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FINITE ELEMENT METHOD AND ITS ANALYSIS FOR A NONLINEAR HELMHOLTZ EQUATION WITH HIGH WAVE NUMBERS*

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Abstract. The well-posedness of a nonlinear Helmholtz equation with an impedance boundary condition is established for high frequencies in two and three dimensions. Stability estimates are derived with explicit dependence on the wave number. Linear finite elements are considered for the discretization of the nonlinear Helmholtz equation, and the well-posedness of the finite element systems is analyzed. Stability and preasymptotic error estimates of the finite element solutions are achieved with explicit dependence on the wave number. Numerical examples are also presented to demonstrate the effectiveness and accuracies of the proposed finite element method for solving the nonlinear Helmholtz equation.

Key words. nonlinear Helmholtz equation, high wave number, finite element method, stability, error estimates

AMS subject classifications. 65N15, 65N12, 65N30, 78A40

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1. Introduction. The propagation of electromagnetic waves may be modeled by Maxwell's equations in physical media with various medium responses. In this work we are interested in high intensity radiation, where the medium quantities may depend on the location, frequency, and magnitude of the propagating field, resulting in a nonlinear electromagnetic wave propagation. We shall consider the time-harmonic propagation in a homogeneous background medium in $\mathbb{R}^d$ $(d = 2, 3)$, where some nonlinear medium, say, the Kerr medium, is sitting inside and occupying a domain Ω_0. Let ω_0 and c be the angular frequency and the speed of light in vacuum, and let n_0 and n_2 be the index of refraction of the homogeneous medium and the Kerr coefficient of the nonlinear medium, respectively. Then under linear polarization for the electric field, we may come to the following nonlinear Helmholtz (NLH) equation for the electric field after eliminating the magnetic field from the Maxwell system [8, 9, 12, 22, 23]:

$$-\Delta u - k^2(1 + \varepsilon\, \mathbf{1}_{\Omega_0} |u|^2)u = f \quad \text{in } \mathbb{R}^d, \tag{1.1}$$

where $k = \omega_0 n_0/c$ is the wave number, $\varepsilon(x) = 2n_2(x)/n_0$ represents the Kerr constant satisfying $0 < \varepsilon \ll 1$, and $\mathbf{1}_{\Omega_0}$ is the characteristic function of Ω_0.

There are already many mathematical and numerical studies in the literature for the NLH equation (1.1) under various boundary conditions imposed on the finitely

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truncated subregion of the entire domain $\mathbb{R}^d$. A variational framework was developed in [20] to prove the existence of nontrivial solutions for the NLH equation $\Delta u + k^2 u = f$, where f is a nonlinear function that meets five specific conditions, under two special asymptotic decay assumptions on the solution itself and its radial second-order derivative. In addition, the radial symmetric case is also considered, in which infinitely many solutions are shown to exist for different nonlinearities.

For the one-dimensional periodic dielectric photonic bandgap structures composed of alternatively arranged nonlinear Kerr material layers, the well-posedness of the NLH equation was established in [6], under the two-way boundary conditions that are derived from the standard jump conditions. A numerical scheme was proposed in [37] for solving the NLH equation, based on the existence of a stable steady-state solution to a nonlinear Schrödinger type equation and an operator splitting technique.

A fourth-order finite difference scheme was proposed in [22] for solving (1.1) for the case where the nonlinear medium domain Ω_0 occupies the domain formed by two parallel infinite planes by using a nonlocal two-way artificial boundary condition set on the boundary of Ω_0. In [23], an improved scheme was proposed by introducing some Sommerfeld-type local radiation boundary conditions that are constructed directly in the discrete setting. The high-order scheme was then extended in [7] to a three-dimensional setting with cylindrical symmetry under both boundary conditions from [22, 23].

In this work we shall carry out a systematical mathematical and numerical study of the NLH system (1.2)–(1.3) in a general setting for both two and three dimensions. We first make a general truncation of the homogeneous medium $\mathbb{R}^d \setminus \Omega_0$ by a finite domain Ω with $\Omega_0 \subset \Omega$. Then we consider the lowest order absorbing boundary condition on the boundary Γ of Ω to arrive at the following NLH system of our interest in this work:

$$-\Delta u - k^2(1 + \varepsilon \mathbf{1}_{\Omega_0} |u|^2)u = f \quad \text{in } \Omega, \tag{1.2}$$

$$\frac{\partial u}{\partial n} + \mathbf{i}ku = g \quad \text{on } \Gamma, \tag{1.3}$$

where $\mathbf{i} = \sqrt{-1}$ denotes the imaginary unit and n denotes the unit outer normal to $\partial\Omega$. One may note that g depends on the incident wave u_{inc}, that is, $g = \frac{\partial u_{\text{inc}}}{\partial n} + \mathbf{i}ku_{\text{inc}}$. Other kinds of more accurate boundary conditions such as PML may be considered but are much more complicated to analyze both mathematically and numerically and will not be studied in this work. For ease of presentation, we shall assume that k is constant on Ω and consider only the case $k \gg 1$ since we are mainly interested in high frequencies in this work, though most of our results are naturally true for low frequencies.

The main focus of this work is to study the well-posedness of the NLH system (1.2)–(1.3) and its finite element approximation as well as the error estimates of the finite element solutions for high wave numbers in two and three dimensions. The well-posedness of both the NLH system and its linear finite element approximation is established. Particularly, we emphasize that the stability estimates of the continuous NLH solutions and their finite element solutions are achieved with explicit dependence on the wave number, and the preasymptotic optimal error estimates of the finite element solutions are also derived. To the best of our knowledge, these results and analyses are new, and there are still no similar studies and results available for NLH equations in the literature.

We like to point out that the whole analysis in this work is focused on the NLH system (1.2)–(1.3) that is expressed in terms of the total field u. But with some natural modifications, all our results and analyses can be extended to the case when the NLH system (1.2)–(1.3) is expressed in terms of the scattered field $u_{\rm sc} := u - u_{\rm inc}$ that satisfies

$$-\Delta u_{\rm sc} - k^2 u_{\rm sc} - k^2\varepsilon \mathbf{1}_{\Omega_0}\big(\left|u_{\rm sc}+u_{\rm inc}\right|^2(u_{\rm sc}+u_{\rm inc}) - \left|u_{\rm inc}\right|^2 u_{\rm inc}\big) = \tilde f \quad \text{in } \Omega, \tag{1.4}$$

$$\frac{\partial u_{\rm sc}}{\partial n} + \mathbf{i}k u_{\rm sc} = 0 \quad \text{on } \Gamma, \tag{1.5}$$

where

$$\tilde f := f + \Delta u_{\rm inc} + k^2(1+\varepsilon \mathbf{1}_{\Omega_0}\left|u_{\rm inc}\right|^2)u_{\rm inc}.$$

We shall provide further illustrations about this extension in Remarks 2.1 and 3.1.

Throughout the paper, we shall use the standard Sobolev space $H^s(\Omega)$, its norm and inner product, and refer to [10, 15] for their definitions. But $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ are used for the L^2-inner product on the complex-valued spaces $L^2(\Omega)$ and $L^2(\Gamma)$, respectively. We will write by $\|\cdot\|_s$ and $|\cdot|_s$ the norm and seminorm of the space $H^s(\Omega)$ and by $\|\cdot\|_{s,\Gamma}$ and $|\cdot|_{s,\Gamma}$ the norm and seminorm of $H^s(\Gamma)$. In particular, we will often use the weighted energy norm $|||w||| = (\|\nabla w\|_0^2 + k^2\|w\|_0^2)^{1/2}$ for any $w \in H^1(\Omega)$.

For the simplicity of notation, we shall frequently use C for a generic positive constant in most of the subsequent estimates, which is independent of the parameters ε, k and functions f, g in (1.2)–(1.3), as well as the penalty parameters involved in the subsequent continuous interior penalty FEM. We will also often write $A \lesssim B$ and $B \gtrsim A$ for the inequalities $A \le CB$ and $B \ge CA$, respectively. $A \eqsim B$ is used for an equivalent statement when both $A \lesssim B$ and $B \lesssim A$ hold. We shall consider only the case where the domain Ω is convex, so it is "strictly star-shaped," which means that there exists a point $x_\Omega \in \Omega$ and a positive constant c_Ω depending only on Ω such that

$$(x - x_\Omega)\cdot n \ge c_\Omega \quad \forall x \in \partial\Omega. \tag{1.6}$$

We shall assume $\operatorname{dist}(\partial\Omega,\Omega_0) \gtrsim \operatorname{diam}(\Omega_0)$ and frequently use the special function

$$\alpha(x) = x - x_\Omega \quad \forall\, x \in \bar\Omega.$$

We end this section with a detailed plan for the rest of the paper. Section 2 is devoted to the stability estimate of the continuous NLH equation in L^2-, H^1-, H^2-, and L^∞-norm. We will discuss the piecewise linear continuous finite element approximation of the NLH equation and its error estimates in section 3. To reduce the pollution errors of the linear finite elements, we introduce the continuous interior penalty FEM in section 4. Finally in section 5, we present two numerical examples to demonstrate the effectiveness and accuracies of our proposed finite element method for solving the NLH equation.

2. Stability estimates of the continuous problem. In this section we will establish some stability estimates of the solutions to the continuous NLH equation (1.2)–(1.3), with their bounds depending on the wave number explicitly. We first cite the following two integral identities [21, Lemma 4.1], which will play an important role in our subsequent analysis.

LEMMA 2.1. *It holds for $v \in H^2(\Omega)$ that*

$$d\|v\|_0^2 + 2\mathrm{Re}(v, \alpha\cdot\nabla v) = \int_\Gamma \alpha\cdot n|v|^2, \tag{2.1}$$

$$(d-2)\|\nabla v\|_0^2 + 2\mathrm{Re}(\nabla v, \nabla(\alpha\cdot\nabla v)) = \int_\Gamma \alpha\cdot n|\nabla v|^2. \tag{2.2}$$

We remark that the identity (2.2) can be viewed as a local version of the Rellich identity for the Laplacian $\triangle$ (cf. [16]).

