Mixed Finite Element Method with Gauss' Law Enforced for Maxwell Eigenproblem

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Abstract

A mixed finite element method is proposed for Maxwell eigenproblem under the general setting. The method is based on a modification of the Kikuchi mixed formulation in terms of the electric field and the multiplier, with a mesh-dependent Gauss' law of the electric field enforced in the formulation. The electric field is discretized by discontinuous elements and the multiplier always by the lowest-order continuous nodal element (e.g., linear element). The method renders four key features: the discrete de Rham complex exact sequence is not required, replaced by a gradient inclusion condition of a low order scalar element, i.e., the finite element space of the electric field includes the gradient of an auxiliary scalar H^1-conforming finite element space of low order; the discrete compactness property holds; the strong convergence of the Gauss' law is ensured globally for the finite element solution; the method converges nearly optimally for both singular and smooth solutions. With these features, we develop a general analysis to prove that whether or not the discrete eigenmodes are spurious-free and spectral-correct attributes essentially to the first-order approximation property in the $H(\mathbf{curl}\,;\Omega)$ norm. As a direct application, except three lowest-order elements that do not have the first-order approximation property on nonaffine meshes, the first-kind Nédélec elements on nonaffine quadrilateral and hexhedral meshes and the second-kind Nédélec elements on affine and nonaffine quadrilateral and hexhedral meshes, including their discontinuous versions, are spurious-free and spectral-correct in the new mixed method, while these Nédélec elements generate spurious and incorrect discrete eigenmodes in the classical methods.

Keywords. Maxwell eigenproblem, Gauss' law, mixed finite elements, Nédélec elements, discontinuous elements.
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1 Introduction

In this work, we shall propose and analyse a mixed finite element method for the Maxwell eigenproblem, for which the following two classical variational formulations are frequently used:

Find $(\omega^2, \mathbf{u} \neq \mathbf{0}) \in \mathbb{R} \times H_0(\mathbf{curl}\,;\Omega)$ such that

$$(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) = \omega^2(\boldsymbol{\varepsilon}\mathbf{u}, \mathbf{v}) \qquad \forall \mathbf{v} \in H_0(\mathbf{curl}\,;\Omega)\,; \tag{1.1}$$

Find $(\omega^2, \mathbf{u} \neq \mathbf{0}) \in \mathbb{R} \times H_0(\mathbf{curl}\,;\Omega)$ and $p \in H_0^1(\Omega)$ such that

$$\begin{cases} (\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) + (\boldsymbol{\varepsilon}\mathbf{v}, \nabla p) = \omega^2(\boldsymbol{\varepsilon}\mathbf{u}, \mathbf{v}) & \forall \mathbf{v} \in H_0(\mathbf{curl}\,;\Omega), \\ (\boldsymbol{\varepsilon}\mathbf{u}, \nabla q) = 0 & \forall q \in H_0^1(\Omega). \end{cases} \tag{1.2}$$

These two formulations are known to be equivalent for all nonzero eigenvalues, when the domain is simply connected with a connected boundary. Formulation (1.2) is known as the Kikuchi's method [31] [32], and provides a compact operator. Formulation (1.1) provides a noncompact operator, incurring a zero eigenvalue with an infinite dimensional eigenspace, which is the kernel space of the $\mathbf{curl}$ operator. When both classical formulations (1.1) and (1.2) are discretized by a conforming finite element method, the corresponding finite element space $\mathbf{X}_h \subset H_0(\mathbf{curl}\,;\Omega)$ must meet the so-called discrete de Rham complex exact sequence, i.e., there exist two discrete spaces $M_h \subset H_0^1(\Omega)$ and $\mathbf{Y}_h \subset H_0(\mathrm{div}\,;\Omega)$ such that

$$0 \longrightarrow M_h \xrightarrow{\nabla} \mathbf{X}_h \xrightarrow{\mathbf{curl}} \mathbf{Y}_h, \tag{1.3}$$

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which exactly mimics the continuous de Rham complex exact sequence

$$0 \longrightarrow H_0^1(\Omega) \xrightarrow{\nabla} H_0(\mathbf{curl}\,;\Omega) \xrightarrow{\mathbf{curl}} H_0(\mathrm{div}\,;\Omega). \tag{1.4}$$

There are very rich references about the numerical solutions of the Maxwell equations with $H(\mathbf{curl}\,;\Omega)$-conforming and discontinuous finite elements under the sequence (1.3), just to name a few, [10] [30] [14] [19] [31] [15] [32] [39] and references therein.

As classic $H(\mathbf{curl}\,;\Omega)$-conforming elements, most of the Nédélec elements ([36] [37]) satisfy (1.3), e.g., the Nédélec elements of first kind and second kind on simplexes (e.g., triangles, tetrahedra) and the Nédélec elements of first kind on parallelograms and parallelepipeds. Unexpectedly, as the full tensor product spaces, the Nédélec elements of second kind on parallelograms and parallelepipeds fail to meet the discrete exact sequence (1.3), and consequently, all of these elements produce spurious and incorrect approximations by the classical formulations. When the meshes are nonaffine quadrilaterals and hexahedra, the application of the Nédélec elements to the classical formulations becomes difficult. This is because the discrete exact sequence (1.3) may fail for Nédélec elements of both second kind and first kind, hence these elements are generally not spurious-free and not spectral-correct. Up to now, it is still unclear how to generally construct hexahedral elements to satisfy (1.3), although there exist some families of quadrilateral elements (e.g., [2] [21]). Even if the discrete de Rham complex exact sequence (1.3) holds, the analysis for the Maxwell eigenproblem is by no means easy. The key to the analysis hinges on the so-called discrete compactness property in [32] which is nowadays well-known to be the key property for spurious-free and spectral-correct approximations.

The discontinuous Galerkin (DG) method may violate the sequence (1.3). But as far as we know, all the existing DG methods either assume the smooth solutions (e.g., see [33]), or actually still require to obey the discrete exact sequence (1.3), see, e.g., [10] (where the discrete compactness property and the discrete Friedrichs' inequality property of the $H(\mathbf{curl}\,;\Omega)$-conforming edge elements play a key role in the analysis, while both properties rely on the continuous and discrete exact sequences (1.4) and (1.3); see, e.g., [29, Pages 270-272] and [40, Lemma 2 and Theorem 1]). It is well-known that the assumption of smooth solutions may not be realistic, because different from most other problems, Maxwell equations often have singular solutions in the sense that $\mathbf{u}$ and $\mathbf{curl}\,\mathbf{u}$ lie only in some fractional order Sobolev space H^r with $r < 1$. The singular solutions may be caused by a nonsmooth domain with reentrant corners and edges or by the heterogeneous media (cf. [16] [18]). The situation may even be more challenging for the Maxwell eigenproblem, since there may exist infinitely many singular eigenfunctions, as well as infinitely many smooth eigenfunctions with H^1 or higher regularities, and it is generally unlikely to know which eigenfunctions are singular or smooth a priori. Therefore it would be practically very important for a finite element method for the Maxwell eigenproblem to well approximate both the singular and smooth solutions.

There is another physically and mathematically important issue to take care of when the Maxwell system is solved numerically, that is, whether the finite element method could ensure the global strong convergence of the Gauss law for the finite element solution. In the case of the Maxwell eigenproblem, the Gauss law reads as

$$\mathrm{div}\,(\boldsymbol{\varepsilon}\mathbf{u}) = 0. \tag{1.5}$$

In [15] [39], a convergence of the Gauss' law in Sobolev space $H^{-(1-\delta)}(\Omega)$ was studied for the Maxwell source problem, but only the lowest-order Nédélec elements on triangles or tetrahedra were considered. We are still not aware of any similar results for the Maxwell eigenproblem, neither for other Nédélec elements nor for the discontinuous elements. The theory developed in [15] [39] relies crucially on a key property, which is actually closely related to the discrete compactness property. But we know only those edge elements which meet the discrete exact sequence (1.3) possess this key property. Clearly, this fundamental issue deserves our full attention both physically and mathematically when the Maxwell eigenproblem is solved numerically, and it will be studied in this work.

Now, there gives rise to a very important and fundamental question: how to remedy those elements for which the sequence (1.3) does not hold, but the desired spurious-free, spectral-correct approximations are still ensured, and the strong convergence of the Gauss law of the finite element solution also holds? More precisely, the question is how to design a finite element scheme for the Maxwell eigenproblem so that (1.3) can be essentially avoided but the relevant discrete compactness property is easy to realize so that more elements and more meshes can be accommodated, the strong convergence of the Gauss law can be ensured, and full accuracies can be attained by the finite element solution. It is this fundamental question that motivates the current work in developing a new mixed finite element method.

In this paper, we start from the Kikuchi's formulation (1.2) to propose a new mixed finite element method for the Maxwell eigenproblem, with a modification of (1.2) to directly and locally enforce the Gauss law at the discrete level in the finite element formulation. As a result, the new mixed method renders the four key features as stated in the abstract of this paper.

For the new mixed method, the lowest-order continuous nodal element (e.g., linear element), denoted by Q_h, is always used for the multiplier while discontinuous finite elements of any order, denoted by $\mathbf{U}_h$, can be used for the electric field. Under such favorable conditions, the resulting finite element systems are always truly spurious-free (because there is no zero eigenvalue). This is in sharp contrast to the edge element method of the original Kikuchi's method for (1.2), where M_h in (1.3) must be of high order due to the higher order of $\mathbf{X}_h$. It is practically important to note that the discrete system associated with the new mixed method is much smaller than that with the edge element mixed method under (1.2) and (1.3), in terms of the degrees of freedom. On the other hand, in the new mixed method, although Q_h is the lowest-order element such as linear element, the quasi-optimal approximation on $\mathbf{U}_h$ which can be any order holds independent of Q_h. The new mixed method has the four key features as stated earlier while the classical methods may not have. We further discuss these features in more details below. To avoid the sequence (1.3), being partially mimicking $\nabla M_h \subset \mathbf{X}_h$ in (1.3), we propose a similar gradient inclusion condition but for a low order scalar element(actually, the lowest-order nodal element suffices), that is, we ask for a low order scalar finite element space, denoted by $W_h \subset Q$ (an H^1 space), such that

$$\nabla W_h \subset \mathbf{U}_h. \tag{1.6}$$

The main role of the condition (1.6) is to well approximate the singular solutions which do not have the H^1 regularity. For most choices of $\mathbf{U}_h$, a general choice of W_h is Q_h itself; but there are spaces $\mathbf{U}_h$ for which W_h may be different from Q_h, i.e., $\nabla Q_h \not\subset \mathbf{U}_h$ allows, sharply in contrast to $\nabla M_h \subset \mathbf{X}_h$. We emphasize that W_h is only for the theoretical purpose and does not enter the actual discretization of the Maxwell eigenproblem, i.e., we are working with Q_h and $\mathbf{U}_h$ only. For smooth solutions, condition (1.6) is not needed, and all come down to the standard inf-sup condition and the standard approximation properties of $(\mathbf{U}_h, Q_h)$. Since W_h is of low order element, condition (1.6) can be easily realized for various popular shapes of elements, such as simplexes, quadrilaterals, hexahedra, prisms, etc., no matter if $\mathbf{U}_h$ is discontinuous or tangential continuous (e.g., edge elements).

Clearly, all those Nédélec elements who satisfy (1.3) fulfill (1.6) naturally, by simply taking $W_h := M_h$. In other words, all the Nédélec elements that are spurious-free and spectral-correct in the classical methods are still so in our new mixed method. More importantly, as we show in Section 7, a direct application of the new mixed method demonstrates that many existing important elements which are impossible to be spurious-free and spectral-correct in classical methods are now spurious-free and spectral-correct. These include the Nédélec elements of second kind on affine meshes such as rectangles, parallelograms and parallelepipeds, and on nonaffine quadrilateral and hexahedral meshes (excluding only the lowest-order element on nonaffine hexahedral meshes), as well as the Nédélec elements of first kind on nonaffine quadrilateral and hexahedral meshes (excluding only the two lowest-order elements on non-affine quadrilateral and hexahedral meshes). This is the first time to prove that all these elements can be used for obtaining spurious-free and spectral-correct eigenmodes for the Maxwell eigenproblem.

Now, we turn to the enforcement of the Gauss' law in the finite element formulation. We have learned from [15] [39] that $H^{-(1-\delta)}(\Omega)$ is used to study the convergence of the Gauss' law at the discrete level. Some relevant works are referred to [7, 6] and references therein. It turns out that $H^{-(1-\delta)}(\Omega)$ is suitable for both edge element spaces and other finite element spaces including discontinuous elements; a reasonable discrete norm of $H^{-(1-\delta)}(\Omega)$ for measuring the Gauss' law can be taken as

$$|\mathbf{v}_h|^2_{h,\mathrm{div}} := \sum_{K\in\mathcal{T}_h} h_K^{2-2\delta}||\mathrm{div}\,(\boldsymbol{\varepsilon}\mathbf{v}_h)||^2_{0,K} + \sum_{F\in\mathcal{F}_h^{\mathrm{int}}} h_F^{1-2\delta}\int_F |[\![(\boldsymbol{\varepsilon}\mathbf{v}_h)\cdot\mathbf{n}]\!]|^2. \tag{1.7}$$

As a matter of fact, we observe that $||\mathrm{div}\,(\boldsymbol{\varepsilon}\cdot)||_{-(1-\delta)}$ is equivalent to $|\cdot|_{h,\mathrm{div}}$ on $\mathbf{K}_h$ (the space of discrete divergence-free functions). Such observation motivates us to enforce the divergence-free Gauss' law (1.5) in the finite element discretization, i.e., to augment the Kikuchi's formulation (1.2) by a mesh-dependent bilinear form

$$\sum_{K\in\mathcal{T}_h} h_K^{2-2\delta}(\mathrm{div}\,(\boldsymbol{\varepsilon}\mathbf{u}), \mathrm{div}\,(\boldsymbol{\varepsilon}\mathbf{v}))_{0,K} + \sum_{F\in\mathcal{F}_h^{\mathrm{int}}} h_F^{1-2\delta}\int_F [\![(\boldsymbol{\varepsilon}\mathbf{u})\cdot\mathbf{n}]\!][\![(\boldsymbol{\varepsilon}\mathbf{v})\cdot\mathbf{n}]\!]. \tag{1.8}$$

This augmentation is consistent, since (1.8) vanishes for exact $\mathbf{u}$ of (1.5). We shall prove that this augmentation plays the key to the discrete compactness property and also ensures the kernel-ellipticity. All these hold, without resorting to the discrete de Rham complex exact sequence.

With those four features, we develop a general analysis, revealing that from the new mixed method, whether or not $(\mathbf{U}_h, Q_h)$ is spurious-free and spectral-correct essentially attributes to a first-order approximation property of $\mathbf{U}_h$ for a piecewise smooth solution $\mathbf{z}$ (cf. (7.10) and (7.11)). Moreover, we prove, by the abstract theory of saddle-point problems([9] [29]) and the abstract spectral theory of the compact operator([3] [34]), that the new mixed method is stable, and converges essentially optimally for both singular and smooth solutions, up to the above parameter δ, and that the resulting finite element solution converges to the Gauss' law in both norms (1.7) and $||\mathrm{div}\,(\boldsymbol{\varepsilon}\cdot)||_{-(1-\delta)}$.

The rest of the paper is arranged as follows. In Section 2, we review the Maxwell eigenproblem and introduce the Hilbert spaces. The new mixed finite element method is introduced in Section 3, and then a general theory for stability and error estimates for the source problem corresponding to the eigenproblem is developed in Section 4. We obtain the error estimates in Section 5. In Section 6, the previous error estimates are applied to the eigenproblem. In Section 7, the gradient inclusion condition and the standard approximation properties are generally verified for both Nédélec elements and discontinuous elements. In particular, both affine and nonaffine quadrilateral and hexahedral Nédélec elements and discontinuous elements are studied. In the last section Section 8, numerical results are presented, including concluding remarks.

2 Maxwell eigenproblem, its formulation and preliminaries

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a bounded Lipschitz domain, with a polygonal or polyhedral boundary Γ and $\mathbf{n}$ being its outward unit normal. For the description of the topology of Ω and Γ, we follow [1] [26] to introduce a set of polygonal cutting surfaces $\Sigma_1, \cdots, \Sigma_N$ so that the domain $\overset{\circ}{\Omega} = \Omega \setminus \Sigma$, with $\Sigma = \bigcup_{j=1}^N \Sigma_j$, is simply-connected. We denote the connected components of Γ by $\Gamma_0, \cdots, \Gamma_n$, with Γ_0 being the outermost boundary. For modelling different inhomogeneous anisotropic materials occupying Ω, we use $\boldsymbol{\mu} = (\mu_{ij})_{(2d-3)\times(2d-3)} \in \mathbb{R}^{(2d-3)\times(2d-3)}$ and $\boldsymbol{\varepsilon} = (\varepsilon_{ij})_{d\times d} \in \mathbb{R}^{d\times d}$ for two tensor-valued functions defined in Ω, representing the magnetic permeability and the electric permittivity, respectively. We assume $\boldsymbol{\mu} \in (L^\infty(\Omega))^{(2d-3)\times(2d-3)}$, $\boldsymbol{\varepsilon} \in (L^\infty(\Omega))^{d\times d}$, are both symmetric, and satisfy the elliptic conditions: $\boldsymbol{\xi}\cdot\boldsymbol{\mu}\boldsymbol{\xi} \geq \mu_{\min}|\boldsymbol{\xi}|^2$ a.e. in Ω for all $\boldsymbol{\xi} \in \mathbb{R}^{(2d-3)\times(2d-3)}$ and $\boldsymbol{\xi}\cdot\boldsymbol{\varepsilon}\boldsymbol{\xi} \geq \varepsilon_{\min}|\boldsymbol{\xi}|^2$ a.e. in Ω for all $\boldsymbol{\xi} \in \mathbb{R}^{d\times d}$, where $\mu_{\min}$ and $\varepsilon_{\min}$ are two positive constants. We further assume that $\boldsymbol{\mu}$ and $\boldsymbol{\varepsilon}$ are both piecewise smooth with respect to a partition of Ω into a set of polygonal or polyhedral subdomains $\Omega_1, \cdots, \Omega_m$, namely, $\boldsymbol{\mu}|_{\Omega_k} \in (W^{1,\infty}(\Omega_k))^{(2d-3)\times(2d-3)}$ and $\boldsymbol{\varepsilon}|_{\Omega_k} \in (W^{1,\infty}(\Omega_k))^{d\times d}$ for $k = 1, 2, \cdots, m$. The Maxwell eigenproblem under consideration reads as follows: Find $(\omega^2, \mathbf{u} \neq \mathbf{0})$ such that

$$\mathbf{curl}\,(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{u}) = \omega^2\boldsymbol{\varepsilon}\mathbf{u} \quad \text{in } \Omega, \tag{2.1}$$

$$\mathrm{div}\,(\boldsymbol{\varepsilon}\mathbf{u}) = 0 \quad \text{in } \Omega, \tag{2.2}$$

$$\mathbf{n}\times\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma, \tag{2.3}$$

$$\int_{\Gamma_i} (\boldsymbol{\varepsilon}\mathbf{u})\cdot\mathbf{n} = 0, \quad 1 \leq i \leq n. \tag{2.4}$$

As usual, by $H^1(\Omega)$, $H(\mathrm{div}\,;\Omega)$ and $H(\mathbf{curl}\,;\Omega)$ we denote the spaces of square integrable functions, with square integrable grad, div, curl, respectively, and by $H(\mathrm{div}\,;\boldsymbol{\varepsilon};\Omega)$ we denote the space of functions with $(\boldsymbol{\varepsilon}\mathbf{v}) \in H(\mathrm{div}\,;\Omega)$, and by $H(\mathrm{div}\,^0;\Omega)$, $H(\mathrm{div}\,^0;\boldsymbol{\varepsilon};\Omega)$, and $H(\mathbf{curl}\,^0;\Omega)$ the divergence-free and curl-free subspaces, respectively. We shall frequently need the Sobolev spaces $H_0^1(\Omega) = \{q \in H^1(\Omega) : q|_\Gamma = 0\}$, $H_0(\mathrm{div}\,;\Omega) = \{\mathbf{v} \in H(\mathrm{div}\,;\Omega) : \mathbf{n}\cdot\mathbf{v}|_\Gamma = 0\}$, $H_0(\mathbf{curl}\,;\Omega) = \{\mathbf{v} \in H(\mathbf{curl}\,;\Omega) : \mathbf{n}\times\mathbf{v}|_\Gamma = \mathbf{0}\}$, and $H_0(\mathrm{div}\,^0;\Omega) = H_0(\mathrm{div}\,;\Omega) \cap H(\mathrm{div}\,^0;\Omega)$ and $H_0(\mathbf{curl}\,^0;\Omega) = H_0(\mathbf{curl}\,;\Omega) \cap H(\mathbf{curl}\,^0;\Omega)$. We further need

$$H_{\mathrm{flux},\Gamma}(\mathrm{div}\,^0;\boldsymbol{\varepsilon};\Omega) = \left\{\mathbf{v} \in H(\mathrm{div}\,^0;\boldsymbol{\varepsilon};\Omega) : \textstyle\int_{\Gamma_i}(\boldsymbol{\varepsilon}\mathbf{v})\cdot\mathbf{n} = 0, \quad 1 \leq i \leq n\right\}, \tag{2.5}$$

$$H_{0,\mathrm{flux},\Sigma}(\mathrm{div}\,^0;\Omega) = \left\{\mathbf{v} \in H_0(\mathrm{div}\,^0;\Omega) : \textstyle\int_{\Sigma_j}\mathbf{v}\cdot\mathbf{n} = 0, \quad 1 \leq j \leq N\right\}, \tag{2.6}$$

$$Q = \left\{q \in H^1(\Omega) : \quad q|_{\Gamma_0} = 0, \quad q|_{\Gamma_i} = \text{constant}, 1 \leq i \leq n\right\}. \tag{2.7}$$

By using the L^2-norm $||\cdot||_0$, we define the norms $||q||_1^2 = ||q||_0^2 + ||\nabla q||_0^2$, $||\mathbf{v}||_{0,\mathbf{curl}}^2 = ||\mathbf{v}||_0^2 + ||\mathbf{curl}\,\mathbf{v}||_0^2$, $||\mathbf{v}||_{0,\mathrm{div}}^2 = ||\mathbf{v}||_0^2 + ||\mathrm{div}\,\mathbf{v}||_0^2$, and equip $H(\mathrm{div}\,;\Omega)$ and $H(\mathbf{curl}\,;\Omega)$, as well as their subspaces, with the norms $||\cdot||_{0,\mathrm{div}}$ and $||\cdot||_{0,\mathbf{curl}}$, respectively. In addition, we write by $(\cdot,\cdot)$ and $(\cdot,\cdot)_{\boldsymbol{\varepsilon}}$ the L^2- and $\boldsymbol{\varepsilon}$-weighted L^2- inner products, with the norms $||\cdot||_0$ and $||\cdot||_{0,\boldsymbol{\varepsilon}}$, respectively. We shall need the L^2 orthogonal decompositions with respect to the $(\cdot,\cdot)_{\boldsymbol{\varepsilon}}$ inner product (cf. [26] [1]). In three dimensions, there holds

$$(L^2(\Omega))^3 \quad = \quad \nabla Q + \boldsymbol{\varepsilon}^{-1}\mathbf{curl}\,(H(\mathbf{curl}\,;\Omega)\cap H_{0,\text{flux},\Sigma}(\text{div}\,^0;\Omega)).$$

More specifically, for any $\mathbf{v}\in(L^2(\Omega))^3$, we have the decomposition under the $(\cdot,\cdot)_{\boldsymbol{\varepsilon}}$ inner product

$$\mathbf{v}=\nabla q+\boldsymbol{\varepsilon}^{-1}\mathbf{curl}\,\boldsymbol{\psi},\quad q\in Q,\quad \boldsymbol{\psi}\in H(\mathbf{curl}\,;\Omega)\cap H_{0,\text{flux},\Sigma}(\text{div}\,^0;\Omega), \tag{2.8}$$

$$||\mathbf{v}||^2_{0,\boldsymbol{\varepsilon}}=||\nabla q||^2_{0,\boldsymbol{\varepsilon}}+||\boldsymbol{\varepsilon}^{-1/2}\mathbf{curl}\,\boldsymbol{\psi}||^2_0, \tag{2.9}$$

with the stability estimate

$$||\boldsymbol{\psi}||_0\le C||\mathbf{curl}\,\boldsymbol{\psi}||_0. \tag{2.10}$$

In addition, we know by the regularity results in [1] for Lipschitz polyhedral domains that

$$\boldsymbol{\psi}\in(H^s(\Omega))^3\quad(s>1/2),\quad ||\boldsymbol{\psi}||_s\le C||\mathbf{curl}\,\boldsymbol{\psi}||_0. \tag{2.11}$$

Similar results hold also in two dimensions.

