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STOCHASTIC CONVERGENCE OF REGULARIZED SOLUTIONS AND THEIR FINITE ELEMENT APPROXIMATIONS TO INVERSE SOURCE PROBLEMS*

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Abstract. In this work, we investigate the regularized solutions and their finite element solutions to the inverse source problems governed by partial differential equations, and we establish the stochastic convergence and optimal finite element convergence rates of these solutions under pointwise measurement data with random noise. The regularization error estimates and the finite element error estimates are derived with explicit dependence on the noise level, regularization parameter, mesh size, and time step size, which can guide practical choices among these key parameters in real applications. The error estimates also suggest an iterative algorithm for determining an optimal regularization parameter. Numerical experiments are presented to demonstrate the effectiveness of the analytical results.

Key words. inverse source problems, regularization, finite element approximation, stochastic error estimates

AMS subject classifications. 35R30, 65J20, 65M60, 65N21, 65N30

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1. Introduction.

This work presents a quantitative understanding of stochastic convergence of the regularized solutions and their finite element approximations to the inverse source problems governed by partial differential equations under the measurement data with random noise. The inverse source problems may arise from very different applications and modeling, e.g., diffusion or groundwater flow processes [1, 4, 6, 21, 5, 27, 28], heat conduction or convection-diffusion processes [3, 20, 21, 31, 39], or acoustic problems [7, 35]. Pollutant source inversion can find many applications, e.g., indoor and outdoor air pollution, and detecting and monitoring underground water pollution. Physical, chemical, and biological measures have been developed for the identification of sources and source strengths [4, 47, 48]. Due to the important applications of ill-posed inverse source problems, stable numerical solutions have been widely studied, both deterministically and statistically [32, 38, 37]. A popular approach for inverse source problems is the least-squares optimization with appropriate regularizations [3, 21, 46], which will also be the formulation we take in this work.

Our first main result is the establishment of the optimal stochastic error estimates

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between the exact solution and the regularized solution in a weaker topology of the regularization space, in terms of the statistical property of the data noise rather than the norm of the noise in some Hilbert space. We remark that regularization and convergence of regularized solutions in the strong norm of the regularization space have been widely studied under various source conditions. The classical source condition requires the existence of a small source function [15]. One source condition was proposed in [16] for an inverse conductivity problem to relax the restrictive requirement on the smallness of the source function in the classical convergence theory [15]. A variational source condition was proposed in [23] and was further extended in [9, 19, 22]. How the classical or variational source conditions can be verified is still a hot topic. It appears that the analytical techniques in all existing verifications of source conditions are quite different for each concrete inverse problem [10, 11, 24, 25, 29]. The current work makes a very promising first attempt to achieve the error estimates of regularized solutions in a weaker topology without any source conditions.

The second main contribution of this work is to derive the stochastic convergence and error estimates of finite element approximations to the inverse source problems. The error estimates of finite element solutions to inverse problems have been known to be quite challenging and still open to most practically important inverse problems. There have been various efforts on error estimates of finite element solutions for inverse problems, especially for inverse elliptic and parabolic equations. But most existing studies have been carried out only for some not so frequently used mathematical formulations of inverse problems; see [44] for a detailed review and related references therein. We are not aware of any error estimates of finite element solutions to the frequently used least-squares formulations with Tikhonov regularizations, especially when the observation data are treated as random variables. We had a recent study in [26] for a modified regularization formulation for an inverse stationary source problem, where error estimates were achieved under some negative norms. One of our main focuses in this work is to make an attempt to fill the gap, to provide error estimates of finite element solutions to the least-squares formulations with Tikhonov regularizations; more importantly, the observation data will be treated fully as random variables in the entire analysis. As we shall demonstrate, the new error estimates are not only optimal but also present explicit dependence on the critical parameters like noise level, regularization parameter, mesh size, and time step size. Results of this type are highly desirable in real applications as they can provide explicit guidance in choosing these key parameters and are also the major challenge and difficulty in error estimates of finite element solutions to regularized inverse problems.

We would like to mention a very important by-product from our convergence analysis, namely, it suggests a deterministic iterative algorithm for finding an effective regularization parameter. The choice of an effective regularization parameter is essential to the success of all output least-squares minimization approaches with Tikhonov regularizations, but finding an effective regularization parameter for most inverse problems has remained a big challenge.

Another feature of this work is that the entire analysis is carried out for a very practical scenario, i.e., the scattered data. We shall assume the measurement data is collected pointwise, with noise; otherwise no additional regularity assumption is made. This is unlike analyses and results in most existing regularization theories. We refer the reader to [29] for the study under the same regularization functional with scattering data as in this paper, but in a deterministic linear setting.

We studied in a recent work [12] the stochastic convergence of a nonconforming finite element method for the thin plate spline smoother for observational data.

The spline model for scattered data has attracted considerable attention in the literature. The convergence rate in expectation of the error between the solution of the spline model and the true solution was established in [41]. Under the condition that measurement noise is sub-Gaussian random variables, the stochastic convergence of the empirical error was obtained by the peeling argument in [42] ($d = 1$) and [12] ($d = 2, 3$). In subsection 2.1 we shall borrow some analytical tools from [12, 41] to study the stochastic convergence in expectation when the measurement noise is random variables having bounded variance. The peeling argument is used in subsection 2.2 to show that the empirical error has an exponential decaying tail when the measurement noise is sub-Gaussian random variables. The discretization and its error estimates are considered in section 2.3, both in the expectation and in the Orlicz norm for sub-Gaussian measurement noise. The general results developed in section 2 are applied to study an inverse nonstationary source problem in section 3. Numerical examples are presented in section 4 to demonstrate the effectiveness of our analytical results.

2. Inverse source problem. Let Ω be a bounded domain in $\mathbf{R}^d$ ($d = 1, 2, 3$), and let X and Y be two real Hilbert spaces such that Y is continuously embedded in $C(\bar{\Omega})$ and compactly embedded in $L^2(\Omega)$. The inner product and the norm of a Hilbert space H are denoted by $(\cdot,\cdot)_H$ and $\|\cdot\|_H$, respectively, but $(\cdot,\cdot)$ is used if $H = L^2(\Omega)$. Throughout the paper, we shall use C, with or without subscript, to denote a generic constant independent of the mesh size h and the time step size τ, and it may take a different value at each occurrence.

Let S be a linear bounded operator from X to Y whose null space $N(S) = \{0\}$, and let $f^* \in X$ be an unknown source. We are interested in the inverse source problem of the general form:

(SIP) Given the measurement data of Sf^*, recover the source f^*.

There are many examples of inverse source problems of this type. Our studies will focus on a very important physical scenario, assuming that the pointwise measurement data is collected on a set of distributed sensors located at $\{x_i\}_{i=1}^n$ ($x_i \neq x_j$ for $i \neq j$) inside the physical domain Ω [3, 20, 5, 27, 33, 35, 36]. We assume that the measurements come with noise and take the form

$$m_i = (Sf^*)(x_i) + e_i, \quad i = 1, 2, \ldots, n, \tag{2.1}$$

where $e = (e_1, e_2, \ldots, e_n)^T$ is the data noise vector, with $\{e_i\}_{i=1}^n$ being independent and identically distributed random variables on a probability space $(\mathfrak{X}, \mathcal{F}, \mathbb{P})$. We shall denote $m = (m_1, m_2, \ldots, m_n)^T$ to be the vector of scattering data. Throughout this work, we write $\mathbb{E}[A]$ for the expectation of a random variable A.

We look for an approximate solution f_n of the unknown source function f^* through the least-squares regularized minimization:

$$\min_{f\in X} \frac{1}{n}\sum_{i=1}^n |(Sf)(x_i) - m_i|^2 + \lambda_n \|f\|_X^2, \tag{2.2}$$

where $\lambda_n > 0$ is called a regularization parameter.

We shall consider that the set of discrete points $\{x_i\}_{i=1}^n$ are scattered but quasi-uniformly distributed in Ω; i.e., there exists a constant $B > 0$ such that $d_{\max}/d_{\min} \leq B$, where $d_{\max}$ and $d_{\min}$ are defined by

$$d_{\max} = \sup_{x\in\Omega} \inf_{1\leq i\leq n} |x - x_i| \quad \text{and} \quad d_{\min} = \inf_{1\leq i\neq j\leq n} |x_i - x_j|. \tag{2.3}$$

For any $u, v \in C(\bar{\Omega})$ and $y \in \mathbb{R}^n$, we define

$$(y, v)_n = \frac{1}{n}\sum_{i=1}^{n} y_i v(x_i), \quad (u, v)_n = \frac{1}{n}\sum_{i=1}^{n} u(x_i)v(x_i),$$

and the empirical seminorm $\|u\|_n = (\sum_{i=1}^n u^2(x_i)/n)^{1/2}$ for any $u \in C(\bar{\Omega})$.

Throughout the work, we consider two kinds of random noises $\{e_i\}_{i=1}^n$,

(R1) $\{e_i\}_{i=1}^n$ are independent random variables satisfying $\mathbb{E}[e_i] = 0$ and $\mathbb{E}[e_i^2] \le \sigma^2$;
(R2) $\{e_i\}_{i=1}^n$ are independent sub-Gaussian random variables with parameter σ,

and provide two different techniques to analyze the stochastic convergence and a practical approach to choose the parameter λ_n in each case. We study the convergence under the expectation $\mathbb{E}$ in the case (R1) and establish a stronger convergence in the case (R2), where the errors have exponential decay tails.

2.1. Stochastic convergence for noisy data of variables with bounded variance. We consider the measurement data of type (R1) in this subsection and study the stochastic convergence of the error under the expectation $\mathbb{E}$.

ASSUMPTION 2.1. *We assume the following:*

(1) *There exists a constant $\beta > 1$ such that for all $u \in Y$,*

$$\|u\|_{L^2(\Omega)}^2 \le C(\|u\|_n^2 + n^{-\beta}\|u\|_Y^2), \quad \|u\|_n^2 \le C(\|u\|_{L^2(\Omega)}^2 + n^{-\beta}\|u\|_Y^2). \tag{2.4}$$

(2) *The eigenvalues, $0 < \eta_1 \le \eta_2 \le \cdots$, of the eigenvalue problem*

$$(\psi, v)_X = \eta\,(S\psi, Sv) \quad \forall v \in X \tag{2.5}$$

satisfy that $\eta_k \ge Ck^\alpha$, $k = 1, 2, \dots,$ for some constant C depending only on the operator $S : X \to Y$. The constant α satisfies $1 < \alpha \le \beta$.

We remark that the eigenvalue problem (2.5) is equivalent to the eigenvalue problem $S^*S\psi = \lambda\psi$ in X with $\lambda = \eta^{-1}$, where $S^* : L^2(\Omega) \to X$ is the adjoint operator of $S : X \to L^2(\Omega)$. Since Y is assumed to be compactly embedded into $L^2(\Omega)$, the operator S is compact. By means of the spectral theorem of compact self-adjoint operators (see, e.g., [30, Theorem 3, sect. 28] and [34, Theorem 2.36]) and the fact that the null space $N(S) = \{0\}$, there exist the eigenvalues $\lambda_1 \ge \lambda_2 \ge \cdots$ for the compact operator $S^*S : X \to X$, counted according to possible multiplicity, and the corresponding eigenfunctions $\{\phi_k\}_{k=1}^\infty$ such that $\{\phi_k\}_{k=1}^\infty$ forms a complete orthonormal basis of X, that is,

$$S^*S\phi_k = \lambda_k\phi_k \text{ in } X, \quad (\phi_k, \phi_l)_X = \delta_{kl}, \quad (S\phi_k, S\phi_l) = \lambda_k\delta_{kl}, \quad k, l = 1, 2, \dots, \tag{2.6}$$

where δ_{kl} is the Kronecker delta function. The condition $\alpha > 1$ in Assumption 2.1 then implies that $S : X \to L^2(\Omega)$ is a Hilbert–Schmidt operator (see, e.g., [30, section 30.8]).

We also remark that the restriction $\alpha \le \beta$ in Assumption 2.1 can be removed by checking the detailed proof of Theorem 2.3 since we only need a lower bound of $\rho_k \ge C\min\{k^\alpha, n^\beta\}$ from Lemma 2.2 to prove the variance bound. In this case, Theorem 2.3 will depend on β consequently.

The following observation is inspired by [41], where it was shown that the solution of a thin plate spline smoother model is attained in a finite-dimensional subset.

LEMMA 2.1. *For a given $m \in \mathbb{R}^n$, let f be the solution to the optimization problem*

$$\min_{f \in X, (Sf)(x_i) = m_i} \|f\|_X^2; \tag{2.7}$$

then $f \in V_n$, where V_n is an n-dimensional subset of X.

Proof. Let V be a subset of X such that

$$V = \{v \in X : (Sv)(x_i) = 0, i = 1, 2, \dots, n\}.$$

Define the projection operator $P_V : X \to V$,

$$(P_V[f], v)_X = (f, v)_X \quad \forall v \in V.$$

Choose $\phi_i \in X$ such that $(S\phi_i)(x_j) = \delta_{ij}$. Let $\psi_i = -P_V[\phi_i] + \phi_i$ and $V_n = \text{span}\{\psi_1, \dots, \psi_n\}$. We can easily check that $(S\psi_i)(x_j) = \delta_{ij}$ also holds. For any $f \in X$, define the interpolation operator I:

$$If = \sum_{i=1}^{n} (Sf)(x_i)\psi_i.$$

We can easily see that $If \in V_n$ and $f - If \in V$, and hence we derive

$$\begin{aligned}(f - If, If)_X &= (f - If, \sum_{i=1}^{n} (Sf)(x_i)(\phi_i - P_V[\phi_i]))_X \\ &= \sum_{i=1}^{n} (Sf)(x_i)(f - If, \phi_i - P_V[\phi_i])_X = 0,\end{aligned}$$

where we have used the fact that $(v, \phi_i - P_V[\phi_i])_X = 0$ for all $v \in V$.