2.1. An auxiliary problem. For the establishment of the well-posedness of the nonlinear system (1.2)–(1.3), we shall use a linearized process to construct a sequence of approximate solutions to linearized Helmholtz equations, show the uniform bounds of these approximate solutions with respect to the wave number, then verify that the limiting solution of this sequence solves our desired NLH system. To do so, we first study the following linear auxiliary problem for a given function $\phi \in L^\infty(\Omega)$:

$$-\Delta u - k^2(1+\varepsilon \mathbf{1}_{\Omega_0} |\phi|^2)u = f \quad \text{in } \Omega\,, \tag{2.3}$$

$$\frac{\partial u}{\partial n} + \mathbf{i}ku = g \quad \text{on } \Gamma. \tag{2.4}$$

Clearly, the variational formulation of the auxiliary problem reads as

$$(\nabla u, \nabla v) - k^2(u,v) - k^2\varepsilon(\mathbf{1}_{\Omega_0} |\phi|^2 u, v) + \mathbf{i}k \langle u, v\rangle = (f,v) + \langle g, v\rangle \;\; \forall v \in H^1(\Omega). \tag{2.5}$$

For convenience, we introduce two constants,

$$M(f,g) := \|f\|_0 + \|g\|_{0,\Gamma}\,, \quad \widehat{M}(f,g) := M(f,g) + k^{-1}\|g\|_{\frac{1}{2},\Gamma}\,.$$

Then similarly to the stability estimates for the linear Helmholtz equations [16, 28, 30], we may derive stability estimates for the auxiliary problem (2.3)–(2.4).

LEMMA 2.2. *If* $k\varepsilon \|\phi\|^2_{L^\infty(\Omega_0)} \le \theta_0$ *for a positive constant* θ_0*, then we have*

$$|||u||| \lesssim M(f,g) \quad \textit{and} \quad \|u\|_2 \lesssim k\widehat{M}(f,g). \tag{2.6}$$

Proof. We first take $v = u$ in (2.5), then compute the imaginary and real parts of the resulting equation to obtain

$$k\|u\|^2_{0,\Gamma} = \mathrm{Im}((f,u) + \langle g,u\rangle) \le |(f,u)| + \frac{1}{2k}\|g\|^2_{0,\Gamma} + \frac{k}{2}\|u\|^2_{0,\Gamma}\,, \tag{2.7}$$

$$\|\nabla u\|^2_0 - k^2\|u\|^2_0 - k^2\varepsilon\|\phi u\|^2_{0,\Omega_0} \le |(f,u)| + \frac{1}{2k}\|g\|^2_{0,\Gamma} + \frac{k}{2}\|u\|^2_{0,\Gamma}\,, \tag{2.8}$$

which imply immediately

$$k\|u\|^2_{0,\Gamma} \le 2|(f,u)| + \frac{1}{k}\|g\|^2_{0,\Gamma}\,, \tag{2.9}$$

$$\|\nabla u\|^2_0 - k^2\|u\|^2_0 - k^2\varepsilon\|\phi u\|^2_{0,\Omega_0} \le 2|(f,u)| + \frac{1}{k}\|g\|^2_{0,\Gamma}\,. \tag{2.10}$$

Then we take $v = 2\alpha \cdot \nabla u$ in (2.5) and compute the real part of the resulting equation to derive by using Lemma 2.1 that

$$\begin{aligned}&\int_\Gamma \alpha\cdot n|\nabla u|^2 - (d-2)\|\nabla u\|^2_0 - k^2\Big(\int_\Gamma \alpha\cdot n|u|^2 - d\|u\|^2_0\Big)\\ &\quad - 2k^2\varepsilon\,\mathrm{Re}(\mathbf{1}_{\Omega_0}|\phi|^2 u, \alpha\cdot\nabla u) - 2k\,\mathrm{Im}\,\langle u, \alpha\cdot\nabla u\rangle\\ &= 2\,\mathrm{Re}\big((f,\alpha\cdot\nabla u) + \langle g, \alpha\cdot\nabla u\rangle\big).\end{aligned}$$

Using this relation, (2.9)–(2.10) and (1.6) we can deduce as follows:

$$
\begin{aligned}
&\|\nabla u\|_0^2 + k^2\|u\|_0^2 = -\int_\Gamma \alpha\cdot n|\nabla u|^2 + k^2\int_\Gamma \alpha\cdot n|u|^2 + (d-1)\left(\|\nabla u\|_0^2 - k^2\|u\|_0^2\right)\\
&\quad + 2k^2\varepsilon\operatorname{Re}(\mathbf{1}_{\Omega_0}|\phi|^2 u, \alpha\cdot\nabla u) + 2k\operatorname{Im}\langle u, \alpha\cdot\nabla u\rangle + 2\operatorname{Re}\left((f,\alpha\cdot\nabla u) + \langle g, \alpha\cdot\nabla u\rangle\right)\\
&\le -c_\Omega\|\nabla u\|_{0,\Gamma}^2 + Ck\left(2|(f,u)| + \frac{1}{k}\|g\|_{0,\Gamma}^2\right) + 2k^2\varepsilon\left(\|\phi u\|_{0,\Omega_0}^2 + \operatorname{Re}(\mathbf{1}_{\Omega_0}|\phi|^2 u, \alpha\cdot\nabla u)\right)\\
&\quad + \frac{c_\Omega}{2}\|\nabla u\|_{0,\Gamma}^2 + \frac{1}{2}\|\nabla u\|_0^2 + C\|f\|_0^2 + C\|g\|_{0,\Gamma}^2\\
&\le -\frac{c_\Omega}{2}\|\nabla u\|_{0,\Gamma}^2 + 2k^2\varepsilon\left(\|\phi u\|_{0,\Omega_0}^2 + \left|(\mathbf{1}_{\Omega_0}|\phi|^2 u, \alpha\cdot\nabla u)\right|\right)\\
&\quad + \frac{1}{2}\|\nabla u\|_0^2 + \frac{k^2}{2}\|u\|_0^2 + C\|f\|_0^2 + C\|g\|_{0,\Gamma}^2,
\end{aligned}
$$

which implies

$$
\|\!|u|\!\|^2 \le 4k^2\varepsilon\big(\|\phi u\|_{0,\Omega_0}^2 + \left|(\mathbf{1}_{\Omega_0}|\phi|^2 u, \alpha\cdot\nabla u)\right|\big) + C\|f\|_0^2 + C\|g\|_{0,\Gamma}^2. \tag{2.11}
$$

We can easily see

$$
\begin{aligned}
4k^2\varepsilon\big(\|\phi u\|_{0,\Omega_0}^2 + \left|(\mathbf{1}_{\Omega_0}|\phi|^2 u, \alpha\cdot\nabla u)\right|\big) &\lesssim k^2\varepsilon\|\phi\|_{L^\infty(\Omega_0)}^2\left(\|u\|_0^2 + \|u\|_0\|\nabla u\|_0\right)\\
&\lesssim k\varepsilon\|\phi\|_{L^\infty(\Omega_0)}^2\|\!|u|\!\|^2.
\end{aligned} \tag{2.12}
$$

Clearly we see the existence of a positive constant θ_0 such that the first estimate in (2.6) follows from (2.11) and (2.12) if $k\varepsilon\|\phi\|_{L^\infty(\Omega_0)}^2 \le \theta_0$.

On the other hand, for the second estimate in (2.6) we may first apply the following standard a priori estimate for elliptic equations and (2.3)–(2.4):

$$
\begin{aligned}
\|u\|_2 &\lesssim \|\Delta u\|_0 + \|u\|_0 + \left\|\frac{\partial u}{\partial n}\right\|_{\frac12,\Gamma}\\
&\lesssim k^2\|u\|_0 + k^2\varepsilon\left\||\phi|^2 u\right\|_{0,\Omega_0} + \|f\|_0 + \|g - \mathbf{i}ku\|_{\frac12,\Gamma}.
\end{aligned}
$$

Then the desired estimate is a consequence of the first one in (2.6). ∎

The following L^∞ estimate will be crucial to our subsequent analysis.

LEMMA 2.3. *Under the same condition of Lemma* 2.2*, it holds that*

$$
\|u\|_{L^\infty(\Omega_0)} \lesssim k^{\frac{d-3}{2}} M(f,g). \tag{2.13}
$$

Proof. Let $G(x-y)$ be the Green's function of the linear Helmholtz equation $-\Delta u - k^2 u = f$ with the standard radiation condition. Let $r = |x-y|$; then we know

$$
G(x-y) = \begin{cases} \dfrac{\mathbf{i}}{4}H_0^{(1)}(kr) & \text{for } d=2,\\[2mm] \dfrac{e^{\mathbf{i}kr}}{4\pi r} & \text{for } d=3. \end{cases}
$$

Some simple calculations show that the solution u to (2.3)–(2.4) meets the following integral representation:

$$
\begin{aligned}
u(x) &= \int_\Omega G(x-y)\big(f(y) + k^2\varepsilon\mathbf{1}_{\Omega_0}|\phi(y)|^2 u(y)\big)\mathrm{d}y\\
&\quad + \int_\Gamma (g - \mathbf{i}ku)G(x-y)\mathrm{d}s_y - \int_\Gamma u(y)\frac{\partial G(x-y)}{\partial n_y}\mathrm{d}s_y.
\end{aligned} \tag{2.14}
$$

Clearly,

$$|G(x-y)| \lesssim \frac{1}{r} \quad \text{if } d=3. \tag{2.15}$$

For $d = 2$, we know from [35, p. 211] that

$$|G(x-y)| \lesssim \frac{1}{\sqrt{kr}} \quad \text{if } d=2. \tag{2.16}$$

Then by using a standard technique to remove a small ball centered at x we derive

$$\int_\Omega |G(x-y)|^2 \mathrm{d}y \lesssim k^{d-3}. \tag{2.17}$$

On the other hand, we know

$$\nabla_y G(x-y) = \begin{cases} -\frac{\mathbf{i}k}{4} H_1^{(1)}(kr) \frac{y-x}{r} & \text{for } d=2, \\ \frac{e^{\mathbf{i}kr}}{4\pi r} \left(\mathbf{i}k - r^{-1}\right) \frac{y-x}{r} & \text{for } d=3. \end{cases}$$

Suppose $x \in \Omega_0$. Since $\operatorname{dist}(\partial\Omega, \Omega_0) \gtrsim \operatorname{diam}(\Omega_0)$, we have for any $y \in \Gamma$

$$|G(x-y)| \lesssim k^{\frac{d-3}{2}} \quad \text{and} \quad |\nabla_y G(x-y)| \lesssim k^{\frac{d-1}{2}}.$$

Then it follows from these estimates, (2.14), and (2.17) that

$$|u(x)| \lesssim k^{\frac{d-3}{2}} \left(\|f\|_0 + k^2 \varepsilon \|\phi\|^2_{L^\infty(\Omega_0)} \|u\|_0 + \|g\|_{0,\Gamma} + k \|u\|_{0,\Gamma} \right) \lesssim k^{\frac{d-3}{2}} M(f,g),$$

where we have used Lemma 2.2, (2.9), and the assumption that $k\varepsilon \|\phi\|^2_{L^\infty(\Omega_0)} \leq \theta_0$ to derive the last inequality. This completes the proof of the desired estimate (2.13). ∎

2.2. Existence and stability estimates of the NLH solutions. We consider an iterative procedure to establish the existence and stability of the solutions to the NLH system (1.2)–(1.3).