With the above preparations, we are ready to state the variational formulation of the system (2.1)-(2.4):
Find $(\omega^2,\mathbf{u}\neq\mathbf{0})\in\mathbb{R}\times H_0(\mathbf{curl}\,;\Omega)\cap H_{\text{flux},\Gamma}(\text{div}\,^0;\boldsymbol{\varepsilon};\Omega)$ such that

$$(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{u},\mathbf{curl}\,\mathbf{v})=\omega^2(\boldsymbol{\varepsilon}\mathbf{u},\mathbf{v})\qquad\forall\mathbf{v}\in H_0(\mathbf{curl}\,;\Omega)\cap H_{\text{flux},\Gamma}(\text{div}\,^0;\boldsymbol{\varepsilon};\Omega). \tag{2.12}$$

Equivalently, we have the following mixed formulation:
Find $(\omega^2,\mathbf{u}\neq\mathbf{0})\in\mathbb{R}\times H_0(\mathbf{curl}\,;\Omega)$ and $p\in Q$ such that

$$\left\{\begin{array}{rcl}(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{u},\mathbf{curl}\,\mathbf{v})+(\boldsymbol{\varepsilon}\mathbf{v},\nabla p)&=&\omega^2(\boldsymbol{\varepsilon}\mathbf{u},\mathbf{v})\qquad\forall\mathbf{v}\in H_0(\mathbf{curl}\,;\Omega),\\(\boldsymbol{\varepsilon}\mathbf{u},\nabla q)&=&0\qquad\forall q\in Q.\end{array}\right. \tag{2.13}$$

The role of the multiplier p is to relax the constraints from $H_{\text{flux},\Gamma}(\text{div}\,^0;\boldsymbol{\varepsilon};\Omega)$, and p is identically zero. It follows from [1] [26] that $H_0(\mathbf{curl}\,;\Omega)\cap H_{\text{flux},\Gamma}(\text{div}\,^0;\boldsymbol{\varepsilon};\Omega)$ is compactly imbedded into $(L^2(\Omega))^d$ and the bilinear form

$$a(\mathbf{u},\mathbf{v}):=(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{u},\mathbf{curl}\,\mathbf{v}) \tag{2.14}$$

is coercive and bounded over $H_0(\mathbf{curl}\,;\Omega)\cap H_{\text{flux},\Gamma}(\text{div}\,^0;\boldsymbol{\varepsilon};\Omega)$. Consequently, the self-adjoint operator of (2.12) and the counterpart of (2.13) are both compact, admitting a countable set of real and positive eigenvalues that exclude the zero eigenvalue. We shall focus on finite element solutions to the mixed problem (2.13) in this paper.

3 Mixed finite element method

Let $\mathcal{T}_h$ be a shape-regular and conforming triangulation of Ω into elements (cf., [12] [29] [28]), e.g., simplexes, quadrilaterals, hexahedra, etc, where $h:=\max_{K\in\mathcal{T}_h}h_K$, and h_K is the diameter of element K. We assume that the triangulation aligns with the discontinuities of $\boldsymbol{\mu}$ and $\boldsymbol{\varepsilon}$ and all components Γ_i $(0\le i\le n)$ of Γ. In other words, each interface between Ω_ℓ and Ω_k where $\boldsymbol{\mu}$ and $\boldsymbol{\varepsilon}$ are discontinuous, $1\le k\neq\ell\le m$, is a union of elemental edges $(d=2)$ or faces $(d=3)$, and so are Γ_i $(0\le i\le n)$. We denote by $\mathcal{F}_h^{\text{int}}$ the set of all interior elemental faces (edges, if $d=2$) of $\mathcal{T}_h$ and by $\mathcal{F}_h^{\Gamma}$ the set of all boundary faces of $\mathcal{T}_h$, and set $\mathcal{F}_h=\mathcal{F}_h^{\text{int}}\cup\mathcal{F}_h^{\Gamma}$. For a piecewise smooth function $\mathbf{v}$, we introduce the jump and the average. For each $F\in\mathcal{F}_h^{\text{int}}$, shared by two neighbouring elements $K^{(+)}$ and $K^{(-)}$, with outward unit normals $\mathbf{n}^{(+)}$ and $\mathbf{n}^{(-)}=-\mathbf{n}^{(+)}$ respectively, we write $\mathbf{v}^{(+)}$ and $\mathbf{v}^{(-)}$ the traces of $\mathbf{v}$ taken from within $K^{(+)}$ and $K^{(-)}$ respectively, and we define the tangential jump and the average across F by $[\![\mathbf{n}\times\mathbf{v}]\!]:=\mathbf{n}^{(+)}\times\mathbf{v}^{(+)}+\mathbf{n}^{(-)}\times\mathbf{v}^{(-)}$ and $\{\!\{\mathbf{v}\}\!\}:=(\mathbf{v}^{(+)}+\mathbf{v}^{(-)})/2$, respectively, where the notation $\mathbf{n}$ stands for $\mathbf{n}^{(+)}$ or $\mathbf{n}^{(-)}$ but is fixed. For a boundary face $F\in\mathcal{F}_h^{\Gamma}$, $[\![\mathbf{n}\times\mathbf{v}]\!]:=\mathbf{n}\times\mathbf{v}$ and $\{\!\{\mathbf{v}\}\!\}:=\mathbf{v}$. We also define the jump and the average for a scalar function across F by $[\![q]\!]:=q^{(+)}-q^{(-)}$ and $\{\!\{q\}\!\}=(q^{(+)}+q^{(-)})/2$, where $q^{(+)}$ and $q^{(-)}$ denote the traces taken from within $K^{(+)}$ and $K^{(-)}$ respectively, and for a boundary face $F\in\mathcal{F}_h^{\Gamma}$, $[\![q]\!]:=q$ and $\{\!\{q\}\!\}=q$. Let $\mathbf{U}(K)\subset(H^1(K))^d$ and $Q(K)\subset H^1(K)$ denote two finite element spaces on $K\in\mathcal{T}_h$, which are to be defined (See Section 7). We define two finite element spaces for approximating the electric field solution $\mathbf{u}$ and the multiplier p in (2.13) as follows:

$$\mathbf{U}_h:=\{\mathbf{v}\in(L^2(\Omega))^d:\mathbf{v}|_K\in\mathbf{U}(K)\quad\forall K\in\mathcal{T}_h\},\quad Q_h:=\{q\in Q:q|_K\in Q(K)\quad\forall K\in\mathcal{T}_h\}.$$

As usual, we introduce a mesh function $\mathrm{h} \in L^\infty(\mathcal{F}_h)$ by $\mathrm{h}(\mathbf{x}) := h_F \mathrm{c}(\mathbf{x}), \forall \mathbf{x} \in F, \forall F \in \mathcal{F}_h$, where h_F denotes the diameter of the face F (the length of the edge for $d = 2$), and the function $\mathrm{c}(\mathbf{x})$ is defined in the following way: if $\boldsymbol{\mu}_K$ denotes the extension of $\boldsymbol{\mu}|_K$ to ∂K, and $|\boldsymbol{\mu}_K|$ denotes the spectral norm of $\boldsymbol{\mu}_K$, then $\mathrm{c}(\mathbf{x}) := \min(|\boldsymbol{\mu}_{K^{(+)}}(\mathbf{x})|, |\boldsymbol{\mu}_{K^{(-)}}(\mathbf{x})|)$ if $\mathbf{x}$ is in the interior of $\partial K^{(+)} \cap \partial K^{(-)}$ and $\mathrm{c}(\mathbf{x}) := |\boldsymbol{\mu}_K(\mathbf{x})|$ if $\mathbf{x}$ is in the interior of $\partial K \cap \Gamma$. Let $\delta \geq 0$ be a parameter that will be specified later, and $\alpha > 0$ be a penalty parameter that is independent of the mesh size h and the materials $\boldsymbol{\mu}$ and $\boldsymbol{\varepsilon}$. Then we define four bilinear forms $a_{h,\mathbf{curl}}(\cdot,\cdot), a_{h,\mathrm{div}}(\cdot,\cdot), a_h(\cdot,\cdot) : \mathbf{U}_h \times \mathbf{U}_h \to \mathbb{R}$ and $b(\cdot,\cdot) : \mathbf{U}_h \times Q_h \to \mathbb{R}$ as follows:

$$\begin{aligned} a_{h,\mathbf{curl}}(\mathbf{u},\mathbf{v}) \quad := \quad & \sum_{K\in\mathcal{T}_h} (\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v})_{0,K} - \sum_{F\in\mathcal{F}_h}\int_F [\![\mathbf{n}\times\mathbf{v}]\!]\cdot\{\!\{\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{u}\}\!\} \\ & - \sum_{F\in\mathcal{F}_h}\int_F [\![\mathbf{n}\times\mathbf{u}]\!]\cdot\{\!\{\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{v}\}\!\} + \alpha\sum_{F\in\mathcal{F}_h}\int_F \mathrm{h}^{-1}[\![\mathbf{n}\times\mathbf{u}]\!]\cdot[\![\mathbf{n}\times\mathbf{v}]\!], \end{aligned} \tag{3.1}$$

$$a_{h,\mathrm{div}}(\mathbf{u},\mathbf{v}) \quad := \quad \sum_{K\in\mathcal{T}_h} h_K^{2-2\delta}(\mathrm{div}\,(\boldsymbol{\varepsilon}\mathbf{u}), \mathrm{div}\,(\boldsymbol{\varepsilon}\mathbf{v}))_{0,K} + \sum_{F\in\mathcal{F}_h^{\mathrm{int}}} h_F^{1-2\delta}\int_F [\![(\boldsymbol{\varepsilon}\mathbf{u})\cdot\mathbf{n}]\!][\![(\boldsymbol{\varepsilon}\mathbf{v})\cdot\mathbf{n}]\!], \tag{3.2}$$

$$a_h(\mathbf{u},\mathbf{v}) \quad := \quad a_{h,\mathbf{curl}}(\mathbf{u},\mathbf{v}) + a_{h,\mathrm{div}}(\mathbf{u},\mathbf{v}), \tag{3.3}$$

$$b(\mathbf{v},q) \quad := \quad (\boldsymbol{\varepsilon}\mathbf{v}, \nabla q). \tag{3.4}$$

Now we are ready to formulate a mixed finite element approximation of the continuous eigenproblem (2.13):

Find $(\omega_h^2, \mathbf{u}_h \neq \mathbf{0}) \in \mathbb{R}\times\mathbf{U}_h$ and $p_h \in Q_h$ such that

$$\left\{ \begin{array}{rcl} a_h(\mathbf{u}_h,\mathbf{v}_h) + b(\mathbf{v}_h,p_h) & = & \omega_h^2(\boldsymbol{\varepsilon}\mathbf{u}_h,\mathbf{v}_h) \qquad \forall \mathbf{v}_h\in\mathbf{U}_h, \\ b(\mathbf{u}_h,q_h) & = & 0 \qquad \forall q_h \in Q_h. \end{array} \right. \tag{3.5}$$

To study the discrete eigenproblem (3.5), we shall first study the corresponding source problem and its finite element method. The source problem is for any given $\mathbf{f}\in(L^2(\Omega))^d$, to find $\mathbf{u}$ and p such that

$$\left\{ \begin{array}{l} \mathbf{curl}\,(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{u}) + \boldsymbol{\varepsilon}\nabla p = \boldsymbol{\varepsilon}\mathbf{f} \quad \text{in } \Omega, \\ \mathrm{div}\,(\boldsymbol{\varepsilon}\mathbf{u}) = 0 \quad \text{in } \Omega, \\ \mathbf{n}\times\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma, \\ \int_{\Gamma_i}(\boldsymbol{\varepsilon}\mathbf{u})\cdot\mathbf{n} = 0, \quad 1\leq i\leq n, \\ p = 0 \quad \text{on } \Gamma_0, \\ p = \text{constant} \quad \text{on } \Gamma_i, \quad 1\leq i\leq n. \end{array} \right. \tag{3.6}$$

The variational formulation of problem (3.6) is to find $\mathbf{u}\in H_0(\mathbf{curl}\,;\Omega)$ and $p\in Q$ such that

$$\left\{ \begin{array}{rcl} a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) & = & (\boldsymbol{\varepsilon}\mathbf{f},\mathbf{v}) \qquad \forall\mathbf{v}\in H_0(\mathbf{curl}\,;\Omega), \\ b(\mathbf{u},q) & = & 0 \qquad \forall q\in Q. \end{array} \right. \tag{3.7}$$

Its finite element approximation is to find $\mathbf{u}_h\in\mathbf{U}_h$ and $p_h\in Q_h$ such that

$$\left\{ \begin{array}{rcl} a_h(\mathbf{u}_h,\mathbf{v}_h) + b(\mathbf{v}_h,p_h) & = & (\boldsymbol{\varepsilon}\mathbf{f},\mathbf{v}_h) \qquad \forall\mathbf{v}_h\in\mathbf{U}_h, \\ b(\mathbf{u}_h,q_h) & = & 0 \qquad \forall q_h\in Q_h. \end{array} \right. \tag{3.8}$$

Let $(\mathbf{u},p)$ be the exact solution to problem (3.7) and $(\mathbf{u}_h,p_h)$ be the finite element solution to problem (3.8), then it is easy to see the consistency or error orthogonality property:

$$\left\{ \begin{array}{c} a_h(\mathbf{u}-\mathbf{u}_h,\mathbf{v}_h) + b(\mathbf{v}_h,p-p_h) = 0 \qquad \forall\mathbf{v}_h\in\mathbf{U}_h, \\ b(\mathbf{u}-\mathbf{u}_h,q_h) = 0 \qquad \forall q_h\in Q_h. \end{array} \right. \tag{3.9}$$

Before closing this section, we review some formulations in the literature which are related to the one (3.8).

1) The weighted method (cf. [11] [17]), where $w(\mathbf{x})$ is a weight function defined over Ω:

$$\begin{aligned} (\mathbf{curl}\,\mathbf{u},\mathbf{curl}\,\mathbf{v}) + (w(\mathbf{x})\,\mathrm{div}\,\mathbf{u},\mathrm{div}\,\mathbf{v}) + (w(\mathbf{x})\,\mathrm{div}\,\mathbf{v},p) &= (\mathbf{f},\mathbf{v}), \\ (w(\mathbf{x})\,\mathrm{div}\,\mathbf{u},q) &= 0. \end{aligned}$$

2) The L^2-projection method (cf. [20] [22] [23]):

$$
\begin{aligned}
(\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) + \sum_{K\in\mathcal{T}_h} h_K^2(\mathrm{div}\,\mathbf{u}, \mathrm{div}\,\mathbf{v})_{0,K} + (\mathrm{div}\,\mathbf{v}, p) &= (\mathbf{f}, \mathbf{v}),\\
(\mathrm{div}\,\mathbf{u}, q) - (p, q) &= 0.
\end{aligned}
$$

3) The mixed $H^{-\alpha}$-weighted method (cf. [4] [6] [8] [7]):

$$
\begin{aligned}
(\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) + h^{2\alpha}(\mathrm{div}\,\mathbf{u}, \mathrm{div}\,\mathbf{v}) + (\mathbf{v}, \nabla p) &= (\mathbf{f}, \mathbf{v}),\\
(\mathbf{u}, \nabla q) - h^{2(1-\alpha)}(\nabla p, \nabla q) &= 0.
\end{aligned}
$$

4) The mixed method [25, 24]:

$$
\begin{aligned}
(\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) + \sum_{K\in\mathcal{T}_h} h_K^{2-2\delta}(\mathrm{div}\,\mathbf{u}, \mathrm{div}\,\mathbf{v})_{0,K} + (\mathrm{div}\,\mathbf{v}, p) &= (\mathbf{f}, \mathbf{v}),\\
(\mathrm{div}\,\mathbf{u}, q) &= 0.
\end{aligned}
$$

All these methods are based on the continuous Lagrange nodal elements and do not deal with the convergence of the discrete Gauss' law in strong form. Although some of them may be used for inhomogeneous anisotropic and discontinuous media, but may be far from the optimal convergence. Among all these methods, only the method in 4) is somehow closer to our current work, hence we add a couple of more remarks here, in addition to the aforementioned differences. The method in 4) deals only with the eigenproblem in homogeneous media, and it is inappropriate for inhomogeneous anisotropic and discontinuous media. It uses the piecewise constant element to approximate the multiplier, and is restricted to simplexes, not suitable for other meshes, such as hexahedra, because its underlying theory relies on the C^1 elements available on simplexes. Moreover, this method makes use of some special meshes such as Clough-Tocher/Alfeld macro meshes. Our current work develops a general framework for various elements, including various discontinuous elements and edge elements, and handles the strong convergence of the Gauss' law as well. In particular, the current work deals with inhomogeneous anisotropic and discontinuous media in a topologically nontrivial domain, and is suitable for the commonly used meshes, including simplexes, quadrilaterals and hexahedra. One of the significant discoveries in the current work is about the validity of the nonaffine quadrilateral and hexahedral Nédélec elements and discontinuous elements for the Maxwell eigenproblem.

4 Discrete kernel-ellipticity, discrete compactness, and error estimates

In this section, we provide a general theory of stability and error estimates for the mixed finite element problem (3.8) of the Maxwell source problem (3.6). These results provide the essential bridging tools for our subsequent analysis of the discrete eigenproblem (3.5). We shall reach our aim by following the classical framework of the saddle-point problems ([9] [29]), which mainly consists of the discrete kernel-ellipticity and the discrete inf-sup condition. In addition, we also derive the important discrete compactness property, see Proposition 4.1. We start to introduce three mesh-dependent semi-norms:

$$
\begin{aligned}
|\mathbf{v}|^2_{h,\mathbf{curl}} &:= \sum_{K\in\mathcal{T}_h} ||\boldsymbol{\mu}^{-1/2}\mathbf{curl}\,\mathbf{v}||^2_{0,K} + \sum_{F\in\mathcal{F}_h}\int_F \mathrm{h}^{-1}|[\![\mathbf{n}\times\mathbf{v}]\!]|^2, &(4.1)\\
|\mathbf{v}|^2_{h,\mathrm{div}} := a_{h,\mathrm{div}}(\mathbf{v},\mathbf{v}) &= \sum_{K\in\mathcal{T}_h} h_K^{2-2\delta}||\mathrm{div}\,(\boldsymbol{\varepsilon}\mathbf{v})||^2_{0,K} + \sum_{F\in\mathcal{F}_h^{\mathrm{int}}} h_F^{1-2\delta}\int_F |[\![(\boldsymbol{\varepsilon}\mathbf{v})\cdot\mathbf{n}]\!]|^2, &(4.2)\\
||\mathbf{v}||^2_h &:= ||\mathbf{v}||^2_{0,\boldsymbol{\varepsilon}} + |\mathbf{v}|^2_{h,\mathbf{curl}} + |\mathbf{v}|^2_{h,\mathrm{div}}\ , &(4.3)
\end{aligned}
$$

then $||\cdot||_h$ will serve as a norm of the space $\mathbf{U}_h$. Now we can define the discrete kernel space:

$$
\mathbf{K}_h := \{\mathbf{v}_h \in \mathbf{U}_h : b(\mathbf{v}_h, q_h) = 0 \quad \forall q_h \in Q_h\}. \tag{4.4}
$$

To study the kernel-ellipticity, we assume the finite element space Q_h has the first-order approximation property: for any $p \in Q$, there exists an interpolation operator J_h such that $J_h p \in Q_h$, and it holds for $0 \le t \le 1$ that

$$
\Big(\sum_{K\in\mathcal{T}_h}\Big(h_K^{-2t}||p - J_h p||^2_{0,K} + \sum_{F\subset\partial K} h_F^{-(2t-1)}\int_F |p - J_h p|^2\Big)\Big)^{1/2} + h^{1-t}||J_h p||_1 \le Ch^{1-t}||p||_1. \tag{4.5}
$$

We know the averaging type interpolation operators, such as the Scott-Zhang interpolation [38] [13], the Clément interpolation [29] and the Bernardi-Girault interpolation [5], possess this first-order approximation.

