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A NONOVERLAPPING DOMAIN DECOMPOSITION METHOD FOR MAXWELL'S EQUATIONS IN THREE DIMENSIONS*

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Abstract. In this paper, we propose a nonoverlapping domain decomposition method for solving the three-dimensional Maxwell equations, based on the edge element discretization. For the Schur complement system on the interface, we construct an efficient preconditioner by introducing two special coarse subspaces defined on the nonoverlapping subdomains. It is shown that the condition number of the preconditioned system grows only polylogarithmically with the ratio between the subdomain diameter and the finite element mesh size but possibly depends on the jumps of the coefficients.

Key words. Maxwell's equations, Nédélec finite elements, nonoverlapping domain decomposition, condition numbers

AMS subject classifications. 65N30, 65N55

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1. Introduction.

In the numerical solution of the Maxwell equations, one needs to repeatedly solve the following system [9], [12], [17], [21], [28], [30]:

$$\nabla\times(\alpha\,\nabla\times\mathbf{u})+\beta\mathbf{u}=\mathbf{f}\quad\text{in}\quad\Omega, \tag{1.1}$$

where Ω is an open polyhedral domain in $\mathbf{R}^3$ and the coefficients $\alpha(x)$ and $\beta(x)$ are two positive bounded functions in Ω. Among various boundary conditions for (1.1), we shall consider the perfect conductor condition

$$\mathbf{u}\times\mathbf{n}=0\quad\text{on}\quad\partial\Omega, \tag{1.2}$$

where $\mathbf{n}$ is the unit outward normal vector on $\partial\Omega$.

Both the nodal and edge finite element methods have been widely used for solving the system (1.1)–(1.2); see, for example, [5], [10], [11], [12], [22], [24]. However, the algebraic systems arising from the discretization by the edge element methods are very different from the ones arising from the discretization by the standard nodal finite element methods. So the nonoverlapping domain decomposition theory for the nodal element systems, which has been well developed for second order elliptic problems in the past two decades (see the survey articles [13] [33]), does not work for the edge element systems in general, especially in three dimensions. During the last five years, there has been a rapidly growing interest in domain decomposition methods (DDMs) for solving the system (1.1)–(1.2). Some substructuring DDMs were studied for two-dimensional Maxwell equations in [29], [30] and for a different three dimensional model problem in [31]. Overlapping Schwarz methods were investigated in

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[15], [28], [16] for three-dimensional Maxwell equations. As far as the nonoverlapping DDMs are concerned, very few works can be found in the literature. A nonoverlapping DDM with two subdomains was proposed in [3] for Maxwell equations in three dimensions. The current work represents some initial efforts in the construction of efficient nonoverlapping DDMs for the case with general multiple subdomains. As we shall see, not only the construction of the coarse subspaces but also the estimates of the condition numbers of the preconditioned systems for the three-dimensional case with multiple nonoverlapping subdomains are much more difficult and tricky than in the two-dimensional case or the three-dimensional case with overlapping subdomains.

In this paper, we will propose an efficient preconditioner for the Schur complement system arising from the nonoverlapping DDM based on the edge element discretization. For the analysis of our new method, some important inequalities will be established for discrete functions in edge element spaces. We believe these inequalities should also be useful to the future developments in the field. It will be shown that the resulting preconditioned system has a nearly optimal condition number; namely, the condition number grows only polylogarithmically with the ratio between the subdomain diameter and the finite element mesh size. Unlike the optimal nonoverlapping domain decomposition preconditioners for elliptic problems [13], [25], [33], we are still unable to conclude whether the condition number of the preconditioned system generated by our nonoverlapping DDM is independent of the jumps of the coefficients. This is an important problem that we are currently working on.

The paper is arranged as follows. The edge element discretization of the system (1.1)–(1.2) and some basic formulae and definitions will be described in section 2. The construction of nonoverlapping domain decomposition preconditioners and the main results of the paper are discussed in section 3. Section 4 presents some auxiliary lemmas, which are needed in section 5 to deal with the technical difficulties in the estimates of the condition numbers.

2. Domain decompositions and discretizations. This section is devoted to the introduction of the nonoverlapping domain decomposition and the weak form and the edge element discretization of the system (1.1)–(1.2) as well as some discrete operators.

Domain decomposition. We decompose the physical domain Ω into N nonoverlapping tetrahedral subdomains $\{\Omega_i\}_i^N$, with each Ω_i of size d (see [7], [33]). The faces and vertices of the subdomains are always denoted by $\textsc{f}$ and $\textsc{v}$, while the common (open) face of the subdomains Ω_i and Ω_j are denoted by Γ_{ij}, and the union of all such common faces is denoted by Γ, i.e., $\Gamma = \cup\bar{\Gamma}_{ij}$. Γ will be called *the interface.* By Γ_i we denote the intersection of Γ with the boundary of the subdomain Ω_i. So we have $\Gamma_i = \partial\Omega_i$ if Ω_i is an interior subdomain of Ω.

Finite element triangulation. Further, we divide each subdomain Ω_i into smaller tetrahedral elements of size h so that elements from the neighboring two subdomains have an intersection which is either empty or a single nodal point or an edge or a face on the interface Γ. The resulting triangulation of the domain Ω is denoted by $\mathcal{T}_h$, which is assumed to be quasi-uniform (cf. [33]), while the set of edges and the set of nodes in $\mathcal{T}_h$ are denoted by $\mathcal{E}_h$ and $\mathcal{N}_h$, respectively.

Weak formulation. The primary goal of this paper is to construct an efficient nonoverlapping DDM for solving the discrete system arising from the edge element discretization of (1.1). For this, we first introduce its weak form and then the edge element discretization of the weak form. Let $H(\mathbf{curl};\Omega)$ be the Sobolev space consisting of all square integrable functions whose $\mathbf{curl}$'s are also square integrable in Ω, and let $H_0(\mathbf{curl};\Omega)$ be a subspace of $H(\mathbf{curl};\Omega)$ with all functions whose tangential components vanish on $\partial\Omega$, i.e., $\mathbf{v}\times\mathbf{n}=0$ on $\partial\Omega$ for all $\mathbf{v}\in H_0(\mathbf{curl};\Omega)$. Then, by integration by parts, one derives immediately the variational problem associated with the system (1.1)–(1.2).

Find $\mathbf{u}\in H_0(\mathbf{curl};\Omega)$ such that

$$A(\mathbf{u},\mathbf{v})=(\mathbf{f},\mathbf{v})\quad \forall \mathbf{v}\in H_0(\mathbf{curl};\Omega), \tag{2.1}$$

where $A(\cdot,\cdot)$ is a bilinear form given by

$$A(\mathbf{u},\mathbf{v})=(\alpha\nabla\times\mathbf{u},\nabla\times\mathbf{v})+(\beta\mathbf{u},\mathbf{v}),\quad \mathbf{u},\mathbf{v}\in H(\mathbf{curl};\Omega).$$

Here and in what follows, $(\cdot,\cdot)$ denotes the scalar product in $L^2(\Omega)$ or $L^2(\Omega)^3$.

Edge element discretization. The Nédélec edge element space, of the lowest order, is a subspace of piecewise linear polynomials defined on $\mathcal{T}_h$ (cf. [14] and [23]):

$$V_h(\Omega)=\Big\{\mathbf{v}\in H_0(\mathbf{curl};\Omega);\ \mathbf{v}\,|_K\in R(K)\ \forall K\in\mathcal{T}_h\Big\},$$

where $R(K)$ is a subset of all linear polynomials on the element K of the form

$$R(K)=\Big\{\mathbf{a}+\mathbf{b}\times\mathbf{x};\ \mathbf{a},\mathbf{b}\in\mathbf{R}^3,\ \mathbf{x}\in K\Big\}.$$

It is known [14], [23] that the tangential components of any edge element function $\mathbf{v}$ of $V_h(\Omega)$ are continuous on all edges of every element in the triangulation $\mathcal{T}_h$, and $\mathbf{v}$ is uniquely determined by its moments on edges of $\mathcal{T}_h$:

$$\Big\{\lambda_e(\mathbf{v})=\int_e \mathbf{v}\cdot\mathbf{t}_e ds;\ e\in\mathcal{E}_h\Big\},$$

where $\mathbf{t}_e$ denotes the unit vector on the edge e. Let $\{L_e;\ e\in\mathcal{E}_h\}$ be the edge element basis functions of $V_h(\Omega)$ satisfying

$$\lambda_{e'}(L_e)=\begin{cases}1 & \text{if } e'=e\,,\\ 0 & \text{if } e'\neq e\,;\end{cases}$$

then the edge element basis function L_e associated with the edge e has the representation

$$L_e=c_e\left(\lambda_1^e\nabla\lambda_2^e-\lambda_2^e\nabla\lambda_1^e\right), \tag{2.2}$$

where c_e is a constant independent of h, and λ_1^e and λ_2^e are two barycentric basis functions at the two endpoints of e. Furthermore, each function $\mathbf{v}$ of $V_h(\Omega)$ can be expressed as

$$\mathbf{v}(\mathbf{x})=\sum_{e\in\mathcal{E}_h}\lambda_e(\mathbf{v})L_e(\mathbf{x}),\quad \mathbf{x}\in\Omega\,.$$

With the above notation, the edge element approximation to the variational problem (2.1) can be formulated as follows: Find $\mathbf{u}_h\in V_h(\Omega)$ such that

$$A(\mathbf{u}_h,\mathbf{v}_h)=(\mathbf{f},\mathbf{v}_h)\quad \forall\mathbf{v}_h\in V_h(\Omega), \tag{2.3}$$

where $A_h(\cdot,\cdot)$ is a bilinear form given by

$$A_h(\mathbf{u}_h, \mathbf{v}_h) = \sum_{i=1}^{N} A_i(\mathbf{u}_h, \mathbf{v}_h)$$

with each $A_i(\cdot,\cdot)$ defined only on the subdomain Ω_i:

$$A_i(\mathbf{u}, \mathbf{v}) = (\alpha\, \nabla \times \mathbf{u}, \nabla \times \mathbf{v})_{\Omega_i} + (\beta\, \mathbf{u}, \mathbf{v})_{\Omega_i}, \quad i = 1, 2, \dots, N.$$

Some edge element subspaces. In section 3, we will formulate our DDM for solving the edge element system (2.3). Before doing so, we need to introduce more notation, subspaces, and discrete operation tools.

We will often use G to represent a subset of Γ, which may be the entire interface Γ or the local interface Γ_i or a face F of Γ_i. The notation e, with $e \subset G$, always means that e is an edge of $\mathcal{T}_h$ and lies on G. By restricting $V_h(\Omega)$ on G, we generate a subspace of $L^2(G)^3$:

$$V_h(G) = \Big\{\psi \in L^2(G)^3;\ \psi = \mathbf{v} \times \mathbf{n} \ \text{ on } \ G \ \text{ for some } \ \mathbf{v} \in V_h(\Omega)\Big\}.$$

By $V_h(\Omega_i)$ we denote the restriction of $V_h(\Omega)$ on the subdomain Ω_i. The following two local subspaces of $V_h(\Omega_i)$ and $V_h(\mathrm{F})$ will be important to our subsequent analysis:

$$V_h^0(\Omega_i) = \Big\{\mathbf{v} \in V_h(\Omega_i);\ \mathbf{v} \times \mathbf{n} = 0 \ \text{ on } \ \Gamma_i\Big\},$$

$$V_h^0(\mathrm{F}) = \Big\{\Phi = \mathbf{v} \times \mathbf{n} \in V_h(\mathrm{F});\ \lambda_e(\mathbf{v}) = 0 \ \ \forall\, e \subset \partial\mathrm{F} \cap \mathcal{E}_h\Big\}.$$

Discrete operators. We will often use the natural restriction operator from $V_h(\Gamma)$ onto $V_h(G)$, denoted by $\mathbf{I}_G$, and the natural zero extension operator from $V_h(G)$ into $L^2(\Gamma)^3$, denoted by $\mathbf{I}_G^t$. By definition it is clear that for a face F, $\mathbf{I}_{\mathrm{F}}^t \mathbf{v} \in V_h(\Gamma)$ if and only if $\mathbf{v} \in V_h^0(\mathrm{F})$, and $\mathbf{I}_G$ and $\mathbf{I}_G^t$ satisfy

$$\langle \mathbf{I}_G \Psi, \Phi \rangle_G = \langle \Psi, \mathbf{I}_G^t \Phi \rangle \quad \forall\, \Psi \in V_h(\Gamma),\ \Phi \in V_h(G),$$

where $\langle \cdot,\cdot \rangle_G$ stands for the L^2-inner product in $L^2(G)$ or $L^2(G)^3$, and the subscript G will be dropped when $G = \Gamma$. Also, we shall write $\mathbf{I}_i = \mathbf{I}_{\Gamma_i}$ and $\mathbf{I}_{ij}^t = \mathbf{I}_{\Gamma_{ij}}^t$.

For any face F of Ω_i, we use F_b to denote the union of all $\mathcal{T}_h$-induced (closed) triangles on F which have at least one edge lying on $\partial\mathrm{F}$ and F_∂ to denote the open set $\mathrm{F} \backslash \mathrm{F}_b$.