Find $u^l \in H^1(\Omega)$ for $l = 1, 2, \ldots$ by solving the linearized Helmholtz equation:

$$-\Delta u^l - k^2 \left(1 + \varepsilon \mathbf{1}_{\Omega_0} \left|u^{l-1}\right|^2\right) u^l = f \quad \text{in } \Omega\,, \tag{2.18}$$

$$\frac{\partial u^l}{\partial n} + \mathbf{i}ku^l = g \quad \text{on } \Gamma. \tag{2.19}$$

We first derive the following stability estimates of the sequence $\{u^l\}$.

LEMMA 2.4. *There exists a positive constant θ_1 such that the following estimates hold for $l = 1, 2, \ldots$ if $k\varepsilon \left\|u^0\right\|^2_{L^\infty(\Omega_0)} \leq k^{d-2}\varepsilon M(f,g)^2 \leq \theta_1$:*

$$|||u^l|||_0 \lesssim M(f,g), \quad \left\|u^l\right\|_2 \lesssim k\,\widehat{M}(f,g), \quad \left\|u^l\right\|_{L^\infty(\Omega_0)} \lesssim k^{\frac{d-3}{2}} M(f,g). \tag{2.20}$$

Proof. If $k\varepsilon \left\|u^{l-1}\right\|^2_{L^\infty(\Omega_0)} \leq \theta_0$, then Lemma 2.2 implies the estimates in (2.20). Therefore $k\varepsilon \left\|u^l\right\|^2_{L^\infty(\Omega_0)} \leq \theta_0$ if $k^{d-2}\varepsilon M(f,g)^2$ is small enough. Then the proof of the lemma follows by the induction. ∎

Next we prove the well-posedness of the NLH problem (1.2)–(1.3) under certain conditions by showing the convergence of the sequence $\{u^l\}$.

THEOREM 2.5. *There exists a constant* $\theta_2 > 0$ *such that if* $k^{d-2}\varepsilon M(f,g)^2 \le \theta_2$, *then the NLH system* (1.2)–(1.3) *attains a unique solution* u *satisfying the estimates*

$$\|\nabla u\|_0 + k\,\|u\|_0 \lesssim M(f,g),\ \|u\|_2 \lesssim k\widehat{M}(f,g),\ \|u\|_{L^\infty(\Omega_0)} \lesssim k^{\frac{d-3}{2}} M(f,g). \tag{2.21}$$

Proof. Let the sequence u^l for $l \ge 1$ be defined by (2.18) and (2.19). Then it is easy to check that the difference $v^l = u^{l+1} - u^l$ satisfies

$$-\Delta v^l - k^2\left(1+\varepsilon \mathbf{1}_{\Omega_0}\left|u^l\right|^2\right)v^l = k^2\varepsilon\,\mathbf{1}_{\Omega_0}u^l\left(\left|u^l\right|^2 - \left|u^{l-1}\right|^2\right) \quad \text{in } \Omega\,,$$
$$\frac{\partial v^l}{\partial n} + \mathbf{i}kv^l = 0 \quad \text{on } \Gamma.$$

Now we suppose that the conditions of Lemma 2.4 are satisfied; then it follows from Lemmas 2.2 and 2.4 that

$$\begin{aligned}
|||v^l||| &\le Ck^2\varepsilon\left\|u^l\left(\left|u^l\right|^2 - \left|u^{l-1}\right|^2\right)\right\|_{0,\Omega_0}\\
&\le Ck^2\varepsilon\left\|u^l\right\|_{L^\infty(\Omega_0)}\left(\left\|u^l\right\|_{L^\infty(\Omega_0)} + \left\|u^{l-1}\right\|_{L^\infty(\Omega_0)}\right)\left\|v^{l-1}\right\|_{0,\Omega_0}\\
&\le Ck^2\varepsilon\left(k^{\frac{d-3}{2}}M(f,g)\right)^2\left\|v^{l-1}\right\|_0 \le Ck^{d-2}\varepsilon M(f,g)^2\,|||v^{l-1}|||\,.
\end{aligned}$$

Clearly there exists a constant $\tilde\theta_2$ satisfying $0 < \tilde\theta_2 < \theta_1$ (with θ_1 from Lemma 2.4) such that if $k^{d-2}\varepsilon M(f,g)^2 \le \tilde\theta_2$ we have

$$|||v^l||| \le \frac{1}{2}\,|||v^{l-1}|||\,,$$

which implies that $|||v^l||| \le 2^{-l}|||v^0|||$. Hence $\{u^l\}$ is a Cauchy sequence with respect to the energy norm. Moreover, we have

$$\left\|v^l\right\|_2 \lesssim k^3\varepsilon\left\|u^l\left(\left|u^l\right|^2 - \left|u^{l-1}\right|^2\right)\right\|_{0,\Omega_0} \lesssim k^{d-1}\varepsilon M(f,g)^2\,|||v^{l-1}||| \lesssim k\,|||v^{l-1}|||\,,$$

which shows that $\{u^l\}$ is also a Cauchy sequence in the H^2-norm. As a consequence, $u := \lim_{l\to\infty} u^l$ satisfies the NLH equation (1.2)–(1.3) and the stability estimates in (2.21).

It remains to prove the uniqueness. Suppose w is another solution to (1.2)–(1.3) satisfying the estimates in (2.21). Let $v = u - w$; then we can easily see

$$-\Delta v - k^2(1+\varepsilon\mathbf{1}_{\Omega_0}|u|^2)v = k^2\varepsilon w\left(|u|^2 - \mathbf{1}_{\Omega_0}|w|^2\right) \quad \text{in } \Omega\,,$$
$$\frac{\partial v}{\partial n} + \mathbf{i}kv = 0 \quad \text{on } \Gamma.$$

Applying Lemma 2.2, we have

$$|||v||| \le Ck^2\varepsilon\left\|w\left(|u|^2 - |w|^2\right)\right\|_{0,\Omega_0} \le Ck^{d-2}\varepsilon M(f,g)^2\,|||v|||\,.$$

Then it is easy to see that there exists a constant θ_2 satisfying $0 < \theta_2 < \tilde\theta_2$ such that $|||v||| \le \frac{1}{2}|||v|||$ if $k^{d-2}\varepsilon M(f,g)^2 \le \theta_2$. This implies the uniqueness of the solutions to the NLH system (1.2)–(1.3) and so completes the proof of the theorem. ∎

Remark 2.1. If we consider the NLH system (1.4)–(1.5) in terms of the scattered field $u_{\rm sc} := u - u_{\rm inc}$, instead of the NLH system (1.2)–(1.3) in terms of the total field u, then the iterative scheme (2.18)–(2.19) is equivalent to the following one:

$$-\Delta u_{\rm sc}^l - k^2 u_{\rm sc}^l - k^2\varepsilon \mathbf{1}_{\Omega_0}\left(\left|u_{\rm sc}^{l-1}+u_{\rm inc}\right|^2 (u_{\rm sc}^l + u_{\rm inc}) - \left|u_{\rm inc}\right|^2 u_{\rm inc}\right) = \tilde f \text{ in } \Omega, \tag{2.22}$$

$$\frac{\partial u_{\rm sc}^l}{\partial n} + \mathbf{i}k u_{\rm sc}^l = 0 \text{ on } \Gamma, \tag{2.23}$$

where

$$\tilde f := f + \Delta u_{\rm inc} + k^2(1+\varepsilon \mathbf{1}_{\Omega_0}\left|u_{\rm inc}\right|^2)u_{\rm inc}.$$

By following our previous analysis for (2.18)–(2.19), one may show that the sequence $u_{\rm sc}^l$ converges if $|||u_{\rm sc}^0||| \lesssim \|\tilde f\|_0$, $\left\|u_{\rm sc}^0\right\|_{L^\infty(\Omega_0)} \le k^{\frac{d-3}{2}}\|\tilde f\|_0$, and

$$\max\left(k^{d-2}\varepsilon\|\tilde f\|_0^2,\ k\varepsilon\left\|u_{\rm inc}\right\|_{L^\infty(\Omega_0)}^2\right) \le \theta \tag{2.24}$$

for some constant θ sufficiently small. And as a consequence, the following estimates hold under the above conditions:

$$k\left\|u_{\rm sc}\right\|_0 + \left\|\nabla u_{\rm sc}\right\|_0 + k^{-1}\left\|u_{\rm sc}\right\|_2 + k^{\frac{3-d}{2}}\left\|u_{\rm sc}\right\|_{L^\infty(\Omega_0)} \lesssim \|\tilde f\|_0. \tag{2.25}$$

We omit the details.

3. Finite element methods and error estimates. We now discuss the finite element approximation of the NLH system (1.2)–(1.3). Let $\mathcal{T}_h$ be a quasi-uniform family of triangulations of size h with simplicial elements over the domain Ω. For any element $K \in \mathcal{T}_h$, we define $h_K := \operatorname{diam}(K)$ and $h = \max_{K\in\mathcal{T}_h} h_K$. Let V_h be the continuous piecewise linear finite element space associated with the triangulation $\mathcal{T}_h$:

$$V_h := \left\{v_h \in H^1(\Omega) :\ v_h|_K \in P_1(K) \quad \forall K \in \mathcal{T}_h\right\},$$

where $P_1(K)$ denotes the set of all linear polynomials on K.

Now we propose to approximate the solution to the NLH system (1.2)–(1.3) by the finite element solution $u_h \in V_h$ that solves the following equation for any $v_h \in V_h$:

$$(\nabla u_h, \nabla v_h) - k^2\big((1+\varepsilon\mathbf{1}_{\Omega_0}\left|u_h\right|^2)u_h, v_h\big) + \mathbf{i}k\left\langle u_h, v_h\right\rangle = (f, v_h) + \left\langle g, v_h\right\rangle. \tag{3.1}$$

For our subsequent analysis, we need an elliptic projection $P_h : H^1(\Omega) \mapsto V_h$ defined by

$$(\nabla v_h, \nabla P_h w) + \mathbf{i}k\left\langle v_h, P_h w\right\rangle = (\nabla v_h, \nabla w) + \mathbf{i}k\left\langle v_h, w\right\rangle \quad \forall v_h \in V_h. \tag{3.2}$$

The standard finite element error estimates for elliptic problems give the optimal approximation accuracies of the projection P_h in H^1- and L^2-norm (see [36, 10, 39]):

$$\left\|w - P_h w\right\|_0 \lesssim h\,|||w - P_h w||| \lesssim h^2\left|w\right|_2. \tag{3.3}$$

3.1. Discrete Nirenberg inequality. In this subsection, we derive a very important discrete version of the following Nirenberg inequality [33] for our subsequent analysis:

$$\left\|u\right\|_{L^\infty} \lesssim \left\|u\right\|_0^{1-\frac{d}{4}}\left|u\right|_2^{\frac{d}{4}} + \left\|u\right\|_0. \tag{3.4}$$

For this purpose, we first introduce the discrete Laplacian operator $A_h : V_h \mapsto V_h$:

$$(A_h v_h, w_h) = (\nabla v_h, \nabla w_h) + \mathbf{i}k \langle v_h, w_h \rangle \quad \forall w_h \in V_h. \tag{3.5}$$

Note that A_h can be viewed as a discrete version of the Laplacian operator $-\Delta$ under the impedance boundary condition $\frac{\partial u}{\partial n} + \mathbf{i}ku = 0$ on Γ and $\|A_h v_h\|_0$ as a discrete H^2-norm of any $v_h \in V_h$.