Lemma 4.1. *Under the approximation property (4.5) of Q_h, it holds that*

$$\sup_{\mathbf{0}\neq\mathbf{v}_h\in\mathbf{K}_h}\frac{b(\mathbf{v}_h,p)}{||\mathbf{v}_h||_h}\leq\sup_{\mathbf{0}\neq\mathbf{v}_h\in\mathbf{K}_h}\frac{b(\mathbf{v}_h,p)}{|\mathbf{v}_h|_{h,\mathrm{div}}}\leq Ch^{\delta}||p||_1\qquad\forall p\in Q. \tag{4.6}$$

Proof. For any fixed $p\in Q$, we know $J_hp\in Q_h$. First,

$$\begin{aligned}b(\mathbf{v}_h,p) &= b(\mathbf{v}_h,p-J_hp)=(\boldsymbol{\varepsilon}\mathbf{v}_h,\nabla(p-J_hp))\\ &= -\sum_{K\in\mathcal{T}_h}(\mathrm{div}\,(\boldsymbol{\varepsilon}\mathbf{v}_h),p-J_hp)_{0,K}+\sum_{K\in\mathcal{T}_h}\sum_{F\subset\partial K}\int_F(p-J_hp)(\boldsymbol{\varepsilon}\mathbf{v}_h)\cdot\mathbf{n}\\ &= -\sum_{K\in\mathcal{T}_h}(\mathrm{div}\,(\boldsymbol{\varepsilon}\mathbf{v}_h),p-J_hp)_{0,K}+\sum_{F\in\mathcal{F}_h^{\mathrm{int}}}\int_F[\![(\boldsymbol{\varepsilon}\mathbf{v}_h)\cdot\mathbf{n}]\!](p-J_hp).\end{aligned}$$

Taking $t:=1-\delta$ in (4.5), for any $\mathbf{v}_h\in\mathbf{K}_h$ by the Cauchy-Schwarz inequality, from (4.3), we can obtain (4.6). □

The semi-norm $|\cdot|_{h,\mathrm{div}}$ ensures (4.6), which implies the discrete compactness property in Proposition 4.1.

We next recall a coercivity result of $a_{h,\mathbf{curl}}(\cdot,\cdot)$ when the penalty parameter α is sufficiently large. The result can be proven in a standard manner for the DG method (e.g., cf. [10] [30]).

Lemma 4.2. *For a sufficiently large penalty parameter α, we have, for all $\mathbf{v}_h\in\mathbf{U}_h$,*

$$a_{h,\mathbf{curl}}(\mathbf{v}_h,\mathbf{v}_h)\geq C\Big(\sum_{K\in\mathcal{T}_h}||\boldsymbol{\mu}^{-1/2}\mathbf{curl}\,\mathbf{v}_h||_{0,K}^2+\sum_{F\in\mathcal{F}_h}\int_F\mathrm{h}^{-1}|[\![\mathbf{n}\times\mathbf{v}_h]\!]|^2\Big).$$

Theorem 4.1. *Under the same conditions as in Lemma 4.1 and Lemma 4.2, we have the $\mathbf{K}_h$ kernel-ellipticity:*

$$a_h(\mathbf{v}_h,\mathbf{v}_h)\geq C||\mathbf{v}_h||_h^2\qquad\forall\mathbf{v}_h\in\mathbf{K}_h. \tag{4.7}$$

Proof. First, we know from (3.1)-(3.3), (4.1)-(4.3) and Lemma 4.2 that

$$a_h(\mathbf{v}_h,\mathbf{v}_h)\geq C(|\mathbf{v}_h|_{h,\mathbf{curl}}^2+|\mathbf{v}_h|_{h,\mathrm{div}}^2). \tag{4.8}$$

Next, applying the L^2-orthogonal decomposition with respect to $(\cdot,\cdot)_{\boldsymbol{\varepsilon}}$ (see (2.8)-(2.11)) we can write

$$\mathbf{v}_h=\nabla p_0+\boldsymbol{\varepsilon}^{-1}\mathbf{curl}\,\boldsymbol{\psi},\quad p_0\in Q,\quad\boldsymbol{\psi}\in H(\mathbf{curl}\,;\Omega)\cap H_{0,\mathrm{flux},\Sigma}(\mathrm{div}\,^0;\Omega),$$

with the stability estimates

$$||\mathbf{v}_h||_{0,\boldsymbol{\varepsilon}}^2=||\nabla p_0||_{0,\boldsymbol{\varepsilon}}^2+||\boldsymbol{\varepsilon}^{-1/2}\mathbf{curl}\,\boldsymbol{\psi}||_0^2,$$

$$||\boldsymbol{\psi}||_0\leq C||\mathbf{curl}\,\boldsymbol{\psi}||_0\leq C||\boldsymbol{\varepsilon}^{-1/2}\mathbf{curl}\,\boldsymbol{\psi}||_0\leq C||\mathbf{v}_h||_{0,\boldsymbol{\varepsilon}},$$

and the regularity that $\boldsymbol{\psi}\in(H^s(\Omega))^d$ for some $1/2<s<1$, satisfying

$$||\boldsymbol{\psi}||_s\leq C||\mathbf{curl}\,\boldsymbol{\psi}||_0\leq C||\mathbf{v}_h||_{0,\boldsymbol{\varepsilon}}.$$

Since $\mathbf{v}_h\in\mathbf{K}_h$, we can further write

$$||\mathbf{v}_h||_{0,\boldsymbol{\varepsilon}}^2=(\boldsymbol{\varepsilon}\mathbf{v}_h,\mathbf{v}_h)=(\boldsymbol{\varepsilon}\mathbf{v}_h,\nabla p_0+\boldsymbol{\varepsilon}^{-1}\mathbf{curl}\,\boldsymbol{\psi})=(\boldsymbol{\varepsilon}\mathbf{v}_h,\nabla(p_0-J_hp_0))+(\mathbf{v}_h,\mathbf{curl}\,\boldsymbol{\psi}), \tag{4.9}$$

where the last term can be estimated by the Cauchy-Schwarz inequality and the local trace theorem on $F\in\mathcal{F}_h$,

$$\begin{aligned}(\mathbf{v}_h,\mathbf{curl}\,\boldsymbol{\psi}) &= \sum_{K\in\mathcal{T}_h}(\mathbf{curl}\,\mathbf{v}_h,\boldsymbol{\psi})_{0,K}-\sum_{F\in\mathcal{F}_h}\int_F[\![\mathbf{n}\times\mathbf{v}_h]\!]\cdot\boldsymbol{\psi}\\ &\leq C\Big(\sum_{K\in\mathcal{T}_h}||\boldsymbol{\mu}^{-1/2}\mathbf{curl}\,\mathbf{v}_h||_{0,K}^2\Big)^{1/2}||\boldsymbol{\psi}||_0+C\Big(\sum_{F\in\mathcal{F}_h}\int_F\mathrm{h}^{-1}|[\![\mathbf{n}\times\mathbf{v}_h]\!]|^2\Big)^{1/2}||\boldsymbol{\psi}||_s\\ &\leq C\Big(\sum_{K\in\mathcal{T}_h}||\boldsymbol{\mu}^{-1/2}\mathbf{curl}\,\mathbf{v}_h||_{0,K}^2+\sum_{F\in\mathcal{F}_h}\int_F\mathrm{h}^{-1}|[\![\mathbf{n}\times\mathbf{v}_h]\!]|^2\Big)^{1/2}||\mathbf{v}_h||_{0,\boldsymbol{\varepsilon}}\\ &= C|\mathbf{v}_h|_{h,\mathbf{curl}}||\mathbf{v}_h||_{0,\boldsymbol{\varepsilon}}\,.\end{aligned}$$

On the other hand, the same proof of Lemma 4.1 leads to the estimate of the last second term in (4.9):

$$(\boldsymbol{\varepsilon}\mathbf{v}_h, \nabla(p_0 - J_h p_0)) \le Ch^\delta |\mathbf{v}_h|_{h,\text{div}} ||p_0||_1 \le Ch^\delta |\mathbf{v}_h|_{h,\text{div}} ||\mathbf{v}_h||_{0,\boldsymbol{\varepsilon}}.$$

Now it follows readily from the above two estimates and (4.8) that

$$||\mathbf{v}_h||_{0,\boldsymbol{\varepsilon}}^2 \quad \le \quad C||\mathbf{v}_h||_{0,\boldsymbol{\varepsilon}}(|\mathbf{v}_h|_{h,\mathbf{curl}} + |\mathbf{v}_h|_{h,\text{div}}) \le C||\mathbf{v}_h||_{0,\boldsymbol{\varepsilon}}\Big(a_h(\mathbf{v}_h, \mathbf{v}_h)\Big)^{1/2},$$

that is,

$$a_h(\mathbf{v}_h, \mathbf{v}_h) \ge C||\mathbf{v}_h||_{0,\boldsymbol{\varepsilon}}^2.$$

This, along with (4.3) and (4.8), proves the $\mathbf{K}_h$ kernel-ellipticity. □

By the definitions, we can easily see the following continuity of $a_h(\cdot,\cdot)$ and $(\boldsymbol{\varepsilon}\mathbf{f},\cdot)$:

$$|a_h(\mathbf{u}_h, \mathbf{v}_h)| \le C||\mathbf{u}_h||_h||\mathbf{v}_h||_h \qquad \forall \mathbf{u}_h, \mathbf{v}_h \in \mathbf{U}_h, \tag{4.10}$$

$$|(\boldsymbol{\varepsilon}\mathbf{f}, \mathbf{v}_h)| \le ||\boldsymbol{\varepsilon}^{1/2}\mathbf{f}||_0||\boldsymbol{\varepsilon}^{1/2}\mathbf{v}_h||_0 \le C||\mathbf{f}||_0||\mathbf{v}_h||_h \qquad \forall \mathbf{v}_h \in \mathbf{U}_h. \tag{4.11}$$

However, the continuity of $b(\cdot,\cdot)$ is more subtle, since it does not necessarily have the *uniform* continuity under the norm $||\cdot||_{Q_h}$. To better understand this, we first note that

$$|b(\mathbf{v}_h, q)| = |(\boldsymbol{\varepsilon}\mathbf{v}_h, \nabla q)| \le ||\boldsymbol{\varepsilon}^{1/2}\mathbf{v}_h||_0||\boldsymbol{\varepsilon}^{1/2}\nabla q||_0 \le C||\mathbf{v}_h||_h||\nabla q||_0 \qquad \forall \mathbf{v}_h \in \mathbf{U}_h, \forall q \in Q. \tag{4.12}$$

Then, since all the norms are equivalent on the finite dimensional space, there exists a constant $C^*(h)$, which may depend on h, such that, where $||\cdot||_{Q_h}$ denoting the norm of Q_h will be defined later (cf. Theorem 4.2),

$$||\nabla q_h||_0 \le C^*(h)||q_h||_{Q_h} \qquad \forall q_h \in Q_h,$$

which, along with (4.12), implies

$$|b(\mathbf{v}_h, q_h)| \le C^*(h)||\mathbf{v}_h||_h||q_h||_{Q_h}, \tag{4.13}$$

i.e., $b(\cdot,\cdot)$ is continuous over $\mathbf{U}_h \times Q_h$ with respect to the finite element spaces $(\mathbf{U}_h, ||\cdot||_h)$ and $(Q_h, ||\cdot||_{Q_h})$. Clearly, this continuity is not necessarily uniform in h.

Remark 4.1. If either of $\mathbf{u}$ and $\mathbf{v}$ is not a finite element function, one might think that the continuity (4.10) still holds for $a_h(\mathbf{u}, \mathbf{v})$ in the norm $||\cdot||_h$. However, this is generally not true unless either of $\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{u}$ and $\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{v}$ is more regular than $(L^2(\Omega))^d$, mainly due to the fact that the trace theorem does not hold for functions with only L^2-regularity. Unfortunately, for the purpose of error estimates, we must have certain continuity of $a_h(\cdot,\cdot)$ for non-finite element functions. It turns out that this is not a trivial issue in the context of the Maxwell equations; see Lemma 4.3.

Next, we are ready to establish the first main result of stability.

Theorem 4.2. *Assume that*

$$||q_h||_{Q_h} := \sup_{\mathbf{0} \ne \mathbf{v}_h \in \mathbf{U}_h} \frac{b(\mathbf{v}_h, q_h)}{||\mathbf{v}_h||_h} \qquad \forall\, q_h \in Q_h, \tag{4.14}$$

is a norm over Q_h. Then under the same conditions as in Theorem 4.1, for any $\mathbf{f} \in (L^2(\Omega))^d$, problem (3.8) admits a unique solution $(\mathbf{u}_h, p_h) \in \mathbf{U}_h \times Q_h$, satisfying

$$||\mathbf{u}_h||_h + ||p_h||_{Q_h} \le C||\mathbf{f}||_0. \tag{4.15}$$

Proof. With the $\mathbf{K}_h$ kernel-ellipticity (4.7), the inf-sup condition (4.14), and the continuities (4.10)-(4.11) and (4.13), the conclusion is a direct consequence of the classical saddle-point theory (cf. [9] [29]) applied to problem (3.8). □

Now, we are in a position to give our second main result, error estimates.

From the abstract theory of spectral approximation ([3] [34]) of the compact operator, we need to consider the finite element methods of two source problems and their error estimates. The first source problem takes the continuous eigenfunction of the eigenproblem as the right-hand side, while the second source problem takes the discrete eigenfunction of the discrete eigenproblem as the right-hand side. For the first source problem, we consider a general right-hand side (cf. (2.5))

$$\mathbf{f} \in H_{\text{flux},\Gamma}(\text{div}\,^0; \boldsymbol{\varepsilon}; \Omega)\,. \tag{4.16}$$

With this right-hand side, the source problem (3.6) has a multiplier $p = 0$. For the second source problem, we likewise consider a general right-hand side, $\mathbf{f} \in (L^2(\Omega))^d$ only. The multiplier satisfies

$$p \in Q, \quad ||p||_1 \le C||\mathbf{f}||_0. \tag{4.17}$$

But p does not have more regularity than Q itself.

Before studying the error estimates, we recall (cf. Remark 4.1) a result from the DG literature ([10, Appendix], [7, Appendix]). For that purpose, we will use fractional order Sobolev space $H^s(D)$ for any open set D and any real number s, with semi-norm $|\cdot|_{s,D}$ and norm $||\cdot||_{s,D}$. When $D = \Omega$, we write $|\cdot|_s$ and $||\cdot||_s$.

Lemma 4.3. *For any* $\mathbf{z}$ *satisfying* $\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{z}|_K \in H(\mathbf{curl}\,; K) \cap (H^\sigma(K))^d$ $(0 < \sigma < 1/2)$ *for* $K \in \mathcal{T}_h$*, we have*

$$\sum_{F\in\mathcal{F}_h}\int_F [\![\mathbf{n}\times\mathbf{v}_h]\!]\cdot\{\!\{\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{z}\}\!\} \le C\Big(\sum_{F\in\mathcal{F}_h}\int_F \mathrm{h}^{-1}|[\![\mathbf{n}\times\mathbf{v}_h]\!]|^2\Big)^{1/2}$$
$$\times\Big(\sum_{K\in\mathcal{T}_h} h_K^{2\sigma}||\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{z}||_{\sigma,K}^2 + h_K^2||\mathbf{curl}\,(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{z})||_{0,K}^2 + ||\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{z}||_{0,K}^2\Big)^{1/2}.$$

For the convenience of the subsequent analysis, we introduce a notation

$$|\mathbf{v}|_{\sigma,h}^2 := \sum_{K\in\mathcal{T}_h}(h_K^{2\sigma}||\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{v}||_{\sigma,K}^2 + h_K^2||\mathbf{curl}\,(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{v})||_{0,K}^2 + ||\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{v}||_{0,K}^2). \tag{4.18}$$

Let $||\cdot||_h$ be the norm defined in (4.3), then we can show by the local inverse estimates and the condition on $\boldsymbol{\mu}$ that

$$|\mathbf{v}_h|_{\sigma,h}^2 \le C||\mathbf{v}_h||_h^2 \quad \forall\,\mathbf{v}_h \in \mathbf{U}_h, \tag{4.19}$$

The first result of the error estimates is for the source problem (3.6)/(3.7) with the source in (4.16) and the multiplier $p = 0$.

Theorem 4.3. *Let* $(\mathbf{u}, p)$ *be the exact solution of the source problem (3.6)/(3.7) with the source in (4.16) and the multiplier* $p = 0$*, with* $\mathbf{u}$ *satisfying the condition in Lemma 4.3, and* $(\mathbf{u}_h, p_h) \in \mathbf{U}_h \times Q_h$ *be the finite element solution to the problem (3.8), then the following error estimate holds under the same conditions as in Theorem 4.1:*

$$||\mathbf{u}-\mathbf{u}_h||_h + ||p_h||_{Q_h} \le C\inf_{\mathbf{v}_h\in\mathbf{U}_h}\Big(||\mathbf{u}-\mathbf{v}_h||_h + |\mathbf{u}-\mathbf{v}_h|_{\sigma,h} + \sup_{0\ne\mu_h\in Q_h}\frac{b(\mathbf{u}-\mathbf{v}_h,\mu_h)}{||\mu_h||_{Q_h}}\Big). \tag{4.20}$$

Proof. Under the $\mathbf{K}_h$ kernel-ellipticity in Theorem 4.1 and the inf-sup condition in Theorem 4.2, the estimate (4.20) can be established by adapting the standard abstract saddle-point theory (see, e.g., [9]). □

The second result of the error estimates is for the source problem (3.6)/(3.7) with the right-hand side $\mathbf{f} \in (L^2(\Omega))^d$ and the estimate (4.17).

Theorem 4.4. *Let* $(\mathbf{u}, p)$ *be the exact solution of the source problem (3.6)/(3.7) with the right-hand side* $\mathbf{f} \in (L^2(\Omega))^d$ *and (4.17), with* $\mathbf{u}$ *satisfying the condition in Lemma 4.3, and* $(\mathbf{u}_h, p_h)$ *be finite element solution of problem (3.8) in* $\mathbf{U}_h \times Q_h$*. Then under the same conditions as in Theorem 4.1, the following error estimate holds*

$$||\mathbf{u}-\mathbf{u}_h||_h \le C\inf_{\mathbf{v}_h\in\mathbf{U}_h}\Big(||\mathbf{u}-\mathbf{v}_h||_h + |\mathbf{u}-\mathbf{v}_h|_{\sigma,h} + \sup_{0\ne\mu_h\in Q_h}\frac{b(\mathbf{u}-\mathbf{v}_h,\mu_h)}{||\mu_h||_{Q_h}}\Big) + Ch^\delta||p||_1. \tag{4.21}$$

Proof. Again by adapting the standard saddle-point theory (see, e.g., [9]) (but with $p \ne 0$ now), we can achieve

$$\begin{aligned}||\mathbf{u}-\mathbf{u}_h||_h \quad &\le \quad C\inf_{\mathbf{v}_h\in\mathbf{U}_h}\Big(||\mathbf{u}-\mathbf{v}_h||_h + |\mathbf{u}-\mathbf{v}_h|_{\sigma,h} + \sup_{0\ne\mu_h\in Q_h}\frac{b(\mathbf{u}-\mathbf{v}_h,\mu_h)}{||\mu_h||_{Q_h}}\Big)\\ &\quad + C\inf_{q_h\in Q_h}\sup_{\mathbf{0}\ne\mathbf{v}_h\in\mathbf{K}_h}\frac{b(\mathbf{v}_h, p-q_h)}{||\mathbf{v}_h||_h}.\end{aligned}$$

Now applying Lemma 4.1 leads to the desired estimate. □

We can obtain the similar error estimate for the multiplier to the one (4.21), but those are not needed for the eigenproblem. One might think that the factor h^{δ} in (4.21) would mean very low convergence rate. This is not the case. As a matter of fact, Theorem 4.4 is used for the uniform convergence so that spurious-free and spectral-correct approximations of eigenmodes are ensured, while the convergence rate is determined by Theorem 4.3.

From Lemma 4.1, we now prove the well-known the discrete compactness property, as stated in the following proporsition. This property, which was firstly used by Kikuchi([31] [32]) for the Nédélec elements for Maxwell eigenproblem in homogeneous media of topologically trivial domains, mimics the continuous compactness property that $H_0(\mathbf{curl}\,;\Omega) \cap H(\mathrm{div}\,^0;\Omega)$ is compactly imbedded into $(L^2(\Omega))^d$ [1].