By definition, for any $\Phi \in V_h(\Gamma_i)$, there exists a $\mathbf{v} \in V_h(\Omega_i)$ such that $\Phi = \mathbf{v} \times \mathbf{n}$ on Γ_i. So Φ has the representation of the form

$$\Phi(\mathbf{x}) = \sum_{e \subset \Gamma_i} \lambda_e(\mathbf{v})(L_e \times \mathbf{n})(\mathbf{x}), \quad \mathbf{x} \in \Gamma_i. \tag{2.4}$$

For any open face F on Γ_i, we define an operator $\mathbf{I}_{\mathrm{F}_\partial}^0 : V_h(\Gamma_i) \to \mathbf{I}_{\mathrm{F}}^t V_h^0(\mathrm{F})$ by

$$(\mathbf{I}_{\mathrm{F}_\partial}^0 \Phi)(\mathbf{x}) = \sum_{e \subset \mathrm{F}_\partial} \lambda_e(\mathbf{v})(L_e \times \mathbf{n})(\mathbf{x}), \quad \mathbf{x} \in \Gamma_i, \tag{2.5}$$

and an operator $\mathbf{I}^0_{\mathrm{F}_b}$ by

$$(\mathbf{I}^0_{\mathrm{F}_b}\Phi)(\mathbf{x}) = \sum_{e\subset \mathrm{F}_b} \lambda_e(\mathbf{v})\,\mathbf{I}^t_{\mathrm{F}}(L_e\times\mathbf{n})(\mathbf{x}), \quad \mathbf{x}\in\Gamma_i.$$

Some nodal element spaces. From time to time, we shall also need some nodal element spaces in the analyses—for example, the continuous piecewise linear finite element space $Z_h(\Omega)$ of $H^1_0(\Omega)$, its restriction $Z_h(\Gamma)$ on Γ and $Z_h(\Omega_i)$ on any subdomain Ω_i, and the restriction $Z_h(\Gamma_i)$ of $Z_h(\Omega_i)$ on the local interface Γ_i and $Z_h(\mathrm{F})$ on a face F.

The operator $\mathrm{I}^t_{\mathrm{F}} : Z_h(\mathrm{F}) \to L^2(\Gamma)$ is defined similarly to $\mathbf{I}^t_{\mathrm{F}}$.

For a subset G of Γ_i, we introduce a "local" subspace

$$Z^0_h(G) = \{v \in Z_h(\Gamma_i);\ v = 0 \text{ at all nodes on } \Gamma_i\backslash G\}.$$

For any open face $\mathrm{F}\subset\Gamma_i$, we will use $\mathrm{I}^0_{\mathrm{F}} : Z_h(\Gamma_i)\to Z^0_h(\mathrm{F})$ and $\mathrm{I}^0_{\partial\mathrm{F}} : Z_h(\Gamma_i)\to Z^0_h(\partial\mathrm{F})$ to denote the natural restriction operators (see [33]).

curl- and harmonic extension operators. The next two extension operators will play an important role in the subsequent analysis. The first is the discrete **curl**-extension operator $\mathbf{R}^i_h : V_h(\Gamma_i)\to V_h(\Omega_i)$ defined as follows: For any $\Phi\in V_h(\Gamma_i)$, $\mathbf{R}^i_h\Phi\in V_h(\Omega_i)$ satisfies $\mathbf{R}^i_h\Phi\times\mathbf{n} = \Phi$ on Γ_i and solves

$$A_i(\mathbf{R}^i_h\Phi, \mathbf{v}_h) = 0 \quad \forall \mathbf{v}_h\in V^0_h(\Omega_i).$$

The second is the discrete harmonic extension operator $R^i_h : Z_h(\Gamma_i)\to Z_h(\Omega_i)$ defined as follows: For any $v_h\in Z_h(\Gamma_i)$, $R^i_h v_h\in Z_h(\Omega_i)$ satisfies $R^i_h v_h = v_h$ on Ω_i and

$$(\nabla R^i_h v_h, \nabla w_h) = 0 \quad \forall\, w_h\in Z_h(\Omega_i)\cap H^1_0(\Omega_i)\,.$$

3. Nonoverlapping DDMs.

In this section, we propose a nonoverlapping DDM for solving the edge element system (2.3). The notation $\langle\cdot,\cdot\rangle_{\Gamma_i}$ and $(\cdot,\cdot)_{\Omega_i}$ shall be used for the scalar products in $L^2(\Gamma_i)$ and $L^2(\Omega_i)$, respectively.

3.1. The interface equation.

For the solution $\mathbf{u}_h$ to the system (2.3), we write $\mathbf{u}_{hi} = \mathbf{u}_h|_{\Omega_i}$. It follows from (2.3) that

$$A_i(\mathbf{u}_{hi}, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h)_{\Omega_i} \quad \forall\mathbf{v}_h\in V^0_h(\Omega_i). \tag{3.1}$$

This indicates that if the tangential components $\mathbf{u}_{hi}\times\mathbf{n}_i$ are known on Γ_i the "local" unknown $\mathbf{u}_{hi}$ can be obtained by solving the local equation (3.1).

Next, we will establish an equation for the interface quantity $\Phi = \mathbf{u}_h\times\mathbf{n}$ on Γ. To do so, we introduce a "local" interface operator $\mathbf{S}_i : V_h(\Gamma_i)\to V_h(\Gamma_i)^*$ by

$$\langle\mathbf{S}_i\Phi_i, \Psi_i\rangle_{\Gamma_i} = A_i(\mathbf{R}^i_h\Phi_i, \mathbf{R}^i_h\Psi_i) \quad \forall\Psi_i, \Phi_i\in V_h(\Gamma_i).$$

Using the obvious decomposition

$$\mathbf{u}_{hi} = \mathbf{u}^0_{hi} + \mathbf{R}^i_h(\mathbf{u}_{hi}\times\mathbf{n}_i)$$

with $\mathbf{u}^0_{hi}\in V^0_h(\Omega_i)$, solving (3.1), (2.3) reduces to the interface equation (cf. [27])

$$\sum_{i=1}^N \langle\mathbf{S}_i\mathbf{I}_i\Phi, \mathbf{I}_i\Psi\rangle_{\Gamma_i} = \sum_{i=1}^N (\mathbf{f}, \mathbf{R}^i_h\mathbf{I}_i\Psi)_{\Omega_i} \quad \forall\Psi\in V_h(\Gamma). \tag{3.2}$$

Let $\mathbf{g} \in V_h(\Gamma)^*$ be defined by

$$\langle \mathbf{g}, \Psi \rangle_\Gamma = \sum_{i=1}^{N} (\mathbf{f}, \mathbf{R}_h^i \mathbf{I}_i \Psi)_{\Omega_i} \quad \forall \Psi \in V_h(\Gamma),$$

and let $\mathbf{S} = \sum_{i=1}^{N} \mathbf{I}_i^t \mathbf{S}_i \mathbf{I}_i$; then (3.2) may be written as

$$\langle \mathbf{S}\Phi, \Psi \rangle = \langle \mathbf{g}, \Psi \rangle \quad \forall \Psi \in V_h(\Gamma). \tag{3.3}$$

With $\Phi = \mathbf{u}_h \times \mathbf{n}$ available on Γ, the solution of (2.3) can be obtained by solving one subproblem, (3.1), on each subdomain Ω_i. Therefore, the solution of (2.3) reduces to the one of the interface problem (3.3). However, it is very expensive to solve this interface equation directly. Instead, we will construct an efficient preconditioner for $\mathbf{S}$; then (3.3) can be solved by the preconditioned CG method.

3.2. Preconditioners for the interface operator S. We now start to construct a preconditioner for $\mathbf{S}$. As usual, a good preconditioner should involve both local solvers and global coarse solvers.

First, the local solvers can be constructed on each local face Γ_{ij}. For each Γ_{ij}, we define a "local" operator $\mathbf{S}_{ij} : V_h^0(\Gamma_{ij}) \to V_h^0(\Gamma_{ij})^*$ by

$$\langle \mathbf{S}_{ij}\Phi_{ij}, \Psi_{ij} \rangle_{\Gamma_{ij}} = A_i(\mathbf{R}_h^i \mathbf{I}_{ij}^t \Phi_{ij}, \mathbf{R}_h^i \mathbf{I}_{ij}^t \Psi_{ij}) + A_j(\mathbf{R}_h^j \mathbf{I}_{ij}^t \Phi_{ij}, \mathbf{R}_h^j \mathbf{I}_{ij}^t \Psi_{ij})$$
$$\forall \Phi_{ij}, \Psi_{ij} \in V_h^0(\Gamma_{ij}),$$

and $\mathbf{S}_{ij}^{-1}$ will be our desired local solvers. The construction of the global coarse solvers is much more tricky and technical. Before doing this, we would like to illustrate our main idea about the construction. The essential difficulty in the construction of a coarse solver lies in two facts: (1) The edge element space $V_h(\Omega)$, different from the nodal element space, is not a subspace of $H^1(\Omega)^3$; (2) for any $\mathbf{v}_h \in V_h(\Omega)$, its tangential components are continuous on all *cross-edges*, namely, the edges which are shared by more than two fine elements (tangential components make no sense at the *cross-points* in two dimensions), but the moments on the *cross-edges* are not sufficient to determine the values of the tangential trace $\mathbf{v}_h \times \mathbf{n}$ on these edges. As one will see, we have the Helmholtz decomposition

$$V_h(\Omega) = \mathbf{grad}\, Z_h(\Omega) + \tilde{V}_h(\Omega),$$

where $\tilde{V}_h(\Omega)$ corresponds to the divergence-free part and is closely related to the space $H^1(\Omega)^3$. Thus it seems necessary to construct two coarse subspaces and coarse solvers, corresponding to the **curl**-free and divergence-free subspaces $\nabla Z_h(\Omega)$ and $\tilde{V}_h(\Omega)$, respectively.

For the construction of the coarse subspaces, we introduce some more notation below. For any subdomain Ω_i, by $\mathcal{W}_i$ we denote the set of the edges of Ω_i, which belong to at least two other local interfaces Γ_j, $j \neq i$. On each $\mathcal{W}_i$, we define the discrete L^2-scalar product

$$\langle \varphi, \psi \rangle_{h,\mathcal{W}_i} = h \sum_{\mathbf{x} \in \mathcal{N}_h \cap \mathcal{W}_i} \varphi(\mathbf{x})\psi(\mathbf{x}) \quad \forall\, \varphi, \psi \in Z_h(\Gamma_i)\,;$$

the corresponding norm is denoted by $\|\cdot\|_{h,\mathcal{W}_i}$. Let

$$\Delta_i = \bigcup_{\mathrm{F} \subset \Gamma_i} \mathrm{F}_b, \quad i = 1, \ldots, N.$$

We introduce a norm $\|\cdot\|_{*,\Delta_i}$ that is induced from the following inner product in $L^2(\Delta_i)^3$:

$$\langle \mathbf{v}\times\mathbf{n},\mathbf{w}\times\mathbf{n}\rangle_{*,\Delta_i}=\sum_{K\subset\Delta_i}\langle \mathbf{v}\times\mathbf{n},\mathbf{w}\times\mathbf{n}\rangle_{\partial K}\quad \forall\,\mathbf{v}\times\mathbf{n},\mathbf{w}\times\mathbf{n}\in V_h(\Gamma_i),$$

where the summation is over all triangles K in Δ_i.

For any given subset G of Ω and function φ in $L^2(G)$, we use $\gamma_{_G}(\varphi)$ for the average value of φ on G. Similarly, for a vector $\mathbf{v}=(v_1,v_2,v_3)$ in $L^2(G)^3$, we use $\Upsilon_G(\mathbf{v})$ for the constant vector with three average values $\gamma_{_G}(v_1)$, $\gamma_{_G}(v_2)$, and $\gamma_{_G}(v_3)$ as its components.

Now we define two discrete operators in $Z_h(\Gamma)$ and $V_h(\Gamma)$ which will generate two coarse subspaces. For any $\varphi\in Z_h(\Gamma)$, we define $\pi_0\varphi\in Z_h(\Gamma)$ by

$$\pi_0\varphi(\mathbf{x})=\begin{cases}\varphi(\mathbf{x}) & \text{for } \mathbf{x}\in\mathcal{W}_i\cap\mathcal{N}_h\ (i=1,\dots,N),\\ \gamma_{_{\partial\mathrm{F}}}(\varphi) & \text{for } \mathbf{x}\in\mathrm{F}\cap\mathcal{N}_h\ (\mathrm{F}\subset\Gamma).\end{cases}\tag{3.4}$$

Similarly, for each $\mathbf{v}\times\mathbf{n}\in V_h(\Gamma)$, we define $\Pi_0\mathbf{v}\times\mathbf{n}\in V_h(\Gamma)$ such that

$$\lambda_e(\Pi_0\mathbf{v})=\begin{cases}\lambda_e(\mathbf{v}) & \text{for } e\subset\Delta_i\cup\Omega_i\ (i=1,\dots,N),\\ \lambda_e(\Upsilon_{\partial\mathrm{F}}(\mathbf{v})) & \text{for } e\subset\mathrm{F}_\partial\ (\mathrm{F}\subset\Gamma).\end{cases}$$

Note that although $\Pi_0\mathbf{v}$ involves the degrees of freedom inside Ω_i, $\Pi_0\mathbf{v}\times\mathbf{n}$ is determined on Γ uniquely by the moments $\lambda_e(\mathbf{v})$ for all $e\subset\Gamma$. Thus $\Pi_0\mathbf{v}\times\mathbf{n}\in V_h(\Gamma)$ can also be defined directly by

$$\Pi_0\mathbf{v}\times\mathbf{n}=\begin{cases}\mathbf{v}\times\mathbf{n} & \text{on } \Delta_i\ (i=1,\dots,N),\\ \Upsilon_{\partial\mathrm{F}}(\mathbf{v}\times\mathbf{n}) & \text{on } \mathrm{F}_\partial\ (\mathrm{F}\subset\Gamma),\end{cases}$$

where we have used the fact that the normal vector $\mathbf{n}$ is constant on any face $\mathrm{F}\subset\Gamma$ and

$$\Upsilon_{\partial\mathrm{F}}(\mathbf{v})\times\mathbf{n}\big|_{\mathrm{F}}=\Upsilon_{\partial\mathrm{F}}(\mathbf{v}\times\mathbf{n}).$$

Now, we can define the two coarse subspaces:

$$V_h^{01}(\Gamma)=\Big\{\Phi_0\in V_h(\Gamma);\ \mathbf{I}_i\Phi_0=\mathbf{grad}(R_0^i\mathrm{I}_i\pi_0\varphi)\times\mathbf{n}\text{ on }\Gamma_i\text{ for some }\varphi\in Z_h(\Gamma)\Big\},$$

$$V_h^{02}(\Gamma)=\Big\{\mathbf{v}_0\times\mathbf{n}\in V_h(\Gamma);\ \mathbf{v}_0=\Pi_0\mathbf{v}\text{ for some }\mathbf{v}\times\mathbf{n}\in V_h(\Gamma)\Big\}.$$

