - \mathcal{I}_h v^H\|^2_{0,\tau^h} \leq C d^2(\tau^H) |v^H|^2_{1,N(\tau^H)},$$

$$(l2) \quad \sum_{\substack{\tau^h \cap \tau^H \neq \emptyset \\ \tau^h \subset \bar{\Omega}^H}} |\mathcal{I}_h v^H|_{1,\tau^h} \leq C |v^H|_{1,N(\tau^H)},$$

$$(l3) \quad \sum_{\tau^h \in \Omega(x_l^H, x_r^H)} \|v^H - \mathcal{I}_h v^H\|^2_{0,\tau^h} \leq C d^2(\tau^H_{lr}) \sum_{\tau^H \in N(\tau^H_{lr})} |v^H|^2_{1,N(\tau^H)},$$

$$(l4) \quad \sum_{\tau^h \in \Omega(x_l^H, x_r^H)} |\mathcal{I}_h v^H|^2_{1,\tau^h} \leq C \sum_{\tau^H \in N(\tau^H_{lr})} |v^H|^2_{1,N(\tau^H)},$$

where $\Omega(x_l^H, x_r^H)$ *is any region as introduced in section* 3.

Proof. The inequalities ($l1$) and ($l2$) correspond to the parts where the fine grid domain is completely contained in the coarse grid domain. Their proofs can be found in [7, 8]. The last two inequalities ($l3$) and ($l4$) correspond to the fine grid parts which are not covered by the coarse grid and which we shall prove here. We give the proofs only for the cases (C1)–(C2); the other case can be proved similarly.

We first prove inequality ($l3$), i.e., L^2-optimal approximation. For any fine element τ^h in $\Omega(x_l^H, x_r^H)$, as $\mathcal{I}_h v^H$ is linear on τ^h we can express

$$\mathcal{I}_h v^H(x) = \sum_{i=1}^{3} \mathcal{I}_h v^H(x_i^h) \phi_i^h,$$

where x_i^h $(i = 1, 2, 3)$ are the three vertices of τ^h and ϕ_i^h $(i = 1, 2, 3)$ are the corresponding basis functions of V^h at these three nodes. Then by definition of $\mathcal{I}_h$ and the boundedness of θ_i $(i = 1, 2, 3)$, we have

$$\begin{aligned}\|\mathcal{I}_h v^H\|^2_{0,\tau^h} &\leq C d^2(\tau^h) \sum_{i=1}^{3} \left(\mathcal{I}_h v^H(x_i^h)\right)^2 \\ &\leq C d^2(\tau^h) \Big\{ (v^H(x_l^H))^2 + (v^H(x_r^H))^2 + (v^H(x_i^H))^2 \Big\}.\end{aligned}$$

Summing over all $\tau^h \in \Omega(x_l^H, x_r^H)$ and using (H1),

$$\begin{aligned}\sum_{\tau^h \in \Omega(x_l^H, x_r^H)} \|\mathcal{I}_h v^H\|^2_{0,\tau^h} &\leq C \Big\{ (v^H(x_l^H))^2 + (v^H(x_r^H))^2 + (v^H(x_i^H))^2 \Big\} \sum_{\tau^h} d^2(\tau^h) \\ &\leq C \Big\{ (v^H(x_l^H))^2 + (v^H(x_r^H))^2 + (v^H(x_i^H))^2 \Big\} |\tau^H_{lr}| \\ &\leq C \|v^H\|^2_{0,\tau^H_{lr}}.\end{aligned}$$

Using this and inequality $(a + b)^2 \leq 2(a^2 + b^2)$ $\forall\, a,\, b \in R^1$, we obtain

$$\begin{aligned}\sum_{\tau^h \in \Omega(x_l^H, x_r^H)} \|v^H - \mathcal{I}_h v^H\|^2_{0,\tau^h} &\leq 2 \sum_{\tau^h \in \Omega(x_l^H, x_r^H)} \|v^H\|^2_{0,\tau^h} + C \|v^H\|^2_{0,\tau^H_{lr}} \\ &\leq C \|v^H\|^2_{0,N(\tau^H_{lr})}.\end{aligned}$$