Proposition 4.1. *Lemma 4.1 with $\delta > 0$ yields the discrete compactness property, i.e., for any uniformly bounded sequence $\{\mathbf{v}_h\} \subset \mathbf{K}_h$, i.e., $||\mathbf{v}_h||_h \le 1$ for all h, there exists a $\mathbf{v} \in (L^2(\Omega))^d$(which is actually in $H_{\text{flux},\Gamma}(\mathrm{div}\,^0;\boldsymbol{\varepsilon};\Omega)$) such that $||\mathbf{v}_h - \mathbf{v}||_0 \to 0$ as $h \to 0$.*

Proof. We first apply the L^2-orthogonal decomposition with respect to $(\cdot,\cdot)_{\boldsymbol{\varepsilon}}$ to $\mathbf{v}_h$ to write $\mathbf{v}_h = \mathbf{z} + \nabla q$, $\mathbf{z} \in H_{\text{flux},\Gamma}(\mathrm{div}\,^0;\boldsymbol{\varepsilon};\Omega)$ and $q \in Q$, where $||\mathbf{v}_h||^2_{0,\boldsymbol{\varepsilon}} = ||\mathbf{z}||^2_{0,\boldsymbol{\varepsilon}} + ||\nabla q||^2_{0,\boldsymbol{\varepsilon}}$. Using the bound $||\mathbf{v}_h||_h \le 1$, we can derive

$$||q||_1 \le C||\nabla q||_0 \le C||\mathbf{v}_h||_{0,\boldsymbol{\varepsilon}} \le C||\mathbf{v}_h||_h \le C,$$

where the constant C is independent of h. Taking $\mathbf{v} := \mathbf{z} \in H_{\text{flux},\Gamma}(\mathrm{div}\,^0;\boldsymbol{\varepsilon};\Omega)$, we can write

$$||\mathbf{v}_h - \mathbf{v}||^2_{0,\boldsymbol{\varepsilon}} = (\boldsymbol{\varepsilon}(\mathbf{v}_h - \mathbf{v}), \mathbf{v}_h - \mathbf{v}) = (\boldsymbol{\varepsilon}(\mathbf{v}_h - \mathbf{z}), \nabla q) = (\boldsymbol{\varepsilon}\mathbf{v}_h, \nabla q) = b(\mathbf{v}_h, q)\,. \tag{4.22}$$

But we know from Lemma 4.1 that

$$|b(\mathbf{v}_h, q)| \le Ch^{\delta}||q||_1 \le Ch^{\delta},$$

which, along with (4.22), leads readily to the desired discrete compactness property. □

Up to now, the main assumption is the inf-sup condition (4.14), while other assumptions can be easily satisfied, and all the results are very general. ¿From Theorems 4.3 and 4.4, the problem boils down to the approximation property of $\mathbf{U}_h$ and the inf-sup condition (4.14), which will be studied in Sections 5 and 7, respectively.

5 Fortin interpolation and error bounds

In this section, we shall specify the assumptions on $\mathbf{U}_h$ and Q_h so that we can obtain the desired error bounds between the exact solution and the finite element solution from Theorems 4.3 and 4.4. We first state three basic assumptions.

Assumption 1. *There exists a scalar finite element space $W_h \subset Q$ satisfying the gradient inclusion condition*

$$\nabla W_h \subset \mathbf{U}_h. \tag{5.1}$$

Assumption 2. *For any sufficiently smooth $\theta \in Q$, with $\theta \in \prod_{k=1}^m H^{1+r}(\Omega_k)$ for some $r \ge \delta$ and $\mathrm{div}\,(\boldsymbol{\varepsilon}\nabla\theta) \in L^2(\Omega)$, there exists an interpolation operator I_h of W_h such that $I_h\theta \in W_h$ and satisfies*

$$||\nabla(\theta - I_h\theta)||_h \le Ch^{r-\delta}(\sum_{k=1}^m ||\theta||_{1+r,\Omega_k} + ||\mathrm{div}\,(\boldsymbol{\varepsilon}\nabla\theta)||_0 + ||\theta||_1), \tag{5.2}$$

$$\sup_{0\ne\mu_h\in Q_h} \frac{b(\nabla(\theta - I_h\theta),\mu_h)}{||\mu_h||_{Q_h}} \le Ch^{r-\delta}(\sum_{k=1}^m ||\theta||_{1+r,\Omega_k} + ||\mathrm{div}\,(\boldsymbol{\varepsilon}\nabla\theta)||_0 + ||\theta||_1). \tag{5.3}$$

Note that $|\nabla(\theta - I_h\theta)|_{\sigma,h} = 0$. Here the norms $||\cdot||_h$, $||\cdot||_{Q_h}$ and the notation $|\cdot|_{\sigma,h}$ are defined by (4.3), (4.14) and (4.18), respectively, while the parameter δ appears in the definition of the norm $|\cdot|_{h,\mathrm{div}}$ in (4.2).

Assumption 3. *For any sufficiently smooth $\mathbf{z} \in H_0(\mathbf{curl}\,;\Omega)$, satisfying $\mathbf{z} \in H(\mathrm{div}\,;\boldsymbol{\varepsilon};\Omega)$, $\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{z} \in H(\mathbf{curl}\,;\Omega)$, and $\mathbf{z} \in \prod_{k=1}^m (H^{1+r}(\Omega_k))^d$ for some $r \ge \delta$, there exists an interpolation operator $\mathbf{\Pi}_h$ associated with the space $\mathbf{U}_h$ such that $\mathbf{\Pi}_h\mathbf{z} \in \mathbf{U}_h$ and satisfies*

$$||\mathbf{z}-\mathbf{\Pi}_h\mathbf{z}||_h + |\mathbf{z}-\mathbf{\Pi}_h\mathbf{z}|_{\sigma,h} \le Ch^{r-\delta}\Big(\sum_{k=1}^m ||\mathbf{z}||_{1+r,\Omega_k} + ||\mathrm{div}\,(\boldsymbol{\varepsilon}\mathbf{z})||_0 + ||\mathbf{curl}\,(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{z})||_0 + ||\mathbf{z}||_{0,\mathbf{curl}}\Big), \tag{5.4}$$

$$\sup_{0\ne\mu_h\in Q_h} \frac{b(\mathbf{z}-\mathbf{\Pi}_h\mathbf{z},\mu_h)}{||\mu_h||_{Q_h}} \le Ch^{r-\delta}\Big(\sum_{k=1}^m ||\mathbf{z}||_{1+r,\Omega_k} + ||\mathrm{div}\,(\boldsymbol{\varepsilon}\mathbf{z})||_0 + ||\mathbf{curl}\,(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{z})||_0 + ||\mathbf{z}||_{0,\mathbf{curl}}\Big). \tag{5.5}$$

We shall use Assumption 1 to establish the inf-sup condition and to well approximate all the singular solutions. As will be seen later, (5.3) and (5.5) can be achieved from the inf-sup condition and the approximation properties (5.2) and (5.4). Since both θ and $\mathbf{z}$ are smooth, with H^{1+r}-regularity, the approximation properties (5.2) and (5.4) in Assumptions 2 and 3 are standard or classical. Thus, the gradient inclusion condition in Assumption 1 is one of the two most essential ingredients in our proposed method. The other is the Gauss' law that is built in the finite element variational formulation.

Based on the existing results, e.g., from [17] [18] [16] (see also [35] [15]), we can make the following assumptions on the regularity of the solution to the source problem (3.6) with a right-hand side $\mathbf{f} \in (L^2(\Omega))^d$. Assume that the solution $\mathbf{u} \in H_0(\mathbf{curl}\,;\Omega) \cap H_{\text{flux},\Gamma}(\text{div}\,^0;\boldsymbol{\varepsilon};\Omega)$ has the regularity for some $0 < r < 1$,

$$\mathbf{u} \in \prod_{k=1}^{m} (H^r(\Omega_k))^d, \quad \mathbf{curl}\,\mathbf{u} \in \prod_{k=1}^{m} (H^r(\Omega_k))^{2d-3}, \tag{5.6}$$

and further admits regular-singular decomposition

$$\mathbf{u} = \mathbf{u}^{\text{reg}} + \nabla p^{\text{sing}}, \tag{5.7}$$

$$\mathbf{u}^{\text{reg}} \in H_0(\mathbf{curl}\,;\Omega) \cap H(\text{div}\,;\boldsymbol{\varepsilon};\Omega) \cap \prod_{k=1}^{m} (H^{1+r}(\Omega_k))^d, \tag{5.8}$$

$$p^{\text{sing}} \in Q \cap \prod_{k=1}^{m} H^{1+r}(\Omega_k), \quad \text{div}\,(\boldsymbol{\varepsilon}\nabla p^{\text{sing}}) \in L^2(\Omega), \tag{5.9}$$

where $\mathbf{u}^{\text{reg}}$ is the regular part and ∇p^{sing} is the singular part, satisfying

$$\begin{aligned}&\sum_{k=1}^{m} (||\mathbf{u}^{\text{reg}}||_{1+r,\Omega_k} + ||p^{\text{sing}}||_{1+r,\Omega_k}) + ||\text{div}\,(\boldsymbol{\varepsilon}\nabla p^{\text{sing}})||_0 + ||\text{div}\,(\boldsymbol{\varepsilon}\mathbf{u}^{\text{reg}})||_0 \\ &\le C \sum_{k=1}^{m} (||\mathbf{u}||_{r,\Omega_k} + ||\mathbf{curl}\,\mathbf{u}||_{r,\Omega_k}) + C||\mathbf{curl}\,(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{u})||_0 + C||\mathbf{u}||_{0,\mathbf{curl}}\,.\end{aligned} \tag{5.10}$$

Moreover, we assume that

$$\sum_{k=1}^{m} (||\mathbf{u}||_{r,\Omega_k} + ||\mathbf{curl}\,\mathbf{u}||_{r,\Omega_k}) + ||\mathbf{curl}\,(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{u})||_0 + ||\mathbf{u}||_{0,\mathbf{curl}} \le C||\mathbf{f}||_0. \tag{5.11}$$

As for the multiplier $p \in Q$ of the source problem (3.6), we shall not need any more regularity. In fact, we can not have higher regularity about p than Q, because $\mathbf{f}$ lies in $(L^2(\Omega))^d$ only. But we note that the computation and the approximation of the multiplier may not be of practical interest here, since the multiplier for our eigenproblem is identically zero.

Now, for the solution $\mathbf{u}$ of the source problem (3.6) with a right-hand side $\mathbf{f} \in (L^2(\Omega))^d$, satisfying (5.6)-(5.11), we define a Fortin-type interpolation operator

$$\boldsymbol{\pi}_h\mathbf{u} := \boldsymbol{\Pi}_h\mathbf{u}^{\text{reg}} + \nabla(I_h p^{\text{sing}}). \tag{5.12}$$

From Assumptions 1 to 3, we know $\boldsymbol{\pi}_h\mathbf{u} \in \mathbf{U}_h$, and we further find from (5.10)-(5.12) that

$$\sup_{0\neq\mu_h\in Q_h} \frac{b(\mathbf{u}-\boldsymbol{\pi}_h\mathbf{u},\mu_h)}{||\mu_h||_{Q_h}} \le Ch^{r-\delta}||\mathbf{f}||_0, \tag{5.13}$$

$$\begin{aligned}||\mathbf{u}-\boldsymbol{\pi}_h\mathbf{u}||_h + |\mathbf{u}-\boldsymbol{\pi}_h\mathbf{u}|_{\sigma,h} &\le ||\mathbf{u}^{\text{reg}} - \boldsymbol{\Pi}_h\mathbf{u}^{\text{reg}}||_h + |\mathbf{u}^{\text{reg}} - \boldsymbol{\Pi}_h\mathbf{u}^{\text{reg}}|_{\sigma,h} + ||\nabla(p^{\text{sing}} - I_h p^{\text{sing}})||_h \\ &\le Ch^{r-\delta}||\mathbf{f}||_0.\end{aligned} \tag{5.14}$$

The estimates (5.13) and (5.14) imply directly the following result.

Lemma 5.1. *Let $\mathbf{u}$ be the solution of the source problem (3.6) with a right-hand side $\mathbf{f} \in (L^2(\Omega))^d$, satisfying (5.6)-(5.11). Under Assumptions 1 to 3, the Fortin-type operator $\boldsymbol{\pi}_h$ (cf. (5.12)) has the following approximation:*

$$||\mathbf{u}-\boldsymbol{\pi}_h\mathbf{u}||_h + |\mathbf{u}-\boldsymbol{\pi}_h\mathbf{u}|_{\sigma,h} + \sup_{0\neq\mu_h\in Q_h} \frac{b(\mathbf{u}-\boldsymbol{\pi}_h\mathbf{u},\mu_h)}{||\mu_h||_{Q_h}} \le Ch^{r-\delta}||\mathbf{f}||_0. \tag{5.15}$$

Using Lemma 5.1, we readily deduce the following convergence rate and uniform convergence from Theorems 4.3-4.4. And the results will be applied to the error estimates of the finite element solutions to the eigenproblem in the next section.

Theorem 5.1. *Under the same conditions as in Theorem 4.3 and in Lemma 5.1, let* $(\mathbf{u}, p)$ *be the exact solution of the source problem (3.6) or (3.7) with the source in (4.16) and the multiplier* $p = 0$*, and let* $(\mathbf{u}_h, p_h)$ *be the finite element solution of problem (3.8) in* $\mathbf{U}_h \times Q_h$*, then*

$$||\mathbf{u} - \mathbf{u}_h||_h + ||p_h||_{Q_h} \leq Ch^{r-\delta}||\mathbf{f}||_0.$$

Theorem 5.2. *Under the same conditions as in Theorem 4.4 and in Lemma 5.1, let* $(\mathbf{u}, p)$ *denote the exact solution of the source problem (3.6) or (3.7) with the right-hand side* $\mathbf{f} \in (L^2(\Omega))^d$ *and the stability (4.17), and let* $(\mathbf{u}_h, p_h)$ *be the finite element solution of problem (3.8) in* $\mathbf{U}_h \times Q_h$*, then*

$$||\mathbf{u} - \mathbf{u}_h||_h \leq C(h^{r-\delta} + h^{\delta})||\mathbf{f}||_0.$$

Remark 5.1. ¿From Theorem 5.2, with $\delta > 0$, we obtain the uniform convergence with respect to $\mathbf{f} \in (L^2(\Omega))^d$, i.e., $\sup_{\mathbf{0}\neq\mathbf{f}\in(L^2(\Omega))^d} ||\mathbf{u} - \mathbf{u}_h||_h/||\mathbf{f}||_0 \to 0$ as $h \to 0$. It is this uniform convergence that ensures spurious-free and spectral-correct approximations of eigenmodes (eigenvalues and eigenfunctions) of the compact operator.

Now, the role of the parameter $0 < \delta < r$ is clear: it helps to establish the discrete compactness property (cf. Lemma 4.1 and Proposition 4.1) and the uniform convergence in Theorem 5.2. And δ should also be chosen in the range $(0, r)$.

Remark 5.2. If the solution $\mathbf{u}$ is sufficiently smooth piecewisely, say $\mathbf{u} \in \prod_{k=1}^{m}(H^{1+\ell}(\Omega_k))^d$, where the integer $\ell \geq 1$ is the order of approximation of $\mathbf{U}_h$, we can obtain $||\mathbf{u} - \mathbf{u}_h||_h + ||p_h||_{Q_h} \leq Ch^{\ell-\delta}\sum_{k=1}^{m}||\mathbf{u}||_{1+\ell,\Omega_k}$. Moreover, we do not need to resort to Assumption 1 and the regular-singular decomposition (5.7), as long as the kernel-coercivity, the inf-sup condition and the approximation properties hold for $(\mathbf{U}_h, Q_h)$.

6 Eigenproblem and strong convergence of Gauss' law

This section is devoted to the analysis of the finite element method for solving the Maxwell eigenproblem. ¿From Theorem 5.1 and Theorem 5.2, from the abstract theory of the compact operator ([3] [34]), the theoretical results in this section can be proven, and the details will be omitted. Further, we prove the strong convergence of Gauss' law, in the spirit of [15].

Let $A : (L^2(\Omega))^d \to \mathbf{U} := H_0(\mathbf{curl}\,; \Omega)$ denote the solution operator of the continuous source problem (3.7) corresponding to the continuous eigenproblem (2.13), and define the multiplier as $B\mathbf{f} \in Q$, where the solution operator B maps from $(L^2(\Omega))^d$ onto Q. In other words, for any given $\mathbf{f} \in (L^2(\Omega))^d$, $A\mathbf{f} \in \mathbf{U}$ and $B\mathbf{f} \in Q$ are determined by

$$\begin{cases} a(A\mathbf{f}, \mathbf{v}) + b(\mathbf{v}, B\mathbf{f}) = (\boldsymbol{\varepsilon}\mathbf{f}, \mathbf{v}) & \forall \mathbf{v} \in \mathbf{U}, \\ b(A\mathbf{f}, q) = 0 \qquad \forall q \in Q. \end{cases} \tag{6.1}$$

In fact, $A\mathbf{f} \in \mathbf{K} := H_0(\mathbf{curl}\,; \Omega) \cap H_{\text{flux},\Gamma}(\text{div}\,^0; \boldsymbol{\varepsilon}; \Omega)$. As stated in Section 5, we may assume that the solution $A\mathbf{f}$ satisfies the regularity (5.6). The solution operator A is compact from $(L^2(\Omega))^d$ to $\mathbf{U}$, since $\mathbf{K}$ is compactly imbedded into $(L^2(\Omega))^d$. In addition, the multiplier $B\mathbf{f} \in Q$ satisfies

$$B\mathbf{f} = 0 \;\text{ for }\; \mathbf{f} \in H_{\text{flux},\Gamma}(\text{div}\,^0; \boldsymbol{\varepsilon}; \Omega). \tag{6.2}$$

In terms of A, the continuous eigenproblem (2.13) can be written in the equivalent form (with $\lambda = \omega^{-2}$)

$$A\mathbf{u} = \lambda\mathbf{u}\,. \tag{6.3}$$

On the other hand, the discrete source problem (3.8) corresponding to the discrete eigenproblem (3.5) defines the discrete solution operator A_h from $(L^2(\Omega))^d$ onto $\mathbf{U}_h$ and the discrete operator B_h from $(L^2(\Omega))^d$ onto Q_h, i.e., for any given $\mathbf{f} \in (L^2(\Omega))^d$, $A_h\mathbf{f} \in \mathbf{U}_h$ and $B_h\mathbf{f} \in Q_h$ are determined by

$$\begin{cases} a_h(A_h\mathbf{f}, \mathbf{v}_h) + b(\mathbf{v}_h, B_h\mathbf{f}) = (\boldsymbol{\varepsilon}\mathbf{f}, \mathbf{v}_h) & \forall \mathbf{v}_h \in \mathbf{U}_h, \\ b(A_h\mathbf{f}, q_h) = 0 \qquad \forall q_h \in Q_h. \end{cases} \tag{6.4}$$

In terms of A_h, the discrete eigenproblem (3.5) can be written in the form (with $\lambda_h = \omega_h^{-2}$)

$$A_h \mathbf{u}_h = \lambda_h \mathbf{u}_h \,. \tag{6.5}$$

From the $\mathbf{K}_h$ kernel-coercivity of the corresponding source problem, we can easily show that all the discrete eigenfunctions must belong to $\mathbf{K}_h$, and all the discrete eigenvalues ω_h^2 must satisfy

$$\omega_h^2 \geq C > 0 \,. \tag{6.6}$$

Using Theorem 5.2, we have the following approximation of the discrete solution operator A_h to the continuous solution operator A.

Theorem 6.1. *Assume the same conditions in Theorem 5.2. For any $\mathbf{f} \in (L^2(\Omega))^d$, letting $A\mathbf{f} \in \mathbf{U}$ and $\mathbf{A}_h\mathbf{f} \in \mathbf{U}_h$ be defined by (6.1) and (6.4), respectively, then it holds that*

$$||(A - A_h)\mathbf{f}||_h \leq C(h^{r-\delta} + h^{\delta})||\mathbf{f}||_0 \,, \quad ||(A - A_h)\mathbf{f}||_0 \leq C(h^{r-\delta} + h^{\delta})||\mathbf{f}||_0. \tag{6.7}$$

From Remark 5.1, we can see that Theorem 6.1 ensures that the discrete eigenproblem (3.5) provides the spurious-free and spectral-correct approximation of the continuous eigenproblem (2.13). In what follows, we study the order of this convergence.