Now we can establish a discrete Nirenberg inequality that plays a crucial role in the analysis of the finite element solutions in (3.1) to the NLH system (1.2)–(1.3).

LEMMA 3.1. *It holds for $kh \lesssim 1$ that*

$$\|v_h\|_{L^\infty} \lesssim \|v_h\|_0^{1-\frac{d}{4}} \|A_h v_h\|_0^{\frac{d}{4}} + \|v_h\|_0 \quad \forall v_h \in V_h. \tag{3.6}$$

Proof. For any $v_h \in V_h$, let v be the solution to the elliptic problem:

$$-\Delta v = A_h v_h \quad \text{in } \Omega; \qquad \frac{\partial v}{\partial n} + \mathbf{i}kv = 0 \quad \text{on } \Gamma. \tag{3.7}$$

Clearly, we see that $v \in H^1(\Omega)$ satisfies

$$(\nabla v, \nabla w_h) + \mathbf{i}k \langle v, w_h \rangle = (A_h v_h, w_h) = (\nabla v_h, \nabla w_h) + \mathbf{i}k \langle v_h, w_h \rangle \quad \forall w_h \in V_h,$$

which indicates that v_h is the finite element approximation to the elliptic problem (3.7). Using the regularity and finite element theory for elliptic PDEs we know $v \in H^2(\Omega)$ and $\|v\|_2 \lesssim \|A_h v_h\|_0$ and the error estimates

$$h \|v - v_h\|_1 + \|v - v_h\|_0 \lesssim h^2 |v|_2 \lesssim h^2 \|A_h v_h\|, \tag{3.8}$$

whose proof is omitted (by using the fact that $v_h = \overline{P_h \overline{v}}$). Let $I_h v$ be the finite element interpolant of v. It follows from the inverse inequality and the interpolation error estimate that

$$\begin{aligned} \|v_h\|_{L^\infty} &\le \|v_h - I_h v\|_{L^\infty} + \|I_h v\|_{L^\infty} \lesssim h^{-\frac{d}{2}} \|v_h - I_h v\|_0 + \|v\|_{L^\infty} \\ &\lesssim h^{2-\frac{d}{2}} \|A_h v_h\|_0 + \|v\|_{L^\infty}. \end{aligned}$$

From (3.4),

$$\|v_h\|_{L^\infty} \lesssim h^{2-\frac{d}{2}} \|A_h v_h\|_0 + \|v\|_0^{1-\frac{d}{4}} \|A_h v_h\|_0^{\frac{d}{4}} + \|v\|_0. \tag{3.9}$$

By taking $w_h = A_h v_h$ in (3.5) and using the inverse inequality we have

$$\|A_h v_h\|_0^2 = (\nabla v_h, \nabla A_h v_h) + \mathbf{i}k \langle v_h, A_h v_h \rangle \lesssim \left(h^{-2} + kh^{-1}\right) \|v_h\|_0 \|A_h v_h\|_0,$$

which implies for $kh \lesssim 1$ that

$$\|A_h v_h\|_0 \lesssim h^{-2} \|v_h\|_0.$$

Then using this estimate and (3.8), we can get

$$\begin{aligned} \|v\|_0 &\le \|v_h\|_0 + \|v - v_h\|_0 \lesssim \|v_h\|_0 + h^2 \|A_h v_h\|_0 \lesssim \|v_h\|_0, \\ h^{2-\frac{d}{2}} \|A_h v_h\|_0 &= \left(h^2 \|A_h v_h\|_0\right)^{1-\frac{d}{4}} \|A_h v_h\|_0^{\frac{d}{4}} \lesssim \|v_h\|_0^{1-\frac{d}{4}} \|A_h v_h\|_0^{\frac{d}{4}}. \end{aligned}$$

Now the discrete Nirenberg inequality (3.6) follows by combining these two estimates and (3.9). This completes the proof of the lemma. ∎

3.2. A discrete auxiliary problem. We shall follow our analyses of the continuous NLH system (1.2)–(1.3) to study its finite element solutions in (3.1). So for a given function $\phi \in L^\infty(\Omega)$, we introduce the finite element approximation to the auxiliary problem (2.5). Find $u_h \in V_h$ that solves the equation for any $v_h \in V_h$:

$$(\nabla u_h, \nabla v_h) - k^2\big((1+\varepsilon \mathbf{1}_{\Omega_0} |\phi|^2)u_h, v_h\big) + \mathbf{i}k\langle u_h, v_h\rangle = (f, v_h) + \langle g, v_h\rangle\,. \tag{3.10}$$

Let $M(f,g)$ and θ_0 be two constants introduced in Lemma 2.2; then we can establish the stability of the finite element problem (3.10).

LEMMA 3.2. *If $k\varepsilon \|\phi\|^2_{L^\infty(\Omega_0)} \le \theta_0$, there exists a constant $C_0 > 0$ such that the finite element solutions u_h to the approximation system* (3.10) *are stable for $k^3h^2 \le C_0$:*

$$|||u_h||| \lesssim M(f,g)\,. \tag{3.11}$$

Proof. Just like the analysis we did for the continuous problem in subsection 2.1, we first take $v_h = u_h$ in (3.10) to obtain

$$k\|u_h\|^2_{0,\Gamma} \le 2|(f,u_h)| + \frac{1}{k}\|g\|^2_{0,\Gamma}\,, \tag{3.12}$$

$$\|\nabla u_h\|_0^2 - k^2\|u_h\|_0^2 - k^2\varepsilon\|\phi u_h\|^2_{0,\Omega_0} \le 2|(f,u_h)| + \frac{1}{k}\|g\|^2_{0,\Gamma}\,. \tag{3.13}$$

But the second test function we took in the analysis for the continuous case can not be copied now due to the fact that $\alpha \cdot \nabla u_h \notin V_h$. We circumvent this difficulty by a duality argument. Let $w \in H^2(\Omega)$ be the solution of the following problem:

$$-\Delta w - k^2\big(1+\varepsilon \mathbf{1}_{\Omega_0}|\phi|^2\big)w = u_h \quad \text{in } \Omega, \tag{3.14}$$

$$\frac{\partial w}{\partial n} - \mathbf{i}kw = 0 \quad \text{on } \Gamma. \tag{3.15}$$

Since the conjugate of w is the solution to (2.3)–(2.4) with $f = \overline{u_h}$ and $g = 0$, the following regularity estimate holds under the conditions of Lemma 2.2:

$$|||w||| \lesssim \|u_h\|_0 \quad \text{and} \quad \|w\|_2 \lesssim k\|u_h\|_0\,. \tag{3.16}$$

Now we multiply (3.14) by u_h and then apply (3.2) and (3.10) to obtain

$$\begin{aligned}
\|u_h\|_0^2 &= (\nabla u_h, \nabla w) - k^2\big((1+\varepsilon\mathbf{1}_{\Omega_0}|\phi|^2)u_h, w\big) + \mathbf{i}k\langle u_h, w\rangle \\
&= (\nabla u_h, \nabla P_h w) + \mathbf{i}k\langle u_h, P_h w\rangle - k^2\big((1+\varepsilon\mathbf{1}_{\Omega_0}|\phi|^2)u_h, w\big) \\
&= (f,w) + \langle g,w\rangle + (f, P_hw - w) + \langle g, P_hw - w\rangle \\
&\quad - k^2(u_h, w - P_hw) - k^2\varepsilon\big(\mathbf{1}_{\Omega_0}|\phi|^2u_h, w - P_hw\big).
\end{aligned}$$

Using the solution u to the auxiliary problem (2.3)–(2.4), we know from (2.5) and (3.14)–(3.15) that $(f,w) + \langle g,w\rangle = (u,u_h)$. Then we can further derive from (3.3), Lemma 2.2, and (3.16) that

$$\begin{aligned}
\|u_h\|_0^2 \lesssim{}& |(u,u_h)| + \|f\|_0\|w - P_hw\|_0 + \|g\|_{0,\Gamma}\|w - P_hw\|_{0,\Gamma} + k^2\|u_h\|_0\|w-P_hw\|_0 \\
&+ k\theta_0\|u_h\|_0\|w - P_hw\|_0 \\
\lesssim{}& k^{-1}M(f,g)\big(\|u_h\|_0 + kh^{\frac{3}{2}}|w|_2\big) + \big(k^2h^2 + k\theta_0h^2\big)\|u_h\|_0|w|_2 \\
\lesssim{}& k^{-1}M(f,g)\big(1 + k^2h^{\frac{3}{2}}\big)\|u_h\|_0 + \big(k^3h^2 + \theta_0k^2h^2\big)\|u_h\|_0^2,
\end{aligned}$$

and after canceling the common factor $\|u_h\|_0$, we come to

$$\|u_h\|_0 \lesssim k^{-1}M(f,g) + \big(k^3h^2 + \theta_0 k^2h^2\big)\, \|u_h\|_0 \,.$$

This indicates the existence of a constant $C_0 > 0$ such that

$$\|u_h\|_0 \lesssim k^{-1}M(f,g) \tag{3.17}$$

if $k^3h^2 \le C_0$. Next we estimate $\|\nabla u_h\|_0$. It follows from (3.13) that

$$\begin{aligned} \|\nabla u_h\|_0^2 &\le k^2\, \|u_h\|_0^2 + k^2\varepsilon\, \|\phi u_h\|_{0,\Omega_0}^2 + 2\,|(f,u_h)| + \frac{1}{k}\, \|g\|_{0,\Gamma}^2 \\ &\lesssim k^2\, \|u_h\|_0^2 + k^2\varepsilon\, \|\phi\|_{L^\infty(\Omega_0)}^2\, \|u_h\|_0^2 + \frac{1}{k^2}\, \|f\|_0^2 + \frac{1}{k}\, \|g\|_{0,\Gamma}^2 \\ &\lesssim M(f,g)^2. \end{aligned} \tag{3.18}$$

This, along with (3.17), yields the desired estimate (3.11). ∎

The next lemma provides the estimates of the error between the solution to the continuous problem (2.5) and its finite element solution to the discretization (3.10), where θ_0 and C_0 are two constants introduced in Lemmas 2.2 and 3.2, respectively.