We see directly from the above equality that $(If, If)_X \le (f, f)_X$, and hence we have

$$\min_{f \in V_n, (Sf)(x_i) = m_i} \|f\|_X^2 = \min_{f \in X, Sf(x_i) = m_i} \|f\|_X^2.$$

This completes the proof. ∎

LEMMA 2.2. *Let Assumption* 2.1 *be fulfilled, and let V_n be defined as in Lemma* 2.1. *Then the eigenvalue problem*

$$(\psi, v)_X = \rho\,(S\psi, Sv)_n \quad \forall\, v \in V_n \tag{2.8}$$

has n eigenvalues $\rho_1 \le \rho_2 \le \cdots \le \rho_n$, and all the eigenfunctions form an orthogonal basis of V_n with respect to the norm $\|S \cdot\|_n$. Moreover, there exists a constant $C > 0$ independent of k such that $\rho_k \ge Ck^\alpha$ for $k = 1, 2, \dots, n$.

Proof. Consider $V_n = \text{span}\{\psi_i\}_{i=1}^n$ as defined in the proof of Lemma 2.1, and $(S\psi_i)(x_j) = \delta_{ij}$. We can write $\psi = \sum_{i=1}^n (S\psi)(x_i)\psi_i$ for any $\psi \in V_n$. This implies $\|S \cdot\|_n$ is a norm of V_n. Therefore, the eigenvalue problem (2.8) is equivalent to a matrix eigenvalue problem $\mathbb{A}\Psi = \rho\,\mathbb{B}\Psi$ for $\Psi \in \mathbf{R}^n$, where $\mathbb{A}, \mathbb{B} \in \mathbf{R}^{n\times n}$ are two symmetric positive definite matrices. Thus the eigenvalue problem (2.8) has n finite eigenvalues $\rho_1 \le \rho_2 \le \cdots \le \rho_n$ and all eigenfunctions form an orthogonal basis of V_n with respect to the norm $\|S \cdot\|_n$.

We are now ready to give a lower bound of the eigenvalues ρ_k. Using the min-max principle of the Rayleigh quotient for the eigenvalues and (2.4), we can derive

$$\begin{aligned}\rho_k &= \min_{dim(X)=k, X\subset V_n} \max_{u\in X} \frac{(u,u)_X}{(Su,Su)_n} \\ &\geq C \min_{dim(X)=k, X\subset V_n} \max_{u\in X} \frac{(u,u)_X}{(Su,Su)+n^{-\beta}(u,u)_X} \\ &\geq C \min_{dim(X)=k, X\subset L^2(\Omega)} \max_{u\in X} \frac{(u,u)_X}{(Su,Su)+n^{-\beta}(u,u)_X} \\ &= C\frac{1}{\eta_k^{-1}+n^{-\beta}} \geq C\frac{1}{k^{-\alpha}+n^{-\beta}},\end{aligned}$$

where we have used the fact that $\eta_k \geq Ck^\alpha$ by Assumption 2.1. Now $k^\alpha n^{-\beta} \leq n^{\alpha-\beta} \leq 1$ for all $k \leq n$ and $\alpha \leq \beta$. We conclude that $\rho_k \geq Ck^\alpha$. This completes the proof. ∎

THEOREM 2.3. *Let Assumption* 2.1 *be fulfilled, and let $f_n \in X$ be the unique solution of* (2.2). *Then there exist constants $\lambda_0 > 0$ and $C > 0$ such that for any $\lambda_n \leq \lambda_0$,*

$$\mathbb{E}\big[\|Sf_n - Sf^*\|_n^2\big] \leq C\lambda_n\|f^*\|_X^2 + C\sigma^2/(n\lambda_n^{1/\alpha}), \tag{2.9}$$

$$\mathbb{E}\big[\|f_n - f^*\|_X^2\big] \leq C\|f^*\|_X^2 + C\sigma^2/(n\lambda_n^{1+1/\alpha}). \tag{2.10}$$

Proof. By deriving the necessary condition of the quadratic minimization (2.2), we can readily see that the unique minimizer $f_n \in X$ satisfies the variational equation

$$\lambda_n(f_n, v)_X + (Sf_n, Sv)_n = (m, Sv)_n \quad \forall v \in X. \tag{2.11}$$

For any $v \in X$, we introduce the energy norm $|||v|||_{\lambda_n}^2 := \lambda(v,v)_X + \|Sv\|_n^2$. By taking $v = f_n - f^*$ in (2.11), along with (2.1), we obtain

$$|||f_n - f^*|||_{\lambda_n} \leq \lambda_n^{1/2}\|f^*\|_X + \sup_{v\in L^2(\Omega)} \frac{(e,Sv)_n}{|||v|||_{\lambda_n}}. \tag{2.12}$$

It remains to estimate the supremum term in (2.12). Using Lemma 2.1, we can rewrite this supremum term equivalently as

$$\begin{aligned}\sup_{v\in X}\frac{(e,Sv)_n^2}{|||v|||_{\lambda_n}^2} &= \sup_{v\in X}\frac{(e,Sv)_n^2}{\lambda_n(v,v)_X+\|Sv\|_n^2} \\ &\leq \sup_{v\in X}\frac{(e,Sv)_n^2}{\lambda_n \min_{u\in X, Su(x_i)=Sv(x_i)}(u,u)_X+\|Sv\|_n^2} \\ &= \sup_{v\in X}\frac{(e,Sv)_n^2}{\lambda_n \min_{u\in V_n, Su(x_i)=Sv(x_i)}(u,u)_X+\|Sv\|_n^2} \\ &= \sup_{v\in V_n}\frac{(e,Sv)_n^2}{\lambda_n(v,v)_X+\|Sv\|_n^2}.\end{aligned}$$

Let $\rho_1 \leq \rho_2 \leq \cdots \leq \rho_n$ be the eigenvalues of the problem

$$(\psi, v)_X = \rho(S\psi, Sv)_n \quad \forall v \in V_n, \tag{2.13}$$

with the corresponding eigenfunctions $\{\psi_k\}_{k=1}^n$, which is an orthonormal basis of V_n under the inner product $(S\cdot, S\cdot)_n$. Thus $(S\psi_k, S\psi_l)_n = \delta_{kl}$ and, consequently, $(\psi_k, \psi_l)_X = \rho_k \delta_{kl}$, $k, l = 1, 2, \ldots, n$.

Now for any $v \in V_n$, we have the expansion $v(x) = \sum_{k=1}^n v_k \psi_k(x)$, where $v_k = (Sv, S\psi_k)_n$ for $k = 1, 2, \ldots, n$. Thus $|||v|||_{\lambda_n}^2 = \sum_{k=1}^n (\lambda_n \rho_k + 1) v_k^2$. By the Cauchy–Schwarz inequality we can readily get

$$
\begin{aligned}
(e, Sv)_n^2 &= \frac{1}{n^2} \sum_{i=1}^n e_i \left(\sum_{k=1}^n v_k \psi_k(x_i) \right) = \frac{1}{n^2} \sum_{k=1}^n v_k \left(\sum_{i=1}^n e_i \psi_k(x_i) \right) \\
&\le \frac{1}{n^2} \sum_{k=1}^n (1 + \lambda_n \rho_k) v_k^2 \cdot \sum_{k=1}^n (1 + \lambda_n \rho_k)^{-1} \left(\sum_{i=1}^n e_i (S\psi_k)(x_i) \right)^2 .
\end{aligned}
$$

This, along with the fact that $\|S\psi_k\|_n = 1$, implies

$$
\begin{aligned}
\mathbb{E} \left[\sup_{v \in V_n} \frac{(e, Sv)_n^2}{|||v|||_{\lambda_n}^2} \right] &\le \frac{1}{n^2} \sum_{k=1}^n (1 + \lambda_n \rho_k)^{-1} \mathbb{E} \left(\sum_{i=1}^n e_i (S\psi_k)(x_i) \right)^2 \\
&\le \sigma^2 n^{-1} \sum_{k=1}^n (1 + \lambda_n \rho_k)^{-1},
\end{aligned}
$$

where we have used in the last estimate the fact that the random variables $\{e_i\}_{i=1}^n$ are independent and identically distributed, i.e., $\mathbb{E}[e_i e_j] = \delta_{ij}$.

Now by Assumption 2.1 we readily derive

$$
\mathbb{E} \left[\sup_{v \in X} \frac{(e, Sv)_n^2}{|||v|||_{\lambda_n}^2} \right] \le C\sigma^2 n^{-1} \sum_{k=1}^n (1 + \lambda_n k^\alpha)^{-1} \le C\sigma^2 n^{-1} \int_1^\infty (1 + \lambda_n t^\alpha)^{-1} dt.
$$

It is easy to see that

$$
\int_1^\infty (1 + \lambda_n t^\alpha)^{-1} dt = \lambda_n^{-1/\alpha} \int_{\lambda_n^{1/\alpha}}^\infty (1 + s^\alpha)^{-1} ds \le C\lambda_n^{-1/\alpha}.
$$

This completes the proof by using (2.12). ∎

We can observe that Theorem 2.3 presents the expectational convergence of the output error $Sf_n - Sf^*$, but only the expectational boundedness of the source error $f_n - f^*$ in the X-norm. Next, we shall show that we can achieve the expectational convergence of the source error $f_n - f^*$ in a weaker topology. To do so, we consider the eigensystem $\{(\lambda_k = \eta_k^{-1}, \phi_k)\}_{k=1}^\infty$ of the compact operator $S^*S : X \to X$ satisfying (2.6) and define a subspace of X:

$$
W = \left\{ v \in X : \ \sum_{k=1}^\infty \eta_k^{1/2} (v, \phi_k)_X^2 < \infty \right\} \tag{2.14}
$$

with the norm $\|v\|_W := (\sum_{k=1}^\infty \eta_k^{1/2} (v, \phi_k)_X^2)^{1/2}$ for all $v \in W$. One can see that $W = R[(S^*S)^{1/4}]$, i.e., the range of the operator $(S^*S)^{1/4}$. We recall that for $\theta > 0$, $X_\theta = R[(S^*S)^\theta]$ is called the source sets in the literature [14, p. 58].

COROLLARY 2.4. *Let Assumption* 2.1 *be satisfied and let* $\lambda_n \ge n^{-\beta}$ *for all* $n \ge 1$. *Then*

$$
\mathbb{E}\big[\|f_n - f^*\|_{W'}^2\big] \le C\lambda_n^{1/2} \|f^*\|_X^2 + C\sigma^2/(n\lambda_n^{1/2+1/\alpha}),
$$

where W' is the dual space of W with respect to X.

Proof. By (2.6), for any $v \in X$, we have the expansion $v = \sum_{k=1}^{\infty} v_k \phi_k$ with $v_k = (v, \phi_k)_X$. We can directly check that $\|v\|_X^2 = \sum_{k=1}^{\infty} v_k^2$ and $\|Sv\|_{L^2(\Omega)}^2 = \sum_{k=1}^{\infty} \eta_k^{-1} v_k^2$. Then for any $g \in X$, $g = \sum_{k=1}^{\infty} g_k \phi_k$, with $g_k = (g, \phi_k)_X$, we can obtain by the Cauchy–Schwarz inequality that

$$\|g\|_{W'} = \sup_{0 \neq v \in Z} \frac{|(g,v)_X|}{\|g\|_W} = \sup_{0 \neq g \in Z} \frac{|\sum_{k=1}^{\infty} g_k v_k|}{\|g\|_W} \leq \left(\sum_{k=1}^{\infty} \eta_k^{-1/2} v_k^2 \right)^{1/2} \leq \|Sv\|_{L^2(\Omega)}^{1/2} \|v\|_X^{1/2}.$$

Taking $g = f^* - f_n$ in the above inequality, we obtain

$$\|f^* - f_n\|_{W'}^2 \leq \|Sf^* - Sf_n\|_{L^2(\Omega)} \|f^* - f_n\|_X. \tag{2.15}$$

From Assumption 2.1 (1), the boundedness of the operator $S : X \to Y$, and the assumption that $\lambda_n \geq n^{-\beta}$, we deduce

$$\begin{aligned} \|Sf^* - Sf_n\|_{L^2(\Omega)}^2 &\leq C(\|Sf^* - Sf_n\|_n^2 + n^{-\beta} \|f^* - f_n\|_X^2) \\ &\leq C(\|Sf^* - Sf_n\|_n^2 + \lambda_n \|f^* - f_n\|_X^2). \end{aligned}$$

Using this estimate, we derive from (2.15) that

$$\|f^* - f_n\|_{W'}^2 \leq C\lambda_n^{1/2} \|f^* - f_n\|_X^2 + C\lambda_n^{-1/2} \|Sf^* - Sf_n\|_n^2, \tag{2.16}$$

which, together with (2.9) and (2.10), completes the proof of the corollary. □

2.2. Stochastic convergence for noisy data being sub-Gaussian random variables. We consider in this section the case (R2) for the data (2.1), that is,

$$\mathbb{E}\Big[\exp(\lambda(e_i - \mathbb{E}[e_i]))\Big] \leq \exp\Big(\frac{1}{2}\sigma^2\lambda^2\Big) \quad \forall \lambda \in \mathbb{R}, \tag{2.17}$$

and study the stochastic convergence of the error $\|Sf^* - Sf_n\|_n$ and $\|f^* - f_n\|_{W'}$.

We first give a brief introduction of sub-Gaussian random variables and the theory of empirical processes that will be used in our subsequent analysis; see [12, 43, 42] for more details. The probability distribution function of a sub-Gaussian random variable Z has an exponentially decaying tail, that is,

$$\mathbb{P}(|Z - \mathbb{E}[Z]| \geq z) \leq 2\exp\Big(-\frac{z^2}{2\sigma^2}\Big) \quad \forall z > 0. \tag{2.18}$$

We shall also use the Orlicz norm. For a monotonically increasing convex function ψ satisfying $\psi(0) = 0$, the Orlicz norm $\|Z\|_\psi$ of a random variable Z is defined as

$$\|Z\|_\psi = \inf\left\{C > 0 : \mathbb{E}\left[\psi\left(\frac{|X|}{C}\right)\right] \leq 1\right\}. \tag{2.19}$$

For most of our analyses, we will use the Orlicz norm $\|Z\|_{\psi_2}$, with $\psi_2(t) = e^{t^2} - 1$ for $t > 0$. Through some calculations, we have the estimate (see, e.g., [12, (4.5)])

$$\mathbb{P}(|Z| \geq z) \leq 2\exp\left(-\frac{z^2}{\|Z\|_{\psi_2}^2}\right) \quad \forall z > 0. \tag{2.20}$$

Consider a semimetric space $\mathbb{T}$ with a semimetric d and the random process $\{Z_t : t \in \mathbb{T}\}$ indexed by $\mathbb{T}$. The random process $\{Z_t : t \in \mathbb{T}\}$ is called sub-Gaussian if

$$\mathbb{P}(|Z_s - Z_t| > z) \leq 2\exp\left(-\frac{z^2}{2\,\mathsf{d}(s,t)^2}\right) \quad \forall\, s,t \in \mathbb{T}, \quad z > 0. \tag{2.21}$$

For a semimetric space $(\mathbb{T}, \mathsf{d})$ and $\varepsilon > 0$, the covering number $N(\varepsilon, \mathbb{T}, \mathsf{d})$ is the minimum number of ε-balls that cover $\mathbb{T}$, and $\log N(\varepsilon, \mathbb{T}, \mathsf{d})$ is called the covering entropy that is a crucial quantity to characterize the complexity of space $\mathbb{T}$. We assume the following.