Let λ be an eigenvalue of multiplicity L, with $\mathbf{E} \subset \mathbf{U}$ being the corresponding eigenspace. Obviously, we have $\mathbf{E} \subset \mathbf{K}$. We denote by $\lambda_{1,h}, \cdots, \lambda_{L,h}$ the discrete eigenvalues converging to λ and by $\mathbf{E}_h \subset \mathbf{K}_h$ the direct sum of the corresponding eigenspaces. Introduce the gaps between the spaces of continuous and discrete eigenfunctions in L^2-norm: $\Delta_0^*(\mathbf{E}, \mathbf{E}_h) = \max(\Delta_0(\mathbf{E}, \mathbf{E}_h), \Delta_0(\mathbf{E}_h, \mathbf{E}))$, where $\Delta_0(\mathbf{E}, \mathbf{E}_h)$ and $\Delta_0(\mathbf{E}_h, \mathbf{E})$ are given by $\Delta_0(\mathbf{E}, \mathbf{E}_h) = \sup_{\mathbf{u}\in\mathbf{E}, ||\mathbf{u}||_0=1} \inf_{\mathbf{v}_h\in\mathbf{E}_h} ||\mathbf{u} - \mathbf{v}_h||_0$ and $\Delta_0(\mathbf{E}_h, \mathbf{E}) = \sup_{\mathbf{u}_h\in\mathbf{E}_h, ||\mathbf{u}_h||_0=1} \inf_{\mathbf{v}\in\mathbf{E}} ||\mathbf{u}_h - \mathbf{v}||_0$. Similarly, we can introduce the gaps between the spaces of continuous and discrete eigenfunctions in the norm $||\cdot||_h$: $\Delta_h^*(\mathbf{E}, \mathbf{E}_h) = \max(\Delta_h(\mathbf{E}, \mathbf{E}_h), \Delta_h(\mathbf{E}_h, \mathbf{E}))$, where $\Delta_h(\mathbf{E}, \mathbf{E}_h)$ and $\Delta_h(\mathbf{E}_h, \mathbf{E})$ are given by $\Delta_h(\mathbf{E}, \mathbf{E}_h) = \sup_{\mathbf{u}\in\mathbf{E}, ||\mathbf{u}||_h=1} \inf_{\mathbf{v}_h\in\mathbf{E}_h} ||\mathbf{u}-\mathbf{v}_h||_h$ and $\Delta_h(\mathbf{E}_h, \mathbf{E}) = \sup_{\mathbf{u}_h\in\mathbf{E}_h, ||\mathbf{u}_h||_h=1} \inf_{\mathbf{v}\in\mathbf{E}} ||\mathbf{u}_h-\mathbf{v}||_h$. We also introduce $||(A-A_h)_{|\mathbf{E}}||_h := \sup_{\mathbf{0}\neq\mathbf{v}\in\mathbf{E}} ||(A-A_h)\mathbf{v}||_h/||\mathbf{v}||_h$, $||(B_h)_{|\mathbf{E}}||_{Q_h} := \sup_{\mathbf{0}\neq\mathbf{v}\in\mathbf{E}} ||(B_h)\mathbf{v}||_{Q_h}/||\mathbf{v}||_h$, $||(A-A_h)_{|\mathbf{E}}||_0 := \sup_{\mathbf{0}\neq\mathbf{v}\in\mathbf{E}} ||(A-A_h)\mathbf{v}||_0/||\mathbf{v}||_0$.

By Theorem 5.1 we can obtain the error estimates of the gaps $\Delta_0^*(\mathbf{E}, \mathbf{E}_h)$ and $\Delta_h^*(\mathbf{E}, \mathbf{E}_h)$ as stated in Theorem 6.2 and Theorem 6.3 below.

Theorem 6.2. *Under the same hypotheses of Theorem 5.1, there exists a constant C such that*

$$\Delta_0^*(\mathbf{E}, \mathbf{E}_h) \leq C||(A - A_h)_{|\mathbf{E}}||_0 \leq Ch^{r-\delta}, \quad \Delta_h^*(\mathbf{E}, \mathbf{E}_h) \leq C||(A - A_h)_{|\mathbf{E}}||_h \leq Ch^{r-\delta}. \tag{6.8}$$

Theorem 6.3. *Under the same hypotheses of Theorem 5.1, there exists a constant C such that*

$$|\lambda - \lambda_{i,h}| \leq C\Big(||(A - A_h)_{|\mathbf{E}}||_h^2 + ||(A - A_h)_{|\mathbf{E}}||_h ||(B_h)_{|\mathbf{E}}||_{Q_h}\Big) \leq Ch^{2(r-\delta)} \quad \textit{for } \; i = 1, 2, \cdots, L. \tag{6.9}$$

Remark 6.1. If some eigenfunctions are sufficiently smooth, say $\mathbf{u} \in \prod_{k=1}^m (H^{1+\ell}(\Omega_k))^d$ for $\ell \geq 1$, and the order of approximation of $\mathbf{U}$ is ℓ, then we can obtain from Remark 5.2 (with $\Delta^* := \Delta_0^*$ or $\Delta^* := \Delta_h^*$)

$$\Delta^*(\mathbf{E}, \mathbf{E}_h) \leq Ch^{\ell-\delta}\,; \quad |\lambda - \lambda_{i,h}| \leq Ch^{2(\ell-\delta)} \quad \text{for } \; i = 1, 2, \cdots, L.$$

We have seen that the error estimates are essentially optimal, only up to an arbitrarily small positive constant δ, relative to the regularity of the solution and the order of approximation.

Now, we turn to the strong convergence of the Gauss' law of the finite element solution in the norm $||\cdot||_{-(1-\delta)}$ of $H^{-(1-\delta)}(\Omega)$. For that purpose, we first note that for a given eigenvalue λ, by means of Theorem 6.2, we know the existence of a discrete eigenfunction $\mathbf{u}_h$ such that, where $\mathbf{u} \in \mathbf{E}$ is the eigenfunction of λ,

$$||\mathbf{u} - \mathbf{u}_h||_h \leq Ch^{r-\delta}. \tag{6.10}$$

Theorem 6.4. *Under the same hypotheses of Theorem 5.1, for any given eigenvalue $\lambda = (\omega^2)^{-1}$, with its convergent discrete eigenvalue λ_h, letting an eigenfunction be $\mathbf{u}$ corresponding to λ, normalized with $||\mathbf{u}||_h = 1$, and the discrete eigenfunction be $\mathbf{u}_h$ corresponding to λ_h satisfying (6.10), then we have for $0 \leq \delta < 1/2$ that*

$$||\mathrm{div}\,(\varepsilon(\mathbf{u} - \mathbf{u}_h))||_{-(1-\delta)} \leq C||\mathbf{u} - \mathbf{u}_h||_h \leq Ch^{r-\delta}. \tag{6.11}$$

Proof. By the definition of the weak divergence, we can easily see for any $q \in \mathscr{D}(\Omega)$ that

$$\langle \mathrm{div}\,(\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}_h)), q\rangle = -(\nabla q, \boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}_h)).$$

Since $\mathbf{u}_h$ is the finite element solution, satisfying

$$b(\mathbf{u}-\mathbf{u}_h, q_h) = (\nabla q_h, \boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}_h)) = 0 \qquad \forall q_h \in Q_h.$$

Particularly, we can take $q_h \in Q_h \cap H_0^1(\Omega)$ from (4.5) which satisfies for $0 \le t \le s$ and $1/2 < s \le 1$,

$$\Big(\sum_{K\in\mathcal{T}_h} h_K^{-2t}||q-q_h||_{0,K}^2\Big)^{1/2} + \Big(\sum_{F\in\mathcal{F}_h^{\mathrm{int}}} h_F^{-(2t-1)}\int_F |q-q_h|^2\Big)^{1/2} \le Ch^{s-t}||q||_s. \tag{6.12}$$

Now, we can obtain by integration by parts,

$$\begin{aligned} -(\nabla q, \boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}_h)) &= -(\nabla(q-q_h), \boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}_h)) \\ &= \sum_{K\in\mathcal{T}_h}(q-q_h, \mathrm{div}\,(\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}_h)))_{0,K} - \sum_{K\in\mathcal{T}_h}\int_{\partial K}(q-q_h)(\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}_h))\cdot\mathbf{n}, \end{aligned}$$

where the second term can be rewritten as

$$-\sum_{K\in\mathcal{T}_h}\int_{\partial K}(q-q_h)(\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}_h))\cdot\mathbf{n} = -\sum_{F\in\mathcal{F}_h^{\mathrm{int}}}\int_F [\![(\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}_h))\cdot\mathbf{n}]\!](q-q_h).$$

Now taking $t := 1-\delta$ and $s := 1-\delta$ in (6.12), we can deduce immediately

$$\begin{aligned} |\langle \mathrm{div}\,(\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}_h)), q\rangle| \;\le\; & C||q||_{1-\delta}\Big(\sum_{K\in\mathcal{T}_h} h_K^{2-2\delta}||\mathrm{div}\,(\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}_h))||_{0,K}^2 \\ & + \sum_{F\in\mathcal{F}_h^{\mathrm{int}}} h_F^{1-2\delta}\int_F |[\![(\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}_h))\cdot\mathbf{n}]\!]|^2\Big)^{1/2} \le C||q||_{1-\delta}|\mathbf{u}-\mathbf{u}_h|_{h,\mathrm{div}}. \end{aligned}$$

Hence, we find that

$$||\mathrm{div}\,(\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}_h))||_{-(1-\delta)} \le C|\mathbf{u}-\mathbf{u}_h|_{h,\mathrm{div}},$$

and further obtain from (6.10) and the definition (4.3) of $||\cdot||_h$ that

$$||\mathrm{div}\,(\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}_h))||_{-(1-\delta)} \le C|\mathbf{u}-\mathbf{u}_h|_{h,\mathrm{div}} \le C||\mathbf{u}-\mathbf{u}_h||_h \le Ch^{r-\delta}.$$

□

Equation (6.11) shows that he strong convergence of the Gauss law is implied from the convergence of the finite element solution. Following the same argument as for (6.11), we have

$$||\mathrm{div}\,(\boldsymbol{\varepsilon}\mathbf{v}_h)||_{-(1-\delta)} \le C|\mathbf{v}_h|_{h,\mathrm{div}} \qquad \forall \mathbf{v}_h \in \mathbf{K}_h.$$

On the other hand, we can prove the converse is also true, and $|\cdot|_{h,\mathrm{div}}$ is equivalent to $||\mathrm{div}\,(\boldsymbol{\varepsilon}\cdot)||_{-(1-\delta)}$ on $\mathbf{K}_h$.

Remark 6.2. To clarify what assumptions and conditions are the main ingredients for the proposed mixed finite element method, we now give a brief summary for a later use; also see the comment we made at the end of Section 4. We have seen that the main assumptions for the finite element method (3.5) are

- The inf-sup condition (4.14) in Theorem 4.1.
- The three assumptions: Assumption 1, Assumption 2, Assumption 3.

All other assumptions and conditions can be easily verified. In the sequel, we shall mainly focus on the verification of the above assumptions and the inf-sup condition. We shall further see that the crucial assumption is the gradient inclusion condition in Assumption 1. It essentially implies the inf-sup condition (4.14) and the approximation properties in Assumptions 2 and 3.

7 Verification of assumptions and general finite elements

In this section, we carry out a verification of Assumptions 1 to 3 and the inf-sup condition (4.14) for general finite elements, and then apply to some specific elements such as discontinuous elements and tangential continuous Nédélec elements on quadrilateral and hexahedral meshes.

We assume that every element $K \in \mathcal{T}_h$ in the physical coordinate system $x_1, \cdots, x_d$ of $\mathbb{R}^d$ is obtained from a single reference element $\hat{K}$ in the reference coordinate system $\hat{x}_1, \cdots, \hat{x}_d$ of $\mathbb{R}^d$ through a diffeomorphism $F_K : \hat{K} \to \mathbb{R}^d$, i.e., $K = F_K(\hat{K})$. We write the Jacobian matrix of F_K as DF_K and its determinant as JF_K. With $\mathbf{x} = F_K(\hat{\mathbf{x}})$, we have

$$\nabla q(\mathbf{x}) = (DF_K)^{-T} \hat{\nabla} \hat{q}(\hat{\mathbf{x}}). \tag{7.1}$$

We will call the transformation $\mathbf{v}(\mathbf{x}) = (DF_K)^{-T} \hat{\mathbf{v}}(\hat{\mathbf{x}})$ as the Piola-like transformation from $\hat{\mathbf{v}}$ on $\hat{K}$ to $\mathbf{v}$ on K.

Let $\hat{\mathcal{R}}_\ell(\hat{K})$ denote the scalar space of polynomials of degree $\ell \geq 1$ on $\hat{K}$, and define the local element space $\mathcal{R}_\ell(K) := \{q : q(F_K(\hat{\mathbf{x}})) = \hat{q}(\hat{\mathbf{x}}), \hat{\mathbf{x}} \in \hat{K}, \hat{q} \in \hat{\mathcal{R}}_\ell(\hat{K})\}$. For example, if we use simplexes, $\mathcal{R}_\ell(K)$ will be the image $\mathcal{P}_\ell(K)$ of $\hat{\mathcal{P}}_\ell(\hat{K})$ (the space of polynomials of total degree in variables $\hat{x}_1 \cdots \hat{x}_d$ not greater than ℓ on $\hat{K}$) through the affine isomorphism F_K; if we use quadrilaterals and hexahedra, $\mathcal{R}_\ell(K)$ will be the image $\mathcal{Q}_\ell(K)$ of $\hat{\mathcal{Q}}_\ell(\hat{K})$ (the space of polynomials of separate degree in variables $\hat{x}_1 \cdots \hat{x}_d$ not greater than ℓ on $\hat{K} = [0, 1]^d$) through the bilinear or trilinear isomorphism F_K. We define the scalar finite element space Q_h for approximating the multiplier in $\mathcal{R}_\ell(K)$ with $\ell = 1$:

$$Q_h = \{q_h \in Q : q_h|_K \in Q(K) \quad \forall K \in \mathcal{T}_h\}, \quad Q(K) := \mathcal{R}_1(K). \tag{7.2}$$

To define the space $\mathbf{U}_h$ for approximating the electric field, we let $\hat{\mathbf{E}}_\ell(\hat{K})$ be the vector-valued space of polynomials, belonging to some space of complete polynomials of degree $\ell \geq 1$ on $\hat{K}$, and define the local element space $\mathbf{E}_\ell(K) := \{\mathbf{v} : \mathbf{v}(F_K(\hat{\mathbf{x}})) = (D\mathbf{F}_K)^{-T}(\hat{\mathbf{x}})\hat{\mathbf{v}}(\hat{\mathbf{x}}), \hat{\mathbf{x}} \in \hat{K}, \hat{\mathbf{v}} \in \hat{\mathbf{E}}_\ell(\hat{K})\}$. For example, on tetrahedral elements, $\hat{\mathbf{E}}_\ell(\hat{K})$ can be chosen as the Nédélec element of first kind, i.e.,

$$\hat{\mathbf{E}}_\ell(\hat{K}) := \mathbf{N}_\ell(\hat{K}) := (\hat{\mathcal{P}}_{\ell-1}(\hat{K}))^3 + \{\hat{\mathbf{v}} \in (\tilde{\mathcal{P}}_\ell(\hat{K}))^3 : \hat{\mathbf{v}}(\hat{\mathbf{x}}) \cdot \hat{\mathbf{x}} = 0\}, \tag{7.3}$$

where $\tilde{\mathcal{P}}_\ell(\hat{K})$ denotes the space of homogeneous polynomials of degree not greater than ℓ on $\hat{K}$; $\hat{\mathbf{E}}_\ell(\hat{K})$ may also be chosen as the second kind, i.e.,

$$\hat{\mathbf{E}}_\ell(\hat{K}) := (\hat{\mathcal{P}}_\ell(\hat{K}))^3. \tag{7.4}$$

We now define the vector-valued (discontinuous) finite element space $\mathbf{U}_h$ for the electric field in $\mathbf{E}_\ell(K)$ ($\ell \geq 1$):

$$\mathbf{U}_h = \{\mathbf{v}_h \in (L^2(\Omega))^d : \mathbf{v}_h|_K \in \mathbf{U}(K) \quad \forall K \in \mathcal{T}_h\}, \quad \mathbf{U}(K) := \mathbf{E}_\ell(K) \tag{7.5}$$

We can also choose $\mathbf{U}_h$ to be tangential continuous (e.g., Nédélec elements), and the theory and results of this work are equally applicable.

Next we discuss the verifications of Assumptions 1 to 3. First for Assumption 1, from (7.1) and the definition of $\mathbf{E}_\ell(K)$ that $\hat{\nabla}\hat{\mathcal{R}}_1(\hat{K}) \subset \hat{\mathbf{E}}_\ell(\hat{K})$, we have

$$\nabla \mathcal{R}_1(K) \subset \mathbf{E}_\ell(K), \tag{7.6}$$

and consequently, $\nabla Q_h \subset \mathbf{U}_h$, and then $W_h := Q_h$. Thus we arrive at the following result.

Lemma 7.1. *If the general finite element spaces Q_h and $\mathbf{U}_h$ are constructed by means of (7.2), (7.5), and the local gradient inclusion (7.6), then the gradient inclusion condition in Assumption 1 holds with $W_h := Q_h$.*

With the above general finite element spaces Q_h and $\mathbf{U}_h$, we can establish the inf-sup condition from the local gradient inclusion condition (the proof is given later in this section):

$$\sup_{\mathbf{0} \neq \mathbf{v}_h \in \mathbf{U}_h} \frac{b(\mathbf{v}_h, q_h)}{||\mathbf{v}_h||_h} \geq C||q_h||_{\delta, Q_h} := C\Big(\sum_{K \in \mathcal{T}_h} h_K^{2\delta} ||\nabla q_h||_{0,K}^2\Big)^{1/2} \qquad \forall\, q_h \in Q_h. \tag{7.7}$$

This implies immediately the inf-sup condition (4.14) in Theorem 4.1, satisfying

$$||q_h||_{Q_h} \geq C||q_h||_{\delta, Q_h} \qquad \forall q_h \in Q_h. \tag{7.8}$$

It remains to verify Assumptions 2 and 3. For that purpose, corresponding to $\mathbf{U}_h$ and Q_h defined by (7.5) and (7.2), respectively, we require the standard approximation properties of $I_h\theta$ and $\mathbf{\Pi}_h \mathbf{z}$ for $0 \leq t \leq r$, as follows:

$$\Big(\sum_{K\in\mathcal{T}_h} h_K^{-2t}||\nabla(\theta-\nabla(I_h\theta))||_{0,K}^2\Big)^{1/2} \quad\leq\quad Ch^{r-t}\sum_{k=1}^m ||\theta||_{1+r,\Omega_k}, \tag{7.9}$$

$$\Big(\sum_{K\in\mathcal{T}_h} h_K^{-2t}||\mathbf{z}-\mathbf{\Pi}_h\mathbf{z}||_{0,K}^2\Big)^{1/2} \quad\leq\quad Ch^{r-t}\sum_{k=1}^m ||\mathbf{z}||_{1+r,\Omega_k}, \tag{7.10}$$

$$\Big(\sum_{K\in\mathcal{T}_h} ||\mathbf{curl}\,(\mathbf{z}-\mathbf{\Pi}_h\mathbf{z})||_{0,K}^2\Big)^{1/2} \quad\leq\quad Ch^{r}\sum_{k=1}^m ||\mathbf{z}||_{1+r,\Omega_k}. \tag{7.11}$$

Note that the approximation properties in Assumptions 2 and 3 are only needed for piecewise *smooth functions*. We shall show that Assumption 2 follows from (7.9), while Assumption 3 follows from (7.10) and (7.11). These approximation properties are classical results, since $\mathbf{z}$ and θ are piecewise smooth, and I_h and $\mathbf{\Pi}_h$ can be constructed locally.