The operator R_0^i used in $V_h^{01}(\Gamma)$ is the zero extension into the interior of Ω_i; namely, for any $v_h\in Z_h(\Gamma_i)$, $R_0^iv_h\in Z_h(\Omega_i)$ takes the same values as v_h on Γ_i and vanishes at all interior nodes of Ω_i. We can define two coarse solvers $\mathbf{S}_{0k}:V_h^{0k}(\Gamma)\to V_h^{0k}(\Gamma)^*$, $k=1,2$, associated with these coarse subspaces. For any $\Phi_0,\Psi_0\in V_h^{01}(\Gamma)$, there exist $\varphi,\ \psi\in Z_h(\Gamma)$ such that on Γ_i,

$$\mathbf{I}_i\Phi_0=\mathbf{grad}(R_0^i\mathrm{I}_i\pi_0\varphi)\times\mathbf{n},\quad \mathbf{I}_i\Psi_0=\mathbf{grad}(R_0^i\mathrm{I}_i\pi_0\psi)\times\mathbf{n}\,.$$

Then $\mathbf{S}_{01}$ is defined by

$$\langle\mathbf{S}_{01}\Phi_0,\Psi_0\rangle=[1+\log(d/h)]\sum_{i=1}^N\langle\pi_0\varphi-\gamma_{_{\mathcal{W}_i}}(\pi_0\varphi),\pi_0\psi-\gamma_{_{\mathcal{W}_i}}(\pi_0\psi)\rangle_{h,\mathcal{W}_i}.$$

Similarly, for any $\Phi_0, \Psi_0 \in V_h^{02}(\Gamma)$, there exist $\mathbf{v}$, $\mathbf{w} \in V_h(\Omega)$ such that on Γ_i,

$$\mathbf{I}_i\Phi_0 = \Pi_0\mathbf{v} \times \mathbf{n}, \quad \mathbf{I}_i\Psi_0 = \Pi_0\mathbf{w} \times \mathbf{n}.$$

Then $\mathbf{S}_{02}$ is defined by

$$\langle \mathbf{S}_{02}\Phi_0, \Psi_0\rangle = [1 + \log(d/h)] \sum_{i=1}^{N} \langle \Phi_0 - \Upsilon_{\Delta_i}(\mathbf{v}) \times \mathbf{n}, \Psi_0 - \Upsilon_{\Delta_i}(\mathbf{w}) \times \mathbf{n}\rangle_{*,\Delta_i} + d^2\langle \Phi_0, \Psi_0\rangle_{*,\Delta_i}.$$

Hereafter, $\Upsilon_{\Delta_i}(\mathbf{v})$ is the constant vector satisfying

$$\|\Phi_0 - \Upsilon_{\Delta_i}(\mathbf{v}) \times \mathbf{n}\|^2_{*,\Delta_i} = \min_{C_{\Delta_i} \in \mathcal{R}^3} \|\Phi_0 - C_{\Delta_i} \times \mathbf{n}\|^2_{*,\Delta_i},$$

which can be viewed as some average of Φ_0 on Δ_i. And the average is well defined.

Finally, the preconditioner for the interface operator $\mathbf{S}$ can be defined as follows:

$$\mathbf{M}^{-1} = \mathbf{S}_{01}^{-1} + \mathbf{S}_{02}^{-1} + \sum_{\Gamma_{ij}} \mathbf{I}_{ij}^t \mathbf{S}_{ij}^{-1} \mathbf{I}_{ij}. \tag{3.5}$$

For this preconditioner, we have the following theorem.

THEOREM 3.1. *The condition number of the preconditioned system can be estimated by*

$$\mathrm{cond}(\mathbf{M}^{-1}\mathbf{S}) \le C[1 + \log(d/h)]^3. \tag{3.6}$$

Remark 3.1. A simple algorithm to implement the coarse solver $\mathbf{S}_{01}$ can be found in [33]. By the minimum property of the average $\Upsilon_{\Delta_i}(\Phi_0)$, we can also derive a simple algorithm for implementing the coarse solver $\mathbf{S}_{02}$, which is similar to the one in [33]. Note that one may also use the inner product $h^{-1}\langle\cdot,\cdot\rangle_{\Delta_i}$ in the definition of $\mathbf{S}_{02}$ instead of the inner product $\langle\cdot,\cdot\rangle_{*,\Delta_i}$. Furthermore, one may use the discrete $L^2(\Delta_i)^3$-inner product

$$\langle\langle \mathbf{v} \times \mathbf{n}, \mathbf{w} \times \mathbf{n}\rangle\rangle_{h,\Delta_i} = \sum_{e \subset \Delta_i} \lambda_e(\mathbf{v})\lambda_e(\mathbf{w}) \quad \forall\, \mathbf{v} \times \mathbf{n}, \mathbf{w} \times \mathbf{n} \in V_h(\Gamma_i),$$

to define the coarse solver $\mathbf{S}_{02}$, but we do not know yet how to verify the existence of the corresponding average.

Remark 3.2. The "local" operator $\mathbf{S}_{ij}$ may be replaced by any other spectrally equivalent operator, for example, the operator defined by

$$\langle \mathbf{S}_{ij}^i \Phi_{ij}, \Psi_{ij}\rangle_{\Gamma_{ij}} = A_i(\mathbf{R}_h^i \mathbf{I}_{ij}^t \Phi_{ij}, \mathbf{R}_h^i \mathbf{I}_{ij}^t \Psi_{ij}) \quad \forall \Psi_{ij} \in V_h^0(\Gamma_{ij}).$$

$\mathbf{S}_{ij}^i$ is easier to implement than $\mathbf{S}_{ij}$, but it loses the symmetry with respect to the face Γ_{ij}.

Remark 3.3. Based on our current analysis in section 5, the constant C in the condition number estimate (3.6) may have a factor $\gamma_{\max}/\gamma_{\min}$ related to the coefficients in (1.1), where $\gamma_{\max}$ is the supremum of $\beta(x)$ and $\alpha^2(x)$ over $\bar{\Omega}$, and $\gamma_{\min}$ is the infimum of $\beta(x)$ and $\alpha^2(x)$ over $\bar{\Omega}$. It is possible to improve such dependence on the coefficients if a more localized and sharper analysis can be found.

Remark 3.4. The nodal element coarse interpolant π_0 is widely used in nonoverlapping DDMs for second order elliptic problems [13], [33]. The new edge element coarse interpolant Π_0 is very similar to π_0 but with some essential differences. For a $H^1(\Omega)^3$ vector-valued function $\mathbf{v}$, there is no trace on the wirebasket set $\mathcal{W}_i$, and the coarse interpolants $\pi_0\mathbf{v}$ and $\Pi_0\mathbf{v}$ make no sense. However, it is known that π_0 is stable in the nodal element space $Z_h(\Gamma_i)$ [13], [33]. Likewise, we shall show in section 4 that Π_0 is stable in the edge element space $V_h(\Gamma_i)$, with the stability constants growing only polylogarithmically with d/h. This explains somewhat why we can achieve a logarithmical bound (3.6) on the condition number.

4. Some auxiliary lemmas. As we shall see, the estimate (3.6) of the condition number $\text{cond}(\mathbf{M}^{-1}\mathbf{S})$ for the preconditioned system is rather technical. This section presents some basic properties of Sobolev spaces and auxiliary lemmas, which are needed to deal with the technical difficulties in the estimate of the condition number. The proofs will be provided in the appendix. The constant C will be used often in what follows for the generic constant that may take different values at different occasions.

4.1. The scaled norms. A large part of the condition number estimate will be carried out on the subdomains, for which we need some scaled norms. For the space $H^1(\Omega_i)^3$, we define a scaled norm by

$$\|\mathbf{v}\|_{1,\Omega_i} = (|\mathbf{v}|^2_{1,\Omega_i} + d^{-2}\|\mathbf{v}\|^2_{0,\Omega_i})^{\frac{1}{2}} \quad \forall\, \mathbf{v} \in H^1(\Omega_i)^3,$$

while for the space $H(\mathbf{curl};\Omega_i)$, the restriction of $H_0(\mathbf{curl};\Omega)$ on the subdomain Ω_i, and the interface space $H^{-\frac{1}{2}}(\Gamma_i)$, we define their scaled norms by

$$\|\mathbf{v}\|_{\mathbf{curl};\Omega_i} = \left(\|\mathbf{curl}\,\mathbf{v}\|^2_{0,\Omega_i} + d^{-2}\|\mathbf{v}\|^2_{0,\Omega_i}\right)^{\frac{1}{2}} \quad \forall\, \mathbf{v} \in H(\mathbf{curl};\Omega_i)\,,$$

$$\|\lambda\|_{-\frac{1}{2},\Gamma_i} = \sup_{v\in H^{\frac{1}{2}}(\Gamma_i)} \frac{|\langle \lambda, v\rangle_{\Gamma_i}|}{\|v\|_{\frac{1}{2},\Gamma_i}} \quad \forall\, \lambda \in H^{-\frac{1}{2}}(\Gamma_i),$$

where

$$\|v\|_{\frac{1}{2},\Gamma_i} = (|v|^2_{\frac{1}{2},\Gamma_i} + d^{-1}\|v\|^2_{0,\Gamma_i})^{\frac{1}{2}}.$$

For any $\Phi \in V_h(\Gamma_i)$, we use $\text{div}_\tau\Phi$ to denote the tangential divergence of Φ; see [2] and [3] for the definition of $\text{div}_\tau\Phi$. It is known that $\text{div}_\tau\Phi \in H^{-\frac{1}{2}}(\Gamma_i)$, so it makes sense to define the norm

$$\|\Phi\|_{\mathcal{X}_{\Gamma_i}} = d^{-1}\|\Phi\|_{-\frac{1}{2},\Gamma_i} + \|\text{div}_\tau\Phi\|_{-\frac{1}{2},\Gamma_i}.$$

The next two estimates on this norm $\|\cdot\|_{\mathcal{X}_{\Gamma_i}}$ can be found in [3].

LEMMA 4.1. *The discrete* **curl**-*extension* $\mathbf{R}^i_h\Phi \in V_h(\Omega_i)$ *satisfies*

$$\|R^i_h\Phi\|_{\mathbf{curl};\Omega_i} \le C\|\Phi\|_{\mathcal{X}_{\Gamma_i}}. \tag{4.1}$$

LEMMA 4.2. *Let* $\mathbf{u} \in V_h(\Omega_i)$, *which satisfies* $\mathbf{u}\times\mathbf{n} = \Phi$ *on* Γ_i. *Then*

$$\|\Phi\|_{\mathcal{X}_{\Gamma_i}} \le C\|\mathbf{u}\|_{\mathbf{curl};\Omega_i}. \tag{4.2}$$

4.2. Estimates with the norm $\|\cdot\|_{1/2,\Gamma_i}$ and the edge element interpolant. The results in Lemma 4.3 can be found in [7] and [33].

LEMMA 4.3. *For any $\varphi \in Z_h(\Gamma)$, we have*

$$C\,|\pi_0\varphi|^2_{\frac{1}{2},\Gamma_i} \le [1+\log(d/h)]\|\varphi - \gamma_{\mathcal{W}_i}(\varphi)\|^2_{h,\mathcal{W}_i} \le C[1+\log(d/h)]^2|\varphi|^2_{\frac{1}{2},\Gamma_i} \tag{4.3}$$

and for any face F $\subset \Gamma_i$,

$$\|\mathrm{I}^0_{\mathrm{F}}(\varphi - \pi_0\varphi)\|^2_{\frac{1}{2},\Gamma_i} \le C[1+\log(d/h)]^2|\varphi|^2_{\frac{1}{2},\Gamma_i}\,. \tag{4.4}$$

Now we define an interpolation operator $\mathbf{r}_h$ associated with the space $V_h(\Omega)$. For any appropriately smooth $\mathbf{v}$, $\mathbf{r}_h\mathbf{v} \in V_h(\Omega)$ is a function in $V_h(\Omega)$ which has the same moments on the edges of $\mathcal{T}_h$ as $\mathbf{v}$, namely,

$$\int_e \mathbf{r}_h\mathbf{v}\cdot\mathbf{t}_e ds = \int_e \mathbf{v}\cdot\mathbf{t}_e ds \quad \forall\, \mathbf{v}\in H^1(\Omega) \text{ and } \ e\in\mathcal{E}_h.$$

The interpolant $\mathbf{r}_h\mathbf{v}$ is well defined on each element K for all $\mathbf{v}$ lying in the space

$$\Big\{\mathbf{w}\in L^p(K)^3;\ \mathbf{curl}\,\mathbf{v}\in L^p(K)^3 \text{ and } \mathbf{v}\times\mathbf{n}\in L^p(\partial K)^3\Big\}$$

with $p > 2$; see Lemma 4.7 in [4]. From this we immediately know that $\mathbf{r}_h\mathbf{v}$ is well defined for all $\mathbf{v}$ in $H^1(\Omega)^3$ whose **curl** is in $L^p(K)^3$.

The following three lemmas present some estimates on the interpolation operator r_h. The proof of the first lemma below is quite similar to the proofs of Lemma 4.7 in [4] and Lemma 3.2 in [12], and details can be found in [20].