ASSUMPTION 2.2. *For a unit ball SY in Y and any $\varepsilon > 0$, there exists a constant $\gamma < 2$ such that the covering entropy is controlled by*

$$\log N(\varepsilon, SY, \|\cdot\|_{L^\infty(\Omega)}) \leq C\varepsilon^{-\gamma}\,.$$

Important estimates of the covering entropy for Sobolev spaces can be found in [8]. We shall often need the following maximal inequality [43, section 2.2.1].

LEMMA 2.5. *If $\{Z_t : t \in \mathbb{T}\}$ is a separable sub-Gaussian random process, then it holds for some constant $K > 0$ that*

$$\|\sup_{s,t\in\mathbb{T}} |Z_s - Z_t|\|_{\psi_2} \leq K \int_0^{\operatorname{diam}\mathbb{T}} \sqrt{\log N\Big(\frac{\varepsilon}{2}, \mathbb{T}, \mathsf{d}\Big)}\; d\varepsilon\,.$$

The useful results in the following two lemmas can be found in [12].

LEMMA 2.6. *$\{E_n(f) := (e, Sf)_n : f \in X\}$ is a sub-Gaussian random process with respect to the semidistance $\mathsf{d}(f,v) = \sigma n^{-1/2}\|Sf - Sv\|_n$ for any $f, v \in X$.*

LEMMA 2.7. *Let $C_1 > 0$ and $K_1 > 0$ be two constants, and let Z be any random variable satisfying*

$$\mathbb{P}(|Z| > \alpha(1+z)) \leq C_1 \exp\left(-\frac{z^2}{K_1^2}\right) \quad \forall\, \alpha > 0, \quad z \geq 1\,.$$

Then there exists a constant $C(C_1, K_1) > 0$ depending on C_1 and K_1 such that

$$\|Z\|_{\psi_2} \leq C(C_1, K_1)\,\alpha\,.$$

THEOREM 2.8. *Let Assumption 2.2 be fulfilled, let $\rho_0 = \|f^*\|_X + \sigma n^{-1/2}$, and let $f_n \in X$ be the solution of problem (2.2). If we take $\lambda_n^{1/2+\gamma/4} = O(\sigma n^{-1/2}\rho_0^{-1})$, then there exists a constant $C > 0$ such that*

$$\mathbb{P}(\|Sf_n - Sf^*\|_n \geq \lambda_n^{1/2}\rho_0 z) \leq 2\,e^{-Cz^2} \quad \textit{and} \quad \mathbb{P}(\|f_n\|_X \geq \rho_0 z) \leq 2\,e^{-Cz^2}.$$

Proof. By using the estimate (2.20), it suffices to prove

$$\|\|Sf_n - Sf^*\|_n\|_{\psi_2} \leq C\lambda_n^{1/2}\rho_0 \quad \text{and} \quad \|\|f_n\|_X\|_{\psi_2} \leq C\rho_0. \tag{2.22}$$

Because of their similarity, we will prove only the first estimate in (2.22) by the peeling argument. It follows from (2.2) that

$$\|Sf_n - Sf^*\|_n^2 + \lambda_n\|f_n\|_X^2 \leq 2(e, Sf_n - Sf^*)_n + \lambda_n\|f^*\|_X^2. \tag{2.23}$$

Let $\delta > 0,\ \rho > 0$ be two constants to be determined later, and we set for $i, j \geq 1$

$$A_0 = [0, \delta), \quad A_i = [2^{i-1}\delta, 2^i\delta), \quad B_0 = [0, \rho), \quad B_j = [2^{j-1}\rho, 2^j\rho)\,. \tag{2.24}$$

For $i, j \geq 0$, we further define

$$F_{ij} = \{v \in X : \ \|Sv\|_n \in A_i\ ,\ \|v\|_X \in B_j\}.$$

Then we can readily see

$$\mathbb{P}(\|Sf_n - Sf^*\|_n > \delta) \leq \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} \mathbb{P}(f_n - f^* \in F_{ij}). \tag{2.25}$$

Now we estimate $\mathbb{P}(f_n - f^* \in F_{ij})$ for each pair $\{i, j\}$. By Lemma 2.6, we know $\{(e, Sv)_n : v \in X\}$ is a sub-Gaussian random process with respect to the semidistance $\mathsf{d}(f, v)$. With this semidistance, it is easy to see that $\operatorname{diam}(F_{ij}) \leq 2\sigma n^{-1/2} \cdot 2^i\delta$. Then we can deduce by using Lemma 2.5 that

$$\begin{aligned}\| \sup_{f-f^*\in F_{ij}} |(e, Sf - Sf^*)_n|\|_{\psi_2} &\leq K \int_0^{\sigma n^{-1/2}\cdot 2^{i+1}\delta} \sqrt{\log N\left(\frac{\varepsilon}{2}, F_{ij}, \mathsf{d}\right)}\, d\varepsilon \\ &= K \int_0^{\sigma n^{-1/2}\cdot 2^{i+1}\delta} \sqrt{\log N\left(\frac{\varepsilon}{2\sigma n^{-1/2}}, F_{ij}, \|S\cdot\|_n\right)}\, d\varepsilon.\end{aligned}$$

By Assumption 2.2, we have the estimate for the covering entropy,

$$\begin{aligned}&\log N\left(\frac{\varepsilon}{2\sigma n^{-1/2}}, F_{ij}, \|S\cdot\|_n\right) \leq \log N\left(\frac{\varepsilon}{2\sigma n^{-1/2}}, F_{ij}, \|S\cdot\|_{L^\infty(\Omega)}\right) \\ &= \log N\left(\frac{\varepsilon}{2\sigma n^{-1/2}}, S(F_{ij}), \|\cdot\|_{L^\infty(\Omega)}\right) \leq C\left(\frac{2\sigma n^{-1/2}\cdot 2^j\rho}{\varepsilon}\right)^{\gamma},\end{aligned}$$

where we have used the fact that $S(F_{ij})$ is included in the ball in Y of radius $C(2^j\rho)$ since $S : X \to Y$ is a bounded operator. Using this, we can further derive

$$\begin{aligned}\| \sup_{f-f^*\in F_{ij}} |(e, Sf - Sf^*)_n\|_{\psi_2} &\leq K \int_0^{\sigma n^{-1/2}\cdot 2^{i+1}\delta} \left(\frac{2\sigma n^{-1/2}\cdot 2^j\rho}{\varepsilon}\right)^{\gamma/2} d\varepsilon \\ &= C\sigma n^{-1/2}(2^j\rho)^{\gamma/2}(2^i\delta)^{1-\gamma/2}.\end{aligned} \tag{2.26}$$

Then by using the estimates (2.23) and (2.20), we have for $i, j \geq 1$,

$$\begin{aligned}\mathbb{P}(f_n - f^* \in F_{ij}) &\leq \mathbb{P}\left(2^{2(i-1)}\delta^2 + \lambda_n 2^{2(j-1)}\rho^2 \leq 2 \sup_{f-f^*\in F_{ij}} |(e, f - f^*)_n| + \lambda_n\rho_0^2\right) \\ &= \mathbb{P}\left(2 \sup_{f-f^*\in F_{ij}} |(e, Sf - Sf^*)_n| \geq 2^{2(i-1)}\delta^2 + \lambda_n 2^{2(j-1)}\rho^2 - \lambda_n\rho_0^2\right) \\ &\leq 2\exp\left[-\frac{1}{C\sigma^2 n^{-1}}\left(\frac{2^{2(i-1)}\delta^2 + \lambda_n 2^{2(j-1)}\rho^2 - \lambda_n\rho_0^2}{(2^i\delta)^{1-\gamma/2}(2^j\rho)^{\gamma/2}}\right)^2\right].\end{aligned}$$

Now for $z \geq 1$, we take $\delta^2 = \lambda_n\rho_0^2(1+z)^2,\ \rho = \rho_0$. Then with the choice that $\lambda_n^{\frac{1}{2}+\frac{\gamma}{4}} = O(\sigma n^{-1/2}\rho_0^{-1})$ and by direct computing, we readily obtain for $i, j \geq 1$ that

$$\mathbb{P}(f_n - f^* \in F_{ij}) \leq 2\exp\left[-C\left(\frac{2^{2(i-1)}z(1+z) + 2^{2(j-1)}}{(2^i(1+z))^{1-\gamma/2}(2^j)^{\gamma/2}}\right)^2\right]. \tag{2.27}$$

To simplify the above estimate, we use Young's inequality that $ab \le a^p/p + b^q/q$ for any $a, b > 0$ and $p, q > 1$ such that $p^{-1} + q^{-1} = 1$ to obtain

$$(2^i(1+z))^{1-\gamma/2}(2^j)^{\gamma/2} \le C((1+z)2^i + 2^j).$$

Therefore we get from (2.27) for $i, j \ge 1$ that

$$\mathbb{P}(f_n - f^* \in F_{ij}) \le 2\exp\left[-C(2^{2i}z^2 + 2^{2j})\right].$$

Similarly, one can show for $i \ge 1, j = 0$ that

$$\mathbb{P}(f_n - f^* \in F_{i0}) \le 2\exp\left[-C(2^{2i}z^2)\right].$$

Collecting the above estimates for all $i, j \ge 0$ and using the facts that

$$\sum_{j=1}^{\infty} \exp\big(-C(2^{2j})\big) \le \exp(-C) < 1 \quad \text{and} \quad \sum_{i=1}^{\infty} \exp\big(-C(2^{2i}z^2)\big) \le \exp(-Cz^2),$$

we come to the conclusion that

$$\sum_{i=1}^{\infty}\sum_{j=0}^{\infty} \mathbb{P}(f_n - f^* \in F_{ij}) \le 2\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \exp(-C(2^{2i}z^2 + 2^{2j})) + 2\sum_{i=1}^{\infty} \exp(-C(2^{2i}z^2)).$$

The above estimate can be further bounded by $4\exp(-Cz^2)$. Using this, we get from (2.25) that

$$\mathbb{P}(\|Sf_n - Sf^*\|_n > \lambda_n^{1/2}\rho_0(1+z)) \le 4\exp(-Cz^2) \quad \forall z \ge 1. \tag{2.28}$$

This, along with Lemma 2.7, implies that $\|\|Sf_n - Sf^*\|_n\|_{\psi_2} \le C\lambda_n^{1/2}\rho_0$, which is the first estimate in (2.22). The second estimate is similar to the first one by taking $i \ge 0$ and $j \ge 1$ in the summation above (2.28). ∎

Using the subspace W defined in (2.14), we can derive the following stochastic convergence of the error $\|f_n - f^*\|_{W'}$.

COROLLARY 2.9. *Let Assumptions* 2.1 *and* 2.2 *be satisfied. If* $\lambda_n \ge n^{-\beta}$, *we have*

$$\mathbb{P}(\|f_n - f^*\|_{W'} \ge \lambda_n^{1/4}\rho_0 z) \le 2\, e^{-Cz^2}.$$

Proof. By (2.16) and (2.22), we readily deduce

$$\|\|f^* - f_n\|_{W'}\|_{\psi_2} \le C\lambda_n^{1/4}\|\|f^* - f_n\|_X\|_{\psi_2} + C\lambda_n^{-1/4}\|\|Sf^* - Sf_n\|_n\|_{\psi_2} \le C\rho_0\lambda_n^{1/4}.$$

Then the desired estimate is a direct consequence of (2.20). ∎

2.3. Convergence of the discrete solutions. In this section we consider the approximation to the optimal control problem (2.2), i.e.,

$$\min_{f \in X} \|Sf - m\|_n^2 + \lambda_n\|f\|_X^2.$$

We can directly verify that the solution $f_n \in X$ satisfies the weak formulation

$$\lambda_n(f_n, v)_X + (Sf_n, Sv)_n = (m, Sv)_n \quad \forall v \in X. \tag{2.29}$$

Let $V_h \subset X$ and $Y_h \subset C(\bar{\Omega})$ be two discrete function spaces (e.g., finite element spaces) with dimensions N_h and M_h, respectively, and let $S_h : X \to Y_h$ be the discrete approximation of the operator $S : X \to Y$. We make the following standard assumptions on the discretization space V_h and the approximation operator S_h.

ASSUMPTION 2.3. *For the discrete operator $S_h : X \to Y_h$, the following hold:*
(1) *There exists an error estimate $e(h)$ such that the discrete operator S_h satisfies*

$$\|Sf - S_h f\|_n^2 \le Ce(h)\|f\|_X^2 \quad \forall\, f \in X\,.$$

(2) *For any $f \in X$, there exists $v_h \in V_h$ such that*

$$\lambda_n\|f - v_h\|_X^2 + \|S_h f - S_h v_h\|_n^2 \le C(\lambda_n + e(h))\|f\|_X^2.$$

We can now look for the discrete solution to problem (2.2):

$$\min_{f_h \in V_h} \|S_h f_h - m\|_n^2 + \lambda_n\|f_h\|_X^2.$$

Obviously, f_h satisfies the weak formulation:

$$\lambda_n(f_h, v_h)_X + (S_h f_h, S_h v_h)_n = (m, S_h v_h)_n \quad \forall v_h \in V_h. \tag{2.30}$$

2.3.1. Convergence for noisy data from random variables with bounded variance. We study in this section the expectational convergence of the discrete solution to (2.30) in the case (R1) for the data (2.1), with the main results stated below.