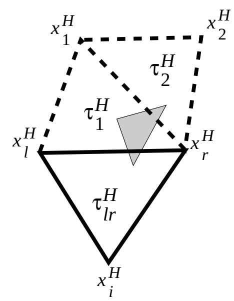

FIG. 6. *Fine grid element (shaded) in* $\Omega(x_l^H, x_r^H)$*, which is covered by* τ_{lr}^H *plus two extended coarse elements,* τ_1^H *and* τ_2^H*.*

Noting that the left-hand side of the inequality doesn't change by replacing v^H with v^H plus any constant, we obtain by Poincaré inequality that

$$\sum_{\tau^h \in \Omega(x_l^H, x_r^H)} \|v^H - \mathcal{I}_h v^H\|_{0,\tau^h}^2 \leq C d^2(\tau_{lr}^H) |v^H|_{1,N(\tau_{lr}^H)}^2.$$

This proves ($l3$).

We next prove ($l4$), i.e., H^1-stability. For the ease of notation, we assume that $\Omega(x_l^H, x_r^H)$ can be covered by τ_{lr}^H plus two extended coarse elements τ_1^H and $\tau_2^H \in \tilde{\Omega}^H$ (see Fig. 6).

Let us define

$$\tilde{\Theta}_h v^H(x) = \begin{cases} v^H(x) & \text{if } x \in \bar{\Omega} \cap \bar{\Omega}^H, \\ \Theta_h v^H(x) & \text{if } x \in \Omega \backslash \Omega^H. \end{cases}$$

It is easy to see that $\tilde{\Theta}_h v^H$ is continuous and belongs to $H^1(\bar{\tau}_{lr}^H \cup \Omega(x_l^H, x_r^H))$, and by definition of $\mathcal{I}_h$ and $\tilde{\Theta}_h$, we have

$$\mathcal{I}_h v^H(x) = \tilde{\mathcal{I}}_h \tilde{\Theta}_h v^H(x) \quad \forall\, x \in \bar{\tau}_{lr}^H \cup \Omega(x_l^H, x_r^H).$$

Here $\tilde{\mathcal{I}}_h$ is the standard nodal value interpolant defined on the finite element space V^h. We have to bound $|\mathcal{I}_h v^H|_{1,\tau^h}$ for all $\tau^h \in \Omega(x_l^H, x_r^H)$. By the triangle inequality,

$$|\mathcal{I}_h v^H|_{1,\tau^h}^2 \leq 2|\tilde{\mathcal{I}}_h \tilde{\Theta}_h v^H - \tilde{\Theta}_h v^H|_{1,\tau^h}^2 + 2|\tilde{\Theta}_h v^H|_{1,\tau^h}^2. \tag{A.1}$$

For the first term in (A.1), we have by standard interpolation theory (see Ciarlet [10]) that

$$(I)_3 \equiv |\tilde{\mathcal{I}}_h \tilde{\Theta}_h v^H - \tilde{\Theta}_h v^H|_{1,\tau^h}^2 \leq C h^2 |\tilde{\Theta}_h v^H|_{1,\infty,\tau^h}^2. \tag{A.2}$$

Let the maximum of $\tilde{\Theta}_h v^H$ be reached at some point x_0, which must belong to either $\tau_1^H \cup \tau_2^H$ or τ_{lr}^H or $N(\tau_{lr}^H) \setminus \tau_{lr}^H$, and denote it by $m(x_0) = |\tilde{\Theta}_h v^H|_{1,\infty,\tau^h}^2$ We consider only the two cases $x_0 \in \tau_1^H \cup \tau_2^H$ or $x_0 \in \tau_{lr}^H$ as the case of $x_0 \in N(\tau_{lr}^H) \setminus \tau_{lr}^H$ is similar to the one for $x_0 \in \tau_{lr}^H$. For either case, we can always construct a shape regular element τ_1^h with x_0 as one of its vertices such that $\tau_1^h \subset \tau_1^H \cup \tau_2^H$ for the former and $\tau_1^h \subset \tau_{lr}^H$ for the latter and $d(\tau_1^h)$ is of the same size as $d(\tau^h)$ (see Fig. 7). Then it follows from the inverse inequality that for $x_0 \in \tau_1^H \cup \tau_2^H$,

$$(I)_3 \leq C d^2(\tau^h) m(x_0) \leq C d^2(\tau^h) |\tilde{\Theta}_h v^H|_{1,\infty,\tau_1^h}^2 \leq C |\tilde{\Theta}_h v^H|_{1,\tau_1^h}^2;$$