LEMMA 4.4. *Let $\mathbf{w}\in H^1(\Omega_i)^3$ and its interpolant $\mathbf{r}_h\mathbf{w}$ be well defined in $V_h(\Omega_i)$. Also, we assume that $\mathbf{curl}\,\mathbf{w} = \mathbf{curl}\,\mathbf{v}_h$ for some $\mathbf{v}_h\in V_h(\Omega_i)$. Then*

$$\|\mathbf{r}_h\mathbf{w}-\mathbf{w}\|_{0,\Omega_i} \le Ch(|\mathbf{w}|^2_{1,\Omega_i} + \|\mathbf{curl}\,\mathbf{v}_h\|^2_{0,\Omega_i})^{\frac{1}{2}}. \tag{4.5}$$

LEMMA 4.5. *Under the same assumptions as in Lemma* 4.4, *for any face* F *of* Γ_i *we have*

$$\|(\mathbf{r}_h\mathbf{w})\times\mathbf{n}\|_{*,\mathrm{F}_b} \le C[1+\log(d/h)]^{\frac{1}{2}}(\|\mathbf{w}\|^2_{1,\Omega_i} + \|\mathbf{curl}\,\mathbf{v}_h\|^2_{0,\Omega_i})^{\frac{1}{2}}. \tag{4.6}$$

LEMMA 4.6. *Under the same assumptions as in Lemma* 4.4, *for any face* F *of* Γ_i *we have*

$$d^{-2}\|\mathbf{r}_h\mathbf{w} - \Upsilon_{\partial\mathrm{F}}(\mathbf{r}_h\mathbf{w})\|^2_{0,\Omega_i} \le C[1+\log(d/h)](|\mathbf{w}|^2_{1,\Omega_i} + \|\mathbf{curl}\,\mathbf{v}_h\|^2_{0,\Omega_i}), \tag{4.7}$$

$$d^{-2}\|\mathbf{w} - \Upsilon_{\partial\mathrm{F}}(\mathbf{r}_h\mathbf{w})\|_{0,\Omega_i} \le C[1+\log(d/h)](|\mathbf{w}|^2_{1,\Omega_i} + \|\mathbf{curl}\,\mathbf{v}_h\|^2_{0,\Omega_i}). \tag{4.8}$$

4.3. Some estimates with the norm$\|\cdot\|_{\mathcal{X}_{\Gamma_i}}$.

LEMMA 4.7. *Let $\mathbf{w}$ and $\mathbf{v}_h$ be the same as specified in Lemma* 4.4, *and $\Phi = \mathbf{r}_h\mathbf{w}\times\mathbf{n}$ on Γ_i. Then for any face* F $\subset\Gamma_i$ *we have*

$$\|\mathbf{I}^0_{\mathrm{F}_\partial}\Phi\|_{\mathcal{X}_{\Gamma_i}} \le C[1+\log(d/h)](\|\Phi\|_{\mathcal{X}_{\Gamma_i}} + \|\mathbf{w}\|_{1,\Omega_i} + \|\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_i}). \tag{4.9}$$

LEMMA 4.8. *Let $\Phi = \mathbf{v}\times\mathbf{n}\in V_h(\Gamma_i)$ on Γ_i, and*

$$\mathbf{I}^0_{\Delta_i}\Phi(\mathbf{x}) = \sum_{e\subset\Delta_i}\lambda_e(\mathbf{v})(L_e\times\mathbf{n}_i)(\mathbf{x}), \quad \mathbf{x}\in\Gamma_i.$$

We have

$$\|\mathbf{I}^0_{\Delta_i}\Phi\|_{\mathcal{X}_{\Gamma_i}} \le C[1+\log(d/h)]^{\frac{1}{2}}\|\Phi\|_{*,\Delta_i}. \tag{4.10}$$

LEMMA 4.9. *Assume that $\mathbf{v}\in V_h(\Omega)$ and* F $\subset\Gamma_k$. *Then*

$$\|\mathbf{I}^0_{\mathrm{F}_\partial}(\Upsilon_{\partial\mathrm{F}}(\Pi_0\mathbf{v})\times\mathbf{n})\|^2_{\mathcal{X}_{\Gamma_k}} \le C[1+\log(d/h)]\|(\Pi_0\mathbf{v})\times\mathbf{n}\|^2_{*,\mathrm{F}_b}. \tag{4.11}$$

5. The estimate of condition number. This section is devoted to the estimate (3.6) of the condition number of the preconditioned system $\mathbf{M}^{-1}\mathbf{S}$. The estimation will be done by using the following additive Schwarz framework [26], [32], whose proof is standard (cf. [18] and [27]).

LEMMA 5.1. *Assume that the following two conditions hold:*

(i) *For any* $\Phi \in V_h(\Gamma)$ *there is a decomposition* $\Phi = \Phi_{01} + \Phi_{02} + \sum_{i<j} \mathrm{I}_{ij}^t \Phi_{ij}$, *with* $\Phi_{0k} \in V_h^{0k}(\Gamma)$ $(k = 1,\ 2)$ *and* $\Phi_{ij} \in V_h^0(\Gamma_{ij})$, *such that*

$$\langle \mathbf{S}_{01}\Phi_{01}, \Phi_{01}\rangle + \langle \mathbf{S}_{02}\Phi_{02}, \Phi_{02}\rangle + \sum_{i<j} \langle \mathbf{S}_{ij}\Phi_{ij}, \Phi_{ij}\rangle_{\Gamma_{ij}} \le C_1 \langle \mathbf{S}\Phi, \Phi\rangle; \tag{5.1}$$

(ii) *For any* $\Psi_{0k} \in V_h^{0k}(\Gamma)$ $(k = 1, 2)$ *and* $\Psi_{ij} \in V_h^0(\Gamma_{ij})$, *we have*

$$\begin{aligned} &\left\langle \mathbf{S}\left(\sum_{i<j} \mathbf{I}_{ij}^t \Psi_{ij} + \Psi_{01} + \Psi_{02}\right),\ \sum_{i<j} \mathbf{I}_{ij}^t \Psi_{ij} + \Psi_{01} + \Psi_{02} \right\rangle \\ &\quad \le C_2 \left\{ \sum_{i<j} \langle \mathbf{S}_{ij}\Psi_{ij}, \Psi_{ij}\rangle_{\Gamma_{ij}} + \langle \mathbf{S}_{01}\Psi_{01}, \Psi_{01}\rangle + \langle \mathbf{S}_{02}\Psi_{02}, \Psi_{02}\rangle \right\}. \end{aligned} \tag{5.2}$$

Then we have $\mathrm{cond}(\mathbf{M}^{-1}\mathbf{S}) \le C_1 C_2$.

The rest of this section applies Lemma 5.1 to show Theorem 3.1, the main result of this paper. First, we construct the important decomposition required in the lemma. For this, we will make use of the so-called regular decomposition instead of the usual $L^2(\Omega)$-orthogonal Helmholtz decomposition [14].

For any $\mathbf{v} \in H_0(\mathbf{curl}; \Omega)$, there exist some $\mathbf{w} \in H_0^1(\Omega)^3$ and $p \in H_0^1(\Omega)$ such that the following regular decomposition holds (cf. [6], [16]):

$$\mathbf{v} = \nabla p + \mathbf{w} \tag{5.3}$$

with the estimates

$$\|w\|_{0,\Omega} + \|p\|_{1,\Omega} \le C\, \|\mathbf{v}\|_{0,\Omega}\,, \quad |\mathbf{v}|_{1,\Omega} \le C \|\mathbf{curl}\,\mathbf{v}\|_{0,\Omega}\,. \tag{5.4}$$

We remark that the use of Helmholtz-type or regular decompositions is a fundamental technique for the analysis of preconditioners for $H(\mathbf{curl};\Omega)$- and $H(\mathrm{div};\Omega)$-elliptic problems [1], [15], [17], [16], [28].

Now, for any $\Phi \in V_h(\Gamma)$, we define a $\mathbf{v}_h \in V_h(\Omega)$ such that $\mathbf{v}_h = \mathbf{R}_h^i \mathbf{I}_i \Phi$ in each subdomain Ω_i. By the regular decomposition (5.3), there exist $p \in H_0^1(\Omega)$ and $\mathbf{w} \in H_0^1(\Omega)^3$ such that

$$\mathbf{v}_h = \mathbf{grad}\ p + \mathbf{w}\,. \tag{5.5}$$

As $\mathbf{w} \in H_0^1(\Omega)^3$ and $\mathbf{curl}\,\mathbf{w} = \mathbf{curl}\,\mathbf{v}_h$, so $\mathbf{r}_h\mathbf{w}$ is well defined (see subsection 4.2). This, with (5.5), implies

$$\mathbf{v}_h = \mathbf{r}_h \mathbf{grad}\ p + \mathbf{r}_h \mathbf{w}\,.$$

By Lemma 5.10 in [14], there exists a function $p_h \in Z_h(\Omega)$ such that

$$\mathbf{v}_h = \mathbf{grad}\ p_h + \mathbf{r}_h\mathbf{w} = \mathbf{grad} p_h + \mathbf{w}_h \tag{5.6}$$

with $\mathbf{w}_h = \mathbf{r}_h \mathbf{w} \in V_h(\Omega)$. By (5.5) and (5.6), we know

$$\mathbf{curl}\ \mathbf{w}_h = \mathbf{curl}\ \mathbf{w} = \mathbf{curl}\ \mathbf{v}_h. \tag{5.7}$$

Now we are ready to show Theorem 3.1 using Lemma 5.1. We divide the proof into four steps.

Step 1. Establish a suitable decomposition for $\Phi \in V_h(\Gamma)$. For ease of notation, we introduce $p_h^0 \in Z_h(\Omega)$ and Φ_{01} by

$$\begin{aligned} p_h^0 &= R_h^i \mathrm{I}_i \pi_0(p_h|_\Gamma) \quad \text{in} \quad \Omega_i, \quad i = 1, \dots, N, \\ \Phi_{01}(\mathbf{x}) &= (\mathbf{grad}\ (p_h^0|_{\Omega_i}) \times \mathbf{n})(\mathbf{x}), \quad \mathbf{x} \in \Gamma_i,\ i = 1, 2, \dots, N. \end{aligned}$$

By direct checking, we can also write

$$\Phi_{01}(\mathbf{x}) = (\mathbf{grad}\ (\tilde{p}_h^0|_{\Omega_i}) \times \mathbf{n})(\mathbf{x}), \quad \mathbf{x} \in \Gamma_i,$$

with $\tilde{p}_h^0 = R_0^i \mathrm{I}_i \pi_0(p_h|_\Gamma)$. So we know $\Phi_{01}(\mathbf{x}) \in V_h^{01}(\Gamma)$. Next, we choose $\mathbf{w}_{02} = \Pi_0 \mathbf{w}_h \in V_h(\Omega)$ and let

$$\Phi_{02} = (\mathbf{w}_{02} \times \mathbf{n})|_\Gamma \in V_h^{02}(\Gamma).$$

Define $\Phi_{ij} \in V_h(\Gamma_{ij})$ by

$$\begin{aligned} \Phi_{ij} &= \mathbf{I}_{ij}((\mathbf{grad}\ p_h + \mathbf{w}_h) \times \mathbf{n}) - \mathbf{I}_{ij}(\Phi_{01} + \Phi_{02}) \\ &= \mathbf{I}_{ij}(\mathbf{grad}\ (p_h - p_h^0) \times \mathbf{n}) + \mathbf{I}_{ij}(\mathbf{w}_h \times \mathbf{n} - \Phi_{02}) \\ &= \mathbf{I}_{ij}(\mathbf{grad}\ (p_h - p_h^0) \times \mathbf{n}) + \mathbf{I}_{ij}((\mathbf{w}_h - \mathbf{w}_{02}) \times \mathbf{n}). \end{aligned}$$

Noting the fact that $p_h^0 - p_h$ vanishes on the wirebasket set $\mathcal{W}_i$, we can easily verify that $\lambda_e(\mathbf{grad}\ (p_h - p_h^0)) = 0$ for any $e \in \mathcal{E}_h \cap \mathcal{W}_i$. Also, we have $\lambda_e(\mathbf{w}_h - \mathbf{w}_{02}) = 0$ for any face e on Δ_i. Thus $\Phi_{ij} \in V_h^0(\Gamma_{ij})$, and the following decomposition holds:

$$\Phi = \Phi_{01} + \Phi_{02} + \sum_{\Gamma_{ij}} \mathbf{I}_{ij}^t \Phi_{ij}. \tag{5.8}$$

Step 2. Prove the estimate

$$\sum_{\Gamma_{ij}} \langle \mathbf{S}_{ij} \Phi_{ij}, \Phi_{ij} \rangle_{\Gamma_{ij}} \le C[1 + \log(d/h)]^3 \langle \mathbf{S}\Phi, \Phi \rangle. \tag{5.9}$$

For any face Γ_{ij} of Γ_i, we define

$$\begin{aligned} p_{ij}^i &= R_h^i \mathrm{I}_{ij}^t [(p_h - p_h^0)|_{\Gamma_{ij}}] \in Z_h(\Omega_i), \\ \mathbf{w}_{ij}^i &= \mathbf{R}_h^i \mathbf{I}_{ij}^t [((\mathbf{w}_h - \mathbf{w}_{02}) \times \mathbf{n})|_{\Gamma_{ij}}] \in V_h(\Omega_i), \\ \mathbf{v}_{ij}^i &= \mathbf{grad}\ p_{ij}^i + \mathbf{w}_{ij}^i \in V_h(\Omega_i). \end{aligned}$$

Using the fact that

$$\mathbf{R}_h^i \mathbf{I}_{ij}^t \Phi_{ij} \times \mathbf{n} = \mathbf{I}_{ij}^t \Phi_{ij} = \mathbf{v}_{ij}^i \times \mathbf{n} \quad \text{on} \quad \Gamma_i,$$

we obtain by the minimum **curl**-energy property of the discrete **curl**-extension that

$$\begin{aligned} A_i(\mathbf{R}_h^i \mathbf{I}_{ij}^t \Phi_{ij}, \mathbf{R}_h^i \mathbf{I}_{ij}^t \Phi_{ij}) &\le A_i(\mathbf{v}_{ij}^i, \mathbf{v}_{ij}^i) = \|\alpha^{\frac{1}{2}} \mathbf{curl}\ \mathbf{w}_{ij}^i\|_{0,\Omega_i}^2 + \|\beta^{\frac{1}{2}} \mathbf{v}_{ij}^i\|_{0,\Omega_i}^2 \\ &\le C(\|\mathbf{grad}\ p_{ij}^i\|_{0,\Omega_i}^2 + \|\mathbf{w}_{ij}^i\|_{\mathbf{curl},\Omega_i}^2). \end{aligned} \tag{5.10}$$

As $p_h^0 = \pi_0(p_h|_\Gamma)$ on Γ, we have

$$\mathrm{I}_{ij}^t[(p_h - p_h^0)|_{\Gamma_{ij}}] = \mathrm{I}_{ij}^0(p_h|_\Gamma - \pi_0(p_h|_\Gamma)).$$