LEMMA 3.3. *Let u and u_h be the solutions to* (2.5) *and* (3.10)*, respectively. Then the error estimates hold under the conditions that $k\varepsilon\, \|\phi\|_{L^\infty(\Omega_0)}^2 \le \theta_0$ and $k^3h^2 \le C_0$:*

$$|||u - u_h||| \lesssim (kh + k^3h^2)\widehat{M}(f,g) \quad \text{and} \quad \|u - u_h\|_0 \lesssim k^2h^2\widehat{M}(f,g). \tag{3.19}$$

Proof. Let $\tilde{u}_h = \overline{P_h\overline{u}}$ be the elliptic projection of u; then it follows readily from the definition (3.2) of P_h that

$$(\nabla\tilde{u}_h, \nabla v_h) + \mathbf{i}k\,\langle\tilde{u}_h, v_h\rangle = (\nabla u, \nabla v_h) + \mathbf{i}k\,\langle u, v_h\rangle \quad \forall v_h \in V_h, \tag{3.20}$$

which, along with (3.3) and Lemma 2.2, implies

$$\|u - \tilde{u}_h\|_0 + h\,|||u - \tilde{u}_h||| \lesssim kh^2\widehat{M}(f,g). \tag{3.21}$$

It remains to estimate $\eta_h := u_h - \tilde{u}_h$. For any $v_h \in V_h$, we can see that η_h solves

$$\begin{aligned} &(\nabla\eta_h, \nabla v_h) - k^2\big(\big(1 + \varepsilon\mathbf{1}_{\Omega_0}\,|\phi|^2\big)\eta_h, v_h\big) + \mathbf{i}k\,\langle\eta_h, v_h\rangle \\ &\quad = (f, v_h) + \langle g, v_h\rangle - (\nabla u, v_h) - \mathbf{i}k\,\langle u, v_h\rangle + k^2\big(\big(1 + \varepsilon\mathbf{1}_{\Omega_0}\,|\phi|^2\big)\tilde{u}_h, v_h\big) \\ &\quad = k^2\big(\big(1 + \varepsilon\mathbf{1}_{\Omega_0}\,|\phi|^2\big)(\tilde{u}_h - u), v_h\big). \end{aligned} \tag{3.22}$$

Applying the stability estimate of Lemma 3.2 to this problem leads to

$$|||\eta_h||| \lesssim k^2\big\|\big(1 + \varepsilon\mathbf{1}_{\Omega_0}\,|\phi|^2\big)(\tilde{u}_h - u)\big\|_0 \lesssim k^2\big\|\tilde{u}_h - u\big\|_0 \lesssim k^3h^2\widehat{M}(f,g).$$

This, combining (3.21), gives the desired estimate (3.19) by the triangle inequality. ∎

We end this subsection with an L^∞ estimate of the finite element solution u_h to the approximation (3.10), and this estimate is also essential to our subsequent analysis.

LEMMA 3.4. *Under the same conditions as in Lemma* 3.3*, the following L^∞ estimate holds for the finite element solution u_h to the system* (3.10)*:*

$$\|u_h\|_{L^\infty(\Omega_0)} \lesssim |\ln h|\, k^{\frac{d-3}{2}}\widehat{M}(f,g).$$

Proof. As in the proof of Lemma 3.3, we let $\tilde{u}_h = \overline{P_h \overline{u}}$ and $\eta_h = u_h - \tilde{u}_h$. Then we have by the triangle inequality that

$$\|u_h\|_{L^\infty(\Omega_0)} \le \|\eta_h\|_{L^\infty(\Omega_0)} + \|\tilde{u}_h - u\|_{L^\infty(\Omega_0)} + \|u\|_{L^\infty(\Omega_0)}. \tag{3.23}$$

Then it remains to estimate the first two terms on the right-hand side. It follows first from the interior maximum-norm estimates for finite element solutions [34, Theorem 5.1], Lemma 2.3, (3.3), and Lemma 2.2 that

$$\begin{aligned} \|u - \tilde{u}_h\|_{L^\infty(\Omega_0)} &\lesssim |\ln h|\, \|u\|_{L^\infty(\Omega)} + \|u - \tilde{u}_h\|_0 \lesssim |\ln h|\, \|u\|_{L^\infty(\Omega)} + h\, \|u\| \\ &\lesssim \Big(|\ln h| k^{\frac{d-3}{2}} + h\Big) M(f,g) \lesssim |\ln h| k^{\frac{d-3}{2}} M(f,g), \end{aligned} \tag{3.24}$$

where we have used $k^3 h^2 \le C_0$ to derive the last inequality.

Now we estimate $\|\eta_h\|_{L^\infty}$. From the definition (3.5) of A_h and (3.22), we have

$$(A_h \eta_h, v_h) = k^2 \big((1 + \varepsilon \mathbf{1}_{\Omega_0} |\phi|^2)(u_h - u), v_h \big) \quad \forall v_h \in V_h.$$

Clearly, we can derive from (3.19) and (3.21) that

$$\|\eta_h\|_0 \lesssim \|u_h - u\|_0 + \|u - \tilde{u}_h\|_0 \lesssim k^2 h^2 \widehat{M}(f,g), \tag{3.25}$$

$$\|A_h \eta_h\|_0 \lesssim k^2 \big\| (1 + \varepsilon \mathbf{1}_{\Omega_0} |\phi|^2)(u_h - u) \big\|_0 \lesssim k^4 h^2 \widehat{M}(f,g). \tag{3.26}$$

Now we define $\eta \in H^1(\Omega)$ by the variational equation:

$$(\nabla \eta, \nabla v) + \mathbf{i} k \langle \eta, v \rangle = k^2 \big((1 + \varepsilon \mathbf{1}_{\Omega_0} |\phi|^2)(u_h - u), v \big) \quad \forall v \in H^1(\Omega). \tag{3.27}$$

Using the regularity estimate of the elliptic PDEs, we have

$$\|\eta\|_2 \lesssim k^2 \big\| (1 + \varepsilon \mathbf{1}_{\Omega_0} |\phi|^2)(u_h - u) \big\|_0 \lesssim k^4 h^2 \widehat{M}(f,g). \tag{3.28}$$

To go further, we can easily verify that

$$(\nabla \eta_h, \nabla v) + \mathbf{i} k \langle \eta_h, v \rangle = (\nabla \eta, \nabla v) + \mathbf{i} k \langle \eta, v \rangle. \tag{3.29}$$

So η_h can be viewed as the finite element approximation of η, and it then follows from the standard arguments for the Céa lemma and the interpolation error estimates that

$$\|\eta - \eta_h\|_0 \lesssim h^2 |\eta|_2 \lesssim k^4 h^4 \widehat{M}(f,g). \tag{3.30}$$

Now we take a subdomain Ω_1 in Ω such that $\Omega_0 \subset \Omega_1$ and

$$\operatorname{dist}(\partial\Omega_0, \partial\Omega_1) \eqsim \operatorname{dist}(\partial\Omega_1, \partial\Omega) \eqsim 1;$$

then we get from (3.29) and the interior maximum-norm estimates [34, Theorem 5.1]

$$\begin{aligned} \|\eta - \eta_h\|_{L^\infty(\Omega_0)} &\lesssim |\ln h|\, \|\eta - I_h \eta\|_{L^\infty(\Omega_1)} + \|\eta - \eta_h\|_0 \\ &\lesssim |\ln h| h^2 |\eta|_{W^{2,\infty}(\Omega_1)} + k^4 h^4 \widehat{M}(f,g). \end{aligned}$$

From (3.27), the Schauder interior estimates for the elliptic equations [24], and the Nirenberg and discrete Nirenberg inequalities (3.4) and (3.6), we conclude that

$$\begin{aligned} \|\eta\|_{W^{2,\infty}(\Omega_1)} &\lesssim \|\eta\|_{L^\infty(\Omega)} + k^2 \|\eta_h\|_{L^\infty(\Omega)} + k^2 \|\tilde{u}_h - u\|_{L^\infty(\Omega)} \\ &\lesssim \|\eta\|_{L^\infty(\Omega)} + k^2 \|\eta_h\|_{L^\infty(\Omega)} + k^2 \|\tilde{u}_h - I_h u\|_{L^\infty(\Omega)} + k^2 \|u\|_{L^\infty(\Omega)} \\ &\lesssim \|\eta\|_2 + k^2 \|\eta_h\|_0^{1-\frac{d}{4}} \|A_h \eta_h\|_0^{\frac{d}{4}} \\ &\qquad + k^2 \|\tilde{u}_h - I_h u\|_0^{1-\frac{d}{4}} \|A_h(\tilde{u}_h - I_h u)\|_0^{\frac{d}{4}} + k^2 \|u\|_0^{1-\frac{d}{4}} \|u\|_2^{\frac{d}{4}} \\ &\lesssim k^2 k^{\frac{d}{2}+2} h^2 \widehat{M}(f,g) + k^{\frac{d}{2}+1} \widehat{M}(f,g), \end{aligned}$$

where we have used (3.28), (3.25)–(3.26), and Lemma 2.2 to derive the last inequality. By combining the above two estimates and using $k^3h^2 \le C_0$, we have

$$\begin{aligned} \|\eta - \eta_h\|_{L^\infty(\Omega_0)} &\lesssim |\ln h| \left(k^{\frac{d}{2}+4}h^4 + k^{\frac{d}{2}+1}h^2\right) \widehat{M}(f,g) + k^4h^4\widehat{M}(f,g) \\ &= |\ln h| h^{\frac{1}{3}} \left(k^{\frac{11}{2}}h^{\frac{11}{3}} + k^{\frac{5}{2}}h^{\frac{5}{3}}\right) k^{\frac{d-3}{2}}\widehat{M}(f,g) + k^4h^4\widehat{M}(f,g) \\ &\lesssim k^{\frac{d-3}{2}}\widehat{M}(f,g). \end{aligned} \tag{3.31}$$