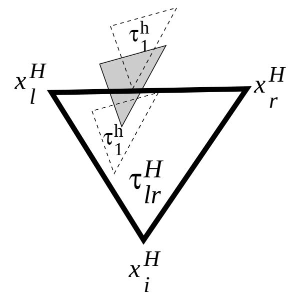

FIG. 7. *A shape regular element, τ_1^h, whose diameter is of the same size as $\tau^h \in \Omega(x_l^H, x_r^H)$ (shaded).*

while for $x_0 \in \tau_{lr}^H$,

$$(I)_3 \le Cd^2(\tau^h)m(x_0) \le Cd^2(\tau^h)|v^H|^2_{1,\infty,\tau_1^h} \le C|v^H|^2_{1,\tau_1^h}.$$

Summing $(I)_3$ over all $\tau^h \in \Omega(x_l^H, x_r^H)$, and using (A.1)–(A.2), we obtain

$$\sum_{\tau^h\in\Omega(x_l^H,x_r^H)} |\mathcal{I}_h v^H|^2_{1,\tau^h} \le C\Big(|\tilde\Theta_h v^H|^2_{1,\tau_1^H} + |\tilde\Theta_h v^H|^2_{1,\tau_2^H} + |v^H|^2_{1,N(\tau_{lr}^H)}\Big). \tag{A.3}$$

, $\tilde\Theta_h v^H$ is linear over $\Omega(x_l^H, x_r^H)$, uniquely determined by values $v^H(x_l^H)$, $v^H(x_r^H)$ and $v^H(x_i^H)$; thus we derive immediately by direct calculations (cf. Fig. 7) that, with $w^H = \tilde\Theta_h v^H$,

$$\begin{aligned}|w^H|^2_{1,\tau_1^H} \le C\{&(w^H(x_l^H) - w^H(x_r^H))^2 + (w^H(x_r^H) - w^H(x_1^H))^2\\ &+(w^H(x_1^H) - w^H(x_l^H))^2\}.\end{aligned}$$

Using the assumption (H2), we know

$$w^H(x_l^H) = v^H(x_l^H), \quad w^H(x_r^H) = v^H(x_r^H).$$

Combining with the definition of Θ_h, the boundedness of θ_i and (3.1) yields

$$\begin{aligned}|w^H|^2_{1,\tau_1^H} &\le C\{(v^H(x_l^H) - v^H(x_r^H))^2 + (v^H(x_r^H) - v^H(x_i^H))^2\\ &\quad +(v^H(x_i^H) - v^H(x_l^H))^2\}\\ &\le C|v^H|^2_{1,\tau_{lr}^H}.\end{aligned}$$

The same result is true for $|w^H|^2_{1,\tau_2^H} \equiv |\tilde\Theta_h v^H|^2_{1,\tau_2^H}$. Thus we obtain from these estimates and (A.3) that

$$\sum_{\tau^h\in\Omega(x_l^H,x_r^H)} |\mathcal{I}_h v^H|^2_{1,\tau^h} \le C|v^H|^2_{1,N(\tau_{lr}^H)}.$$

This proves ($l4$). ∎

The next lemma implies the stability and approximation properties of the interpolant $\mathcal{I}_h u^H$ to u. Let $\hat{\Omega}$ be an open bounded domain in R^2 which is large enough that it contains both Ω and $\tilde{\Omega}^H$, and let $E : H^1(\Omega) \to H^1(\hat{\Omega})$ be a linear extension operator satisfying

$$Ew|_\Omega = w, \qquad \|Ew\|_{1,\hat{\Omega}} \le C\|w\|_{1,\Omega} \quad \forall\, w \in H^1(\Omega).$$