THEOREM 2.10. *Let Assumptions* 2.1 *and* 2.3 *be fulfilled, and let $f_h \in V_h$ be the solution of* (2.30). *Then there exist constants $\lambda_0 > 0$ and $C > 0$ such that for any $\lambda_n \le \lambda_0$,*

$$\mathbb{E}\big[\|Sf^* - S_h f_h\|_n^2\big] \le C(\lambda_n + e(h))\|f^*\|_X^2 + C\left[1 + \frac{e(h)}{\lambda_n} + \frac{N_h e(h)}{\lambda_n^{1-1/\alpha}}\right]\frac{\sigma^2}{n\lambda_n^{1/\alpha}}\,, \tag{2.31}$$

$$\mathbb{E}\big[\|f^* - f_h\|_X^2\big] \le C\frac{\lambda_n + e(h)}{\lambda_n}\|f^*\|_X^2 + C\left[1 + \frac{e(h)}{\lambda_n} + \frac{N_h e(h)}{\lambda_n^{1-1/\alpha}}\right]\frac{\sigma^2}{n\lambda_n^{1+1/\alpha}}. \tag{2.32}$$

In particular, if $e(h) \le C\lambda_n$ and $N_h e(h) \le C\lambda_n^{1-1/\alpha}$, we have

$$\mathbb{E}\big[\|Sf^* - S_h f_h\|_n^2\big] \le C\lambda_n\|f^*\|_X^2 + C\sigma^2/(n\lambda_n^{1/\alpha}), \tag{2.33}$$

$$\mathbb{E}\big[\|f^* - f_h\|_X^2\big] \le C\|f^*\|_X^2 + C\sigma^2/(n\lambda_n^{1+1/\alpha}). \tag{2.34}$$

Proof. For any $f, v \in X$, we denote $a_h(f,v) = \lambda_n(f,v)_X + (S_h f, S_h v)_n$ and $\|f\|_{a_h}^2 = a_h(f,f)$. For any $w_h \in V_h$, by taking $v = w_h$ in (2.29) and $v_h = w_h$ in (2.30), we readily obtain

$$\begin{aligned} a_h(f_h - v_h, w_h) &= a_h(f_n - v_h, w_h) + ((S - S_h)f_n, S_h w_h)_n + (Sf^* - Sf_n, (S_h - S)w_h)_n \\ &\quad + (e, (S_h - S)w_h)_n :\equiv a_h(f_n - v_h, w_h) + F(w_h) \quad \forall v_h, w_h \in V_h. \end{aligned}$$

By the triangle inequality, we can further derive

$$\|f_n - f_h\|_{a_h} \le C \inf_{v_h \in V_h} \|f_n - v_h\|_{a_h} + C \sup_{w_h \in V_h} \frac{|F(w_h)|}{\|w_h\|_{a_h}}\,. \tag{2.35}$$

But from Assumption 2.3 (1), we have

$$\sup_{w_h \in V_h} \frac{|((S - S_h)f_n, S_h w_h)_n|}{\|w_h\|_{a_h}} \le \|Sf_n - S_h f_n\|_n \le Ce(h)^{1/2}\|f_n\|_X, \tag{2.36}$$

$$\sup_{w_h \in V_h} \frac{|(Sf^* - Sf_n, (S_h - S)w_h)_n|}{\|w_h\|_{a_h}} \le C\|Sf^* - Sf_n\|_n \frac{e(h)^{1/2}}{\lambda_n^{1/2}}. \tag{2.37}$$

Now we estimate $\mathbb{E}(\sup_{w_h\in V_h}|(e,Sw_h-S_hw_h)_n|^2/\|w_h\|_{a_h}^2)$. Let $\{\psi_k\}_{k=1}^{N_h}$ be the orthogonal basis of V_h (with $N_h=\dim(V_h)$) such that $(\psi_i,\psi_j)=\delta_{ij}$. Then for any $w_h\in V_h$, we have $w_h=\sum_{j=1}^{N_h}(w_h,\psi_j)\psi_j$, and $\|w_h\|^2_{L^2(\Omega)}=\sum_{j=1}^{N_h}(w_h,\psi_j)^2$. Applying the Cauchy–Schwarz inequality,

$$
\begin{aligned}
(e,(S-S_h)w_h)_n^2 &\le \frac{1}{n^2}\sum_{j=1}^{N_h}(w_h,\psi_j)^2\sum_{j=1}^{N_h}\left(\sum_{i=1}^n e_i(S-S_h)\psi_j(x_i)\right)^2\\
&= \frac{1}{n^2}\|w_h\|^2_{L^2(\Omega)}\sum_{j=1}^{N_h}\left(\sum_{i=1}^n e_i(S-S_h)\psi_j(x_i)\right)^2,
\end{aligned}
$$

we derive

$$
\begin{aligned}
(2.38)\ \mathbb{E}\left(\sup_{w_h\in V_h}\frac{|(e,Sw_h-S_hw_h)_n|^2}{\|w_h\|^2_{a_h}}\right) &\le \frac{1}{\lambda_n n^2}\sum_{j=1}^{N_h}\mathbb{E}\left(\sum_{i=1}^n e_i(S-S_h)\psi_j(x_i)\right)^2\\
&= \frac{1}{\lambda_n n}\sum_{j=1}^{N_h}\sigma^2\|(S-S_h)\psi_j\|_n^2 \le C\frac{\sigma^2}{\lambda_n n}N_h e(h).
\end{aligned}
$$

This completes the desired estimates by substituting (2.36), (2.37), (2.38) into (2.35) and using Assumption 2.3 (2) and Theorem 2.3. ∎

Corollary 2.11. *Let W be defined as in* (2.14). *Then it holds under Assumptions* 2.1–2.3 *and $\lambda_n\ge n^{-\beta}$ that*

$$
\begin{aligned}
\mathbb{E}\big[\|f^*-f_h\|^2_{W'}\big] &\le C(\lambda_n^{1/2}+e^{1/2}(h))\frac{\lambda_n+e(h)}{\lambda_n}\|f^*\|_X^2\\
&\quad + C(\lambda_n^{1/2}+e^{1/2}(h))\left[1+\frac{e(h)}{\lambda_n}+\frac{N_he(h)}{\lambda_n^{1-1/\alpha}}\right]\frac{\sigma^2}{n\lambda_n^{1/\alpha}}.
\end{aligned}
$$

Moreover, if $e(h)\le C\lambda_n$ and $N_he(h)\le C\lambda_n^{1-1/\alpha}$, it holds that

$$
\mathbb{E}\big[\|f^*-f_h\|^2_{W'}\big]\le C\lambda_n^{1/2}\|f^*\|_X^2+C\sigma^2/(n\lambda_n^{1/2+1/\alpha}).
$$

Proof. By (2.15) and Assumption 2.1 (1), we can derive that

$$
\begin{aligned}
\|f^*-f_h\|^2_{W'} &\le \|Sf^*-Sf_h\|_{L^2(\Omega)}\|f^*-f_h\|_X\\
&\le C\left(\|Sf^*-Sf_h\|_n+n^{-\beta/2}\|f^*-f_h\|_X\right)\|f^*-f_h\|_X\\
&\le C\left(\|Sf^*-S_hf_h\|_n+\|S_hf_h-Sf_h\|_n+\lambda_n^{1/2}\|f^*-f_h\|_X\right)\|f^*-f_h\|_X.
\end{aligned}
$$

Then the corollary follows by applying the estimates (2.31), (2.32) and Assumption 2.3 (1) to the above estimate. ∎

2.3.2. Convergence for noisy data being sub-Gaussian random variables. We consider in this subsection the convergence of the discrete solution in the case (R2) for the data (2.1). We start by recalling the following lemma in [42, Corollary 2.6] about the estimation of the covering entropy of finite-dimensional subsets.

Lemma 2.12. *Let G be a finite-dimensional subspace of X of dimension $N_G>0$ and $G_R=\{f\in G:\|f\|_X\le R\}$. Then it holds that*

$$
N(\varepsilon,G_R,\|\cdot\|_X)\le(1+4R/\varepsilon)^{N_G}\quad\forall\varepsilon>0.
$$

LEMMA 2.13. *Let Assumption* 2.3 *be fulfilled, and let* $G_h := \{w_h \in V_h : \|w_h\|_{a_h} \leq 1\}$. *Assume that* $e(h) \leq C\lambda_n$ *and* $N_h e(h) \leq C\lambda_n^{1-\gamma/2}$. *Then it holds that*

$$\| \sup_{w_h \in G_h} |(e, Sw_h - S_h w_h)_n| \, \|_{\psi_2} \leq C\sigma n^{-1/2}\lambda_n^{-\gamma/4}.$$

Proof. By Lemma 2.6 we know that $\{\hat{E}_n(v_h) := (e, Sw_h - S_h w_h)_n \ \forall w_h \in G_h\}$ is a sub-Gaussian random process with respect to the semidistance $\hat{\mathsf{d}}(v_h, w_h) = \sigma n^{-1/2}\|(Sv_h - S_h v_h) - (Sw_h - S_h w_h)\|_n$. By Assumption 2.3 and the condition that $e(h) \leq C\lambda_n$, we derive for any $w_h \in G_h$ that $\|Sw_h - S_h w_h\|_n \leq Ce^{1/2}(h)\|w_h\|_X \leq Ce^{1/2}(h)\lambda_n^{-1/2} \leq C$. This implies that the diameter of G_h is bounded by $C\sigma n^{-1/2}$. Now we deduce by the maximal inequality in Lemma 2.5 that

$$\| \sup_{w_h \in G_h} |(e, Sw_h - S_h w_h)_n| \, \|_{\psi_2} \leq K \int_0^{C\sigma n^{-1/2}} \sqrt{\log N\left(\frac{\varepsilon}{2}, G_h, \hat{\mathsf{d}}\right)}\, d\varepsilon. \tag{2.39}$$

By Assumption 2.3, we know

$$\hat{\mathsf{d}}(v_h, w_h) \leq C\sigma n^{-1/2} e^{1/2}(h)\|v_h - w_h\|_X \quad \forall v_h, w_h \in V_h.$$

Thus we can see that

$$\log N\left(\frac{\varepsilon}{2}, G_h, \hat{\mathsf{d}}\right) = \log N\left(\frac{\varepsilon}{C\sigma n^{-1/2}e^{1/2}(h)}, G_h, \|\cdot\|_X\right). \tag{2.40}$$

Now we estimate the covering entropy of G_h. First, we have $\|w_h\|_X \leq \lambda_n^{-1/2}$ for any $w_h \in G_h$. Noting the dimension N_h of V_h, we obtain by Lemma 2.12 and (2.40) that

$$\log N\left(\frac{\varepsilon}{2}, G_h, \hat{\mathsf{d}}\right) \leq CN_h(1 + \sigma n^{-1/2}e^{1/2}(h)\lambda_n^{-1/2}/\varepsilon).$$

Inserting this estimate into (2.39), we obtain

$$\begin{aligned} \| \sup_{v_h \in G_h} |(e, \hat{v}_h - \Pi_h v_h)_n| \, \|_{\psi_2} &\leq C \int_0^{C\sigma n^{-1/2}} \sqrt{CN_h(1 + \sigma n^{-1/2}e^{1/2}(h)\lambda_n^{-1/2}/\varepsilon)}\, d\varepsilon \\ &\leq C\sqrt{N_h}\sigma n^{-1/2}e^{1/2}(h)\lambda_n^{-1/2}. \end{aligned}$$

This completes the proof using the condition that $N_h e(h) \leq C\lambda_n^{1-\gamma/2}$. ∎

The following theorem presents the main results of this subsection, where W is the subspace defined in (2.14).

THEOREM 2.14. *Let Assumptions* 2.2 *and* 2.3 *be fulfilled, and let* $f_h \in V_h$ *be the solution of* (2.30). *Denote* $\rho_0 = \|f^*\|_X + \sigma n^{-1/2}$. *If we take* $e(h) \leq C\lambda_n$, $N_h e(h) \leq C\lambda_n^{1-\gamma/2}$, *and* $\lambda_n^{1/2+\gamma/4} = O(\sigma n^{-1/2}\rho_0^{-1})$, *then there exists a constant* $C > 0$ *such that for any* $z > 0$,

$$\mathbb{P}(\|S_h f_h - Sf^*\|_n \geq \lambda_n^{1/2}\rho_0 z) \leq 2e^{-Cz^2} \quad \text{and} \quad \mathbb{P}(\|f_h\|_X \geq \rho_0 z) \leq 2e^{-Cz^2}.$$

Moreover, if Assumption 2.1 *is satisfied and* $\lambda_n \geq n^{-\beta}$, *it holds that*

$$\mathbb{P}(\|f_h - f^*\|_{W'} \geq \lambda_n^{1/4}\rho_0 z) \leq 2e^{-Cz^2}.$$

Proof. We first derive from (2.35) that

$$\big\|\|f_n - f_h\|_{a_h}\big\|_{\psi_2} \le C\Big\|\inf_{v_h\in V_h}\|f_n - v_h\|_{a_h}\Big\|_{\psi_2} + C\Big\|\sup_{w_h\in V_h}\frac{|F(w_h)|}{\|w_h\|_{a_h}}\Big\|_{\psi_2}.$$

But we know $\sup_{w_h\in V_h}|F(w_h)|/\|w_h\|_{a_h} = \sup_{w_h\in G_h}|F(w_h)|$ from the proof of Theorem 2.10, and hence it suffices to estimate $\|\sup_{w_h\in G_h}|(e, Sw_h - S_h w_h)_n|\|_{\psi_2}$. Then the first two estimates of the theorem follow readily from (2.22), Lemma 2.13, the assumption that $\sigma n^{-1/2} = O(\lambda_n^{1/2+\gamma/4}\rho_0)$, and (2.20).

To show the estimate of $\|f^* - f_h\|_{W'}$, we use the last inequality in the proof of Corollary 2.11 to obtain

$$\|f^* - f_h\|_{W'} \le C\lambda_n^{1/4}\|f^* - f_h\|_X + C\lambda_n^{-1/4}(\|Sf^* - S_h f_h\|_n + \|S_h f_h - Sf_h\|_n).$$

Then the desired estimate follows by using (2.20). We omit the details. ∎

3. An inverse nonstationary source problem. In this section, we apply the theory developed in the previous section to study the regularized solutions to an inverse nonstationary source problem associated with the heat conduction system

$$\begin{cases} u_t + Lu = F(x,t) & \text{in } \Omega\times(0,T),\\ u(x,t) = 0 \ \text{ on } \partial\Omega\times(0,T), & u(x,0) = 0 \ \text{ in } \Omega, \end{cases} \tag{3.1}$$

where L is a second order elliptic operator of the form $Lu = -\nabla\cdot(a(x)\nabla u) + c(x)u$, and $\Omega\subset\mathbb{R}^d$ $(d = 1,2,3)$ is a bounded domain with C^2 boundary or a convex polyhedral domain. We assume $a\in C^1(\bar\Omega)$, $c\in C(\bar\Omega)$ with $c(x)\ge 0$ in Ω, and that the source is of the separable form $F(x,t) = f(x)g(t)$ for $(x,t)\in\Omega\times(0,T)$, where the temporal component $g\in H^1(0,T)$ is known and satisfies that $g(t)\ge 0$ for all $t\in(0,T)$, while $f(x)$ is an unknown to be recovered.