On the other hand, by rewriting (3.27) we can see that $\eta \in H^1(\Omega)$ solves the equation

$$\begin{aligned} (\nabla\eta, \nabla v) + \mathbf{i}k\langle \eta, v\rangle - k^2(\eta, v) = k^2(\eta_h - \eta, v) + k^2\big(\tilde{u}_h - u, v\big) \\ + k^2\big(\varepsilon \mathbf{1}_{\Omega_0}|\phi|^2(u_h - u), v\big) \quad \forall v \in H^1(\Omega), \end{aligned}$$

so we can apply Lemma 2.3 to obtain

$$\begin{aligned} \|\eta\|_{L^\infty(\Omega_0)} &\lesssim k^{\frac{d-3}{2}}k^2\big(\|\eta_h - \eta\|_0 + \|\tilde{u}_h - u\|_0 + k^{-1}\|u_h - u\|_0\big) \\ &\lesssim k^{\frac{d-3}{2}}k^2\big(k^4h^4 + kh^2\big)\widehat{M}(f,g) \\ &= \big(k^6h^4 + k^3h^2\big)k^{\frac{d-3}{2}}\widehat{M}(f,g) \lesssim k^{\frac{d-3}{2}}\widehat{M}(f,g). \end{aligned}$$

Combining this and (3.31) gives

$$\|\eta_h\|_{L^\infty(\Omega_0)} \lesssim k^{\frac{d-3}{2}}\widehat{M}(f,g). \tag{3.32}$$

Then the desired L^∞ estimate follows readily from (2.13), (3.23)–(3.24), and (3.32). ∎

3.3. Existence of the finite element solution. Following the analysis for the continuous NLH equation in section 2.2, we consider an iterative procedure to establish the existence and stability of the finite element solutions to the discrete NLH system (3.1): for a given $u_h^0 \in V_h$, find $u_h^l \in V_h$ for $l = 1, 2, \ldots$ such that

$$\big(\nabla u_h^l, \nabla v_h\big) - k^2\left(\left(1 + \varepsilon\mathbf{1}_{\Omega_0}\left|u_h^{l-1}\right|^2\right)u_h^l, v_h\right) + \mathbf{i}k\left\langle u_h^l, v_h\right\rangle = (f, v_h) + \langle g, v_h\rangle \quad \forall v_h \in V_h. \tag{3.33}$$

The following lemma gives the stability estimates of this sequence u_h^l for $l \ge 1$.

Lemma 3.5. *There exists a constant $\theta_3 > 0$ such that if $k\varepsilon\left\|u_h^0\right\|^2_{L^\infty(\Omega_0)} \le |\ln h|^2 k^{d-2}\varepsilon\widehat{M}(f,g)^2 \le \theta_3$ and $k^3h^2 \le C_0$ (from Lemma 3.2), then the following stability estimates hold for $l = 1, 2, \ldots$:*

$$|||u_h^l||| \lesssim M(f,g) \quad \text{and} \quad \left\|u_h^l\right\|_{L^\infty(\Omega_0)} \lesssim |\ln h| k^{\frac{d-3}{2}}\widehat{M}(f,g). \tag{3.34}$$

Proof. First we can easily see that if $k\varepsilon\left\|u_h^{l-1}\right\|^2_{L^\infty(\Omega_0)} \le \theta_0$, then (3.34) follows directly from Lemmas 3.2 and 3.4. This implies immediately the existence of a constant $\theta_3 > 0$ such that

$$k\varepsilon\left\|u_h^l\right\|^2_{L^\infty(\Omega_0)} \le C|\ln h|^2k^{d-2}\varepsilon\widehat{M}(f,g)^2 \le \theta_0$$

if $|\ln h|^2\ k^{d-2}\varepsilon\widehat{M}(f,g)^2 \le \theta_3$. Now the proof of the lemma follows by induction. ∎

We are now ready to show the convergence of the sequence $\{u_h^l\}$ to a finite element solution of the discrete NLH system (3.1) under proper conditions.

THEOREM 3.6. *There exists a constant $\widetilde{C} > 0$ such that if $k^3h^2 \le C_0$ (from Lemma 3.2) and $\sigma := \widetilde{C}|\ln h|^2 k^{d-2}\varepsilon\widehat{M}(f,g)^2 < 1$, then the finite element system (3.1) attains a unique solution u_h satisfying the stability estimates:*

$$\|\nabla u_h\|_0 + k\,\|u_h\|_0 \lesssim M(f,g) \quad \text{and} \quad \|u_h\|_{L^\infty(\Omega_0)} \lesssim |\ln h| k^{\frac{d-3}{2}}\widehat{M}(f,g). \tag{3.35}$$

Moreover, if $|||u_h^0||| \le M(f,g)$ and $\|u_h^0\|_{L^\infty(\Omega_0)} \le |\ln h| k^{\frac{d-3}{2}}\widehat{M}(f,g)$, then the iterative scheme (3.33) converges at a rate given by

$$\left|\left|\left|u_h^l - u_h\right|\right|\right| \lesssim \sigma^l M(f,g). \tag{3.36}$$

Proof. It is easy to verify using the iterative scheme (3.33) that the difference $v_h^l := u_h^{l+1} - u_h^l$ solves the following equation:

$$\begin{aligned}(\nabla v_h^l, \nabla v_h) - k^2\left(\left(1+\varepsilon\mathbf{1}_{\Omega_0}\left|u_h^l\right|^2\right)v_h^l, v_h\right) + \mathbf{i}k\left\langle v_h^l, v_h\right\rangle \\ = k^2\varepsilon\left(\mathbf{1}_{\Omega_0}u_h^l\left(\left|u_h^l\right|^2 - \left|u_h^{l-1}\right|^2\right), v_h\right) \quad \forall v_h \in V_h.\end{aligned}$$

Under the conditions of Lemma 3.5, we have $k\varepsilon\left\|u_h^l\right\|^2_{L^\infty(\Omega_0)} \le \theta_0$. Then we conclude from Lemmas 3.2 and 3.5 that

$$\begin{aligned}|||v_h^l||| &\lesssim k^2\varepsilon\left\|u_h^l\left(\left|u_h^l\right|^2 - \left|u_h^{l-1}\right|^2\right)\right\|_{0,\Omega_0} \\ &\lesssim k\varepsilon\left\|u_h^l\right\|_{L^\infty(\Omega_0)}\left(\left\|u_h^l\right\|_{L^\infty(\Omega_0)} + \left\|u_h^{l-1}\right\|_{L^\infty(\Omega_0)}\right)|||v_h^{l-1}||| \\ &\lesssim |\ln h|^2 k^{d-2}\varepsilon\widehat{M}(f,g)^2\left|\left|\left|v_h^{l-1}\right|\right|\right|.\end{aligned}$$

That is, $|||v_h^l||| \le \widetilde{C}|\ln h|^2k^{d-2}\varepsilon\widehat{M}(f,g)^2|||v_h^{l-1}|||$ for some constant $\widetilde{C} > 0$. Therefore $\{u_h^l\}$ is a Cauchy sequence if $\sigma = \widetilde{C}|\ln h|^2k^{d-2}\varepsilon\widehat{M}(f,g)^2 < 1$ and converges, say, to u_h. It is easy to check from (3.33) that u_h solves (3.1), and the estimates (3.35) are a consequence of (3.34). The uniqueness of the solutions u_h to (3.1) can be proved in a similar manner to the one for Theorem 2.5, and the details are omitted.

To see the error estimate (3.36), we recall the previous estimate $|||v_h^l||| \le \sigma|||v_h^{l-1}|||$. Then we readily obtain $|||v_h^l||| \le \sigma^l|||v_h^0||| \lesssim \sigma^l M(f,g)$ and $|||u_h^l - u_h||| \lesssim \sigma^l/(1-\sigma)M(f,g)$. This completes the proof of the theorem. ∎

3.4. Error estimates of finite element solutions. In this subsection we estimate the error between the continuous solution u to the NLH system (1.2)–(1.3) and its finite element solution u_h to the discrete NLH system (3.1).

THEOREM 3.7. *There exist constants $C_0, C_1, C_2, \theta > 0$ such that if $k^3h^2 \le C_0$ and $|\ln h|^2k^{d-2}\varepsilon\widehat{M}(f,g)^2 \le \theta$; then the following error estimate between the finite element solution u_h to (3.1) and the NLH solution u to (1.2)–(1.3) holds:*

$$|||u - u_h||| \le (C_1kh + C_2k^3h^2)\widehat{M}(f,g). \tag{3.37}$$

Proof. We know from Theorems 2.5 and 3.6 that u and u_h are the limits of two sequences $\{u^l\}$ and $\{u_h^l\}$ defined by the systems (2.18)–(2.19) and (3.33), respectively. So it is natural to estimate the error $u^l - u_h^l$.

We know from (2.18)–(2.19) that u^l solves the variational formulation for $l = 1, 2, \dots$ and $v \in H^1(\Omega)$:

$$(\nabla u^l, \nabla v) - k^2\big((1+\varepsilon \mathbf{1}_{\Omega_0} \left|u^{l-1}\right|^2)u^l, v\big) + \mathbf{i}k\left\langle u^l, v\right\rangle = (f, v) + \langle g, v\rangle . \tag{3.38}$$

We now define $\tilde{u}_h^0 = u_h^0$ and $\tilde{u}_h^l \in V_h$ for $l = 1, 2, \dots$ to be the solution to the following problem for all $v_h \in V_h$:

$$(\nabla \tilde{u}_h^l, \nabla v_h) - k^2\big((1+\varepsilon \mathbf{1}_{\Omega_0} \left|u^{l-1}\right|^2)\tilde{u}_h^l, v_h\big) + \mathbf{i}k\left\langle \tilde{u}_h^l, v_h\right\rangle = (f, v_h) + \langle g, v_h\rangle . \tag{3.39}$$

Clearly we can apply Lemma 3.3 with $\phi = u^{l-1}$ to the system (3.38) and its finite element approximation (3.39) to get for $l \geq 1$

$$|||u^l - \tilde{u}_h^l||| \lesssim (kh + k^3h^2)\widehat{M}(f, g). \tag{3.40}$$

Using $u^l - u_h^l = (u^l - \tilde{u}_h^l) + (\tilde{u}_h^l - u_h^l)$, we still need to estimate $\eta_h^l := \tilde{u}_h^l - u_h^l$. We know from (3.33) and (3.39) that $\eta_h^l \in V_h$ solves

$$\begin{aligned}(\nabla \eta_h^l, \nabla v_h) &- k^2\left(\left(1+\varepsilon \mathbf{1}_{\Omega_0} \left|u^{l-1}\right|^2\right)\eta_h^l, v_h\right) + \mathbf{i}k\left\langle \eta_h^l, v_h\right\rangle \\ &= k^2\varepsilon\left(\mathbf{1}_{\Omega_0}\left(\left|u^{l-1}\right|^2 - \left|u_h^{l-1}\right|^2\right)u_h^l, v_h\right) \quad \forall v_h \in V_h.\end{aligned} \tag{3.41}$$