See Stein [22] for the existence of such an extension operator. Let $Q_H : L^2(\tilde{\Omega}^H) \to \tilde{V}^H$ be Clément's interpolant. We refer to Clément [11] for its definition and Chan–Zou [8] and Chan–Smith–Zou [7] for its use in domain decomposition contexts. Evidently,

$$(Q_H w^H)|_{\Omega^H} \in V^H \quad \forall\, w^H \in L^2(\tilde{\Omega}^H).$$

LEMMA A.2. *Given any interpolation operator $\mathcal{I}_h$ satisfying Definition* 3.1, *then for any $u^h \in V^h$ there exists $u^H \in V^H$ such that for all $\tau^H \in \mathcal{T}^H$, we have*

$$(l1) \quad \sum_{\substack{\tau^h \cap \tau^H \neq \emptyset \\ \tau^h \subset \bar{\Omega}^H}} \|u^h - \mathcal{I}_h u^H\|^2_{0,\tau^h} \le C d^2(\tau^H) |Eu^h|^2_{1,N(\tau^H)},$$

$$(l2) \quad \sum_{\substack{\tau^h \cap \tau^H \neq \emptyset \\ \tau^h \subset \bar{\Omega}^H}} |\mathcal{I}_h u^H|_{1,\tau^h} \le C |Eu^h|_{1,N(\tau^H)},$$

$$(l3) \quad \sum_{\tau^h \in \Omega(x_l^H, x_r^H)} \|u^h - \mathcal{I}_h u^H\|^2_{0,\tau^h} \le C d^2(\tau^H_{lr}) \sum_{\tau^H \in N(\tau^H_{lr})} |Eu^h|^2_{1,N(\tau^H)},$$

$$(l4) \quad \sum_{\tau^h \in \Omega(x_l^H, x_r^H)} |\mathcal{I}_h u^H|^2_{1,\tau^h} \le C \sum_{\tau^H \in N(\tau^H_{lr})} |Eu^h|^2_{1,N(\tau^H)}.$$

Proof. As stated in the proof of Lemma A.1, the proof of the inequalities ($l1$) and ($l2$) is easy and can be found in [7, 8]. We next prove ($l3$) and ($l4$).

For any $u^h \in V^h$, we choose $u^H \in V^H$ by

$$u^H = Q_H E u^h|_{\Omega^H} \in V^H.$$

This u^H satisfies the required results. The H^1-stability ($l4$) is an immediate consequence of Lemma A.1 and the H^1-stability of Q_H. We now prove ($l3$).

On the fine domain $\Omega^h = \Omega$, we can split $u^h - \mathcal{I}_h u^H$ into two parts:

$$u^h - \mathcal{I}_h u^H = (Eu^h - Q_H E u^h) + (Q_H E u^h - \mathcal{I}_h Q_H E u^h). \tag{A.4}$$

First term estimate in (A.4). If a Neumann boundary condition is imposed on at least one of the two coarse nodes x_l^H and x_r^H in the space V^H, we derive by assumption on $\Omega(x_l^H, x_r^H)$ and properties of Clément's interpolant Q_H that

$$\sum_{\tau^h \in \Omega(x_l^H, x_r^H)} \|Eu^h - Q_H E u^h\|^2_{0,\tau^h} \le \|Eu^h - Q_H E u^h\|^2_{0,N(\tau^H_{lr})}$$
$$\le C d^2(\tau^H_{lr}) \sum_{\tau^H \in N(\tau^H_{lr})} |Eu^h|^2_{1,N(\tau^H)}.$$

If a Dirichlet boundary condition is imposed on both nodes x_l^H and x_r^H in the space V^H, the result follows from Poincaré inequality.

Second term estimate in (A.4). We obtain from Lemma A.1 that, with $v^h = Eu^h$,

$$\sum_{\tau^h \in \Omega(x_l^H, x_r^H)} \|Q_H v^h - \mathcal{I}_h Q_H v^h\|_{0,\tau^h}^2 \le C d^2(\tau_{lr}^H) |Q_H v^h|_{1,N(\tau_{lr}^H)}^2.$$

Then using the stability of Q_H yields

$$\sum_{\tau^h \in \Omega(x_l^H, x_r^H)} \|Q_H E u^h - \mathcal{I}_h Q_H E u^h\|_{0,\tau^h}^2 \le C d^2(\tau_{lr}^H) \sum_{\tau^H \in N(\tau_{lr}^H)} |E u^h|_{1,N(\tau^H)}^2.$$

Now ($l3$) follows from (A.4) and the above two estimates for the first and second terms in (A.4). □

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