For the subsequent analysis, we first recall some standard results for parabolic equations (cf., e.g., [17, section 7.1]). For $F\in H^1(0,T;L^2(\Omega))$, we know the solution u to (3.1) satisfies $\partial_t u\in C([0,T];L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))$ and the a priori estimate

$$\|\partial_t u\|_{C([0,T];L^2(\Omega))} \le C\|F\|_{H^1(0,T;L^2(\Omega))} \le C\|f\|_{L^2(\Omega)}.$$

It follows then from (3.1) and the regularity theory of elliptic equations that $u\in C([0,T];H^2(\Omega))$ and there exists a constant C such that

$$\|u\|_{C([0,T];H^2(\Omega))} \le C\|f\|_{L^2(\Omega)}. \tag{3.2}$$

Let $X = L^2(\Omega)$, $Y = H^2(\Omega)$, and the forward operator $S: X\to Y$ be defined by $Sf = u(\cdot,T)$. By (3.2) we know that $S: X\to Y$ is a bounded operator

$$\|Sf\|_{H^2(\Omega)} \le C\|f\|_{L^2(\Omega)} \quad \forall\, f\in L^2(\Omega).$$

We are mainly interested in the following inverse nonstationary source problem:

(TIP) Given the measurement data of $u(\cdot,t)$ at the terminal $t = T$, recover the spatial source distribution $f^*(x)$ in the entire domain Ω.

We focus on an important physical scenario, i.e., measurement data is collected pointwise on a set of distributed sensors located at $\{x_i\}_{i=1}^n$ inside the domain Ω [3, 20, 5, 27, 33, 35, 36]. Again, we assume the data is of the noisy form (2.1), where $\{x_i\}_{i=1}^n$ is quasi-uniformly distributed in the sense of (2.3).

We then look for an approximate solution of the true source f^* through the following least-squares regularized minimization:

$$\min_{f\in X}\|Sf - m\|_n^2 + \lambda_n\|f\|_X^2\,. \tag{3.3}$$

3.1. Stochastic convergence for the inverse heat source problem. In this subsection we apply the results in section 2 to study the stochastic convergence of the solution of problem (3.3) to the exact source f^*. We first recall an important property about the eigenvalue distribution for the elliptic operator L [2, 18].

LEMMA 3.1. *Suppose Ω is a bounded domain in $\mathbb{R}^d$ and $a, c \in C^0(\bar{\Omega})$, $c \geq 0$. Then the eigenvalue problem*

$$L\psi = \mu\,\psi \quad \text{in } \Omega, \quad \psi = 0 \quad \text{on } \partial\Omega \tag{3.4}$$

has a countable set of positive eigenvalues $\mu_1 \leq \mu_2 \leq \cdots$, with its corresponding eigenfunctions $\{\phi_k\}_{k=1}^{\infty}$ forming an orthonormal basis of $L^2(\Omega)$. Moreover, there exist constants $C_1, C_2 > 0$ such that $C_1 k^{2/d} \leq \mu_k \leq C_2 k^{2/d}$ for all $k = 1, 2, \ldots$.

With Lemma 3.1, we can derive the important spectral property of operator S.

THEOREM 3.2. *Let $g \in H^1(0,T)$, $g \geq 0$ but $g \not\equiv 0$ in $(0,T)$. Then the null space $N(S) = \{0\}$ and the eigenvalue problem*

$$(\psi, v) = \rho(S\psi, Sv) \quad \forall\, v \in X \tag{3.5}$$

has a countable set of positive eigenvalues $0 < \rho_1 \leq \rho_2 \leq \cdots$. Moreover, there exists a constant $C > 0$ such that $\rho_k \geq C k^{4/d}$ for all $k = 1, 2, \ldots$.

Proof. We first consider the eigenvalue problem

$$\psi = \eta\, S\psi. \tag{3.6}$$

Let $\{\phi_k\}_{k=1}^{\infty}$ be eigenfunctions of problem (3.4) which forms an orthogonal basis of $L^2(\Omega)$. We write $f = \sum_{k=1}^{\infty} f_k \phi_k$ for a set of coefficients f_k. Let $u = \sum_{k=1}^{\infty} u_k(t)\phi_k$ be the solution of problem (3.1). Plugging these two expressions of f and u into the first equation of (3.1), we get by noting the fact that $L\phi_k = \mu_k \phi_k$ and comparing the coefficients of ϕ_k on both sides of the equation that $u_k(0) = 0$ and

$$u_k'(t) + \mu_k u_k = f_k\, g(t) \quad \text{in } (0,T)\,.$$

We can write the solution as $u_k(T) = \alpha_k\, f_k$, with $\alpha_k = e^{-\mu_k T} \int_0^T e^{\mu_k s} g(s) ds$. Since $g \geq 0$ in $(0,T)$, we know $\alpha_1 \geq \alpha_2 \geq \cdots > 0$. Now if $Sf = 0$ for some $f \in L^2(\Omega)$, then $u_k(T) = \alpha_k f_k = 0$ for all $k \geq 1$, which implies $f_k = 0$ for all $k \geq 1$. Hence $f = 0$, that is, the null space of S is zero.

Moreover, we can easily see that $|\alpha_k| \leq C\mu_k^{-1}$. Noting that $Sf = u(\cdot, T) = \sum_{k=1}^{\infty} u_k(T)\phi_k$, we can formally write

$$S\left(\sum_{k=1}^{\infty} f_k \phi_k\right) = \sum_{k=1}^{\infty} \alpha_k f_k \phi_k.$$

Since $\{\phi_k\}_{k=1}^{\infty}$ is an orthogonal basis of $L^2(\Omega)$, we can readily see that the eigenvalue problem (3.6) has a countable set of positive eigenvalues $\{\eta_k = \alpha_k^{-1}\}_{k=1}^{\infty}$, with $\{\phi_k\}_{k=1}^{\infty}$ being their corresponding eigenfunctions. By Lemma 3.1, we have $\eta_k = \alpha_k^{-1} \geq C\mu_k \geq C_1 k^{2/d}$. Therefore, the eigenvalue problem (3.5) has a countable set of eigenvalues $\{\rho_k\}_{k=1}^{\infty}$ that satisfies $\rho_k = \eta_k^2 \geq C k^{4/d}$. This completes the proof. ∎

Within the setting of this section, the abstract subspace W in (2.14) is given by

$$W = \left\{ v \in L^2(\Omega) : \; v = \sum_{k=1}^{\infty} v_k \phi_k, \; v_k = (v, \phi_k), \text{ and } \sum_{k=1}^{\infty} \rho_k^{1/2} v_k^2 < \infty \right\}, \tag{3.7}$$

and $W' = H^{-1}(\Omega)$ if $g \in H^1(0,T)$ satisfying $g > 0$ in $[0,T]$, as indicated below.

LEMMA 3.3. *Let $g \in H^1(0,T)$, $g \geq 0$ but $g \not\equiv 0$ in $(0,T)$. Then W is a subspace of $H_0^1(\Omega)$ and $\|v\|_{H^1(\Omega)} \leq C_1\|v\|_W$ for all $v \in W$. If, in addition, $g > 0$ in $[0,T]$, then $W = H_0^1(\Omega)$ and $\|v\|_W \leq C_2\|v\|_{H^1(\Omega)}$ for all $v \in H_0^1(\Omega)$.*

Proof. Since the eigenfunctions $\{\phi_k\}_{k=1}^\infty$ form an orthonormal basis of $L^2(\Omega)$, any function $v \in L^2(\Omega)$ can be expanded as $v = \sum_{k=1}^\infty v_k\phi_k$, where $v_k = (v, \phi_k)$. From the definition of $\{\phi_k\}_{k=1}^\infty$ in (3.4), we obtain by integrating by parts that

$$a(\phi_k, q) = \mu_k(\phi_k, q) \quad \forall q \in H_0^1(\Omega),$$

where $a(p,q) = (ap, q) + (cp, q)$. Thus we have $a(\phi_k, \phi_l) = \mu_k\delta_{kl}$, and that $\|v\|_{H^1(\Omega)} \leq C\|v\|_W$, which is a consequence of the estimate by the ellipticity of the operator L:

$$\|v\|_{H^1(\Omega)}^2 \leq Ca(v,v) = \sum_{k=1}^\infty \mu_k v_k^2 \leq C\sum_{k=1}^\infty \rho_k^{1/2}v_k^2.$$

Next, since $g \in H^1(0,T)$, we know $g \in C[0,T]$. Thus if $g > 0$ in $[0,T]$, then $g \geq g_{\min} > 0$ in $[0,T]$ for some constant $g_{\min}$. With the same notation as in the proof of Theorem 3.2, we have

$$\alpha_k = e^{-\mu_k T}\int_0^T e^{\mu_k s}g(s)ds \geq g_{\min}\frac{1-e^{-\mu_k T}}{\mu_k} \geq g_{\min}\frac{1-e^{-\mu_1 T}}{\mu_k} \geq C\mu_k^{-1}.$$

Thus $\mu_k \geq C\alpha_k^{-1} = \rho_k^{1/2}$. This yields $\|v\|_W \leq C\|v\|_{H^1(\Omega)}$. $\square$

Verification of Assumptions 2.1 and 2.2. We first know Assumption 2.1 (1) holds with $\beta = 4/d$ from [41, Theorems 3.3 and 3.4]. This, along with Theorem 3.2, verifies Assumption 2.1 (2) with $\alpha = \beta = 4/d$. Assumption 2.2 (with $\gamma = d/2$) is a consequence of the following important estimate about the covering entropy [8].

LEMMA 3.4. *Let Q be the unit cube in $\mathbf{R}^d$, and let $SW^{s,p}(Q)$ be the unit sphere of space $W^{s,p}(Q)$ for $s > 0$ and $p \geq 1$. Then it holds for sufficiently small $\varepsilon > 0$ that*

$$\log N(\varepsilon, SW^{s,p}(Q), \|\cdot\|_{L^q(Q)}) \leq C\varepsilon^{-d/s},$$

where $1 \leq q \leq \infty$ for $sp > d$, and $1 \leq q \leq q^*$ with $q^* = p(1-sp/d)^{-1}$ for $sp \leq d$.

Under Assumptions 2.1 and 2.2, the following two main results are direct consequences of Theorems 2.3 and Corollary 2.4 for the noisy data of type (R1) (random variables with bounded variance) and Theorem 2.8 and Corollary 2.9 for the noisy data of type (R2) (sub-Gaussian random variables), respectively,

THEOREM 3.5. *For the minimizer $f_n \in L^2(\Omega)$ to problem (3.3), there exist constants $\lambda_0 > 0$ and $C > 0$ such that the following estimates hold for any $\lambda_n \leq \lambda_0$:*

$$\mathbb{E}\big[\|Sf_n - Sf^*\|_n^2\big] \leq C\lambda_n\|f^*\|_{L^2(\Omega)}^2 + C\sigma^2/(n\lambda_n^{d/4}),$$
$$\mathbb{E}\big[\|f_n\|_{L^2(\Omega)}^2\big] \leq C\|f^*\|_{L^2(\Omega)}^2 + C\sigma^2/(n\lambda_n^{1+d/4}).$$

Moreover, if $\lambda_n \geq n^{-4/d}$ and $g > 0$ in $[0,T]$, then

$$\mathbb{E}\big[\|f_n - f^*\|_{H^{-1}(\Omega)}^2\big] \leq C\lambda_n^{1/2}\|f^*\|_{L^2(\Omega)}^2 + C\sigma^2/(n\lambda_n^{1/2+d/4}).$$

THEOREM 3.6. *Let $f_n \in L^2(\Omega)$ be the solution of* (3.3) *and $\rho_0 = \|f^*\|_{L^2(\Omega)} + \sigma n^{-1/2}$. If we take λ_n such that $\lambda_n^{1/2+d/8} = O(\sigma n^{-1/2}\rho_0^{-1})$, then the following estimates hold for some constant $C > 0$:*

$$\mathbb{P}(\|Sf_n - Sf^*\|_n \geq \lambda_n^{1/2}\rho_0 z) \leq 2e^{-Cz^2}, \qquad \mathbb{P}(\|f_n\|_{L^2(\Omega)} \geq \rho_0 z) \leq 2e^{-Cz^2}.$$

Moreover, if $\lambda_n \geq n^{-4/d}$ and $g > 0$ in $[0, T]$, then

$$\mathbb{P}(\|f_n - f^*\|_{H^{-1}(\Omega)} \geq \lambda_n^{1/4}\rho_0 z) \leq 2e^{-Cz^2}.$$

We remark that $\lambda_n^{1/2+d/8} = O(\sigma n^{-1/2}\rho_0^{-1})$ implies $\lambda_n \geq Cn^{-4/(d+4)}$. Thus the condition $\lambda_n \geq n^{-4/d}$ is not very restrictive in the applications.

3.2. Finite element method for the inverse heat source problem. In this section we consider a finite element approximation to the optimal control problem (3.3) associated with the inverse heat source problem (TIP). For convenience, we assume Ω is a polygonal or polyhedral domain in $\mathbf{R}^d$ ($d = 2, 3$). Let $\mathcal{M}_h$ be a family of shape-regular and quasi-uniform finite element meshes over the domain Ω, and let $V_h \subset H_0^1(\Omega)$ be the conforming linear finite element space over the mesh $\mathcal{M}_h$. We divide the time interval $(0, T)$ into a uniform grid with time step size $\tau = T/N$ and write $t^i = i\tau$ for $i = 0, 1, \dots, N$.