Thus, using (4.4) and the trace theorem, we obtain

$$\begin{aligned}(5.11)\qquad \|\mathbf{grad}\ p_{ij}^i\|_{0,\Omega_i}^2 &= |p_{ij}^i|_{1,\Omega_i}^2 \le C|\mathrm{I}_{ij}^t[(p_h - p_h^0)|_{\Gamma_{ij}}]|_{\frac{1}{2},\Gamma_i}^2 \\ &\le C[1+\log(d/h)]^2|p_h|_{\frac{1}{2},\Gamma_i}^2 \\ &\le C[1+\log(d/h)]^2|p_h|_{1,\Omega_i}^2.\end{aligned}$$

We next estimate $\mathbf{w}_{ij}^i$. For each (open) common face $\mathrm{F} = \Gamma_{ij}$ shared by Ω_i and Ω_j, it follows from the definition of Π_0 that

$$\lambda_e(\mathbf{w}_h - \mathbf{w}_{02}) = \begin{cases} 0 & \text{if } e \subset \mathrm{F}_b, \\ \lambda_e(\mathbf{w}_h - \Upsilon_{\partial \mathrm{F}}(\mathbf{w}_h)) & \text{if } e \subset \mathrm{F}_\partial\,. \end{cases}$$

Then we derive by using (5.7) and Lemmas 4.1 and 4.7 that

$$\begin{aligned}(5.12)\qquad \|\mathbf{w}_{ij}^i\|_{\mathbf{curl},\Omega_i}^2 &\le C\|\mathbf{I}_{ij}^t[((\mathbf{w}_h - \mathbf{w}_{02})\times\mathbf{n})|_{\Gamma_{ij}}]\|_{\mathcal{X}_{\Gamma_i}}^2 \\ &= C\|\mathbf{I}_{\mathrm{F}_\partial}^0[(\mathbf{w}_h - \Upsilon_{\partial\Gamma_{ij}}(\mathbf{w}_h))\times\mathbf{n}]\|_{\mathcal{X}_{\Gamma_i}}^2 \\ &\le C[1+\log(d/h)]^2(\|(\mathbf{w}_h - \Upsilon_{\partial\Gamma_{ij}}(\mathbf{w}_h))\times\mathbf{n}\|_{\mathcal{X}_{\Gamma_i}}^2 \\ &\quad + \|\mathbf{w} - \Upsilon_{\partial\Gamma_{ij}}(\mathbf{w}_h)\|_{1,\Omega_i}^2 + \|\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_i}^2).\end{aligned}$$

On the other hand, for the term $(\mathbf{w}_h - \Upsilon_{\partial\Gamma_{ij}}(\mathbf{w}_h))\times\mathbf{n}$ we have by Lemma 4.2 and (5.5) that

$$\begin{aligned}&\|(\mathbf{w}_h - \Upsilon_{\partial\Gamma_{ij}}(\mathbf{w}_h))\times\mathbf{n}\|_{\mathcal{X}_{\Gamma_i}}^2 \\ &\le C\|\mathbf{w}_h - \Upsilon_{\partial\Gamma_{ij}}(\mathbf{w}_h)\|_{\mathbf{curl};\Omega_i}^2 \\ &= C(\|\mathbf{curl}\ \mathbf{w}_h\|_{0;\Omega_i}^2 + d^{-2}\|\mathbf{w}_h - \Upsilon_{\partial\Gamma_{ij}}(\mathbf{w}_h)\|_{0;\Omega_i}^2) \\ &= C(\|\mathbf{curl}\ \mathbf{v}_h\|_{0;\Omega_i}^2 + d^{-2}\|\mathbf{w}_h - \Upsilon_{\partial\Gamma_{ij}}(\mathbf{w}_h)\|_{0;\Omega_i}^2).\end{aligned}$$

Combining this with (5.12) and using Lemma 4.6 give

$$\|\mathbf{w}_{ij}^i\|_{\mathbf{curl},\Omega_i}^2 \le C[1+\log(d/h)]^3(|\mathbf{w}|_{1,\Omega_i}^2 + \|\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_i}^2).$$

With this estimate, (5.10), and (5.11), we come to

$$(5.13)\quad A_i(\mathbf{R}_h^i\mathbf{I}_{ij}^t\Phi_{ij}, \mathbf{R}_h^i\mathbf{I}_{ij}^t\Phi_{ij}) \le C[1+\log(d/h)]^3(|p_h|_{1,\Omega_i}^2 + |\mathbf{w}|_{1,\Omega_i}^2 + \|\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_i}^2).$$

Similarly, we have

$$A_j(\mathbf{R}_h^j\mathbf{I}_{ij}^t\Phi_{ij}, \mathbf{R}_h^j\mathbf{I}_{ij}^t\Phi_{ij}) \le C[1+\log(d/h)]^3(|p_h|_{1,\Omega_j}^2 + |\mathbf{w}|_{1,\Omega_j}^2 + \|\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_j}^2).$$

So we have proved

$$\begin{aligned}\langle S_{ij}\Phi_{ij}, \Phi_{ij}\rangle_{\Gamma_{ij}} &\le C[1+\log(d/h)]^3(|p_h|_{1,\Omega_i}^2 + |p_h|_{1,\Omega_j}^2 + |\mathbf{w}|_{1,\Omega_i}^2 \\ &\quad + |\mathbf{w}|_{1,\Omega_j}^2 + \|\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_i}^2 + \|\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_j}^2),\end{aligned}$$

or

$$\sum_{\Gamma_{ij}} \langle \mathbf{S}_{ij}\Phi_{ij}, \Phi_{ij}\rangle_{\Gamma_{ij}} \le C[1+\log(d/h)]^3 \sum_{i=1}^{N} (|p_h|^2_{1,\Omega_i} + |\mathbf{w}|^2_{1,\Omega_i} + \|\mathbf{curl}\ \mathbf{v}_h\|^2_{0,\Omega_i}) \tag{5.14}$$

$$= C[1+\log(d/h)]^3 \left(|p_h|^2_{1,\Omega} + |\mathbf{w}|^2_{1,\Omega} + \sum_{i=1}^{N} \|\mathbf{curl}\ \mathbf{v}_h\|^2_{0,\Omega_i} \right).$$

To prove (5.9), it suffices to show

$$|p_h|^2_{1,\Omega} + |\mathbf{w}|^2_{1,\Omega} \le C(\|\mathbf{v}_h\|^2_{0,\Omega} + \|\mathbf{curl}\ \mathbf{v}_h\|^2_{0,\Omega}), \tag{5.15}$$

as this, with (5.14), implies

$$\sum_{\Gamma_{ij}} \langle \mathbf{S}_{ij}\Phi_{ij}, \Phi_{ij}\rangle_{\Gamma_{ij}} \le C[1+\log(d/h)]^3 \sum_{i=1}^{N} (\|\mathbf{curl}\ \mathbf{v}_h\|^2_{0,\Omega_i} + \|\mathbf{v}_h\|^2_{0,\Omega_i})$$

$$\le C[1+\log(d/h)]^3 \sum_{i=1}^{N} A_i(\mathbf{R}_h^i \mathbf{I}_i \Phi, \mathbf{R}_h^i \mathbf{I}_i \Phi).$$

Next we show (5.15). It follows from (5.4) and (5.7) that

$$|\mathbf{w}|^2_{1,\Omega} \le C\|\mathbf{curl}\ \mathbf{w}\|^2_{0,\Omega} = C\|\mathbf{curl}\ \mathbf{v}_h\|^2_{0,\Omega}\,. \tag{5.16}$$

However, by Lemma 4.4 and (5.4) we obtain that

$$\|\mathbf{r}_h\mathbf{w}\|^2_{0,\Omega} \le C\,(h^2\,\|\mathbf{curl}\,\mathbf{v}_h\|^2_{0,\Omega} + h^2\,|\mathbf{w}|^2_{1,\Omega} + \|\mathbf{w}\|^2_{0,\Omega}) \le C\,\|\mathbf{curl}\,\mathbf{v}_h\|^2_{0,\Omega}.$$

Inequality (5.15) is then a consequence of this estimate, (5.16), and the triangle inequality

$$\|\nabla p_h\|_{0,\Omega} \le \|\mathbf{v}_h\|_{0,\Omega} + \|\mathbf{r}_h\mathbf{w}\|_{0,\Omega}\,.$$

Step 3. Derive the estimate

$$\langle \mathbf{S}_{01}\Phi_{01}, \Phi_{01}\rangle + \langle \mathbf{S}_{02}\Phi_{02}, \Phi_{02}\rangle \le C[1+\log(d/h)]^2 \langle \mathbf{S}\Phi, \Phi\rangle. \tag{5.17}$$

It follows from the definitions of $\mathbf{S}_{01}$ and Φ_{01} that

$$\langle \mathbf{S}_{01}\Phi_{01}, \Phi_{01}\rangle = [1+\log(d/h)] \sum_{i=1}^{N} \|p_h^0 - \gamma_{\Delta_i} p_h^0\|^2_{h,\Delta_i}.$$

Thus, by (4.3) and the trace theorem, we have

$$\langle \mathbf{S}_{01}\Phi_{01}, \Phi_{01}\rangle \le C[1+\log(d/h)]^2 \sum_{i=1}^{N} |p_h|^2_{\frac{1}{2},\Gamma_i} \tag{5.18}$$

$$\le C[1+\log(d/h)]^2 \sum_{i=1}^{N} |p_h|^2_{1,\Omega_i}$$

$$\leq C[1+\log(d/h)]^2|p_h|_{1,\Omega}^2.$$

By the definitions of $\mathbf{S}_{02}$ and Φ_{02}, we know

$$\langle \mathbf{S}_{02}\Phi_{02},\Phi_{02}\rangle = [1+\log(d/h)]\sum_{i=1}^{N}(\|(\mathbf{w}_h-\Upsilon_{\Delta_i}(\mathbf{w}_h))\times\mathbf{n}\|_{*,\Delta_i}^2+d^2\|\mathbf{w}_h\times\mathbf{n}\|_{*,\Delta_i}^2). \tag{5.19}$$

From the definition of $\Upsilon_{\Delta_i}(\mathbf{w}_h)$, we have

$$\|(\mathbf{w}_h-\Upsilon_{\Delta_i}(\mathbf{w}_h))\times\mathbf{n}\|_{*,\Delta_i}^2 \leq \|(\mathbf{w}_h-\Upsilon_{\Gamma_i}(\mathbf{w}))\times\mathbf{n}\|_{*,\Delta_i}^2.$$

This, with Lemma 4.5 and the Poincaré inequality, gives

$$\begin{aligned}\|(\mathbf{w}_h-\Upsilon_{\Delta_i}(\mathbf{w}_h))\times\mathbf{n}\|_{*,\Delta_i}^2 &\leq \sum_{\mathrm{F}\subset\Gamma_i}\|(\mathbf{w}_h-\Upsilon_{\Gamma_i}(\mathbf{w}))\times\mathbf{n}\|_{*,\mathrm{F}_b}^2\\ &\leq C[1+\log(d/h)](\|\mathbf{w}-\Upsilon_{\Gamma_i}(\mathbf{w})\|_{1,\Omega_i}^2\\ &\quad+\|\mathbf{curl}\ (\mathbf{v}_h-\Upsilon_{\Gamma_i}(\mathbf{w}))\|_{0,\Omega_i}^2)\\ &\leq [1+\log(d/h)](|\mathbf{w}|_{1,\Omega_i}^2+\|\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_i}^2).\end{aligned} \tag{5.20}$$

The other terms in (5.19) are estimated by Lemma 4.5 and (5.5) as follows:

$$\begin{aligned}d^2\|\mathbf{w}_h\times\mathbf{n}\|_{*,\Delta_i}^2 &= d^2\sum_{\mathrm{F}\subset\Gamma_i}\|\mathbf{w}_h\times\mathbf{n}\|_{*,\mathrm{F}_b}^2\\ &\leq Cd^2[1+\log(d/h)](\|\mathbf{w}\|_{1,\Omega_i}^2+\|\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega_i}^2)\\ &= C[1+\log(d/h)](d^2|\mathbf{w}|_{1,\Omega_i}^2+\|\mathbf{w}\|_{0,\Omega_i}^2+d^2\|\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega_i}^2)\\ &\leq C[1+\log(d/h)](|\mathbf{w}|_{1,\Omega_i}^2+\|\mathbf{v}_h\|_{0,\Omega_i}^2+\|\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega_i}^2).\end{aligned}$$

So we have proved by (5.19) that

$$\langle \mathbf{S}_{02}\Phi_{02},\Phi_{02}\rangle \leq C[1+\log(d/h)]^2(|\mathbf{w}|_{1,\Omega}^2+\|\mathbf{v}_h\|_{0,\Omega}^2+\|\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega}^2),$$

which, together with (5.18), yields

$$\begin{aligned}&\langle \mathbf{S}_{01}\Phi_{01},\Phi_{01}\rangle+\langle \mathbf{S}_{02}\Phi_{02},\Phi_{02}\rangle\\ &\leq C[1+\log(d/h)]^2(|p_h|_{1,\Omega}^2+|\mathbf{w}|_{1,\Omega}^2+\|\mathbf{v}_h\|_{0,\Omega}^2+\|\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega}^2)\\ &\leq C[1+\log(d/h)]^2(|p_h|_{1,\Omega}^2+|\mathbf{curl}\ \mathbf{w}|_{1,\Omega}^2+\|\mathbf{v}_h\|_{0,\Omega}^2+\|\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega}^2)\\ &\leq C[1+\log(d/h)]^2(|p_h|_{1,\Omega}^2+\|\mathbf{v}_h\|_{0,\Omega}^2+\|\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega}^2)\\ &\leq C[1+\log(d/h)]^2(\|\mathbf{v}_h\|_{0,\Omega}^2+\|\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega}^2)\\ &\leq C[1+\log(d/h)]^2\langle \mathbf{S}\Phi,\Phi\rangle.\end{aligned}$$

The estimates (5.9) and (5.17) indicate that the constant C_1 in (5.1) can be bounded by $C[1+\log(d/h)]^3$.