Although the approximation properties in Assumption 3 are stated for piecewise smooth functions, we still need to be cautious when dealing with the tangential jumps, e.g.,

$$\sum_{F\in\mathcal{F}_h}\int_F \mathrm{h}^{-1}[\![\mathbf{n}\times\mathbf{z}]\!]\cdot[\![\mathbf{n}\times\mathbf{v}]\!]. \tag{7.12}$$

For linear and higher order elements of $\mathbf{U}_h$, there is no problem in dealing with the tangential jumps, because of the following approximation property for $0\leq t\leq 1+r$,

$$\Big(\sum_{K\in\mathcal{T}_h} h_K^{-2t}||\mathbf{z}-\mathbf{\Pi}_h\mathbf{z}||_{0,K}^2\Big)^{1/2} \leq Ch^{1+r-t}\sum_{k=1}^m ||\mathbf{z}||_{1+r,\Omega_k}. \tag{7.13}$$

When $\mathbf{U}_h$ does not contain linear element or a complete space of polynomials (in the reference element), e.g., the lowest-order Nédélec element of first kind on tetrahedra, (7.13) does not hold. Then, to deal with the tangential jump terms such as (7.12), we need to additionally require that $\mathbf{\Pi}_h\mathbf{z}\in H_0(\mathbf{curl}\,;\Omega)$ satisfies

$$\sum_{F\in\mathcal{F}_h}\int_F \mathrm{h}^{-1}[\![\mathbf{n}\times(\mathbf{z}-\mathbf{\Pi}_h\mathbf{z})]\!]\cdot[\![\mathbf{n}\times\mathbf{v}_h]\!]=0 \qquad \forall\mathbf{v}_h\in\mathbf{U}_h. \tag{7.14}$$

With the above preparations, we now verify Assumptions 2 and 3. We give the details only for Assumption 2 as the same applies to Assumption 3. We first introduce an approximation of ε. Let $\mathbf{J}_K\varepsilon$ denote the local canonical Lagrange interpolation of $\varepsilon\in(W^{1,\infty}(K))^{d\times d}$ from any suitable element-local Lagrange finite element space $(\mathcal{L}(K))^{d\times d}$ on K, e.g., $\mathcal{L}(K):=\mathcal{R}_1(K)$. We have from [12] and [29] that

$$||\varepsilon-\mathbf{J}_K\varepsilon||_{0,\infty,K}+h_K|\varepsilon-\mathbf{J}_K\varepsilon|_{1,\infty,K}\leq Ch_K||\varepsilon||_{1,\infty,K}. \tag{7.15}$$

The verification of Assumption 2 is based on (7.9), (7.7), and (7.8). By the definition (4.3) of $||\cdot||_h$,

$$\begin{aligned}&||\nabla(\theta-I_h\theta)||_h^2\\ &=||\nabla(\theta-I_h\theta)||_{0,\varepsilon}^2+\sum_{K\in\mathcal{T}_h}h_K^{2-2\delta}||\mathrm{div}\,(\varepsilon\nabla(\theta-I_h\theta))||_{0,K}^2+\sum_{F\in\mathcal{F}_h^{\mathrm{int}}}h_F^{1-2\delta}\int_F|[\![(\varepsilon\nabla(\theta-I_h\theta))\cdot\mathbf{n}]\!]|^2.\end{aligned} \tag{7.16}$$

For the first term above, we have from (7.9) that

$$||\nabla(\theta-I_h\theta)||_{0,\varepsilon}\leq Ch^r\sum_{k=1}^m||\theta||_{1+r,\Omega_k}. \tag{7.17}$$

It remains to estimate the last two terms in (7.16). For $\varepsilon\nabla\theta\in(H^r(K))^d$ with $r>0$, letting $|K|$ be the volume of K, we define a piecewise constant L^2 projection

$$\boldsymbol{\rho}_K(\varepsilon\nabla\theta)=\frac{\int_K\varepsilon\nabla\theta}{|K|}.$$

It can be shown (e.g., cf. [29]) that

$$||\boldsymbol{\rho}_K(\boldsymbol{\varepsilon}\nabla\theta) - \boldsymbol{\varepsilon}\nabla\theta||_{0,K} \le Ch_K^r||\boldsymbol{\varepsilon}\nabla\theta||_{r,K} \le Ch_K^r||\theta||_{1+r,K}.$$

Also, we know $\mathrm{div}\,(\boldsymbol{\rho}_K(\boldsymbol{\varepsilon}\nabla\theta)) = 0$ on K, hence we can derive

$$h_K^{1-\delta}||\mathrm{div}\,(\boldsymbol{\varepsilon}\nabla(\theta - I_h\theta))||_{0,K} \le h_K^{1-\delta}||\mathrm{div}\,(\boldsymbol{\varepsilon}\nabla\theta)||_{0,K} + h_K^{1-\delta}||\mathrm{div}\,(\boldsymbol{\rho}_K(\boldsymbol{\varepsilon}\nabla\theta)) - \mathrm{div}\,(\boldsymbol{\varepsilon}\nabla(I_h\theta))||_{0,K},$$

where the last term can be estimated, by means of the local inverse estimates and L^2-projection approximation property, as follows:

$$\begin{aligned}
h_K^{1-\delta}||\mathrm{div}\,(\boldsymbol{\rho}_K(\boldsymbol{\varepsilon}\nabla\theta)) - \mathrm{div}\,(\boldsymbol{\varepsilon}\nabla(I_h\theta))||_{0,K} &\le h_K^{1-\delta}||\mathrm{div}\,(\boldsymbol{\rho}_K(\boldsymbol{\varepsilon}\nabla\theta) - (\mathbf{J}_K\boldsymbol{\varepsilon})\nabla(I_h\theta))||_{0,K} \\
&\quad + h_K^{1-\delta}||\mathrm{div}\,((\boldsymbol{\varepsilon} - \mathbf{J}_K\boldsymbol{\varepsilon})\nabla(I_h\theta))||_{0,K}, \\
h_K^{1-\delta}||\mathrm{div}\,(\boldsymbol{\rho}_K(\boldsymbol{\varepsilon}\nabla\theta) - (\mathbf{J}_K\boldsymbol{\varepsilon})\nabla(I_h\theta))||_{0,K} &\le Ch_K^{-\delta}||\boldsymbol{\rho}_K(\boldsymbol{\varepsilon}\nabla\theta) - (\mathbf{J}_K\boldsymbol{\varepsilon})\nabla(I_h\theta)||_{0,K} \\
&\le Ch_K^{-\delta}||\boldsymbol{\rho}_K(\boldsymbol{\varepsilon}\nabla\theta) - \boldsymbol{\varepsilon}\nabla\theta||_{0,K} + h_K^{-\delta}||\boldsymbol{\varepsilon}\nabla\theta - (\mathbf{J}_K\boldsymbol{\varepsilon})\nabla(I_h\theta)||_{0,K} \\
&\le Ch_K^{r-\delta}||\theta||_{1+r,K} + h_K^{-\delta}||(\boldsymbol{\varepsilon} - \mathbf{J}_K\boldsymbol{\varepsilon})\nabla\theta||_{0,K} + h_K^{-\delta}||(\mathbf{J}_K\boldsymbol{\varepsilon})\nabla(\theta - I_h\theta)||_{0,K} \\
&\le Ch_K^{r-\delta}||\theta||_{1+r,K} + Ch_K^{-\delta}||\boldsymbol{\varepsilon} - \mathbf{J}_K\boldsymbol{\varepsilon}||_{0,\infty,K}||\theta||_{1,K} \\
&\quad + Ch_K^{-\delta}||\mathbf{J}_K\boldsymbol{\varepsilon}||_{0,\infty,K}||\nabla(\theta - I_h\theta)||_{0,K} \\
&\le Ch_K^{r-\delta}||\theta||_{1+r,K} + Ch_K^{1-\delta}||\theta||_{1,K} + Ch_K^{-\delta}||\nabla(\theta - I_h\theta)||_{0,K}, \\
h_K^{1-\delta}||\mathrm{div}\,((\boldsymbol{\varepsilon} - \mathbf{J}_K\boldsymbol{\varepsilon})\nabla(I_h\theta))||_{0,K} &\le Ch_K^{1-\delta}|\boldsymbol{\varepsilon} - \mathbf{J}_K\boldsymbol{\varepsilon}|_{1,\infty,K}||\nabla(I_h\theta)||_{0,K} + Ch_K^{1-\delta}||\boldsymbol{\varepsilon} - \mathbf{J}_K\boldsymbol{\varepsilon}||_{0,\infty,K}|I_h\theta|_{2,K} \\
&\le Ch_K^{1-\delta}||I_h\theta||_{1,K} \le Ch_K^{1-\delta}||\theta||_{1,K}.
\end{aligned}$$

Combining all the above, we find that

$$\Big(\sum_{K\in\mathcal{T}_h} h_K^{2-2\delta}||\mathrm{div}\,(\boldsymbol{\varepsilon}\nabla(\theta - I_h\theta))||_{0,K}^2\Big)^{1/2} \le Ch^{r-\delta}\Big(\sum_{k=1}^m ||\theta||_{1+r,\Omega_k} + ||\mathrm{div}\,(\boldsymbol{\varepsilon}\nabla\theta)||_0 + ||\theta||_1\Big), \tag{7.18}$$

To estimate the jump term in (7.16), we make use of the Raviart-Thomas elements. We first note that

$$\begin{aligned}
& h_F^{1-2\delta}\int_F |[\![(\boldsymbol{\varepsilon}\nabla(\theta - I_h\theta)) \cdot \mathbf{n}]\!]|^2 = h_F^{1-2\delta}\int_F |[\![(\boldsymbol{\varepsilon}\nabla(I_h\theta)) \cdot \mathbf{n}]\!]|^2 \\
& \le Ch_F^{1-2\delta}\Big(\int_F |((\boldsymbol{\varepsilon} - \mathbf{J}_{K^{(+)}}\boldsymbol{\varepsilon})\nabla(I_h\theta)) \cdot \mathbf{n}|^2 + \int_F |((\boldsymbol{\varepsilon} - \mathbf{J}_{K^{(-)}}\boldsymbol{\varepsilon})\nabla(I_h\theta)) \cdot \mathbf{n}|^2\Big) \\
& + Ch_F^{1-2\delta}\int_F |((\mathbf{J}_{K^{(+)}}\boldsymbol{\varepsilon})\nabla(I_h\theta) - (\mathbf{J}_{K^{(-)}}\boldsymbol{\varepsilon})\nabla(I_h\theta)) \cdot \mathbf{n}|^2,
\end{aligned} \tag{7.19}$$

where the first term on the right-hand side can be estimated by the local trace theorem that $h_F\int_F |w_h|^2 \le C||w_h||_{0,K}^2$ for any finite element function w_h,

$$\begin{aligned}
& h_F^{1-2\delta}\Big(\int_F |((\boldsymbol{\varepsilon} - \mathbf{J}_{K^{(+)}}\boldsymbol{\varepsilon})\nabla(I_h\theta)) \cdot \mathbf{n}|^2 + \int_F |((\boldsymbol{\varepsilon} - \mathbf{J}_{K^{(-)}}\boldsymbol{\varepsilon})\nabla(I_h\theta)) \cdot \mathbf{n}|^2\Big) \\
& \le Ch_F^{1-2\delta}(||\boldsymbol{\varepsilon} - \mathbf{J}_{K^{(+)}}\boldsymbol{\varepsilon}||_{0,\infty,K^{(+)}}^2\int_F |\nabla(I_h\theta)|^2 + ||\boldsymbol{\varepsilon} - \mathbf{J}_{K^{(-)}}\boldsymbol{\varepsilon}||_{0,\infty,K^{(-)}}^2\int_F |\nabla(I_h\theta)|^2) \\
& \le C(h_{K^{(+)}}^{2-2\delta}||\nabla(I_h\theta)||_{0,K^{(+)}}^2 + h_{K^{(-)}}^{2-2\delta}||\nabla(I_h\theta)||_{0,K^{(-)}}^2) \\
& \le C(h_{K^{(+)}}^{2-2\delta}||\theta||_{1,K^{(+)}}^2 + h_{K^{(-)}}^{2-2\delta}||\theta||_{1,K^{(-)}}^2).
\end{aligned}$$

Since $\boldsymbol{\varepsilon}\nabla\theta \in (H^r(K))^d$ with $r > 0$ and $\mathrm{div}\,(\boldsymbol{\varepsilon}\nabla\theta) \in L^2(K)$, we can construct a canonical interpolation from any suitable Raviart-Thomas finite element space $\mathbf{RT}(K)$ on K, i.e., there exists an interpolation operator Υ_K such that $\Upsilon_K(\boldsymbol{\varepsilon}\nabla\theta) \in \mathbf{RT}(K)$ satisfies $(\Upsilon_{K^{(+)}}(\boldsymbol{\varepsilon}\nabla\theta) - \Upsilon_{K^{(-)}}(\boldsymbol{\varepsilon}\nabla\theta)) \cdot \mathbf{n} = 0$ on $F = \partial K^{(+)} \cap \partial K^{(-)}$ and

$$||\boldsymbol{\varepsilon}\nabla\theta - \Upsilon_K(\boldsymbol{\varepsilon}\nabla\theta)||_{0,K^{(\pm)}} \le Ch_{K^{(\pm)}}^r||\boldsymbol{\varepsilon}\nabla\theta||_{r,K^{(\pm)}} \le Ch_{K^{(\pm)}}^r||\theta||_{1+r,K^{(\pm)}}.$$

By the local trace theorem again, we can also estimate the second term on the right-hand side of (7.19),

$$\begin{aligned}
& h_F^{1-2\delta}\int_F |((\mathbf{J}_{K^{(+)}}\boldsymbol{\varepsilon})\nabla(I_h\theta) - (\mathbf{J}_{K^{(-)}}\boldsymbol{\varepsilon})\nabla(I_h\theta)) \cdot \mathbf{n}|^2 \\
& = h_F^{1-2\delta}\int_F |((((\mathbf{J}_{K^{(+)}}\boldsymbol{\varepsilon})\nabla(I_h\theta) - \Upsilon_{K^{(+)}}(\boldsymbol{\varepsilon}\nabla\theta)) - (\Upsilon_{K^{(-)}}(\boldsymbol{\varepsilon}\nabla\theta) - (\mathbf{J}_{K^{(-)}}\boldsymbol{\varepsilon})\nabla(I_h\theta))) \cdot \mathbf{n}|^2 \\
& \le Ch_F^{-2\delta}||(\mathbf{J}_{K^{(+)}}\boldsymbol{\varepsilon})\nabla(I_h\theta) - \Upsilon_{K^{(+)}}(\boldsymbol{\varepsilon}\nabla\theta)||_{0,K^{(+)}}^2 + Ch_F^{-2\delta}||(\mathbf{J}_{K^{(-)}}\boldsymbol{\varepsilon})\nabla(I_h\theta) - \Upsilon_{K^{(-)}}(\boldsymbol{\varepsilon}\nabla\theta)||_{0,K^{(-)}}^2,
\end{aligned}$$

which can be further bounded by the triangle inequality and the error estimates of $\mathbf{J}_K$ and Υ_K,

$$||(\mathbf{J}_{K^{(\pm)}}\boldsymbol{\varepsilon})\nabla(I_h\theta)-\Upsilon_{K^{(\pm)}}(\boldsymbol{\varepsilon}\nabla\theta)||_{0,K^{(\pm)}}^2\leq C||\nabla(\theta-I_h\theta)||_{0,K^{(\pm)}}^2+Ch_{K^{(\pm)}}^2||\theta||_{1,K^{(\pm)}}^2+Ch_{K^{(\pm)}}^{2r}||\theta||_{1+r,K^{(\pm)}}^2.$$

Therefore we can come to

$$\begin{aligned}&h_F^{1-2\delta}\int_F|((\mathbf{J}_{K^{(+)}}\boldsymbol{\varepsilon})\nabla(I_h\theta)-(\mathbf{J}_{K^{(-)}}\boldsymbol{\varepsilon})\nabla(I_h\theta))\cdot\mathbf{n}|^2\\&\leq Ch_{K^{(+)}}^{-2\delta}||\nabla(\theta-I_h\theta)||_{0,K^{(+)}}^2+Ch_{K^{(+)}}^{2-2\delta}||\theta||_{1,K^{(+)}}^2+Ch_{K^{(+)}}^{2r-2\delta}||\theta||_{1+r,K^{(+)}}^2\\&+Ch_{K^{(-)}}^{-2\delta}||\nabla(\theta-I_h\theta)||_{0,K^{(-)}}^2+Ch_{K^{(-)}}^{2-2\delta}||\theta||_{1,K^{(-)}}^2+Ch_{K^{(-)}}^{2r-2\delta}||\theta||_{1+r,K^{(-)}}^2,\end{aligned}$$

and then have proved from (7.19) that

$$\Big(\sum_{F\in\mathcal{F}_h^{\rm int}}h_F^{1-2\delta}\int_F|[\![(\boldsymbol{\varepsilon}\nabla(\theta-I_h\theta))\cdot\mathbf{n}]\!]|^2\Big)^{1/2}\leq Ch^{r-\delta}\Big(\sum_{k=1}^m||\theta||_{1+r,\Omega_k}+||\theta||_1\Big).\tag{7.20}$$

Now it is easy to see that the approximation (5.2) in Assumption 2 is a direct consequence of (7.16)-(7.20).

Remark 7.1. For the estimate of the normal jump terms like the one in (7.16), we need only the L^2-norm approximation of the Raviart-Thomas element. For affine or nonaffine quadrilaterals and affine hexahedra, we can choose the original Raviart-Thomas-Nédélec element or the variant in [2]. For nonaffine hexahedra, we can choose the Falk-Gatto-Monk element [28]. However, we can avoid using the Raviart-Thomas element, at the expense of losing some slight accuracy for $r\leq 1/2$, i.e., $r-\delta$ is replaced by $r-\epsilon-\delta$ for any small $\epsilon\in(0,r-\delta)$.

Now we are ready to show (5.3) in Assumption 2. In fact, using the estimates above, we have

$$\begin{aligned}b(\nabla(\theta-I_h\theta),\mu_h)&=(\boldsymbol{\varepsilon}\nabla(\theta-I_h\theta),\nabla\mu_h)\\&\leq C\Big(\sum_{K\in\mathcal{T}_h}h_K^{-2\delta}||\nabla(\theta-I_h\theta)||_{0,K}^2\Big)^{1/2}\Big(\sum_{K\in\mathcal{T}_h}h_K^{2\delta}||\nabla\mu_h||_{0,K}^2\Big)^{1/2}\\&\leq Ch^{r-\delta}||\mu_h||_{\delta,Q_h}\sum_{k=1}^m||\theta||_{1+r,\Omega_k},\end{aligned}$$

which, along with (7.8), yields

$$\sup_{0\neq\mu_h\in Q_h}\frac{b(\nabla(\theta-I_h\theta),\mu_h)}{||\mu_h||_{Q_h}}\leq Ch^{r-\delta}\sum_{k=1}^m||\theta||_{1+r,\Omega_k}.$$

Hence we have proved (5.3), and also completed the verification of Assumption 2. We can do the same to verify Assumption 3, hence have deduced the following results.

Lemma 7.2. *Assumption 2 follows from the same conditions as in Lemma 7.1 and the approximation property (7.9), while Assumption 3 follows from the same conditions as in Lemma 7.1 and the approximation properties (7.10)-(7.11).*

Next, we establish the inf-sup condition (7.7). To do so, using the local inclusion (7.6), we may choose

$$\mathbf{v}_h:=h_K^{2\delta}\nabla q_h,\quad\text{on }K\in\mathcal{T}_h\qquad\forall q_h\in Q_h$$

to derive

$$b(\mathbf{v}_h,q_h)=\sum_{K\in\mathcal{T}_h}h_K^{2\delta}||\varepsilon^{1/2}\nabla q_h||_{0,K}^2\geq C\sum_{K\in\mathcal{T}_h}h_K^{2\delta}||\nabla q_h||_{0,K}^2=C||q_h||_{\delta,Q_h}^2.$$

Then following a similar argument to the one for verifying Assumption 2, we can show $||\mathbf{v}_h||_h\leq C||q_h||_{\delta,Q_h}$, hence the inf-sup condition (7.7) follows, resulting in the following lemma.

Lemma 7.3. *The same conditions as in Lemma 7.1 imply the inf-sup condition (7.7), hence the inf-sup (4.14) follows.*

Collecting the results from Lemmas 7.1 to 7.3 and from Remark 6.2 and Section 6, we can conclude the following results for general finite elements.

Theorem 7.1. *For the spaces $\mathbf{U}_h$ and Q_h defined by (7.5) and (7.2), respectively, the local gradient inclusion condition (7.6) and the standard approximation properties (7.9)-(7.11) guarantee that the mixed finite element approximation (3.5) is spurious-free and spectral-correct for the Maxwell eigenproblem (2.1)-(2.4), with nearly optimal approximation accuracy.*

The error estimates are $\mathcal{O}(h^{r-\delta})$ for the eigenspace gaps and $\mathcal{O}(h^{2(r-\delta)})$ for eigenvalues if the eigenfunctions, along with its curls, are piecewise H^r-regular for some $r > 0$; the error estimates are $\mathcal{O}(h^{\ell-\delta})$ for the eigenspace gaps and $\mathcal{O}(h^{2(\ell-\delta)})$ for eigenvalues if the eigenfunctions are piecewise $H^{1+\ell}$-regular. Here δ is an arbitrarily small positive constant.

Remark 7.2. Theorem 7.1 reveals that the gradient inclusion in Assumption 1 and the standard approximation properties (7.9), (7.10), and (7.11) for piecewise smooth solutions play a fundamental role in generating a spurious-free and spectral-correct finite element solution by the new mixed method (3.5). Furthermore, since the gradient inclusion in Assumption 1 can be easily realized for a lowest-order nodal element $W_h = Q_h$, we know whether the new mixed method (3.5) is spurious-free and spectral-correct relies totally on the standard approximation properties (7.9), (7.10), and (7.11) for piecewise smooth solutions.

We shall demonstrate in the rest of this section that under the new mixed finite element method (3.5), many elements that are impossible to be spurious-free and spectral-correct for the Maxwell eigenproblem in classical formulations are now spurious-free and spectral-correct. These elements include the Nédélec elements of second kind on rectangles, parallelograms and parallelepipeds, the Nédélec elements of second kind on nonaffine quadrilateral and hexahedral meshes (excluding only the lowest-order element on nonaffine hexahedral meshes), as well as the Nédélec elements of first kind on nonaffine quadrilateral and hexahedral meshes (excluding only the lowest-order elements on nonaffine quadrilateral and hexahedral meshes). We refer to Remark 7.3 for more details about the three lowest-order nonaffine elements excluded above.

Affine meshes built on the reference element $\hat{K} = [0,1]^d$. We consider the meshes obtained from $\hat{K} = [0,1]^d$ by an affine mapping $F_K : \hat{K} \to K$, such as elements of rectangles, cuboids, parallelograms, parallelepipeds, etc. For $H(\mathbf{curl}\,;\Omega)$-conforming Nédélec elements of second kind, for any integer $\ell \geq 1$, we define $\mathbf{U}_h$ by choosing

$$\hat{\mathbf{E}}_\ell(\hat{K}) = \hat{\mathbf{Q}}_\ell^{\mathrm{Ned}} := (\hat{\mathcal{Q}}_\ell(\hat{K}))^d, \tag{7.21}$$

while for Q_h, we choose the lowest-order nodal element $\hat{\mathcal{Q}}_1(\hat{K})$. ¿From (7.5)-(7.6) and (7.2), we can easily verify the gradient inclusion in Assumption 1 and the standard approximation properties (7.9)-(7.11) for piecewise smooth solutions. Here, we should note that

$$\mathbf{V}_h := \{\mathbf{v}_h \in (H^1(\Omega))^d \cap H_0(\mathbf{curl}\,;\Omega) : \mathbf{v}_h|_K \in (\mathcal{Q}_\ell(K))^d \,\forall K \in \mathcal{T}_h\} \subset \mathbf{U}_h, \tag{7.22}$$

since DF_K is a constant matrix for the affine mapping F_K. The above inclusion allows the use of the interpolation operator $\mathbf{\Pi}_h$ as the Lagrange interpolation operator for satisfying (7.10) and (7.11). Consequently, the Nédélec elements of second kind on rectangles, parallelograms and parallelepipeds are spurious-free and spectral-correct for solving the Maxwell eigenproblem under (3.5). For the sake of clarification, we formulate a general theorem.