We will use the backward Euler scheme in time and the linear finite element method in space to approximate the heat conduction problem (3.1): Find $u_h^i \in V_h$, $i = 1, 2, \dots, N$, such that

$$\left(\frac{u_h^i - u_h^{i-1}}{\tau}, v_h\right) + a(u_h^i, v_h) = (fg^i, v_h) \quad \forall v_h \in V_h, \tag{3.8}$$

where $a(v, w) = (a\nabla v, \nabla w) + (cv, w)$ for any $v, w \in H_0^1(\Omega)$. We approximate the forward solution Sf by $S_{\tau,h}f = u_h^N$. The inverse problem (3.3) can be approximated by the least-squares problem

$$\min_{f \in V_h} \|S_{\tau,h}f - m\|_n^2 + \lambda_n \|f\|_{L^2(\Omega)}^2. \tag{3.9}$$

We shall make use of the results in section 3.1 to study the stochastic convergence of the solution $f_{\tau,h}$ of the problem (3.9) to the true solution $f^* \in L^2(\Omega)$.

Verification of Assumption 2.3. Let $P_h : L^2(\Omega) \to V_h$ be the orthogonal projection operator in the L^2 inner product. For any $f \in X = L^2(\Omega)$, we know from (3.8) that $S_{\tau,h}f = S_{\tau,h}(P_h f)$. Therefore, Assumption 2.3 (2) is trivially satisfied. It remains to check Assumption 2.3 (1), which amounts to deriving the error estimate of the fully discrete method (3.8). The classical theory for the implicit Euler scheme in time and finite element method in space for solving parabolic equations requires the regularity $\partial_{tt}u \in L^1(0, T; L^2(\Omega))$ of the solution of problem (3.1) (see, e.g., [40, Chapter 1]). This regularity requires the compatibility condition $F(x, 0) = f(x)g(0) = 0$ on $\partial\Omega$, which may not be convenient to meet in practice. Instead, we will derive an error estimate in the remaining part of this section, without this compatibility condition, by adapting some arguments in [40, Chapter 3] for the error estimates of finite element solutions to parabolic equations with rough initial data.

We start with the weak $W^{2,1}(0, T; L^2(\Omega))$ regularity for the solution to (3.1).

LEMMA 3.7. *Let $F(x,t) = f(x)g(t)$ for $(x,t) \in \Omega \times (0,T)$, with $g \in H^2(0,T)$. Then there exists a generic constant C such that the solution u to* (3.1) *satisfies*

$$\|\partial_t u\|_{C([0,T];L^2(\Omega))} \le C\|F(\cdot,0)\|_{L^2(\Omega)} + C\int_0^T \|\partial_t F\|_{L^2(\Omega)} dt,$$

$$\|t\partial_{tt} u\|_{C([0,T];L^2(\Omega))} \le C\|F(\cdot,0)\|_{L^2(\Omega)} + C\int_0^T (\|\partial_t F\|_{L^2(\Omega)} + t\|\partial_{tt} F\|_{L^2(\Omega)}) dt\,.$$

Proof. The proof follows from the standard energy argument, so only an outline is given here. We differentiate the first equation in (3.1) in time to see that $v(x,t) = \partial_t u$ satisfies the conditions that $v = 0$ on $\partial\Omega \times (0,T)$ and $v(x,0) = F(x,0)$ in Ω, and

$$\partial_t v + Lv = \partial_t F(x,t) \quad \text{in } \Omega \times (0,T). \tag{3.10}$$

Then the first estimate in the lemma follows by multiplying both sides of (3.10) by v and integrating by parts.

Next we multiply both sides of (3.10) by $t\partial_t v$, then integrate by parts and apply the first estimate in the lemma to get

$$\int_0^t t\|\partial_t v\|^2_{L^2(\Omega)} dt \le C\|F(\cdot,0)\|^2_{L^2(\Omega)} + C\left(\int_0^T \|\partial_t F\|_{L^2(\Omega)} dt\right)^2 + C\int_0^T t\|\partial_t F\|^2_{L^2(\Omega)} dt. \tag{3.11}$$

Finally, we differentiate (3.10) in time to get

$$\partial_{tt} v + L(\partial_t v) = \partial_{tt} F(x,t) \quad \text{in } \Omega \times (0,T).$$

By multiplying both sides of the equation by $t^2\partial_t v$, integrating by parts again, and applying (3.11), we obtain

$$\begin{aligned} t\|\partial_t v\|_{L^2(\Omega)} \le C\|F(\cdot,0)\|_{L^2(\Omega)} &+ C\int_0^T (\|\partial_t F\|_{L^2(\Omega)} + t\|\partial_{tt} F\|_{L^2(\Omega)}) dt \\ &+ C\left(\int_0^T t\|\partial_t F\|^2_{L^2(\Omega)} dt\right)^{1/2}, \end{aligned}$$

which implies the second estimate of the lemma by noticing that

$$\begin{aligned} \int_0^T t\|\partial_t F\|^2_{L^2(\Omega)} dt &\le \sup_{t\in(0,T)} \|t\partial_t F\|_{L^2(\Omega)} \cdot \int_0^T \|\partial_t F\|_{L^2(\Omega)} dt \\ &= \sup_{t\in(0,T)} \left\|\int_0^t \partial_s(s\partial_s F(s)) ds\right\|_{L^2(\Omega)} \cdot \int_0^T \|\partial_t F\|_{L^2(\Omega)} dt \\ &\le \int_0^T \left(\|\partial_t F\|_{L^2(\Omega)} + t\|\partial_{tt} F\|_{L^2(\Omega)}\right) dt \cdot \int_0^T \|\partial_t F\|_{L^2(\Omega)} dt. \end{aligned}$$

This completes the proof. ◻

LEMMA 3.8. *Let $u_h \in H^1(0,T;V_h)$ be the following semidiscrete finite element solution of problem* (3.1)*:*

$$(\partial_t u_h, v_h) + a(u_h, v_h) = (F, v_h) \quad \forall v_h \in V_h \quad a.e.\ in\ (0,T). \tag{3.12}$$

Then there exists a constant C independent of the mesh size h such that

$$\|u-u_h\|_{C([0,T];L^2(\Omega))}\le Ch^2\max_{t\in[0,T]}(\|\partial_t u\|_{L^2(\Omega)}+\|t\partial_{tt}u\|_{L^2(\Omega)}+\|F\|_{L^2(\Omega)}+\|t\partial_t F\|_{L^2(\Omega)}),$$

where $h=\max_{K\in\mathcal{M}}h_K$ and h_K is the diameter of the element $K\in\mathcal{M}$.

Proof. We follow the argument in [40, Chapter 3]. Define $G: L^2(\Omega)\to H_0^1(\Omega)$ and $G_h: L^2(\Omega)\to V_h$ such that for any $w\in L^2(\Omega)$, $Gw\in H_0^1(\Omega)$ and $G_hw\in V_h$ satisfy

$$a(Gw,v)=(w,v)\quad\forall v\in H_0^1(\Omega);\quad a(G_hw,v_h)=(w,v_h)\quad\forall v_h\in V_h.$$

Equations (3.1) and (3.12) can be reformulated as

$$\partial_t(Gu)+u=GF,\quad \partial_t(G_hu_h)+u_h=G_hF.$$

Writing $e=u-u_h$, then we know e satisfies

$$G_h(\partial_t e)+e=\rho\ \text{ a.e. in } (0,T),\qquad (G_he)(\cdot,0)=0\ \text{ in }\Omega,$$

where $\rho=(G_h-G)(\partial_t u)+(G-G_h)F$. By the argument in the proof of Lemma 3.7 we can obtain (see [40, Lemma 3.4]) that

$$\max_{t\in[0,T]}\|e\|_{L^2(\Omega)}\le C\max_{t\in[0,T]}(\|\rho(t)\|_{L^2(\Omega)}+\|t\partial_t\rho(t)\|_{L^2(\Omega)}).$$

This completes the proof by noting that $\|Gw-G_hw\|_{L^2(\Omega)}\le Ch^2\|w\|_{L^2(\Omega)}$ for all $w\in L^2(\Omega)$, which follows by the Aubin–Nitsche argument since the domain Ω is convex. ∎

The following lemma for the error estimate of the fully discrete finite element method was not covered by the general results in [40, Chapter 8] since we do not have the condition that $F(x,0)=0$ on $\partial\Omega$ here, which was critical in [40].

LEMMA 3.9. *Let $u_h\in H^1(0,T;V_h)$ be the solution of problem* (3.12)*, and let $u_h^i\in V_h, i=1,2,\dots,N$, be the solution of problem* (3.8)*. Then there exists a constant C independent of h,τ such that*

$$\max_{1\le i\le N}\|u_h(\cdot,t_i)-u_h^i\|_{L^2(\Omega)}\le C\tau(1+\ln N)(\|F\|_{C([0,T];L^2(\Omega))}+\|\partial_t F\|_{C([0,T];L^2(\Omega))}).$$

Proof. Let $\{\lambda_j\}_{j=1}^M$ be the eigenvalues of the eigenvalue problem

$$a(\phi_h,v_h)=\lambda(\phi_h,v_h)\qquad\forall v_h\in V_h,$$

and let $\{\phi_j\}_{i=1}^M$ be the corresponding eigenfunctions which form an orthonormal basis of V_h in the $L^2(\Omega)$-norm. By the Poincaré inequality, we know that $\lambda_j\ge C$, $j=1,2,\dots,M$, for some constant C independent of the mesh size h.

We write $u_h(x,t)=\sum_{j=1}^M u_j(t)\phi_j(x)$ and $F(x,t)=\sum_{j=1}^M F_j(t)\phi_j(x)$, where $u_j(t)=(u_h(\cdot,t),\phi_j)$ and $F_j(t)=(F(\cdot,t),\phi_j)$. Then it follows from (3.12) that

$$u_j'(t)+\lambda_ju_j=F_j(t)\qquad\text{a.e. in }(0,T),$$

whose solution can be written as

$$u_j(t^i)=\int_0^{t^i}e^{\lambda_j(s-t^i)}F_j(s)ds=\int_0^{t^i}e^{-\lambda_jt}F_j(t^i-t)dt. \tag{3.13}$$

Similarly, we write $u_h^i = \sum_{j=1}^M U_j^i \phi_j$, where $U_j^i = (u_h^i, \phi_j)$, $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, M$. From (3.8) we know that

$$\frac{1}{\tau}(U_j^i - U_j^{i-1}) + \lambda_j U_j^i = F_j^i := F_j(t^i),\ i = 1, 2, \ldots, N, j = 1, 2, \ldots, M.$$

This implies that $U_j^i = r(\lambda_j)U_j^{i-1} + \tau r(\lambda_j \tau)F_j^i$, where $r(t) = (1+t)^{-1}$ for all $t \geq 0$, and hence

$$U_j^i = \sum_{k=1}^{i} \tau r(\lambda_j \tau)^k F_j^{i-k+1}. \tag{3.14}$$

For any $j = 1, \ldots, M$, we distinguish two cases. If $\lambda_j \tau \geq 1$, we know from (3.13) that

$$|u_j(t^i)| \leq \|F_j\|_{C[0,T]} \int_0^{t^i} e^{-\lambda_j t} dt = \lambda_j^{-1}(1 - e^{-\lambda_j t^i})\|F_j\|_{C[0,T]} \leq \tau \|F_j\|_{C[0,T]}.$$

On the other hand, we obtain from (3.14) that

$$|U_j^i| \leq \left(\sum_{k=1}^{i} 2^{-k} \right) \tau \|F_j\|_{C[0,T]} \leq 2\tau \|F_j\|_{C[0,T]}.$$

Therefore, we derive for $\lambda_j \tau \geq 1$ that

$$|u_j^i(t^i) - U_j^i| \leq C\tau \|F_j\|_{C[0,T]}. \tag{3.15}$$

Now we consider the case when $\lambda_j \tau \leq 1$. By (3.13) we have

$$\begin{aligned} u_j(t^i) &= \sum_{k=1}^{i} \int_{t^{k-1}}^{t^k} e^{-\lambda_j t}(F_j(t^i - t) - F(t^i - t^{k-1}))dt + \sum_{k=1}^{i} \int_{t^{k-1}}^{t^k} e^{-\lambda_j t} F_j^{i-k+1} dt \\ &= \sum_{k=1}^{i} \int_{t^{k-1}}^{t^k} e^{-\lambda_j t}(F_j(t^i - t) - F(t^i - t^{k-1}))dt + \sum_{k=1}^{i} \tau \frac{e^{\lambda_j \tau} - 1}{\lambda_j \tau} e^{-k\lambda_j \tau} F_j^{i-k+1}, \end{aligned}$$

which, together with (3.14), yields

$$\begin{aligned} u_j(t^i) - U_j^i &= \sum_{k=1}^{i} \tau \left(\frac{e^{\lambda_j \tau} - 1}{\lambda_j \tau} e^{-k\lambda_j \tau} - r(\lambda_j \tau)^k \right) F_j^{i-k+1} \\ &\quad + \sum_{k=1}^{i} \int_{t^{k-1}}^{t^k} e^{-\lambda_j t}(F_j(t^i - t) - F(t^i - t^{k-1}))dt := \mathrm{I} + \mathrm{II}. \end{aligned} \tag{3.16}$$

Recalling the following elementary estimate in [40, (7.22)],

$$|e^{-kt} - r(t)^k| \leq Ck^{-1} \quad \forall\, t \geq 0,\ \forall\, k = 1, 2, \ldots,$$

and using the fact that $(t^{-1}(e^t - 1) - 1)/(1 - e^{-t})$ is bounded for $0 \leq t \leq 1$, we obtain

$$\begin{aligned} |\mathrm{I}| &\leq \sum_{k=1}^{i} \tau \left| \left(\frac{e^{\lambda_j \tau} - 1}{\lambda_j \tau} - 1 \right) e^{-k\lambda_j \tau} + (e^{-k\lambda_j \tau} - r(\lambda_j \tau)^k) \right| |F_j^{i-k+1}| \\ &\leq C\tau \left[\left(\frac{e^{\lambda_j \tau} - 1}{\lambda_j \tau} - 1 \right) \frac{1}{1 - e^{-\lambda_j \tau}} + \sum_{k=1}^{i} k^{-1} \right] \|F_j\|_{C[0,T]} \\ &\leq C(1 + \ln i)\tau \|F_j\|_{C[0,T]}. \end{aligned}$$

The term II can be bounded by the standard argument as follows:

$$\mathrm{II} \le C\tau\|\partial_t F_j\|_{C[0,T]} \int_0^{t^i} e^{-\lambda_j t} dt \le C\lambda_j^{-1}\tau\|\partial_t F_j\|_{C[0,T]} \le C\tau\|\partial_t F_j\|_{C[0,T]},$$

where we have used the fact that $\lambda_j \ge C$ for some constant C independent of h.