Step 4. Estimate the constant C_2 in (5.2). It is easy to see that

$$\mathbf{I}_k\left(\sum_{\Gamma_{ij}}\mathbf{I}_{ij}^t\Psi_{ij}+\Psi_{01}+\Psi_{02}\right)=\sum_{\Gamma_{ij}\subset\Gamma_k}\mathbf{I}_{ij}^t\Phi_{ij}+\mathbf{I}_k\Psi_{01}+\mathbf{I}_k\Psi_{02}.$$

Hence

(5.21)
$$\begin{aligned}&\left\langle \mathbf{S}\left(\sum_{\Gamma_{ij}} \mathbf{I}_{ij}^t \Psi_{ij} + \Psi_{01} + \Psi_{02}\right),\ \sum_{\Gamma_{ij}} \mathbf{I}_{ij}^t \Psi_{ij} + \Psi_{01} + \Psi_{02}\right\rangle \\ &\le C \sum_{k=1}^{N} \left\{ \sum_{\Gamma_{ij}\subset\Gamma_k} \langle \mathbf{S}_k \mathbf{I}_{ij}^t \Psi_{ij}, \mathbf{I}_{ij}^t \Phi_{ij}\rangle_{\Gamma_k} + \langle \mathbf{S}_k \mathbf{I}_k \Psi_{01}, \mathbf{I}_k \Psi_{01}\rangle_{\Gamma_k} + \langle \mathbf{S}_k \mathbf{I}_k \Psi_{02}, \mathbf{I}_k \Psi_{02}\rangle_{\Gamma_k} \right\} \\ &\le C \sum_{k=1}^{N} \left\{ \sum_{\Gamma_{ij}\subset\Gamma_k} \langle \mathbf{S}_{ij} \Psi_{ij}, \Psi_{ij}\rangle_{\Gamma_{ij}} + A_k(\mathbf{R}_h^k \mathbf{I}_k \Psi_{01}, \mathbf{R}_h^k \mathbf{I}_k \Psi_{01}) + A_k(\mathbf{R}_h^k \mathbf{I}_k \Psi_{02}, \mathbf{R}_h^k \mathbf{I}_k \Psi_{02}) \right\}.\end{aligned}$$

As each face Γ_{ij} is shared only by two subdomains Ω_i and Ω_j, we have

$$\sum_{k=1}^{N} \sum_{\Gamma_{ij}\subset\Gamma_k} \langle \mathbf{S}_{ij} \Psi_{ij}, \Psi_{ij}\rangle_{\Gamma_{ij}} \le C \sum_{\Gamma_{ij}} \langle \mathbf{S}_{ij} \Psi_{ij}, \Psi_{ij}\rangle_{\Gamma_{ij}}. \tag{5.22}$$

Note that $\Psi_{01} \in V_h^{01}(\Gamma)$ can be written as

$$\mathbf{I}_k \Psi_{01} = \mathbf{grad}(R_h^k \mathrm{I}_k \pi_0 \psi) \times \mathbf{n} \quad \text{on} \ \ \Gamma_k$$

for some $\psi \in Z_h(\Gamma)$, so we have

$$\begin{aligned} A_k(\mathbf{R}_h^k \mathbf{I}_k \Psi_{01}, \mathbf{R}_h^k \mathbf{I}_k \Psi_{01}) &\le A_k(\mathbf{grad}(R_h^k \mathrm{I}_k \pi_0 \psi), \mathbf{grad}(R_h^k \mathrm{I}_k \pi_0 \psi)) \\ &= |\beta^{\frac{1}{2}} R_h^k \mathrm{I}_k \pi_0 \psi|_{1,\Omega_k}^2 \le C |\pi_0 \psi|_{\frac{1}{2},\Gamma_k}^2. \end{aligned}$$

Then it follows from (4.3) that

$$A_k(\mathbf{R}_h^k \mathbf{I}_k \Psi_{01}, \mathbf{R}_h^k \mathbf{I}_k \Psi_{01}) \le C[1 + \log(d/h)] \|\pi_0 \psi - \gamma_{\mathcal{W}_k}(\pi_0 \psi)\|_{h,\mathcal{W}_k}^2.$$

This, with the definition of $\mathbf{S}_{01}$, shows

$$\sum_{k=1}^{N} A_k(\mathbf{R}_h^k \mathbf{I}_k \Psi_{01}, \mathbf{R}_h^k \mathbf{I}_k \Psi_{01}) \le C \langle \mathbf{S}_{01} \Psi_{01}, \Psi_{01}\rangle. \tag{5.23}$$

We next estimate the last term in (5.21). We can write $\Psi_{02} \in V_h^{02}(\Gamma)$ as follows:

$$\mathbf{I}_k \Psi_{02} = \Pi_0 \mathbf{v} \times \mathbf{n} = [\Pi_0 \mathbf{v} - \Upsilon_{\Delta_k}(\Pi_0 \mathbf{v})] \times \mathbf{n} + \Upsilon_{\Delta_k}(\Pi_0 \mathbf{v}) \times \mathbf{n} \quad \text{on} \ \ \Gamma_i$$

for some $\mathbf{v} \in V_h(\Gamma)$. Then, by the triangle inequality, we obtain

$$\begin{aligned} &A_k(\mathbf{R}_h^k \mathbf{I}_k \Psi_{02}, \mathbf{R}_h^k \mathbf{I}_k \Psi_{02}) \\ &\le 2A_k(\mathbf{R}_h^k \mathbf{I}_k [\Pi_0 \mathbf{v} - \Upsilon_{\Delta_k}(\Pi_0 \mathbf{v})] \times \mathbf{n}, \mathbf{R}_h^k \mathbf{I}_k [\Pi_0 \mathbf{v} - \Upsilon_{\Delta_k}(\Pi_0 \mathbf{v}) \times \mathbf{n}]) \\ &\quad + A_k(\mathbf{R}_h^k \mathbf{I}_k [\Upsilon_{\Delta_k}(\Pi_0 \mathbf{v}) \times \mathbf{n}], \mathbf{R}_h^k \mathbf{I}_k [\Upsilon_{\Delta_k}(\Pi_0 \mathbf{v}) \times \mathbf{n}])). \end{aligned}$$

Furthermore, using Lemma 4.1 and the minimum **curl**-energy property of the discrete **curl**-extension, we obtain (note that $\Upsilon_{\Delta_k}(\Pi_0 \mathbf{v})$ is a constant vector)

$$\begin{aligned} &A_k(\mathbf{R}_h^k \mathbf{I}_k \Psi_{02}, \mathbf{R}_h^k \mathbf{I}_k \Psi_{02}) \\ &\le C(\|[\Pi_0 \mathbf{v} - \Upsilon_{\mathcal{W}_k}(\Pi_0 \mathbf{v})] \times \mathbf{n}\|_{\mathcal{X}_{\Gamma_i}}^2 + A_k(\Upsilon_{\Delta_k}(\Pi_0 \mathbf{v}), \Upsilon_{\Delta_k}(\Pi_0 \mathbf{v}))) \\ &= C(\|[\Pi_0 \mathbf{v} - \Upsilon_{\Delta_k}(\Pi_0 \mathbf{v})] \times \mathbf{n}\|_{\mathcal{X}_{\Gamma_k}}^2 + \|\Upsilon_{\Delta_k}(\Pi_0 \mathbf{v})\|_{0,\Omega_k}^2), \end{aligned} \tag{5.24}$$

where the last term can be estimated using the Hölder inequality and direct computation:

(5.25)
$$\|\Upsilon_{\Delta_k}(\Pi_0\mathbf{v})\|^2_{0,\Omega_k} = d^3|\Upsilon_{\Delta_k}(\Pi_0\mathbf{v})|^2 \le Cd^2\|\Upsilon_{\Delta_k}(\Pi_0\mathbf{v})\|^2_{*,\Delta_k} \le Cd^2\|(\Pi_0\mathbf{v})\times\mathbf{n}\|^2_{*,\Delta_k}.$$

Next, we show that the first term in (5.28) has the following bound:

$$\|[\Pi_0\mathbf{v} - \Upsilon_{\Delta_k}(\Pi_0\mathbf{v})]\times\mathbf{n}\|^2_{\mathcal{X}_{\Gamma_k}} \le C[1+\log(d/h)]\|[\Pi_0\mathbf{v} - \Upsilon_{\Delta_k}(\Pi_0\mathbf{v})]\times\mathbf{n}\|^2_{*,\Delta_k}. \quad (5.26)$$

For this, it suffices to prove

$$\|(\Pi_0\mathbf{v})\times\mathbf{n}\|^2_{\mathcal{X}_{\Gamma_k}} \le C[1+\log(d/h)]\|(\Pi_0\mathbf{v})\times\mathbf{n}\|^2_{*,\Delta_k} \quad \forall\mathbf{v}\in V_h(\Omega). \quad (5.27)$$

To see this, using the relation

$$\mathbf{I}_k[(\Pi_0\mathbf{v})\times\mathbf{n}] = \mathbf{I}^0_{\Delta_k}(\Pi_0\mathbf{v}\times\mathbf{n}) + \sum_{\mathrm{F}\subset\Gamma_k}\mathbf{I}^0_{\mathrm{F}_\partial}(\Upsilon_{\partial\mathrm{F}}(\Pi_0\mathbf{v})\times\mathbf{n}),$$

we have

$$\|(\Pi_0\mathbf{v})\times\mathbf{n}\|^2_{\mathcal{X}_{\Gamma_k}} \le C\left(\|\mathbf{I}^0_{\Delta_i}(\Pi_0\mathbf{v}\times\mathbf{n})\|^2_{\mathcal{X}_{\Gamma_k}} + \sum_{\mathrm{F}\subset\Gamma_k}\|\mathbf{I}^0_{\mathrm{F}_\partial}(\Upsilon_{\partial\mathrm{F}}(\Pi_0\mathbf{v})\times\mathbf{n})\|^2_{\mathcal{X}_{\Gamma_k}}\right).$$

This, together with Lemmas 4.8 and 4.9, yields (5.27).

Finally, we obtain using (5.24), (5.25), and (5.26) that

$$A_k(\mathbf{R}^k_h\mathbf{I}_k\Psi_{02}, \mathbf{R}^k_h\mathbf{I}_k\Psi_{02}) \le C([1+\log(d/h)]\|[\mathbf{v}-\Upsilon_{\Delta_k}(\mathbf{v})]\times\mathbf{n}\|^2_{*,\Delta_k} + d^2\|\mathbf{v}\times\mathbf{n}\|^2_{*,\Delta_k}),$$

which implies

$$\sum_{k=1}^N A_k(\mathbf{R}^k_h\mathbf{I}_k\Psi_{02}, \mathbf{R}^k_h\mathbf{I}_k\Psi_{02}) \le C\langle\mathbf{S}_{02}\Psi_{02}, \Psi_{02}\rangle.$$

This estimate with (5.22)–(5.23) indicates that the constant C_2 in (5.2) is bounded by a constant independent of h and d.

6. Appendix. This appendix provides the technical proofs for the auxiliary lemmas in Section 4.

6.1. Proofs of Lemmas 4.5 and 4.6. In this subsection we shall prove Lemmata 4.5 and 4.6. For this, we first give some auxiliary results. The first lemma can be found in [7], [33].

LEMMA 6.1. *Let $v_h \in Z_h(\Gamma_i)$. Then, for any $\mathrm{F}\subset\Gamma_i$, we have*

$$\|v_h\|_{0,\partial\mathrm{F}} \le C[1+\log(d/h)]^{\frac{1}{2}}\|v_h\|_{\frac{1}{2},\Gamma_i}, \quad (6.1)$$

$$\|\mathrm{I}^0_{\mathrm{F}}v_h\|_{\frac{1}{2},\Gamma_i} \le C[1+\log(d/h)]\|v_h\|_{\frac{1}{2},\Gamma_i}, \quad (6.2)$$

$$|\mathrm{I}^0_{\partial\mathrm{F}}v_h|_{\frac{1}{2},\mathrm{F}} \le C[1+\log(d/h)]^{\frac{1}{2}}\|v_h\|_{\frac{1}{2},\Gamma_i}. \quad (6.3)$$

LEMMA 6.2. *Assume that $\mathbf{v}_h \in Z_h(\Omega_i)^3$. Then, for any face F of Γ_i we have*

$$d^{-2}\|\mathbf{v}_h - \Upsilon_{\partial\mathrm{F}}(\mathbf{v}_h)\|^2_{0,\Omega_i} \le C[1+\log(d/h)]|\mathbf{v}_h|^2_{1,\Omega_i}. \quad (6.4)$$

Proof. Since $\Upsilon_{\partial \mathrm{F}}(\cdot)$ is invariant with constant vectors, we have

$$\begin{aligned} & d^{-2}\|\mathbf{v}_h - \Upsilon_{\partial\mathrm{F}}(\mathbf{v}_h)\|_{0,\Omega_i}^2 = d^{-2}\|\mathbf{v}_h - \Upsilon_{\mathrm{F}}(\mathbf{v}_h) - \Upsilon_{\partial\mathrm{F}}(\mathbf{v}_h - \Upsilon_{\mathrm{F}}(\mathbf{v}_h))\|_{0,\Omega_i}^2 \\ & \leq 2d^{-2}(\|\mathbf{v}_h - \Upsilon_{\mathrm{F}}(\mathbf{v}_h)\|_{0,\Omega_i}^2 + \|\Upsilon_{\partial\mathrm{F}}(\mathbf{v}_h - \Upsilon_{\mathrm{F}}(\mathbf{v}_h))\|_{0,\Omega_i}^2). \end{aligned} \tag{6.5}$$

It can be verified, by the Hölder inequality, that

$$\|\Upsilon_{\partial\mathrm{F}}(\mathbf{v}_h - \Upsilon_{\mathrm{F}}(\mathbf{v}_h))\|_{0,\Omega_i}^2 \leq Cd^3|\Upsilon_{\partial\mathrm{F}}(\mathbf{v}_h - \Upsilon_{\mathrm{F}}(\mathbf{v}_h))|^2 \leq Cd^2\|\mathbf{v}_h - \Upsilon_{\mathrm{F}}(\mathbf{v}_h)\|_{0,\partial\mathrm{F}}^2.$$