Theorem 7.2. *The whole family of Nédélec elements of second kind on affine quadrilateral and hexahedral meshes are spurious-free and spectral-correct under the mixed finite element method (3.5).*

Nonaffine meshes built from the reference element $\hat{K} = [0,1]^d$. It is well known that it is very difficult to approximate $H(\mathbf{curl}\,;\Omega)$ over nonaffine quadrilateral or hexahedral elements. For nonaffine hexahedral elements, we are only aware of the work [28], where a low order element is constructed as an enriched version of the lowest-order Nédélec element of first kind, and no results are available about the Nédélec element of second kind. As for the Maxwell eigenproblem, the first difficulty is due to the failure of the discrete de Rham exact sequence (1.3) in general, and the other difficulty is that the distortion of the meshes (the faces of hexahedral elements are nappes of hyperbolic paraboloids) may generate finite element spaces that do not even contain the constant vector-valued functions locally.

However, under the new mixed formulation (3.5), the whole family of Nédélec elements $\hat{\mathbf{Q}}_\ell^{\mathrm{Ned}}$ of second kind on nonaffine quadrilaterals and hexahedra (except for the lowest-order element on nonaffine hexahedra, see Remark 7.3), as well as the whole family of Nédélec elements of first kind, namely,

$$\begin{cases} \hat{\mathbf{Q}}_{\ell-1/\ell}^{\mathrm{Ned}} := \hat{\mathcal{Q}}_{\ell-1,\ell}(\hat{K}) \times \hat{\mathcal{Q}}_{\ell,\ell-1}(\hat{K}) \quad \text{for } d = 2, \\ \hat{\mathbf{Q}}_{\ell-1/\ell/\ell}^{\mathrm{Ned}} := \hat{\mathcal{Q}}_{\ell-1,\ell,\ell}(\hat{K}) \times \hat{\mathcal{Q}}_{\ell,\ell-1,\ell}(\hat{K}) \times \hat{\mathcal{Q}}_{\ell,\ell,\ell-1}(\hat{K}) \quad \text{for } d = 3 \end{cases} \tag{7.23}$$

on nonaffine quadrilaterals and hexahedra (except for the lowest-order elements on nonaffine quadrilaterals and hexahedra, see Remark 7.3), are all spurious-free and spectral-correct. Here $\hat{\mathcal{Q}}_{i,j,k}(\hat{K})$ denotes the tensor of polynomials of different degree at most i, j, k in each variable $\hat{x}_1, \hat{x}_2, \hat{x}_3$ on $\hat{K}$, and similarly for the space $\hat{\mathcal{Q}}_{i,j}(\hat{K})$ in two dimensions.

Lemma 7.4. *For Nédélec elements $\hat{\mathbf{Q}}_\ell^{\rm Ned}$ of second kind on nonaffine quadrilateral and hexahedral meshes (with $\ell \geq 2$), let the canonical interpolation $\mathbf{\Pi}_h$ be defined through the Piola-like transformation $(DF_K)^{-T}$ from the canonical interpolation $\hat{\mathbf{\Pi}}$ on $\hat{K}$ by the degrees of freedom on the reference element $\hat{K}$ as in [37] (see also [9]). Then it holds for $\mathbf{z} \in (H^{1+r}(K))^3$ with $r > 0$ that*

$$||\mathbf{z} - \mathbf{\Pi}_h \mathbf{z}||_{0,K} \leq C h_K^r ||\mathbf{z}||_{1+r,K}, \quad ||\mathbf{curl}\,(\mathbf{z} - \mathbf{\Pi}_h \mathbf{z})||_{0,K} \leq C h_K^r ||\mathbf{z}||_{1+r,K}.$$

Proof. Since $\hat{\mathbf{\Pi}}\hat{\mathbf{Q}}_\ell^{\rm Ned} \equiv \hat{\mathbf{Q}}_\ell^{\rm Ned}$, $\mathcal{P}_1(K) \circ F_K \subset \hat{\mathcal{Q}}_1(\hat{K})$, and $(DF_K)^T(\mathcal{P}_1(K) \circ F_K)^d \subset (\hat{\mathcal{Q}}_2(\hat{K}))^d$, we have

$$(DF_K)^T(\mathcal{P}_0(K) \circ F_K)^d \subset \hat{\mathbf{Q}}_\ell^{\rm Ned} = (\hat{\mathcal{Q}}_\ell(\hat{K}))^d \quad \forall\, \ell \geq 1,$$

$$\mathbf{curl}\,\mathbf{q}_1 = \mathbf{curl}\,\mathbf{\Pi}_h \mathbf{q}_1 \quad \forall \mathbf{q}_1 \in (\mathcal{P}_1(K))^d \quad \forall\, \ell \geq 2,$$

where $\mathcal{P}_\ell(K)$ is the space of polynomials on the hexahedral element K of total degree not greater than ℓ. Then the desired result follows from a similar argument in [28, Theorems 8.1 and 8.2]. □

Lemma 7.4 implies immediately the following result.

Theorem 7.3. *For all $\ell \geq 2$, let $\mathbf{U}_h$ be $H(\mathbf{curl}\,;\Omega)$-conforming space by defining $\hat{\mathbf{E}}_\ell(\hat{K})$ as Nédélec elements $\hat{\mathbf{Q}}_\ell^{\rm Ned} = (\hat{\mathcal{Q}}_\ell(\hat{K}))^d$ of second kind on nonaffine quadrilateral and hexahedral meshes. These finite element spaces are all spurious-free and spectral-correct under the mixed formulation (3.5).*

We have the same conclusion as in Theorem 7.3 for the lowest-order case of Nédélec elements of second kind ($\ell = 1$) on nonaffine quadrilaterals, but it is more convenient to be dealt with separately as it is done below.

Theorem 7.4. *Let $\mathbf{U}_h$ be $H(\mathrm{curl}\,;\Omega)$-conforming finite element space by defining $\hat{\mathbf{E}}_1(\hat{K})$ as the lowest-order Nédélec elements $\hat{\mathbf{Q}}_1^{\rm Ned} = (\hat{\mathcal{Q}}_1(\hat{K}))^2$ of second kind on nonaffine quadrilateral meshes. Then it holds for $\mathbf{z} \in (H^{1+r}(K))^2$ with $r > 0$ that*

$$||\mathbf{z} - \mathbf{\Pi}_h \mathbf{z}||_{0,K} \leq C h_K^r ||\mathbf{z}||_{1+r,K}, \quad ||\mathrm{curl}\,(\mathbf{z} - \mathbf{\Pi}_h \mathbf{z})||_{0,K} \leq C h_K^r ||\mathbf{z}||_{1+r,K}. \tag{7.24}$$

Consequently, $\mathbf{U}_h$ is spurious-free and spectral-correct under the mixed finite element formulation (3.5).

Proof. The first estimate in (7.24) follows readily from the fact that

$$(DF_K)^T(\mathcal{P}_0(K) \circ F_K)^2 \subset \hat{\mathbf{Q}}_1^{\rm Ned} = (\hat{\mathcal{Q}}_1(\hat{K}))^2\,,$$

and the standard interpolation estimate. To establish the second estimate in (7.24), we can first use the fact that $\hat{\mathrm{curl}}\,(\hat{\mathcal{Q}}_1(\hat{K}))^2 \equiv \hat{\mathcal{P}}_1(\hat{K})$ to find that $\hat{\mathrm{curl}}\,\hat{\mathbf{\Pi}}\hat{\mathbf{z}} = \hat{\rho}\hat{\mathrm{curl}}\,\hat{\mathbf{z}}$, where $\hat{\rho}$ is the L^2-projection onto $\hat{\mathcal{P}}_1(\hat{K})$.

For convenience, we now rotate $\mathbf{z}$ by $\pi/2$ to get $\mathbf{z}^* := (-z_2, z_1)$, then $\mathrm{curl}\,\mathbf{z} = \mathrm{div}\,\mathbf{z}^*$. By similar rotations for $\mathbf{\Pi}_h$, etc., we can show that

$$||\mathrm{div}\,(\mathbf{z}^* - \mathbf{\Pi}_h^* \mathbf{z}^*)||_{0,K} \leq C h_K^r |\mathrm{div}\,\mathbf{z}^*|_{r,K}.$$

For any constant $t_0 \in \mathcal{P}_0(K)$, we like to find $\hat{\mathbf{s}}_0 \in (\hat{\mathcal{Q}}_1(\hat{K}))^2$ so that it holds for $\mathbf{s}_0 = (JF_K)^{-1} DF_K \hat{\mathbf{s}}_0$ that

$$t_0 = \mathrm{div}\,\mathbf{s}_0 = (JF_K)^{-1} \hat{\mathrm{div}}\,\hat{\mathbf{s}}_0,$$

where $\hat{\mathbf{s}}_0 \in (\hat{\mathcal{Q}}_1(\hat{K}))^2$ is determined by $JF_K t_0 = \hat{\mathrm{div}}\,\hat{\mathbf{s}}_0$, since $\hat{\mathrm{div}}\,(\hat{\mathcal{Q}}_1(\hat{K}))^2 \equiv \hat{\mathcal{P}}_1(\hat{K})$ and $JF_K t_0 \in \hat{\mathcal{P}}_1(\hat{K})$. Further, we can see

$$\mathrm{div}\,\mathbf{\Pi}_h^* \mathbf{s}_0 = (JF_K)^{-1} \hat{\mathrm{div}}\,\hat{\mathbf{\Pi}}^* \hat{\mathbf{s}}_0 = (JF_K)^{-1} \hat{\mathrm{div}}\,\hat{\mathbf{s}}_0 = \mathrm{div}\,\mathbf{s}_0 = t_0,$$

then we can proceed to write and estimate

$$\begin{aligned}&\textstyle\int_K |\mathrm{div}\,(\mathbf{z}^* - \mathbf{\Pi}_h^* \mathbf{z}^*)|^2 = \int_K |\mathrm{div}\,(\mathbf{z}^* - \mathbf{s}_0 - \mathbf{\Pi}_h^*(\mathbf{z}^* - \mathbf{s}_0))|^2 \\ &= \textstyle\int_{\hat{K}} |(JF_K)^{-1} \hat{\mathrm{div}}\,(\hat{\mathbf{z}}^* - \hat{\mathbf{s}}_0 - \hat{\mathbf{\Pi}}^*(\hat{\mathbf{z}}^* - \hat{\mathbf{s}}_0))|^2 JF_K \leq C h_K^{-2} \int_{\hat{K}} |\hat{\mathrm{div}}\,(\hat{\mathbf{z}}^* - \hat{\mathbf{s}}_0 - \hat{\mathbf{\Pi}}^*(\hat{\mathbf{z}}^* - \hat{\mathbf{s}}_0))|^2 \\ &= \textstyle C h_K^{-2} \int_{\hat{K}} |\hat{\mathrm{div}}\,(\hat{\mathbf{z}}^* - \hat{\mathbf{s}}_0) - \hat{\rho}\hat{\mathrm{div}}\,(\hat{\mathbf{z}}^* - \hat{\mathbf{s}}_0)|^2 \leq C h_K^{-2} |\hat{\mathrm{div}}\,(\hat{\mathbf{z}}^* - \hat{\mathbf{s}}_0)|_{r,\hat{K}}^2,\end{aligned}$$

where we have used the shape-regularity condition $JF_K \geq Ch_K^2$. On the other hand, we can bound

$$|\hat{\text{div}}\,(\hat{\mathbf{z}}^* - \hat{\mathbf{s}}_0)|^2_{r,\hat{K}} = |JF_K(\text{div}\,(\mathbf{z}^* - \mathbf{s}_0)) \circ F_K|^2_{r,\hat{K}} \leq Ch_K^4(||(\text{div}\,(\mathbf{z}^* - \mathbf{s}_0)) \circ F_K||^2_{0,\hat{K}} + |(\text{div}\,(\mathbf{z}^* - \mathbf{s}_0)) \circ F_K|^2_{r,\hat{K}}).$$

We can now choose a suitable $t_0 \in \mathcal{P}_0(K)$ as the approximation of $\text{div}\,\mathbf{z}^*$ on K such that

$$\begin{aligned} ||(\text{div}\,(\mathbf{z}^* - \mathbf{s}_0)) \circ F_K||^2_{0,\hat{K}} \leq Ch_K^{-2}||\text{div}\,(\mathbf{z}^* - \mathbf{s}_0)||^2_{0,K} &= Ch_K^{-2}||\text{div}\,\mathbf{z}^* - t_0||^2_{0,K} \leq C|\text{div}\,\mathbf{z}^*|^2_{r,K}, \\ |\text{div}\,(\mathbf{z}^* - \mathbf{s}_0) \circ F_K|^2_{r,\hat{K}} &\leq Ch_K^{2r}h_K^{-2}|\text{div}\,\mathbf{z}^*|^2_{r,K}, \end{aligned}$$

hence we have deduced

$$\int_K |\text{div}\,(\mathbf{z}^* - \mathbf{\Pi}_h^*\mathbf{z}^*)|^2 \leq Ch_K^{2r}|\text{div}\,\mathbf{z}^*|^2_{r,K}.$$

This, along with the fact that $\text{curl}\,\mathbf{z} = \text{div}\,\mathbf{z}^*$, leads to the second estimate in (7.24). □

Remark 7.3. We note that the argument above for the quadrilateral elements do not apply to the lowest-order hexahedral $\hat{\mathbf{Q}}_1^{\text{Ned}}$ Nédélec element of second kind, because $JF_K(DF_K)^{-1}(\mathcal{P}_0(K)\circ F_K)^3 \not\subset \hat{\mathbf{curl}}\,\hat{\mathbf{Q}}_1^{\text{Ned}} = \hat{\mathbf{curl}}\,(\hat{\mathcal{Q}}_1(\hat{K}))^3$. This means that on the physical hexahedral element K, the $\mathbf{curl}$ of the local approximating space on K does not contain the constant vector-valued functions, and consequently, there is no approximation in the $\mathbf{curl}$ semi-norm no matter how smooth the function $\mathbf{z}$ is, that is, $||\mathbf{curl}\,(\mathbf{z} - \mathbf{\Pi}_h\mathbf{z})||_0 = O(1)$. For this reason, the lowest-order Nédélec element $\hat{\mathbf{Q}}_1^{\text{Ned}}$ of second kind on nonaffine hexahedral meshes should not be used in any case. Similar fact is also well-known for the lowest-order Nédélec elements $\hat{\mathbf{Q}}_{0/1}^{\text{Ned}}$ of first kind on nonaffine quadrilateral meshes and $\hat{\mathbf{Q}}_{0/1/1}^{\text{Ned}}$ of first kind on nonaffine hexahedral meshes, motivating the quadrilateral $H(\text{curl}\,;\Omega)$ elements (by a rotation $\pi/2$ of $H(\text{div}\,;\Omega)$ elements) (cf. [21]) and the low order hexahedral $H(\mathbf{curl}\,;\Omega)$ element (cf. [28]).

We end up this section with the Nédélec elements of first kind on quadrilaterals and hexahedra. Unlike the second kind, the first kind on affine meshes such as rectangles satisfies the discrete de Rham exact sequence (1.3) and is spurious-free and spectral-correct for the classical method (1.1). But this is not valid for nonaffine quadrilateral and hexahedral meshes.

By noticing that

$$(DF_K)^T(\mathcal{P}_1(K)\circ F_K)^2 \subset \hat{\mathcal{Q}}_{1,2}(\hat{K}) \times \hat{\mathcal{Q}}_{2,1}(\hat{K}), \quad (DF_K)^T(\mathcal{P}_1(K)\circ F_K)^3 \subset \hat{\mathcal{Q}}_{1,2,2}(\hat{K}) \times \hat{\mathcal{Q}}_{2,1,2}(\hat{K}) \times \hat{\mathcal{Q}}_{2,2,1}(\hat{K}),$$

we can use the same argument as for Theorem 7.3 to derive the following result.

Theorem 7.5. *Let $\mathbf{U}_h$ be $H(\mathbf{curl}\,;\Omega)$-conforming finite element space by defining $\hat{\mathbf{E}}_\ell(\hat{K})$ ($\ell \geq 2$) as Nédélec elements $\hat{\mathbf{Q}}_{\ell-1/\ell}^{\text{Ned}}$ of first kind on nonaffine quadrilateral meshes and $\hat{\mathbf{Q}}_{\ell-1/\ell/\ell}^{\text{Ned}}$ of first kind on nonaffine hexahedral meshes, then all these spaces $\mathbf{U}_h$ are spurious-free and spectral-correct under the new mixed method (3.5).*

All the discontinuous counterparts of those elements in Theorems 7.3 to 7.5 are spurious-free and spectral-correct under the mixed formulation (3.5).

8 Numerical experiments

In this section, we present several sets of numerical results for the lowest-order Nédélec element of second kind on rectangles and quadrilaterals, i.e., $(\mathcal{Q}_1^{\text{Ned}})^2$ element for the electric field. For the multiplier, we use the lowest-order Lagrange isoparametric bilinear element, i.e., $\mathcal{Q}_1$ element.

8.1 Spurious approximation by the classical method

We review the spurious approximations by the classical method (1.1) with $(\mathcal{Q}_1^{\text{Ned}})^2$ element on rectangles. This is well known in the literature (cf. [27]). We take the example from [27], with the domain $\Omega = (0,\pi)^2$. The first 10 exact eigenvalues are as follows: $1, 1, 2, 4, 4, 5, 5, 8, 9, 9$. We collect the numerical results in Table 1 for the nonzero eigenvalues and in Table 2 for the zero eigenvalue. As it was shown explicitly in [27], we see from Tables 1 and 2: zero is a spurious eigenvalue; the discrete counterparts of the eigenvalues $1, 4, 9$ have wrong multiplicities, while other eigenvalues $2, 5, 8$ have correct multiplicities.

Table 1: Spurious approximations by the classical method

π/h	ω^2 /(multiplicities)	ω_h^2 /(multiplicities)	ω^2 /(multiplicities)	ω_h^2 /(multiplicities)
4		1.05238686203823 /(10)		5.91580367687045 /(2)
8		1.01291604505883 /(18)		5.22246349321185 /(2)
16	1 /(2)	1.00321687435621 /(34)	5 /(2)	5.05488105459235 /(2)
32		1.00080344825532 /(66)		5.01367094568282 /(2)
4		2.10477372407648 /(1)		9.72683362966443 /(1)
8		2.02583209011778 /(1)		8.41909489630593 /(1)
16	2 /(1)	2.00643374871356 /(1)	8 /(1)	8.10332836047114 /(1)
32		2.00160689651227 /(1)		8.02573499485413 /(1)
4		4.86341681483219 /(10)		12.843089751768 /(10)
8		4.20954744815297 /(18)		10.0802909335883 /(18)
16	4 /(2)	4.05166418023506 /(34)	9 /(2)	9.26313055554446 /(34)
32		4.01286749742499 /(66)		9.06524486372812 /(66)

Table 2: Zero eigenvalues by the classical method

π/h	ω^2 /(multiplicities)	ω_h^2 (multiplicities)
4		≈ 0 /(9)
8		≈ 0 /(49)
16	0 /(infinity)	≈ 0 /(225)
32		≈ 0 /(961)

8.2 Spurious-free approximation by the new mixed finite element method

We report that the new mixed finite element method (3.5) can provide spurious-free and spectral-correct approximations for the $(\mathcal{Q}_1^{\mathrm{Ned}})^2$ element on rectangles and nonaffine quadrilaterals. The nonaffine quadrilateral mesh is composed of the trapezoids as in [2]. Since the domain is convex, there are no singular solutions (i.e., all eigenfunctions are H^1-regular). For the pair of the $(\mathcal{Q}_1^{\mathrm{Ned}})^2/\mathcal{Q}_1$ element, the theoretical convergence rate is $2(1-\delta)$ for $0<\delta<1$ for discrete eigenvalues. To investigate the effect of the parameter δ, we test $\delta = 0, 0.1, 0.3, 0.5, 0.7, 0.9, 1$, and compute the corresponding convergence rates. From our numerical results, we observe three points: no discrete zero eigenvalues; all discrete eigenvalues converge correctly, with correct multiplicities; the convergence rate approaches 2 (the optimal convergence rate that is the same as the order of approximation) as $h \to 0$ for every discrete eigenvalue when δ lies between 0 and 1, indicating that δ does not really affect the convergence rate. Due to this last point, we present the results only for $\delta = 0$ in Table 3 and in Table 4. The first two observations are consistent with the theoretical results.

8.3 Discontinuous media

We now study a model problem in a discontinuous media in $\Omega = (-1,1)^2$, with $\boldsymbol{\varepsilon} = \varepsilon\mathbf{I}$, where ε is given by $\varepsilon(\mathbf{x}) = 0.5, \mathbf{x} \in \Omega_1 := (0,1)^2 \cup \Omega_3 := (-1,0)^2$, and $\varepsilon(\mathbf{x}) = 1, \mathbf{x} \in \Omega_2 := ((-1,0)\times(0,1)) \cup \Omega_4 := ((0,1)\times(-1,0))$. The minimal regularity of the solution is about $r = 0.78$ (https://perso.univ-rennes1.fr/monique.dauge/benchmax.html). We compute the first 10 eigenvalues with $\delta = 0, 0.3, 0.7, 0.8$; see Tables 5 and 6. We observe no convergence for $\delta = 0$ and $\delta = 0.8 (> 0.78)$ for some discrete eigenvalues. The case $\delta = 0.3$ shows the best convergence among all the 10 eigenvalues. It appears that the convergence rates follow the theoretical rate $2(r_i - \delta)$, where r_i denotes the *piecewise* regularity of the ith eigenvalue, for instance, $r_1 = 1$ approximately, i.e., the eigenfunction $\mathbf{u}|_{\Omega_j} \in (H^{r_1}(\Omega_j))^2$.