Combining (3.15), (3.16) and the above two estimates, we obtain

$$|u_j(t^i) - U_j^i| \le C\tau(1 + \ln N)(\|F_j\|_{C[0,T]} + \|\partial_t F_j\|_{C[0,T]}).$$

This completes the proof. □

By Lemmas 3.7–3.9, we know that under the condition $g \in H^2(0,T)$,

$$\|S_{\tau,h}f - Sf\|_{L^2(\Omega)} \le C(h^2 + \tau|\ln\tau|)\|f\|_{L^2(\Omega)} \tag{3.17}$$

for some constant C which depends possibly on T, $\|g\|_{H^2(0,T)}$ but is independent of h and τ.

Assumption 2.3 (1) is now a consequence of the following lemma.

LEMMA 3.10. *If $g \in H^2(0,T)$, $S_{\tau,h}f = u_h^N$ with u_h^N being the solution of the problem* (3.8)*, then for any $f \in L^2(\Omega)$, there exists a constant C independent of h and τ such that*

$$\|Sf - S_{\tau,h}f\|_n \le C(h^2 + \tau|\ln\tau|)\|f\|_{L^2(\Omega)}.$$

Proof. Let $\Pi_h : C(\bar\Omega) \to V_h$ be the canonical finite element interpolant. Then we know from the standard interpolation theory of finite element methods [13] that

$$\|Sf - \Pi_h(Sf)\|_{L^\infty(K)} \le Ch^{2-d/2}\|Sf\|_{H^2(K)} \quad \forall K \in \mathcal{M}_h,$$
$$\|Sf - \Pi_h(Sf)\|_{L^2(K)} \le Ch^2\|Sf\|_{H^2(K)} \quad \forall K \in \mathcal{M}_h.$$

Let $\mathbb{T}_K = \{x_i : x_i \in K, 1 \le i \le n\}$. By the assumption that $\{x_i\}_{i=1}^n$ is quasi-uniformly distributed and the mesh $\mathcal{M}_h$ is quasi-uniform, we know that the cardinal $\#\mathbb{T}_K \le Cnh^d$. Thus we have

$$\|Sf - \Pi_h(Sf)\|_n^2 \le \frac{1}{n}\sum_{K\in\mathcal{M}_h} \#\mathbb{T}_K\|Sf - \Pi_h(Sf)\|_{L^\infty(K)}^2 \le Ch^4\|Sf\|_{H^2(\Omega)}^2.$$

On the other hand, we can derive by making use of inverse estimates that

$$\begin{aligned}
\|S_{\tau,h}f - \Pi_h(Sf)\|_n^2 &\le \frac{1}{n}\sum_{K\in\mathcal{M}_h} \#\mathbb{T}_K\|S_{\tau,h}f - \Pi_h(Sf)\|_{L^\infty(K)}^2 \\
&\le \frac{1}{n}\sum_{K\in\mathcal{M}_h} \#\mathbb{T}_K|K|^{-1}\|S_{\tau,h}f - \Pi_h(Sf)\|_{L^2(K)}^2 \\
&\le C\|S_{\tau,h}f - \Pi_h(Sf)\|_{L^2(\Omega)}^2 \\
&\le C\|S_{\tau,h}f - Sf\|_{L^2(\Omega)}^2 + C\|\Pi_h(Sf) - Sf\|_{L^2(\Omega)}^2 \\
&\le C\|S_{\tau,h}f - Sf\|_{L^2(\Omega)}^2 + Ch^4\|Sf\|_{H^2(\Omega)}^2.
\end{aligned}$$

Therefore,

$$\|Sf - S_{\tau,h}f\|_n \le C\|S_{\tau,h}f - Sf\|_{L^2(\Omega)} + Ch^2\|f\|_{L^2(\Omega)}.$$

This completes the proof by (3.17). □

After the verification of Assumption 2.3, the following stochastic convergence of the finite element method to the inverse heat source problem follows readily from Theorem 2.10 and Corollary 2.11.

THEOREM 3.11. *Let $g \in H^2(0,T)$, and let the measurement data* (2.1) *be of the type* (R1). *Then there exist constants $\lambda_0 > 0$ and $C > 0$ such that for any $\lambda_n \le \lambda_0$ and $\tau|\ln\tau| = O(h^2)$, the following estimates hold for the solutions $f_n \in L^2(\Omega)$ to* (3.3) *and $f_h \in V_h$ to* (3.9)*:*

$$\mathbb{E}\big[\|Sf^* - S_{\tau,h}f_h\|_n^2\big] \le C(\lambda_n + h^4)\|f^*\|_{L^2(\Omega)}^2 + C\left(1 + \frac{h^4}{\lambda_n}\right)\frac{\sigma^2}{n\lambda_n^{d/4}},$$
$$\mathbb{E}\big[\|f^* - f_h\|_{L^2(\Omega)}^2\big] \le C\left(1 + \frac{h^4}{\lambda_n}\right)\|f^*\|_{L^2(\Omega)}^2 + C\left(1 + \frac{h^4}{\lambda_n}\right)\frac{\sigma^2}{n\lambda_n^{1+d/4}}.$$

Moreover, if $\lambda_n \ge n^{-4/d}$ and $g > 0$ in $[0,T]$, we have

$$\mathbb{E}\big[\|f^* - f_h\|_{H^{-1}(\Omega)}^2\big] \le C(\lambda_n^{1/2} + h^2)\left(1 + \frac{h^4}{\lambda_n}\right)\|f^*\|_{L^2(\Omega)}^2 + C(\lambda_n^{1/2} + h^2)\left(1 + \frac{h^4}{\lambda_n}\right)\frac{\sigma^2}{n\lambda_n^{1+d/4}}.$$

Proof. Since the mesh is assumed to be quasi-uniform, the dimension N_h of the linear finite element space V_h is bounded by $N_h \le Ch^{-d}$. By Theorem 3.2, we know that $\alpha = 4/d$. Take $\tau|\ln\tau| = O(h^2)$; then we know from Theorem 2.10 that

$$\mathbb{E}\big[\|Sf^* - S_{\tau,h}f_h\|_n^2\big] \le C(\lambda_n + h^4)\|f^*\|_{L^2(\Omega)}^2 + C\left[1 + \frac{h^4}{\lambda_n} + \left(\frac{h^4}{\lambda_n}\right)^{1-\frac{d}{4}}\right]\frac{\sigma^2}{n\lambda_n^{1+d/4}}.$$

We can easily check that $(h^4/\lambda_n)^{1-\frac{d}{4}} \le 1$ for $h^4/\lambda_n \le 1$, and $(h^4/\lambda_n)^{1-\frac{d}{4}} \le h^4/\lambda_n$ for $h^4/\lambda_n \ge 1$. Therefore, we have $(h^4/\lambda_n)^{1-\frac{d}{4}} \le 1 + h^4/\lambda_n$. This leads to the conclusions of Theorem 3.11. ∎

We end this section with the following convergence of the finite element method to the inverse heat source problem (TIP), directly following from Theorem 2.14 by noticing that $N_h \le Ch^{-d} \le C\lambda_n^{-\gamma/2}$ with $\gamma = d/2$.

THEOREM 3.12. *Let $g \in H^2(0,T)$, let the measurement data* (2.1) *be of type* (R2), *and let $\rho_0 = \|f^*\|_{L^2(\Omega)} + \sigma n^{-1/2}$. If we take $h = O(\lambda_n^{1/4}), \tau|\ln\tau| = O(\lambda_n^{1/2})$, and $\lambda_n^{1/2+d/8} = O(\sigma n^{-1/2}\rho_0^{-1})$, then there exists a constant $C > 0$ such that for any $z > 0$,*

$$\mathbb{P}(\|S_{\tau,h}f_h - Sf^*\|_n \ge \lambda_n^{1/2}\rho_0 z) \le 2e^{-Cz^2}, \qquad \mathbb{P}(\|f_h\|_{L^2(\Omega)} \ge \rho_0 z) \le 2e^{-Cz^2}.$$

Moreover, if $\lambda_n \ge n^{-4/d}$ and $g > 0$ in $[0,T]$, it holds that

$$\mathbb{P}(\|f_h - f^*\|_{H^{-1}(\Omega)} \ge \lambda_n^{1/4}\rho_0 z) \le 2e^{-Cz^2}.$$

4. Numerical examples. In this section, we present several numerical examples to confirm the theoretical results in previous sections. We take the domain $\Omega = (0,1)\times(0,1)$ and a set of uniformly distributed measurement locations $\{x_i\}_{i=1}^n$ in Ω. In all examples below, we take the coefficients $a(x) = 1, c(x) = 0$, which fulfills the uniform ellipticity condition, and $g(t) \equiv 1$, $T = 1$. The finite element mesh $\mathcal{M}_h$

of Ω is constructed by first dividing Ω into $h^{-1} \times h^{-1}$ uniform rectangles and then connecting the lower left and upper right vertices of each rectangle. We set the noise $e_1, \ldots, e_n$ in the dataset (2.1) to be the normal random variables with variance σ.

Motivated by Theorem 3.5, we propose a self-consistent algorithm to determine the regularization parameter λ_n in (3.9) based on the following heuristic rule:

$$\lambda_n^{1/2+d/8} = \sigma n^{-1/2} \|f^*\|_{L^2(\Omega)}^{-1}, \tag{4.1}$$

which balances the two terms in the error due to bias and variance, and also balances the error between the exact solution and the reconstructed one in the H^{-1}-norm. This choice requires the knowledge of the true source function f^* and the noise level σ. We now propose a self-consistent algorithm to determine the parameter λ_n, without knowing the true source function f^* and the noise level σ. To do so, we estimate $\|f^*\|_{L^2(\Omega)}$ by $\|f_h\|_{L^2(\Omega}$ and σ by $\|S_{\tau,h} f_h - m\|_n$ since $\|Sf^* - m\|_n = \|e\|_n$. This is expected to yield a good estimate of the variance by the law of large numbers.

ALGORITHM 4.1 (computing an estimate of the regularization parameter λ_n).

1° *Given an initial guess of* $\lambda_{n,0}$*; for* $j = 0, 1, \ldots,$ *do the following:*

2° *Solve* (3.9) *for* f_h *with* λ_n *replaced by* $\lambda_{n,j}$ *over the mesh* $\mathcal{M}_h$*;*

3° *Update* $\lambda_{n,j+1}$*:* $\lambda_{n,j+1}^{1/2+d/8} = n^{-1/2} \|S_{\tau,h} f_h - m\|_n \, \|f_h\|_{L^2(\Omega)}^{-1}$.

A natural choice of the initial guess is $\lambda_{n,0} = n^{-4/(d+4)}$ since f^* and σ are unknown, which is used in our numerical examples. In the following examples, the negative norm $\|f^* - f_h\|_{H^{-1}(\Omega)}$ is estimated using the same technique as developed in [26, section 6] which estimates $\|f^* - f_h\|_{H^{-1}(\Omega)}$ by $\|P_h f^* - f_h\|_{H^{-1}(\Omega)}$, where P_h is the L^2-projection to the finite element space V_h.

EXAMPLE 4.1. *This example is used to verify the near optimality of the choice of the smoothing parameter* λ_n *suggested by* (4.1). *We choose* $n = 10^4$, $\sigma = 0.1$ *or* $\sigma = 0.01$, *and the mesh size* $h = 0.05$ *and the time step size* $\tau = 0.01$, *which are sufficiently small so that the finite element errors are negligible. We take the true source* f^* *to be the function whose surface is given as in Figure* 4.1.

Example 4.1 demonstrates the near optimality of the choice of the smoothing parameter λ_n suggested by (4.1). In fact, we have $\|f^*\|_{L^2(\Omega)} \approx 0.54$; then (4.1) suggests $\lambda_n \approx 2.3 \times 10^{-4}$ (for $\sigma = 0.1$) and $\lambda_n \approx 1.1 \times 10^{-5}$ (for $\sigma = 0.01$). These two approximate λ_n's are indeed very close to the optimal $\lambda_n = 1 \times 10^{-4}$ (for $\sigma = 0.1$) and $\lambda_n = 1 \times 10^{-5}$ (for $\sigma = 0.01$), which we have estimated by computing the errors $\|Sf^* - S_{\tau,h} f_h\|_n$ and $\|f_h - f^*\|_{H^{-1}(\Omega)}$ with 10 different choices of regularization parameter: $\lambda_{n,k} = 10^{-k}$ $(k = 1, 2, \ldots, 10)$. In order to show the near optimality of the choice (4.1) more clearly, we take partial data around the global minimum to plot the dependence of the errors on k; see Figure 4.2.