This, together with (6.1) and the trace theorem, yields

$$\begin{aligned} d^{-2}\|\Upsilon_{\partial\mathrm{F}}(\mathbf{v}_h - \Upsilon_{\mathrm{F}}(\mathbf{v}_h))\|_{0,\Omega_i}^2 &\leq C[1+\log(d/h)]\|\mathbf{v}_h - \Upsilon_{\mathrm{F}}(\mathbf{v}_h)\|_{\frac{1}{2},\Gamma_i}^2 \\ &\leq C[1+\log(d/h)]\|\mathbf{v}_h - \Upsilon_{\mathrm{F}}(\mathbf{v}_h)\|_{1,\Omega_i}^2. \end{aligned}$$

Now (6.4) follows from this, (6.5), and the Friedrich's inequality. ∎

For any face F of Γ_i, we define a quantity (not a norm) on F_b as follows:

$$\|\mathbf{v}\|_{*,\mathrm{F}_b} = \left(\sum_{K\in\mathrm{F}_b} \|\mathbf{v}\|_{0,\partial K}^2\right)^{\frac{1}{2}} \quad \forall\, \mathbf{v} \in Z_h(\Gamma_i)^3 \quad \text{or} \quad \mathbf{v} \in V_h(\Gamma_i).$$

LEMMA 6.3. *Assume that* $\mathbf{v}_h \in Z_h(\Gamma_i)^3$. *Then*

$$\|\mathbf{v}_h\|_{*,\mathrm{F}_b} \leq C[1+\log(d/h)]^{\frac{1}{2}}\|\mathbf{v}_h\|_{\frac{1}{2},\Gamma_i}. \tag{6.6}$$

Proof. Consider a triangle $K \in \mathrm{F}_b$, and let e be one of its edges lying on $\partial\mathrm{F}$. Then we have

$$\|\mathbf{v}_h\|_{0,\partial K}^2 \leq 2(\|\mathbf{v}_h - \Upsilon_e(\mathbf{v}_h)\|_{0,\partial K}^2 + \|\Upsilon_e(\mathbf{v}_h)\|_{0,\partial K}^2). \tag{6.7}$$

By the Poincaré inequality we obtain

$$h^{-1}\|\mathbf{v}_h - \Upsilon_e(\mathbf{v}_h)\|_{0,\partial K}^2 \leq h^{-2}\|\mathbf{v}_h - \Upsilon_e(\mathbf{v}_h)\|_{0,K}^2 \leq C|\mathbf{v}_h|_{1,K}^2.$$

Thus

$$\|\mathbf{v}_h - \Upsilon_e(\mathbf{v}_h)\|_{0,\partial K}^2 \leq Ch|\mathbf{v}_h|_{1,K}^2. \tag{6.8}$$

On the other hand, it can be verified directly that

$$\|\Upsilon_e(\mathbf{v}_h)\|_{0,\partial K}^2 \leq Ch|\Upsilon_e(\mathbf{v}_h)|^2 \leq C\|\mathbf{v}_h\|_{0,e}^2.$$

Substituting this and (6.8) into (6.7) and then summing over all the edges e on K yield

$$\|\mathbf{v}_h\|_{0,\partial K}^2 \leq C(h|\mathbf{v}_h|_{1,\mathrm{F}}^2 + \|\mathbf{v}_h\|_{0,\partial\mathrm{F}}^2) \leq C(|\mathbf{v}_h|_{1/2,\mathrm{F}}^2 + \|\mathbf{v}_h\|_{0,\partial\mathrm{F}}^2).$$

Now, (6.6) follows from (6.1). ∎

Proof of Lemma 4.5. Let $\mathbf{P}_h$: $L^2(\Omega_i)^3 \to Z_h(\Omega_i)^3$ be the L^2-projection operator, which is known to have the following H^s-stability (with $0 \leq s \leq 1$) and estimate [8]:

$$\|\mathbf{P}_h\mathbf{w}\|_{s,\Omega_i} \leq C\|\mathbf{w}\|_{s,\Omega_i}, \quad \|\mathbf{w} - \mathbf{P}_h\mathbf{w}\|_{0,\Omega_i} \leq C\,h\,|\mathbf{w}|_{1,\Omega_i}. \tag{6.9}$$

It is easy to verify that

$$\|(\mathbf{r}_h\mathbf{w})\times\mathbf{n}\|_{*,\mathrm{F}_b} \le C\|\mathbf{r}_h\mathbf{w}\|^2_{*,\mathrm{F}_b} \le C\sum_{e\subset \mathrm{F}_b}(\|\mathbf{P}_h\mathbf{w}\|^2_{0,e}+\|\mathbf{r}_h\mathbf{w}-\mathbf{P}_h\mathbf{w}\|^2_{0,e}). \tag{6.10}$$

Let $K_e \in \mathcal{T}_h$ be an element in Ω_i with e being one of its edges, and $\{\lambda_i\}_{i=1}^4$ the barycentric basis functions at the four vertices of K_e, λ_1, and λ_2, correspond to two end-points of e. By the expression (2.2) of the edge element basis functions, it is easy to verify that $(\mathbf{r}_h\mathbf{w}-\mathbf{P}_h\mathbf{w})$ can be written, in the element K_e, as

$$\mathbf{r}_h\mathbf{w}-\mathbf{P}_h\mathbf{w} = \left(\sum_{i=1}^4 a_i\lambda_i\,,\ \sum_{i=1}^4 b_i\lambda_i\,,\ \sum_{i=1}^4 c_i\lambda_i\right)^T,$$

where $a_i,\ b_i$, and c_i $(i=1,2,3,4)$ are constants which may depend on h. By the standard scaling argument, we obtain

$$\|\mathbf{r}_h\mathbf{w}-\mathbf{P}_h\mathbf{w}\|^2_{0,K_e} \ge \bar{C}h^3\sum_{i=1}^4(a_i^2+b_i^2+c_i^2)\,,$$

$$\|\mathbf{r}_h\mathbf{w}-\mathbf{P}_h\mathbf{w}\|^2_{0,e} \le \tilde{C}h\sum_{i=1}^2(a_i^2+b_i^2+c_i^2).$$

This implies

$$\|\mathbf{r}_h\mathbf{w}-\mathbf{P}_h\mathbf{w}\|^2_{0,e} \le Ch^{-2}\|\mathbf{r}_h\mathbf{w}-\mathbf{P}_h\mathbf{w}\|^2_{0,K_e}\,,$$

and so we have

$$\sum_{e\subset \mathrm{F}_b}\|\mathbf{r}_h\mathbf{w}–\mathbf{P}_h\mathbf{w}\|^2_{0,e} \le Ch^{-2}\sum_{e\subset \mathrm{F}_b}\|\mathbf{r}_h\mathbf{w}–\mathbf{P}_h\mathbf{w}\|^2_{0,K_e} \le Ch^{-2}\|\mathbf{r}_h\mathbf{w}–\mathbf{P}_h\mathbf{w}\|^2_{0,\Omega_i}. \tag{6.11}$$

This with (6.10) leads to

$$\|\mathbf{r}_h\mathbf{w}\|^2_{*,\mathrm{F}_b} \le C(\|\mathbf{P}_h\mathbf{w}\|^2_{*,\mathrm{F}_b}+h^{-2}\|\mathbf{r}_h\mathbf{w}-\mathbf{P}_h\mathbf{w}\|^2_{0,\Omega_i}). \tag{6.12}$$

On the other hand, by (6.6), the trace theorem, and (6.9), we obtain

$$\begin{aligned}\|\mathbf{P}_h\mathbf{w}\|_{*,\mathrm{F}_b} &\le C[1+\log(d/h)]^{\frac12}\|\mathbf{P}_h\mathbf{w}\|_{\frac12,\Gamma_i}\\ &\le C[1+\log(d/h)]^{\frac12}\|\mathbf{P}_h\mathbf{w}\|_{1,\Omega_i}\\ &\le C[1+\log(d/h)]^{\frac12}\|\mathbf{w}\|_{1,\Omega_i},\end{aligned} \tag{6.13}$$

while by the triangle inequality, (4.5), and (6.9), we deduce

$$\begin{aligned}h^{-1}\|\mathbf{r}_h\mathbf{w}-\mathbf{P}_h\mathbf{w}\|_{0,\Omega_i} &\le h^{-1}(\|\mathbf{r}_h\mathbf{w}-\mathbf{w}\|_{0,\Omega_i}+\|\mathbf{P}_h\mathbf{w}-\mathbf{w}\|_{0,\Omega_i})\\ &\le C(|\mathbf{w}|^2_{1,\Omega_i}+\|\mathbf{curl}\,\mathbf{v}_h\|^2_{0,\Omega_i})^{\frac12}.\end{aligned} \tag{6.14}$$

Now, (4.6) follows readily from (6.12)–(6.14). ∎

Proof of Lemma 4.6. We can write

$$\mathbf{r}_h\mathbf{w}-\Upsilon_{\partial\mathrm{F}}(\mathbf{r}_h\mathbf{w}) = (\mathbf{r}_h\mathbf{w}-\mathbf{P}_h\mathbf{w})+(\mathbf{P}_h\mathbf{w}-\Upsilon_{\partial\mathrm{F}}(\mathbf{P}_h\mathbf{w}))+\Upsilon_{\partial\mathrm{F}}(\mathbf{P}_h\mathbf{w}-\mathbf{r}_h\mathbf{w});$$

then, by the triangle inequality,

$$(6.15)\quad \|\mathbf{r}_h\mathbf{w}-\Upsilon_{\partial\mathrm{F}}(\mathbf{r}_h\mathbf{w})\|^2_{0,\Omega_i}\le 3(\|\mathbf{P}_h\mathbf{w}-\Upsilon_{\partial\mathrm{F}}(\mathbf{P}_h\mathbf{w})\|^2_{0,\Omega_i} +\|\mathbf{r}_h\mathbf{w}-\mathbf{P}_h\mathbf{w}\|^2_{0,\Omega_i}+\|\Upsilon_{\partial\mathrm{F}}(\mathbf{P}_h\mathbf{w}-\mathbf{r}_h\mathbf{w})\|^2_{0,\Omega_i}).$$

Using (6.4) and (6.9), we know

$$(6.16)\quad \|\mathbf{P}_h\mathbf{w}-\Upsilon_{\partial\mathrm{F}}(\mathbf{P}_h\mathbf{w})\|^2_{0,\Omega_i}\le Cd^2[1+\log(d/h)]|\mathbf{P}_h\mathbf{w}|^2_{1,\Omega_i}\le Cd^2[1+\log(d/h)]|\mathbf{w}|^2_{1,\Omega_i}.$$

On the other hand, by the definition of $\Upsilon_{\partial\mathrm{F}}$, one can verify directly that

$$\|\Upsilon_{\partial\mathrm{F}}(\mathbf{P}_h\mathbf{w}-\mathbf{r}_h\mathbf{w})\|^2_{0,\Omega_i}\le Cd^3|\Upsilon_{\partial\mathrm{F}}(\mathbf{P}_h\mathbf{w}-\mathbf{r}_h\mathbf{w})|^2\le C\,d^2\|\mathbf{P}_h\mathbf{w}-\mathbf{r}_h\mathbf{w}\|^2_{0,\mathrm{F}_b}.$$

This with (6.11) gives

$$\|\Upsilon_{\partial\mathrm{F}}(\mathbf{P}_h\mathbf{w}-\mathbf{r}_h\mathbf{w})\|^2_{0,\Omega_i}\le Cd^2h^{-2}\,\|\mathbf{r}_h\mathbf{w}-\mathbf{P}_h\mathbf{w}\|^2_{0,\Omega_i},$$

and so we obtain by (6.14) that

$$\begin{aligned}&\|\mathbf{r}_h\mathbf{w}-\mathbf{P}_h\mathbf{w}\|^2_{0,\Omega_i}+\|\Upsilon_{\partial\mathrm{F}}(\mathbf{P}_h\mathbf{w}-\mathbf{r}_h\mathbf{w})\|^2_{0,\Omega_i}\\ &\le C(1+d^2h^{-2})\|\mathbf{r}_h\mathbf{w}-\mathbf{P}_h\mathbf{w}\|^2_{0,\Omega_i}\\ &\le C(h^2+d^2)(|\mathbf{w}|^2_{1,\Omega_i}+\|\mathbf{curl}\,\mathbf{v}_h\|^2_{0,\Omega_i}).\end{aligned}$$

Now (4.7) follows from this, (6.15), and (6.16).

Finally, the relation

$$\mathbf{w}-\Upsilon_{\partial\mathrm{F}}(\mathbf{r}_h\mathbf{w})=(\mathbf{w}-\mathbf{r}_h\mathbf{w})+(\mathbf{r}_h\mathbf{w}-\Upsilon_{\partial\mathrm{F}}(\mathbf{r}_h\mathbf{w})),$$

with (4.7) and Lemma 4.4, leads to (4.8) directly. ∎

6.2. Proofs of Lemmas 4.7, 4.8, and 4.9. The proofs of these lemmas are rather technical, and we will start with some auxiliary results.