Then we can apply the stability estimate in Lemmas 3.2, 2.4, and 3.5 to obtain

$$\begin{aligned}|||\eta_h^l||| &\lesssim k^2\varepsilon\big\|\big(\left|u^{l-1}\right|^2 - \left|u_h^{l-1}\right|^2\big)u_h^l\big\|_{0,\Omega_0} \lesssim k^2\varepsilon\big(|\ln h|k^{\frac{d-3}{2}}\widehat{M}(f,g)\big)^2\big\|u^{l-1} - u_h^{l-1}\big\|_0 \\ &\lesssim |\ln h|^2k^{d-2}\varepsilon\widehat{M}(f,g)^2\big(|||u^{l-1} - \tilde{u}_h^{l-1}||| + |||\eta_h^{l-1}|||\big).\end{aligned}$$

Clearly, if $|\ln h|^2k^{d-2}\varepsilon\widehat{M}(f,g)^2$ is sufficiently small, then

$$|||\eta_h^l||| \leq \frac{1}{2}\,|||u^{l-1} - \tilde{u}_h^{l-1}||| + \frac{1}{2}\,|||\eta_h^{l-1}||| .$$

Noting that $\eta_h^0 = 0$, by induction and using (3.40) we conclude that

$$|||\eta_h^l||| \lesssim \sum_{j=0}^{l-1} 2^{j-l}\,\left|\left|\left|u^j - \tilde{u}_h^j\right|\right|\right| \lesssim (kh + k^3h^2)\widehat{M}(f,g) + 2^{-l}\,|||u^0 - u_h^0||| ; \tag{3.42}$$

now combining (3.40) and (3.42) gives

$$|||u^l - u_h^l||| \lesssim (kh + k^3h^2)\widehat{M}(f,g) + 2^{-l}\,|||u^0 - u_h^0||| .$$

Then (3.37) follows by letting $l \to \infty$. This completes the proof of the theorem. ∎

Remark 3.1. As discussed in Remark 2.1, one may show that the iterative scheme (3.33) still converges if the conditions in Lemma 3.5 are replaced by the following conditions:

$$k^3h^2 \leq C_0, \tag{3.43}$$

$$|||u_h^0 - u_{\rm inc}||| \lesssim \|\tilde{f}\|, \quad \big\|u_h^0 - u_{\rm inc}\big\|_{L^\infty(\Omega_0)} \leq |\ln h|k^{\frac{d-3}{2}}\|\tilde{f}\|_0, \tag{3.44}$$

$$\max\left(|\ln h|^2k^{d-2}\varepsilon\|\tilde{f}\|_0^2,\ k\varepsilon\left\|u_{\rm inc}\right\|_{L^\infty(\Omega_0)}^2\right) \leq \theta \tag{3.45}$$

for some constant θ sufficiently small. As a consequence, the limiting solution u_h satisfies the following error estimate under the conditions (3.43) and (3.45):

$$|||u - u_h||| \leq (C_1kh + C_2k^3h^2)\|\tilde{f}\|_0. \tag{3.46}$$

The details are omitted.

4. Continuous interior penalty finite element method. It is well known that the standard FEMs like we used in (3.1) have the strong pollution effect in approximating the linear Helmholtz equation (i.e., $\varepsilon = 0$ in (1.2)) with high wave number, that is, they do not produce optimal convergence. This has been widely studied in the literature, along with efficient numerical solvers for finite element systems arising from the Helmholtz equations; see [2, 3, 13, 14, 19, 25, 26, 29, 31, 32] and the references therein. There are different finite element strategies to reduce such pollution effects, among which the continuous interior penalty finite element method (CIP-FEM) has been proved to be very effective in reducing pollution errors essentially [36, 39, 18, 27, 11].

We shall now introduce the CIP-FEM, which is done by adding some appropriate penalty terms on the jumps of the fluxes across interior edges/faces to the finite element system (3.1). Let $\mathcal{E}_h^I$ be the set of all interior edges/faces of $\mathcal{T}_h$. For every $e = \partial K \cap \partial K' \in \mathcal{E}_h^I$, let n_e be a unit normal vector to e and $[v]$ be the jump of v on e, given by $[v]\,|_e := v|_{K'} - v|_K$.

We define the "energy" space V and the sesquilinear form $a_\gamma(\cdot,\cdot)$ on $V \times V$ as

$$V := H^1(\Omega) \cap \prod_{K \in \mathcal{T}_h} H^2(K),$$

$$a_\gamma(u,v) := (\nabla u, \nabla v) + J(u,v) \qquad \forall\, u, v \in V, \tag{4.1}$$

$$J(u,v) := \sum_{e \in \mathcal{E}_h^I} \gamma_e\, h_e \left\langle \left[\frac{\partial u}{\partial n_e}\right], \left[\frac{\partial v}{\partial n_e}\right] \right\rangle_e, \tag{4.2}$$

where γ_e for $e \in \mathcal{E}_h^I$ are called the penalty parameters, which are complex numbers with nonnegative imaginary parts. It is clear that $J(u,v) = 0$ if $u \in H^2(\Omega)$ and $v \in V$. Therefore, if $u \in H^2(\Omega)$ is the solution of (1.2)–(1.3), then

$$a_\gamma(u,v) - k^2\big((1 + \varepsilon \mathbf{1}_{\Omega_0} |u|^2)u, v\big) + \mathbf{i}k \langle u, v\rangle = (f, v) + \langle g, v\rangle \quad \forall v \in V.$$

This motivates the definition of the CIP-FEM: Find $u_h \in V_h$ such that

$$a_\gamma(u_h,v_h) - k^2\big((1 + \varepsilon \mathbf{1}_{\Omega_0} |u_h|^2)u_h, v_h\big) + \mathbf{i}k \langle u_h, v_h\rangle = (f, v_h) + \langle g, v_h\rangle \quad \forall v_h \in V_h. \tag{4.3}$$

Similarly to the iteration (3.33), we may consider the iterative method for the CIP-FEM system (4.3): for a given $u_h^0 \in V_h$, find $u_h^l \in V_h$ for $l = 1, 2, \cdots$ such that

$$a_\gamma(u_h^l,v_h) - k^2\left(\left(1 + \varepsilon \mathbf{1}_{\Omega_0} \left|u_h^{l-1}\right|^2\right) u_h^l, v_h\right) + \mathbf{i}k \left\langle u_h^l, v_h\right\rangle = (f, v_h) + \langle g, v_h\rangle \quad \forall v_h \in V_h. \tag{4.4}$$

Compared with our earlier standard FEM (3.1), the CIP-FEM (4.3) has added a bilinear form $J(u,v)$ that collects the so-called penalty terms, one from each interior edge/face of $\mathcal{T}_h$. Clearly, the CIP-FEM reduces to the standard FEM (3.1) when the penalty parameters γ_e in $J(u,v)$ are turned off.

The CIP-FEM (4.3) was analyzed systematically in [36, 39, 18] for the linear Helmholtz problem, i.e., $\varepsilon = 0$ in (1.2) and (4.3), and shown to be absolutely stable for penalty parameters γ_e with positive imaginary parts. Optimal order preasymptotic error estimates were also derived, and the penalty parameters may be tuned to reduce the pollution errors significantly [36, 39, 18, 27, 11]. By following the technical derivations and development in section 3, we can establish the stability estimates in Theorem 3.6 and the error estimates in Theorem 3.7 also for the above CIP-FEM. We omit the tedious technical details here.

Remark 4.1. (1) Penalizing the jumps of normal derivatives across interior edges or faces of a finite element mesh was used by Douglas and Dupont [17] for second-order PDEs, by Babuška and Zlámal [4] for the fourth-order PDEs in the context of C^0 finite element methods, by Baker [5] for the fourth-order PDEs, and by Arnold [1] for second-order parabolic PDEs in the context of interior penalty discontinuous Galerkin methods.

(2) We have considered in this work the scattering problem of the time dependence $e^{\mathbf{i}\omega t}$, which corresponds to the positive sign before $\mathbf{i}$ in (1.3). If the scattering problem of the time dependence $e^{-\mathbf{i}\omega t}$ is considered instead, then the sign before $\mathbf{i}$ in (1.3) should change, and the penalty parameters γ_e in $J(u, v)$ are complex numbers with nonpositive imaginary parts.

Remark 4.2. In [38], Yuan and Lu proposed the following modified Newton's method for the NLH (1.2):

$$-\Delta u^l - k^2 \left(1 + 2\varepsilon \mathbf{1}_{\Omega_0} \left|u^{l-1}\right|^2\right) u^l = f - k^2 \varepsilon \mathbf{1}_{\Omega_0} \left|u^{l-1}\right|^2 u^{l-1} \quad \text{in } \Omega\,. \tag{4.5}$$

The corresponding variant of this iterative method for the CIP-FEM system (4.3) takes the following form: for a given $u_h^0 \in V_h$, find $u_h^l \in V_h$ for $l = 1, 2, \ldots$ such that

$$\begin{aligned} a_\gamma(u_h^l, v_h) - k^2\big(\big(1 + 2\varepsilon \mathbf{1}_{\Omega_0} \left|u_h^{l-1}\right|^2\big)u_h^l, v_h\big) + \mathbf{i}k \left\langle u_h^l, v_h \right\rangle \\ = (f - k^2 \varepsilon \mathbf{1}_{\Omega_0} \left|u_h^{l-1}\right|^2 u_h^{l-1}, v_h) + \langle g, v_h \rangle \quad \forall v_h \in V_h. \end{aligned} \tag{4.6}$$

As we may observe, this iterative formula is quite similar to that of (2.18). So the convergence may be established for both the modified Newton's method (4.5) and its CIP-FE discretization (4.6) by following the same arguments as that for Theorems 2.5 and 3.6. We omit the details.

5. Numerical examples. We consider the NLH (1.2)–(1.3) defined on the domain composed of two regular hexagons with their common center being the origin and radiuses being 1 and $\frac{1}{2}$, respectively. For an even number $n > 0$, let $\mathcal{T}_h$ be the equilateral triangulation of mesh size $h = 1/n$. The penalty parameters for CIP-FEM are chosen as

$$\gamma_e \equiv \gamma = -\frac{\sqrt{3}}{24} - \frac{\sqrt{3}}{1728}(kh)^2,$$

which are able to remove the leading term of the dispersion error [27].

5.1. Accuracy of FEM and CIP-FEM. We examine the accuracy of the two methods FEM and CIP-FEM by taking the Kerr constant to be $\varepsilon = k^{-2}$ and the exact solution (cf. [12])

$$u = \frac{5\sqrt{2}\, e^{\mathbf{i}y\sqrt{k^2+25}}}{\sqrt{\varepsilon}\, k\, \cosh(5\,x)}.$$

Figure 5.1 plots the real part of $I_h u$, $\left|u_h^{\text{FEM}} - I_h u\right|$, and $\left|u_h^{\text{CIP-FEM}} - I_h u\right|$ for $k = 100$ and $h = 1/200$. It was shown that the standard FEM provides a wrong approximation, while the CIP-FEM gives the desired approximation of the exact solution.