EXAMPLE 4.2. *This example is presented to verify whether the probability density functions of the empirical error* $\|Sf^* - S_{\tau,h} f_h\|_n$ *and the error* $\|f_h - f^*\|_{H^{-1}(\Omega)}$ *have exponentially decaying tails. We set the variance* $\sigma = 0.001$, $n = 25 \times 10^4$, *and choose the mesh size* h *and time step size* τ *to be small enough so that the finite element errors are negligible. We take* 10,000 *samples and compute the empirical error* $\|Sf^* - S_{\tau,h} f_h\|_n$ *and the error* $\|f_h - f^*\|_{H^{-1}(\Omega)}$ *for each sampling.*

In Example 4.2, we can compute that $\|Sf^*\|_{L^\infty(\Omega)} \approx 0.04$, so the relative noise level $\sigma / \|Sf^*\|_{L^\infty(\Omega)}$ is about 2.5% for this example. Figures 4.3(a) and (c) show the histogram plot of the corresponding errors, while Figures 4.3(b) and (d) show the quantile-quantile (Q-Q) plot to compare the sample distribution of the error with the

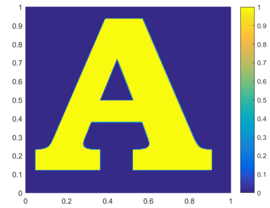


FIG. 4.1. *The surface plot of the exact solution f^*.*

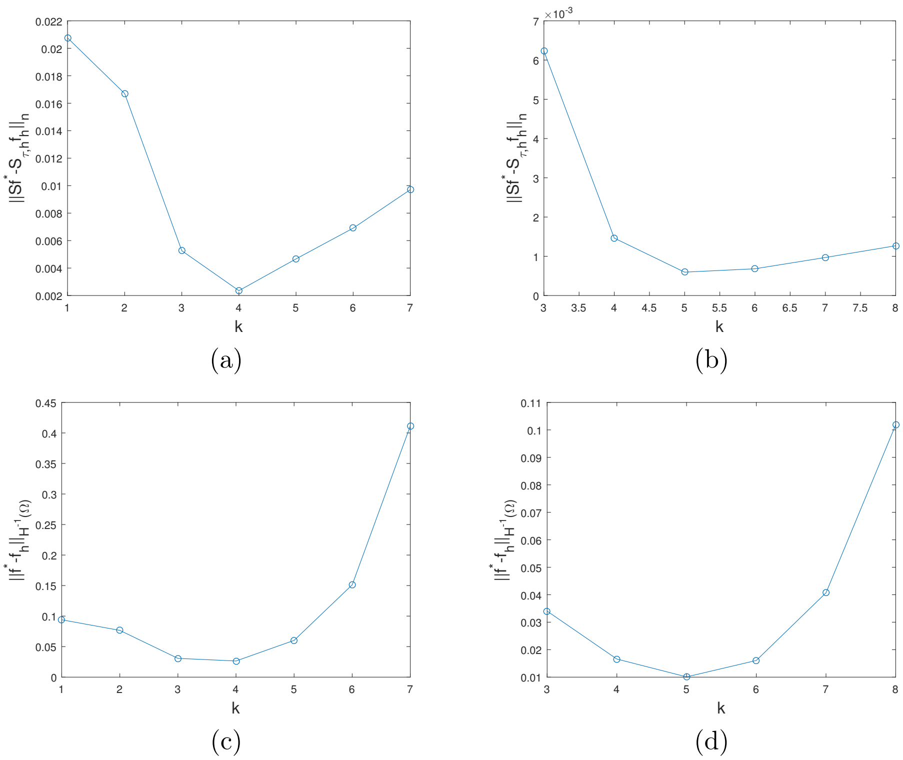


FIG. 4.2. (a) *and* (b) *are the empirical errors* $\|Sf^* - S_{\tau,h}f_h\|_n$ *with* $\lambda_n = 10^{-k}$ $(k = 1, \ldots, 7)$ *for* $\sigma = 0.1$ *(left) and with* $\lambda_n = 10^{-k}$ $(k = 3, \ldots, 8)$ *for* $\sigma = 0.01$ *(right).* (c) *and* (d) *are the errors* $\|f^* - f_h\|_{H^{-1}(\Omega)}$ *with* $\lambda_n = 10^{-k}$ $(k = 1, \ldots, 7)$ *for* $\sigma = 0.1$ *(left) and with* $\lambda_n = 10^{-k}$ $(k = 3, \ldots, 8)$ *for* $\sigma = 0.01$ *(right).*

standard normal distribution. The Q-Q plot is a standard graphic tool in statistics to check the data distribution [45]. If the sample distribution is indeed normal, the Q-Q plot should give a scattered plot, where the points show a linear relationship between the sample and the theoretical quantiles. We can observe from Figure 4.3 (right) that almost all the points are concentrated around the dotted line, which implies that the overall distribution of the error is very close to a normal distribution. Moreover, the points around the two ends are also not far from the line, which indicates that the tail distribution of the error is also close to a Gaussian tail, as indicated in Theorem 3.12. The probability density function is computed by the MATLAB function `qqplot`.

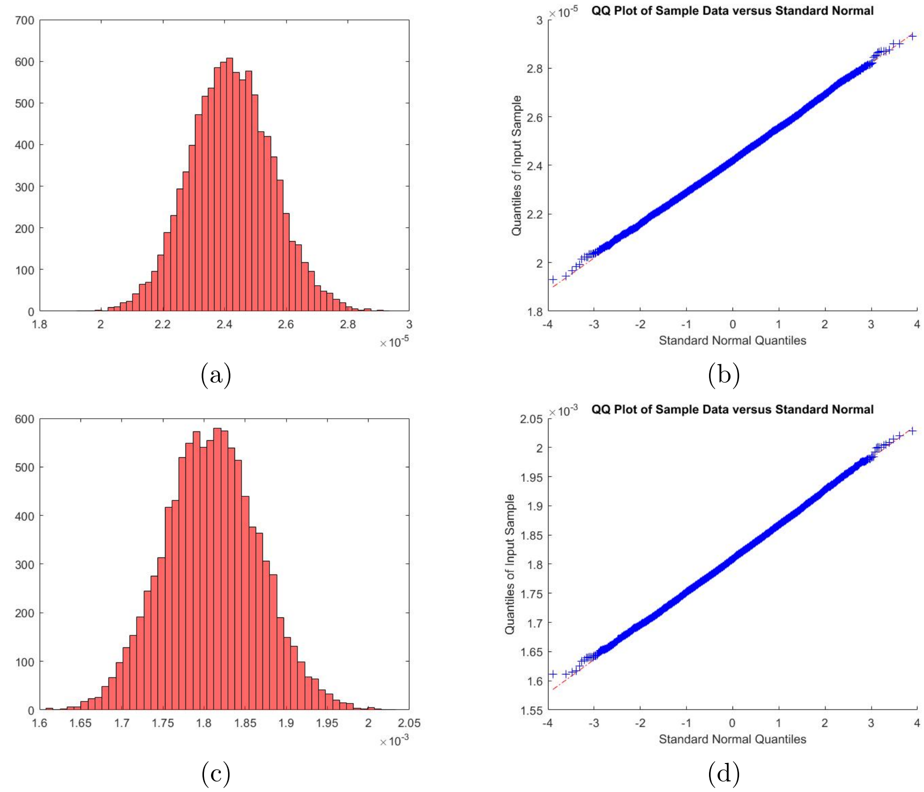


FIG. 4.3. (a) *and* (b) *are the histogram (left) and quantile-quantile (right) plots of the empirical error* $\|S_{\tau,h}f_h - Sf^*\|_n$ *with* 10,000 *samples.* (c) *and* (d) *are the histogram (left) and quantile-quantile (right) plots of the error* $\|f_h - f^*\|_{H^{-1}(\Omega)}$ *with* 10,000 *samples.*

EXAMPLE 4.3. *This example is to confirm Theorems* 3.11 *and* 3.12*, namely, to verify if the empirical error* $\|Sf^* - S_{\tau,h}f_h\|_n$ *and the error* $\|f^* - f_h\|_{H^{-1}(\Omega)}$ *depend linearly on* $\lambda_n^{1/2}$ *when the regularization parameter* λ_n *is taken by the optimal choice* (4.1)*. The mesh size* $h = \lambda_n^{1/4}$ *and the time step size* $\tau|\ln\tau| = \lambda_n^{1/2}$ *are chosen according to Theorems* 3.11 *and* 3.12*. We take the true source* f^* *to be the function given in Figure* 4.1*, and* n *to change from* 25×10^2 *to* 25×10^4*.*

We can see from Figure 4.4 clearly the linear dependences of the empirical error $\|Sf^* - S_{\tau,h}f_h\|_n$ and the error $\|f^* - f_h\|_{H^{-1}(\Omega)}$ on $\lambda_n^{1/2}$ for $\sigma = 0.01$ and 0.04. We can compute that $\|Sf^*\|_{L^\infty(\Omega)} \approx 0.04$, so the relative noise levels $\sigma/\|Sf^*\|_{L^\infty(\Omega)}$ are about 25% and 100% for $\sigma = 0.01$ and 0.04, respectively.

Through the previous 3 examples, we have verified the optimality of the choice rule (4.1) for λ_n, the stochastic convergence (Theorem 3.12), and the convergence order of the finite element method. But we do not know the exact solution and the variance of the noise in most applications, so we use the next example to show the efficiency of Algorithm 4.1 to determine an optimal regularization parameter λ_n iteratively, without the knowledge of f^* and σ.

EXAMPLE 4.4. *We choose* $n = 25 \times 10^4$ *and set the noise* $e_1, \ldots, e_n$ *in the dataset* (2.1) *to be independent normal random variables with variance* $\sigma = 0.001$*. Algorithm*

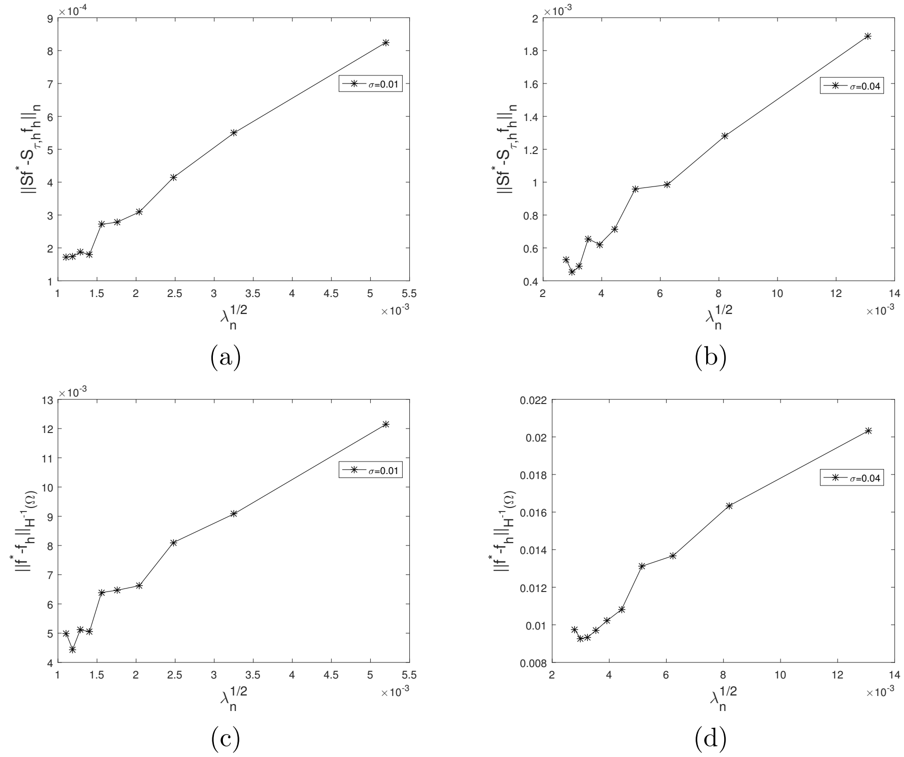


FIG. 4.4. (a) *and* (b) *are the linear dependences of the empirical error* $\|Sf_* - S_{\tau,h}f_h\|_n$ *on* $\lambda_n^{1/2}$ *with* $\sigma = 0.01$ *(left) and* $\sigma = 0.04$ *(right), respectively.* (c) *and* (d) *are the linear dependences of the error* $\|f_* - f_h\|_{H^{-1}(\Omega)}$ *on* $\lambda_n^{1/2}$ *with* $\sigma = 0.01$ *(left) and* $\sigma = 0.04$ *(right), respectively.*

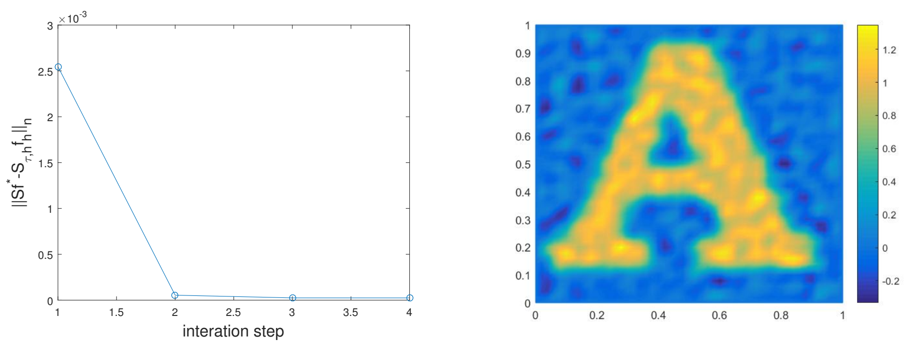
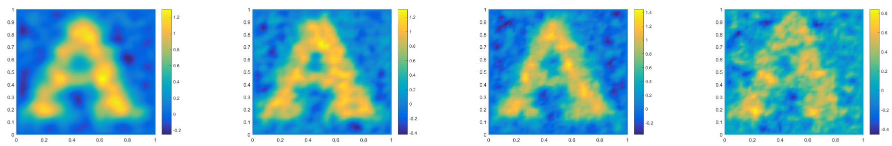


FIG. 4.5. *The relative empirical error* $\|Sf^* - S_{\tau,h}f_h\|_n$ *at each iteration (left); the computed solution* f_h *at the end of iterations (right).*

4.1 *is terminated when the absolute difference between two consecutive iterates* $\lambda_{n,k}$ *and* $\lambda_{n,k+1}$ *is less than* 10^{-10}.

We can compute that $\|Sf^*\|_{L^\infty(\Omega)} \approx 0.04$, so the relative noise level $\sigma/\|Sf^*\|_{L^\infty(\Omega)}$ is about 2.5% in this example. Figure 4.5 shows clearly the convergence of the sequence $\{\lambda_{n,k}\}$ generated by Algorithm 4.1. The numerical computation gives $\lambda_{n,4} =$



(a) (b) (c) (d)

FIG. 4.6. (a)–(d) *are the computed solutions* f_h *when* $T = 1,\ 0.1\ , 0.01,\ 0.001$*, respectively.*

5.53×10^{-8} that agrees very well with the optimal choice 5.33×10^{-8} given by (4.1). Furthermore, $\|m - S_{\tau,h} f_h\|_n = 9.99 \times 10^{-4}$ provides also a good estimate of the variance σ.

EXAMPLE 4.5. *In this example, we show the influence of* T *on the ill-posedness of the inverse problem. We take* $T = 1,\ 0.1,\ 0.01,\ 0.001$*, choose* $n = 25 \times 10^4$*, and set the variance* $\sigma = 0.01$*. We choose the regularization parameter* λ_n *by the optimal rule* (4.1)*.*

We observe from Figure 4.6 that the numerical reconstruction deteriorates as T decreases. This fact can be interpreted by using the notation in Theorem 3.2 as follows: the singular value of $S : L^2(\Omega) \to L^2(\Omega)$, which is the eigenvalue of $(S^*S)^{1/2}$, approaches 0 as $T \to 0$, i.e., $\rho_k^{-1/2} = \alpha_k \le \mu_k^{-1}(1 - e^{-\mu_k T})\|g\|_{C[0,T]} \to 0$ for all $k \ge 1$.

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