8.4 Nonconvex domain

We provide numerical results in a L-shaped $(-1,1)^2 \setminus ([0,1]\times[-1,0])$ and a cracked domain $\Omega = (-1,1)^2 \setminus \{(x_1,x_2) \in \mathbb{R}^2 : 0 \le x < 1, y = 0\}$. The exact eigenvalues are taken from https://perso.univ-rennes1.fr/monique-.dauge/benchmax.html. The first eigenvalue for the L-shaped domain is 1.47562182408, and its eigenfunction has the strongest unbounded singularity, belonging to $(H^{2/3-\epsilon}(\Omega))^2$ for any $\epsilon > 0$. The first eigenvalue for the cracked domain 1.03407400850, whose eigenfunction is very singular, belonging to $(H^{1/2-\epsilon}(\Omega))^2$ for any $\epsilon > 0$. For other eigenvalues whose eigenfunctions are H^1 regular, the parameter δ does not affect the convergence rate, as reported in Section 8.2. We present the numerical results with different values of the parameter δ on rectangular meshes. We can see from Table 7 and Table 8 that the convergence rate behaves like $2(r-\delta)$ as δ varies. We also observe that when δ is bigger than r, the computed first eigenvalue tends to merge the second eigenvalue.

8.5 Discrete Gauss' law

We present numerical results to verify the discrete Gauss' law. ¿From our theory, the finite element solution $\mathbf{u}_h$ converges in the norm $||\mathrm{div}\,(\varepsilon\cdot)||_{-(1-\delta)}$ to the exact solution $\mathbf{u}$ of H^r regularity with convergence order $h^{r-\delta}$. We need only to compute the convergence in the mesh-dependent norm $|\mathbf{u}-\mathbf{u}_h|_{h,\mathrm{div}}$. For the current Maxwell eigenproblem, $\mathrm{div}\,(\varepsilon\mathbf{u}) = 0$ and $[\![(\varepsilon\mathbf{u})\cdot\mathbf{n}]\!] = 0$, so we need actually only to compute $|\mathbf{u}_h|_{h,\mathrm{div}}$. We present only numerical results for the L-shaped domain, with rectangle meshes, and compute the discrete Gauss' law for the first eigenvalue(denoted by $\lambda_{\min}$) with different values of δ. ¿From Table 9, we see that the discrete Gauss' law holds with the convergence $h^{r-\delta}$ approximately ($r = 2/3 - \epsilon$).

Table 3: Rectangle mesh: Relative errors and conver -gence rate of the new method, with $\delta = 0$

ω^2	π/h	ω_h^2	$\lvert\omega^2-\omega_h^2\rvert/\lvert\omega^2\rvert$	Conv. rate
1	4	1.052386862038	5.2387E-02	–
	8	1.012916045059	1.2916E-02	2.0200
	16	1.003216874357	3.2169E-03	2.0054
	32	1.000803448248	8.0345E-04	2.0014
1	4	1.052386862038	5.2387E-02	–
	8	1.012916045059	1.2916E-02	2.0200
	16	1.003216874357	3.2169E-03	2.0054
	32	1.000803448292	8.0345E-04	2.0014
2	4	2.117798387026	5.8899E-02	–
	8	2.016141152150	8.0706E-03	2.8675
	16	2.003199319691	1.5997E-03	2.3349
	32	2.000747585914	3.7379E-04	2.0975
4	4	4.863416814832	2.1585E-01	–
	8	4.209547448153	5.2387E-02	2.0428
	16	4.051664180236	1.2916E-02	2.0200
	32	4.012867497428	3.2169E-03	2.0054
4	4	4.863416814832	2.1585E-01	–
	8	4.209547448153	5.2387E-02	2.0428
	16	4.051664180236	1.2916E-02	2.0200
	32	4.012867497428	3.2169E-03	2.0054
5	4	5.981382804172	1.9628E-01	–
	8	5.175337048473	3.5067E-02	2.4847
	16	5.039480829353	7.8962E-03	2.1509
	32	5.009603696742	1.9207E-03	2.0395
5	4	5.981382804172	1.9628E-01	–
	8	5.175337048473	3.5067E-02	2.4847
	16	5.039480829353	7.8962E-03	2.1509
	32	5.009603696757	1.9207E-03	2.0395
8	4	9.982803144233	2.4785E-01	–
	8	8.209258327497	2.6157E-02	3.2442
	16	8.032410837451	4.0514E-03	2.6907
	32	8.006845438435	8.5568E-04	2.2433
9	4	12.843089751768	4.2701E-01	–
	8	10.080290933588	1.2003E-01	1.8308
	16	9.263130555544	2.9237E-02	2.0376
	32	9.065244863730	7.2494E-03	2.0118
9	4	12.843089751768	4.2701E-01	–
	8	10.080290933588	1.2003E-01	1.8308
	16	9.263130555545	2.9237E-02	2.0376
	32	9.065244863785	7.2494E-03	2.0118

Table 4: Quadrilateral mesh: Relative errors and conver -gence rate of the new method, with $\delta = 0$

ω^2	π/h	ω_h^2	$\lvert\omega^2-\omega_h^2\rvert/\lvert\omega^2\rvert$	Conv. rate
1	4	1.088800832495	8.8801E-02	–
	8	1.016593866630	1.6594E-02	2.4199
	16	1.003583466571	3.5835E-03	2.2112
	32	1.000822706689	8.2271E-04	2.1229
1	4	1.230798335934	2.3080E-01	–
	8	1.093587158873	9.3587E-02	1.3023
	16	1.013460450306	1.3460E-02	2.7976
	32	1.002339680398	2.3397E-03	2.5243
2	4	2.320337045007	1.6017E-01	–
	8	2.090877405767	4.5439E-02	1.8176
	16	2.013848986366	6.9245E-03	2.7141
	32	2.002071811856	1.0359E-03	2.7408
4	4	5.413997793531	3.5350E-01	–
	8	4.267861008813	6.6965E-02	2.4002
	16	4.055837156948	1.3959E-02	2.2622
	32	4.013500832806	3.3752E-03	2.0482
4	4	6.241424874135	5.6036E-01	–
	8	4.487614139165	1.2190E-01	2.2006
	16	4.095773971189	2.3943E-02	2.3480
	32	4.016788732686	4.1972E-03	2.5121
5	4	6.750542215301	3.5011E-01	–
	8	5.306099514632	6.1220E-02	2.5157
	16	5.055089346678	1.1018E-02	2.4742
	32	5.011132699334	2.2265E-03	2.3070
5	4	6.955212026321	3.9104E-01	–
	8	5.480154424885	9.6031E-02	2.0258
	16	5.082393210334	1.6479E-02	2.5429
	32	5.014918352329	2.9837E-03	2.4654
8	4	10.673760260586	3.3422E-01	–
	8	8.592251467340	7.4031E-02	2.1746
	16	8.088253919420	1.1032E-02	2.7465
	32	8.012580236137	1.5725E-03	2.8105
9	4	13.218593939734	4.6873E-01	–
	8	10.366039431151	1.5178E-01	1.6268
	16	9.288481349492	3.2053E-02	2.2434
	32	9.067794393815	7.5327E-03	2.0892
9	4	14.148527375137	5.7206E-01	–
	8	10.538680759916	1.7096E-01	1.7425
	16	9.328232990934	3.6470E-02	2.2289
	32	9.072495460249	8.0551E-03	2.1788

Table 5: Relative errors and convergence rate in nonhomogeneous media

ω^2	$1/h$	$\delta = 0.3$			$\delta = 0.7$		
		ω_h^2	$\frac{\lvert\omega^2-\omega_h^2\rvert}{\lvert\omega^2\rvert}$	Conv. rate	ω_h^2	$\frac{\lvert\omega^2-\omega_h^2\rvert}{\lvert\omega^2\rvert}$	Conv. rate
3.31754876342	4	3.373790330997	1.6953E-02	–	3.393420726989	2.2870E-02	–
	8	3.333046028939	4.6713E-03	1.8596	3.344885745624	8.2401E-03	1.4727
	16	3.322033479731	1.3518E-03	1.7889	3.329002334654	3.4524E-03	1.2551
	32	3.318900288110	4.0739E-04	1.7304	3.322958840759	1.6307E-03	1.0821
3.36632415726	4	3.454932324173	2.6322E-02	–	3.553031255492	5.5463E-02	–
	8	3.403425644698	1.1021E-02	1.2560	3.509333242230	4.2482E-02	0.3847
	16	3.384229077442	5.3188E-03	1.0511	3.491675557791	3.7237E-02	0.1901
	32	3.375611128640	2.7588E-03	0.9471	3.480892659893	3.4034E-02	0.1298
6.18638956249	4	6.254570072979	1.1021E-02	–	6.266662659233	1.2976E-02	–
	8	6.202870552138	2.6641E-03	2.0486	6.205480800053	3.0860E-03	2.0720
	16	6.190494892254	6.6361E-04	2.0052	6.190996258842	7.4465E-04	2.0511
	32	6.187420591039	1.6666E-04	1.9934	6.187514737509	1.8188E-04	2.0336
13.92632333103	4	14.676576363672	5.3873E-02	–	14.688942604189	5.4761E-02	–
	8	14.112175178217	1.3345E-02	2.0132	14.115146794519	1.3559E-02	2.0139
	16	13.972689289927	3.3294E-03	2.0030	13.973328528184	3.3753E-03	2.0061
	32	13.937919218731	8.3266E-04	1.9995	13.938051224857	8.4214E-04	2.0029
15.08299096123	4	15.836349691064	4.9948E-02	–	15.862542934204	5.1684E-02	–
	8	15.271373430179	1.2490E-02	1.9997	15.278478526612	1.2961E-02	1.9956
	16	15.130230327187	3.1320E-03	1.9956	15.132161475598	3.2600E-03	1.9912
	32	15.094856440883	7.8668E-04	1.9932	15.095381335190	8.2148E-04	1.9886
15.77886590819	4	16.350460308391	3.6225E-02	–	16.487901759137	4.4936E-02	–
	8	15.932600032036	9.7430E-03	1.8946	15.963545007066	1.1704E-02	1.9408
	16	15.818793890060	2.5305E-03	1.9450	15.829234109283	3.1921E-03	1.8744
	32	15.789211806927	6.5568E-04	1.9483	15.793982644500	9.5804E-04	1.7364
18.64329693686	4	19.265547904577	3.3377E-02	–	19.459898887434	4.3801E-02	–
	8	18.819221400703	9.4363E-03	1.8225	18.854528858053	1.1330E-02	1.9508
	16	18.689673943004	2.4876E-03	1.9235	18.700535843318	3.0702E-03	1.8838
	32	18.655376951069	6.4795E-04	1.9408	18.662104326135	1.0088E-03	1.6057
25.79753111031	4	24.390190847849	5.4553E-02	–	27.196764801646	5.4239E-02	–
	8	26.035534902772	9.2258E-03	2.5639	26.159754081781	1.4041E-02	1.9497
	16	25.867640143095	2.7177E-03	1.7633	25.887537590825	3.4890E-03	2.0088
	32	25.816271147558	7.2643E-04	1.9035	25.819833263321	8.6451E-04	2.0128
29.85240067684	4	26.667059793475	1.0670E-01	–	33.602270950088	1.2561E-01	–
	8	30.731286700643	2.9441E-02	1.8577	30.771111916146	3.0775E-02	2.0292
	16	30.072538942593	7.3742E-03	1.9973	30.084036270336	7.7594E-03	1.9878
	32	29.907719198975	1.8531E-03	1.9926	29.912414589455	2.0104E-03	1.9485
30.53785871253	4	27.692056035051	9.3189E-02	–	34.231984360219	1.2097E-01	–
	8	31.364373481626	2.7065E-02	1.7837	31.496404670057	3.1389E-02	1.9463
	16	30.751787457670	7.0054E-03	1.9499	30.838299474985	9.8383E-03	1.6738
	32	30.593920472721	1.8358E-03	1.9320	30.672594137224	4.4121E-03	1.1570

Table 6: Relative errors and convergence rate in nonhomogeneous media

ω^2	$1/h$	$\delta = 0$			$\delta = 0.8$		
		ω_h^2	$\frac{\lvert\omega^2 - \omega_h^2\rvert}{\lvert\omega^2\rvert}$	Conv. rate	ω_h^2	$\frac{\lvert\omega^2 - \omega_h^2\rvert}{\lvert\omega^2\rvert}$	Conv. rate
3.31754876342	4	3.366248958742	1.4680E-02	–	3.401272745439	2.5237E-02	–
	8	3.329698177698	3.6622E-03	2.0030	3.350769814651	1.0014E-02	1.3335
	16	3.320613031874	9.2365E-04	1.9873	3.333236659173	4.7288E-03	1.0824
	32	3.318322833918	2.3333E-04	1.9850	3.325929063342	2.5261E-03	0.9046
3.36632415726	4	3.423554358868	1.7001E-02	–	3.596287098015	6.8313E-02	–
	8	3.382036077480	4.6674E-03	1.8649	3.570161834290	6.0552E-02	0.1740
	16	3.370794152961	1.3279E-03	1.8135	3.570329092694	6.0602E-02	-0.0012
	32	3.367632574545	3.8868E-04	1.7725	3.577542345870	6.2744E-02	-0.0501
6.18638956249	4	6.241788464926	8.9550E-03	–	6.270503728102	1.3597E-02	–
	8	6.199433569895	2.1085E-03	2.0865	6.206565292445	3.2613E-03	2.0597
	16	6.189607391509	5.2015E-04	2.0192	6.191263280101	7.8781E-04	2.0495
	32	6.187192103837	1.2973E-04	2.0034	6.187575187702	1.9165E-04	2.0394
13.92632333103	4	13.021391490081	6.4980E-02	–	14.693287465271	5.5073E-02	–
	8	13.035385668436	6.3975E-02	0.0225	14.116368453597	1.3646E-02	2.0128
	16	13.067332334894	6.1681E-02	0.0527	13.973625278181	3.3966E-03	2.0064
	32	13.089406560634	6.0096E-02	0.0376	13.938118006116	8.4693E-04	2.0038
15.08299096123	4	14.668520969081	2.7479E-02	–	15.870101827730	5.2185E-02	–
	8	14.110034645747	6.4507E-02	-1.2311	15.281101773441	1.3135E-02	1.9903
	16	13.972115991998	7.3651E-02	-0.1912	15.132949734990	3.3123E-03	1.9875
	32	13.937762914382	7.5928E-02	-0.0439	15.095601131377	8.3605E-04	1.9862
15.77886590819	4	15.170249716379	3.8572E-02	–	16.517079925341	4.6785E-02	–
	8	15.277040011130	3.1804E-02	0.2783	15.974805343280	1.2418E-02	1.9136
	16	15.131574502674	4.1023E-02	-0.3672	15.834859987226	3.5487E-03	1.8071
	32	15.095147732531	4.3331E-02	-0.0790	15.797347165666	1.1713E-03	1.5992
18.64329693686	4	15.857932054135	1.4940E-01	–	19.491319158026	4.5487E-02	–
	8	15.623096436102	1.6200E-01	-0.1168	18.864808578347	1.1882E-02	1.9367
	16	15.742515583047	1.5559E-01	0.0582	18.706583070793	3.3946E-03	1.8074
	32	15.770301093979	1.5410E-01	0.0139	18.667936944365	1.3217E-03	1.3609
25.79753111031	4	17.134025416776	3.3583E-01	–	27.307802555697	5.8543E-02	–
	8	17.130811241358	3.3595E-01	-0.0005	26.185520245099	1.5040E-02	1.9607
	16	17.076177339536	3.3807E-01	-0.0091	25.893638099976	3.7254E-03	2.0133
	32	17.091352132797	3.3748E-01	0.0025	25.821255863400	9.1965E-04	2.0182
29.85240067684	4	17.538669107976	4.1249E-01	–	33.642193225593	1.2695E-01	–
	8	17.286296227339	4.2094E-01	-0.0293	30.784139569954	3.1212E-02	2.0241
	16	17.460680947272	4.1510E-01	0.0202	30.089646647282	7.9473E-03	1.9735
	32	17.547902020120	4.1218E-01	0.0102	29.915443658327	2.1118E-03	1.9120
30.53785871253	4	20.120531759866	3.4113E-01	–	34.293322995642	1.2298E-01	–
	8	19.101902300399	3.7448E-01	-0.1346	31.550556663024	3.3162E-02	1.8908
	16	18.774780592102	3.8520E-01	-0.0407	30.899081089006	1.1829E-02	1.4872
	32	18.678386863165	3.8835E-01	-0.0118	30.744458870508	6.7654E-03	0.8060

Table 7: Relative errors and convergence rate in L-shaped domain for $\omega^2 = 1.47562182408$

δ	$1/h$	ω_h^2	$\lvert\omega^2 - \omega_h^2\rvert/\lvert\omega^2\rvert$	Conv. rate
0	4	1.56373176117288	5.9710E-02	–
	8	1.51265068392776	2.5094E-02	1.2507
	16	1.49127832611766	1.0610E-02	1.2419
	32	1.48215443760253	4.4270E-03	1.2610
0.1	4	1.59255025816799	7.9240E-02	–
	8	1.53319476789002	3.9016E-02	1.0222
	16	1.50424692779372	1.9399E-02	1.0081
	32	1.48966601020096	9.5175E-03	1.0273
0.3	4	1.67629489053128	1.3599E-01	–
	8	1.60613750576798	8.8448E-02	0.6206
	16	1.56141456021173	5.8140E-02	0.6053
	32	1.53142288003012	3.7815E-02	0.6206
0.5	4	1.81161668840557	2.2770E-01	–
	8	1.75687192270555	1.9060E-01	0.2566
	16	1.71390891708074	1.6148E-01	0.2392
	32	1.67714571668900	1.3657E-01	0.2417

Table 8: Relative errors and convergence rate in cracked domain for $\omega^2 = 1.03407400850$

δ	$1/h$	ω_h^2	$\lvert\omega^2 - \omega_h^2\rvert/\lvert\omega^2\rvert$	Conv. rate
0	4	1.29565551733762	2.5296E-01	–
	8	1.17409151374395	1.3540E-01	0.9017
	16	1.10829350768039	7.1774E-02	0.9157
	32	1.07268848782586	3.7342E-02	0.9427
0.1	4	1.38878730483937	3.4303E-01	–
	8	1.25593297905526	2.1455E-01	0.6770
	16	1.17172217844846	1.3311E-01	0.6887
	32	1.11788869550645	8.1053E-02	0.7157
0.3	4	1.65805430137384	6.0342E-01	–
	8	1.54604603486598	4.9510E-01	0.2854
	16	1.45211537647965	4.0427E-01	0.2924
	32	1.37071725654294	3.2555E-01	0.3124
0.5	4	2.08428961333973	1.0156E+00	–
	8	2.12767428004568	1.0576E+00	-0.0584
	16	2.17194844843256	1.1004E+00	-0.0573
	32	2.21271425146206	1.1398E+00	-0.0508

Table 9: Discrete Gauss' law in L-shaped domain

$1/h$		4	8	16	32	64
$\delta = 0$	$\lambda_{\min}$	1.56373	1.51265	1.49128	1.48215	1.47831
	$\lvert\mathbf{u}_h\rvert_{h,\mathrm{div}}$	0.3038	0.2020	0.1339	0.0876	0.0566
	Conv. rate	0.62	0.59	0.59	0.61	0.63
$\delta = 0.1$	$\lambda_{\min}$	1.59255	1.53319	1.50425	1.48967	1.48239
	$\lvert\mathbf{u}_h\rvert_{h,\mathrm{div}}$	0.3436	0.2437	0.1730	0.1216	0.0845
	Conv. rate	0.53	0.50	0.49	0.51	0.52
$\delta = 0.3$	$\lambda_{\min}$	1.67629	1.60614	1.56141	1.53142	1.51135
	$\lvert\mathbf{u}_h\rvert_{h,\mathrm{div}}$	0.4385	0.3525	0.2856	0.2309	0.1855
	Conv. rate	0.35	0.31	0.30	0.31	0.32
$\delta = 0.5$	$\lambda_{\min}$	1.81162	1.75687	1.71391	1.67715	1.64503
	$\lvert\mathbf{u}_h\rvert_{h,\mathrm{div}}$	0.5562	0.5043	0.4624	0.4258	0.3922
	Conv. rate	0.18	0.14	0.13	0.12	0.12

Concluding remarks

We have proposed a new mixed finite element method for Maxwell eigenproblem, in terms of electric field and a scalar multiplier, where the Gauss' law is enforced in the finite element formulation and discontinuous elements are used for the electric field and the lowest-order nodal element for the multiplier. We have shown the stability and error estimates of the new method and the strong convergence of the discrete Gauss' law. In particular, we prove that holding the well-known discrete compactness property and with no resorting to the discrete de Rham complex exact sequence, the new method is spurious-free and spectral-correct, so long as the first-order approximation in $H(\mathbf{curl}\,;\Omega)$ norm holds for smooth functions. Consequently, all Nédélec elements on affine and nonaffine meshes are spurious-free and spectral-correct in the new method, except only the three lowest-order Nédélec elements that do not hold the first-order approximation on the nonaffine quadrilateral and hexahedral meshes. Hence, the new method is more advantageous than the classical methods that must require the discrete de Rham complex exact sequence and do not satisfy the discrete compactness property or such property is an extremely difficult issue for many Nédélec elements. Numerical results have confirmed the theoretical results of the new method.

Since the new method (in its h-version) can always ensure the discrete compactness property, it would be highly interesting to extend and generalize the new method to the p- and hp-version of the finite element discretization. As is well-known, in the classical methods, the discrete compactness property for the p- and hp-versions has been a far more difficult issue than that for the h-version, even if the discrete de Rham complex exact sequence holds. For example, for the hp-version on simplexes, little work is available in the establishment of the discrete compactness property. Such extension and generalization will be presented elsewhere.

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