Figure 5.2 plots the relative error in energy norm of the interpolant, the FE solution, and the CIP-FE solution for $k = 10 : 10 : 500$ with fixed $kh = 1$ and $kh = \frac{1}{2}$, respectively. It is shown that the interpolant is pollution-free and the FE solution suffers from obvious pollution effect, while the CIP-FE solution is almost pollution-free for k up to 500.

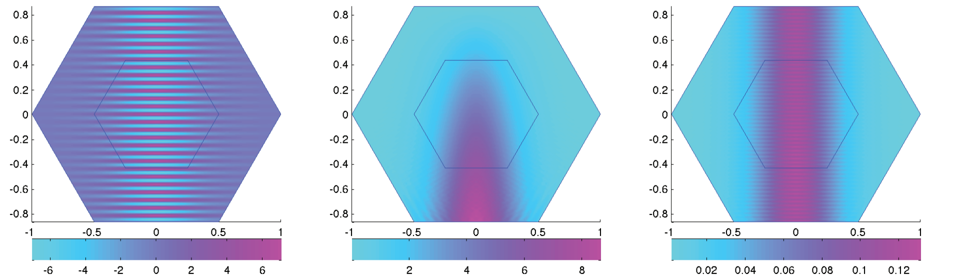

FIG. 5.1. $k = 100, h = 1/200$. *Left: Real part of* $I_h u$*; middle:* $|u_h^{\mathrm{FEM}} - I_h u|$*; right:* $|u_h^{\mathrm{CIP-FEM}} - I_h u|$.

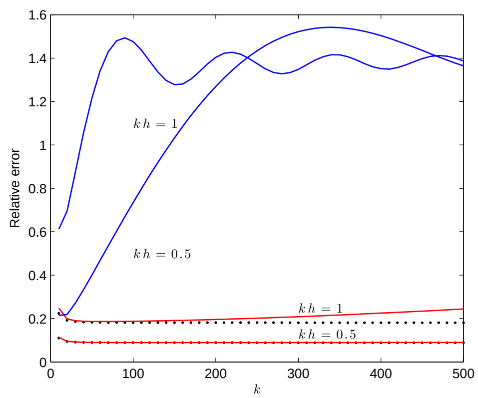

FIG. 5.2. $kh = 1, 0.5$, $k = 10 : 10 : 500$. *Relative error in energy norm. Dotted: interpolation; blue: FEM; red: CIP-FEM.*

5.2. Collision of nonparaxial solitons. Unlike the Schrödinger equation commonly used in the nonlinear optics, the NLH has no preferred direction of propagation. Therefore, it can be used to model the interaction of beams traveling at different angles. To demonstrate this capability, we solve the NLH with $\varepsilon = k^{-2}$ and incident wave

$$u_{\rm inc} = u_{\rm inc}^1 + u_{\rm inc}^2 := \frac{20\sqrt{2}\,\mathrm{e}^{\mathbf{i}y\sqrt{k^2+400}}}{\cosh(20\,x)} + \frac{20\sqrt{2}\,\mathrm{e}^{\mathbf{i}\sqrt{k^2+400}\left(\frac{y}{2}-\frac{\sqrt{3}\,x}{2}\right)}}{\cosh\left(20\left(\frac{1}{2}x+\frac{\sqrt{3}}{2}\,y\right)\right)}.$$

Note that the incident wave $u_{\rm inc}$ consists of two nonparaxial solitons, which are incident from south and southeast, respectively, into the nonlinear medium. We set the source term

$$f = \begin{cases} -\Delta u_{\rm inc} - k^2 u_{\rm inc} & \text{in } \Omega \setminus \Omega_0, \\ 0 & \text{in } \Omega_0. \end{cases}$$

Figure 5.3, left, shows the surface plot of $|u_{\rm inc}|^2$ with $k = 100$, while Figure 5.3, right, plots the square of the amplitude of the CIP-FE solution with $k = 100$ and $h = 1/400$. As expected, the two nonparaxial solitons are almost unchanged by the collision.

For comparison, we now consider only one incident nonparaxial soliton, that is, we solve the NLH with

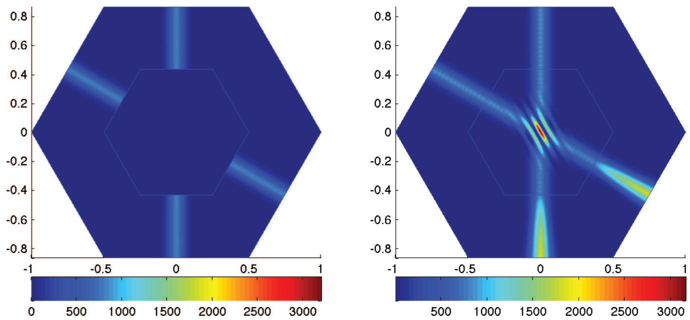

FIG. 5.3. $k = 100, h = 1/400$. *Left:* $|u_{\rm inc}|^2$*; right:* $|u_h|^2$ *of CIP-FEM.*

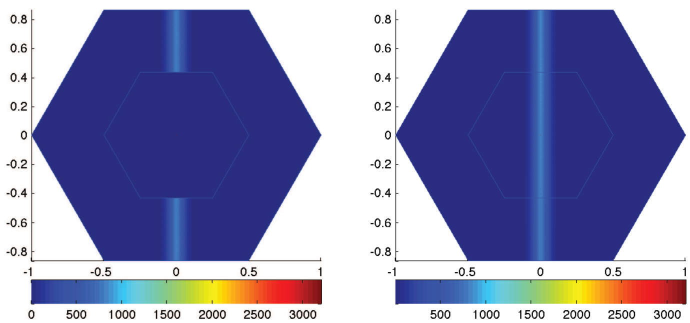

FIG. 5.4. $k = 100, h = 1/400$. *Left:* $|u_{\rm inc}|^2$*; right:* $|u_h|^2$ *of CIP-FEM.*

$$u_{\rm inc} = u^1_{\rm inc}, \qquad f = \begin{cases} -\Delta u^1_{\rm inc} - k^2 u^1_{\rm inc} & \text{in } \Omega \setminus \Omega_0, \\ 0 & \text{in } \Omega_0. \end{cases}$$

Figure 5.4 shows that the total field is almost the same as the incident wave, which means the backward scattering is weak for the case of only one incident nonparaxial soliton, while for the case of two incident nonparaxial solitons as shown in Figure 5.3, the yellow part in $\Omega\setminus\Omega_0$ of total field indicates that the backward scattering is strong.

5.3. Optical bistability. We consider the NLH problem (1.2)–(1.3) with $k = k_0 := 13.8$ in $\Omega \setminus \Omega_0$, $k = 2.5k_0$ in Ω_0, $\varepsilon = 10^{-12}$, and $f = 0$. The incident wave is specified as a plane wave $u_{\rm inc} = Ae^{k_0 \mathbf{i} x}$.

In Figure 5.5, we show the energy norm of u_h versus that of the incident wave. A reference incident wave $u^0_{\rm inc} = A_0 e^{k_0 \mathbf{i} x}$ with $A_0 = 10^5$ is introduced for scaling. The vertical and horizontal axes are $|||u_h|||/|||u^0_{\rm inc}|||$ and $\|u_{\rm inc}\|/|||u^0_{\rm inc}|||$, respectively. Clearly, the larger the amplitude, the stronger the intensity of the incident wave. We set the mesh size $h = 1/100$. The lower branch (solid) is computed by the iterative method (4.4), the upper branch (dotted) is computed by the modified Newton's method with the CIP-FEM (4.6), and the middle branch (dashed) is computed by the standard Newton's method with the CIP-FEM (see, e.g., [38]). The method (4.4) is easier to implement than the other two methods, but it converges only for A small enough ($A \leq 192{,}240$). The method (4.6) is robust for small and large A but it jumps to the upper branch a little earlier at $A = 192{,}020$ and fails in computing the middle branch. We remark that a similar example on a circular domain is computed by the modified Newton's method discretized by a mixed pseudospectral method in

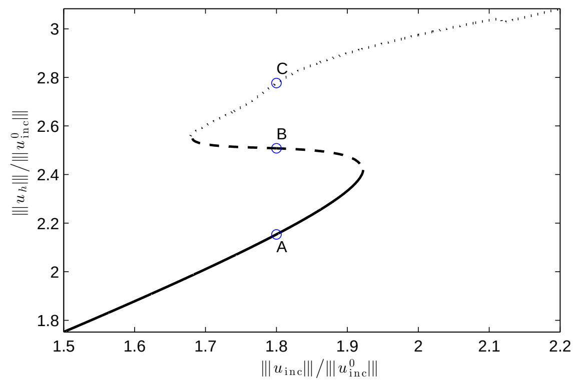

FIG. 5.5. *Normalized scattered energy as the function of the normalized incident wave energy.*

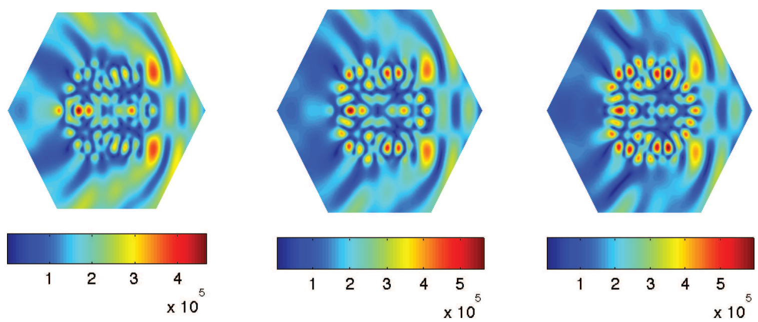

FIG. 5.6. *Wave field patterns (magnitude of* u*) of the three solutions marked as A, B, and C in Figure* 5.5.

[38]. Clearly, the proposed CIP-FEM here works for more general domains and more complicated media. For 167,740 $< A <$ 192,240, the NLH has three solutions with different levels of energy. This corresponds to the optical bistability phenomenon, since the two solutions corresponding to the upper and lower branches in Figure 5.5 are presumably stable, and the solution corresponding to the middle branch is unstable [38]. For $A =$ 180,000, the NLH has three solutions marked as A, B, and C in Figure 5.5. The electric field patterns of these solutions are shown in Figure 5.6. The initial guess for the Newton's method at point B is chosen as 0.9 times the solution corresponding to point C. After that, we can easily find the middle branch by decreasing or increasing the amplitude A slightly in each step.

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