LEMMA 6.4. *For any $\Phi\in V_h(\Gamma_i)$, we have*

$$(6.17)\quad \|\Phi\|_{0,\Gamma_i}\le Ch^{-\frac12}\|\Phi\|_{-\frac12,\Gamma_i},\quad \|\mathbf{I}^0_{\mathrm{F}_b}\Phi\|_{0,\mathrm{F}}\le Ch^{\frac12}\|\Phi\|_{*,\mathrm{F}_b}.$$

Proof. The first estimate was proved in [3]. We prove only the second inequality in (6.17). For any $\Phi\in V_h(\Gamma_i)$, we can write $\Phi=\mathbf{v}\times\mathbf{n}$ on Γ_i for some $\mathbf{v}\in V_h(\Omega_i)$. Using the definitions of $\mathbf{I}^0_{\mathrm{F}_b}$, we deduce

$$(6.18)\quad \|\mathbf{I}^0_{\mathrm{F}_b}\Phi\|^2_{0,\mathrm{F}}\le C\sum_{e\subset\mathrm{F}_b}\lambda^2_e(\mathbf{v})\|L_e\times\mathbf{n}_i\|^2_{0,\mathrm{F}}.$$

It follows by (2.2) that $\|L_e\times\mathbf{n}\|^2_{0,\mathrm{F}}\le C$. This, together with (6.18), yields

$$\|\mathbf{I}^0_{\mathrm{F}_b}\Phi\|^2_{0,\mathrm{F}}\le C\sum_{e\subset\mathrm{F}_b}\lambda^2_e(\mathbf{v}).$$

Now we need only to prove

$$(6.19)\quad \lambda^2_e(\mathbf{v})\le Ch\|\Phi\|^2_{0,e}\quad\forall\,e\subset\mathrm{F}_b\subset\Gamma_i.$$

Noting the fact that $\mathbf{v} = (\mathbf{v}\cdot\mathbf{n})\mathbf{n} + \mathbf{n}\times\mathbf{v}\times\mathbf{n}$ on F, for any $e \subset$ F we have

$$\mathbf{v}|_{\mathrm{F}}\cdot\mathbf{t}_e = (\mathbf{n}\times\mathbf{v}\times\mathbf{n})|_{\mathrm{F}}\cdot\mathbf{t}_e.$$

Thus (6.19) comes readily from the following:

$$\lambda_e^2(\mathbf{v}) = \left|\int_e \mathbf{v}\cdot\mathbf{t}_e ds\right|^2 \le \int_e |\mathbf{n}\times\mathbf{v}\times\mathbf{n}|^2 ds \int_e |\mathbf{t}_e|^2 ds \le Ch\int_e |\mathbf{n}\times\mathbf{v}|^2 ds. \quad \Box \tag{6.20}$$

LEMMA 6.5. *Let* $\Phi \in V_h(\Gamma_i)$*, and let* $\mathbf{I}^0_{\mathrm{F}_\partial}\Phi$ *be defined as in* (2.5)*. Then*

$$\|\mathbf{I}^0_{\mathrm{F}_\partial}\Phi\|_{-\frac12,\Gamma_i} \le C([1+\log(d/h)]\|\Phi\|_{-\frac12,\Gamma_i} + h^{\frac12}\|\Phi\|_{*,\mathrm{F}_b}). \tag{6.21}$$

Proof. The proof is similar to that of Lemma 6 in [19]. However, for the reader's convenience, we still give a complete proof below.

For any $\mathbf{v}\in H^{1/2}(\Gamma_i)^3$, let $\mathbf{v}_h\in Z_h(\Gamma_i)^3$ be the $L^2(\Gamma_i)$-projection of $\mathbf{v}$. Then

$$|\langle\mathbf{I}^0_{\mathrm{F}_\partial}\Phi,\mathbf{v}\rangle_{\Gamma_i}| \le |\langle\mathbf{I}^0_{\mathrm{F}_\partial}\Phi,\mathbf{v}-\mathbf{v}_h\rangle_{\Gamma_i}| + |\langle\mathbf{I}^0_{\mathrm{F}_\partial}\Phi,\mathbf{v}_h\rangle_{\Gamma_i}|. \tag{6.22}$$

It is known that

$$\|\mathbf{v}_h-\mathbf{v}\|_{0,\Gamma_i}\le Ch^{\frac12}\|\mathbf{v}\|_{\frac12,\Gamma_i}, \quad \|\mathbf{v}_h\|_{\frac12,\Gamma_i}\le C\|\mathbf{v}\|_{\frac12,\Gamma_i}. \tag{6.23}$$

This, together with (6.17), leads to

$$\begin{aligned}|\langle\mathbf{I}^0_{\mathrm{F}_\partial}\Phi,\mathbf{v}-\mathbf{v}_h\rangle_{\Gamma_i}| &\le \|\mathbf{I}^0_{\mathrm{F}_\partial}\Phi\|_{0,\Gamma_i}\,\|\mathbf{v}-\mathbf{v}_h\|_{0,\Gamma_i}\\ &\le Ch^{1/2}\|\Phi\|_{0,\Gamma_i}\|\mathbf{v}\|_{\frac12,\Gamma_i} \le C\|\Phi\|_{-\frac12,\Gamma_i}\,\|\mathbf{v}\|_{\frac12,\Gamma_i}.\end{aligned} \tag{6.24}$$

On the other hand, from the definitions of the operators $\mathbf{I}^0_{\mathrm{F}_\partial}$ and $\mathbf{I}^0_{\mathrm{F}_b}$, we have $\mathbf{I}^0_{\mathrm{F}_\partial}\Phi = \Phi - \mathbf{I}^0_{\mathrm{F}_b}\Phi$ on F. Then

$$|\langle\mathbf{I}^0_{\mathrm{F}_\partial}\Phi,\mathbf{v}_h\rangle_{\Gamma_i}| = |\langle\mathbf{I}^0_{\mathrm{F}_\partial}\Phi,\mathbf{v}_h\rangle_{\mathrm{F}}| \le |\langle\Phi,\mathbf{v}_h\rangle_{\mathrm{F}}| + |\langle\mathbf{I}^0_{\mathrm{F}_b}\Phi,\mathbf{v}_h\rangle_{\mathrm{F}}|. \tag{6.25}$$

It follows from (6.17) that

$$|\langle\mathbf{I}^0_{\mathrm{F}_b}\Phi,\mathbf{v}_h\rangle_{\mathrm{F}}| \le \|\mathbf{I}^0_{\mathrm{F}_b}\Phi\|_{0,\mathrm{F}}\,\|\mathbf{v}_h\|_{0,\mathrm{F}} \le Ch^{\frac12}\|\Phi\|_{*,\mathrm{F}_b}\,\|\mathbf{v}_h\|_{\frac12,\Gamma_i}. \tag{6.26}$$

For the term $\langle\Phi,\mathbf{v}_h\rangle_{\mathrm{F}}$ in (6.25), we use the simple decomposition

$$\mathbf{v}_h(\mathbf{x}) = \mathrm{I}^0_{\mathrm{F}}\mathbf{v}_h(\mathbf{x}) + \mathrm{I}^0_{\partial\mathrm{F}}\mathbf{v}_h(\mathbf{x}) \quad \forall\,\mathbf{x}\in\mathrm{F} \tag{6.27}$$

to derive (note that $\mathrm{I}^0_{\mathrm{F}}\mathbf{v}_h(\mathbf{x}) = \mathbf{0}$ on $\Gamma_i\backslash$F)

$$\begin{aligned}|\langle\Phi,\mathbf{v}_h\rangle_{\mathrm{F}}| &\le |\langle\Phi,\mathrm{I}^0_{\mathrm{F}}\mathbf{v}_h\rangle_{\mathrm{F}}| + |\langle\Phi,\mathrm{I}^0_{\partial\mathrm{F}}\mathbf{v}_h\rangle_{\mathrm{F}}|\\ &\le |\langle\Phi,\mathrm{I}^0_{\mathrm{F}}\mathbf{v}_h\rangle_{\Gamma_i}| + \|\Phi\|_{0,\mathrm{F}}\,\|\mathrm{I}^0_{\partial\mathrm{F}}\mathbf{v}_h\|_{0,\mathrm{F}}\\ &\le \|\Phi\|_{-\frac12,\Gamma_i}\,\|\mathrm{I}^0_{\mathrm{F}}\mathbf{v}_h\|_{\frac12,\Gamma_i} + Ch^{\frac12}\|\Phi\|_{0,\Gamma_i}\,\|\mathbf{v}_h\|_{0,\partial\mathrm{F}},\end{aligned}$$

where a direct computation is used to bound the term $\|\mathrm{I}^0_{\partial\mathrm{F}}\mathbf{v}_h\|_{0,\mathrm{F}}$ by $h^{1/2}\|\mathbf{v}_h\|_{0,\partial\mathrm{F}}$ using the discrete L^2-norm. This with (6.17), (6.2), and (6.1) yields

$$|\langle\Phi,\mathbf{v}_h\rangle_{\mathrm{F}}| \le C[1+\log(d/h)]\|\Phi\|_{-\frac12,\Gamma_i}\,\|\mathbf{v}_h\|_{\frac12,\Gamma_i}.$$

Substituting it and (6.26) into (6.25) yields

$$|\langle \mathbf{I}^0_{\mathrm{F}_\partial}\Phi, \mathbf{v}_h\rangle_{\mathrm{F}}| \le C([1+\log(d/h)]\|\Phi\|_{-\frac{1}{2},\Gamma_i} + h^{\frac{1}{2}}\|\Phi\|_{*,\mathrm{F}_b})\|\mathbf{v}_h\|_{\frac{1}{2},\Gamma_i},$$

which, along with (6.22) and (6.24), leads to

$$|\langle \mathbf{I}^0_{\mathrm{F}_\partial}\Phi, \mathbf{v}\rangle_{\Gamma_i}| \le C([1+\log(d/h)]\|\Phi\|_{-\frac{1}{2},\Gamma_i} + h^{\frac{1}{2}}\|\Phi\|_{*,\mathrm{F}_b})\|\mathbf{v}\|_{\frac{1}{2},\Gamma_i}.$$

Now (6.21) follows directly from the definition of the norm $\|\cdot\|_{-1/2,\Gamma_i}$. ∎

Next, we are going to prove Lemma 6.10 on the estimate of $\|\mathrm{div}_\tau(\mathbf{I}^0_{\mathrm{F}}\Phi)\|_{-\frac{1}{2},\Gamma_i}$ for all $\Phi \in V_h(\Gamma_i)$. To do so, we have to present some auxiliary results first (Lemmas 6.6–6.9).

LEMMA 6.6. *Let $\varphi \in L^2(\Gamma_i)$ be piecewise constant with respect to the $\mathcal{T}_h$-induced triangulation $\mathcal{T}_{h,i}$ on Γ_i. Then*

$$\|\varphi\|_{0,\Gamma_i} \le Ch^{-\frac{1}{2}}\|\varphi\|_{-\frac{1}{2},\Gamma_i}. \tag{6.28}$$

Proof. By definition,

$$\|\varphi\|_{-\frac{1}{2},\Gamma_i} = \sup_{\psi\in H^{1/2}(\Gamma_i)} \frac{|\langle\varphi,\psi\rangle_{\Gamma_i}|}{\|\psi\|_{\frac{1}{2},\Gamma_i}}.$$

The inequality (6.28) then follows if we can construct a function $\psi_0 \in H^{\frac{1}{2}}(\Gamma_i)$ such that

$$|\langle\varphi,\psi_0\rangle_{\Gamma_i}| \ge C\|\varphi\|_{0,\Gamma_i}\|\psi_0\|_{0,\Gamma_i}, \quad \|\psi_0\|_{\frac{1}{2},\Gamma_i} \le Ch^{-\frac{1}{2}}\|\psi_0\|_{0,\Gamma_i}. \tag{6.29}$$

To construct the function ψ_0 for each triangle $K \in \mathcal{T}_{h,i}$ and lying on Γ_i, with O_K being its barycenter, we refine K by connecting O_K with three vertices of K. Let a_K denote the (constant) value of φ on the triangle K, and let ψ_0 be a piecewise linear function on K with respect to this subdivision such that ψ_0 equals a_K at O_K and vanishes on the edges of K. It is clear that such a function ψ_0 is in $H^{1/2}(\Gamma_i)$. As ψ_0 is piecewise linear on the entire boundary Γ_i with respect to the subdivision of $\mathcal{T}_h$, the second inequality in (6.29) follows directly from the inverse inequality. Moreover, by the equivalent discrete L^2-norms we have

$$\|\psi_0\|^2_{0,\Gamma_i} \le Ch^2\sum_{K\in\mathcal{T}_{h,i}}|a_K|^2. \tag{6.30}$$

Let S_K be the area of the triangle K. We have

$$\begin{aligned}|\langle\varphi,\psi_0\rangle_{\Gamma_i}| &= \left|\sum_{K\in\mathcal{T}_{h,i}}\langle\varphi,\psi_0\rangle_K\right| = \left|\sum_{K\in\mathcal{T}_{h,i}}a_K\langle 1,\psi_0\rangle_K\right| \\ &= \frac{1}{3}\left|\sum_{K\in\mathcal{T}_{h,i}}a_K^2 S_K\right| \ge Ch^2\sum_{K\in\mathcal{T}_{h,i}}|a_K^2|.\end{aligned}$$

Now the first inequality of (6.29) follows readily from this and (6.30). ∎

The next lemma can be shown similarly as Lemma 6.5 by using Lemma 6.6.

LEMMA 6.7. *Let φ be the same as in Lemma 6.6; then*

$$\|\mathrm{I}^t_{\mathrm{F}}(\varphi|_{\mathrm{F}})\|_{-\frac{1}{2},\Gamma_i} \le C[1+\log(d/h)]\|\varphi\|_{-\frac{1}{2},\Gamma_i}. \tag{6.31}$$

For the proof, we introduce some new functions. For any $\Phi = \mathbf{v} \times \mathbf{n} \in V_h(\Gamma_i)$ and any face $\mathrm{F} \subset \Gamma_i$, we define a function in $L^2(\Gamma_i)$ as follows:

$$\varphi_{\mathrm{F}_b}(\mathbf{x}) = \sum_{e \subset \mathrm{F}_b} \lambda_e(\mathbf{v})(\mathbf{n}_i \cdot \mathbf{curl}\, L_e)(\mathbf{x}), \quad \mathbf{x} \in \bar{\mathrm{F}}; \quad \varphi_{\mathrm{F}_b}(\mathbf{x}) = 0, \quad \mathbf{x} \in \Gamma_i \backslash \bar{\mathrm{F}}, \tag{6.32}$$

where $\{L_e; e \in \mathcal{E}_h\}$ are the edge element basis functions defined in (2.2). One can see that φ_{F_b} is piecewise constant on Γ_i, and it vanishes everywhere except in those triangles which are in F and have a vertex on ∂F at least. We now present two estimates for $\varphi_{\mathrm{F}_b}(\mathbf{x})$ below.

LEMMA 6.8. *For any $\Phi \in V_h(\Gamma_i)$ and any face* F *of Γ_i, we have*

$$\|\varphi_{\mathrm{F}_b}\|_{-\frac{1}{2},\Gamma_i} \le Ch^{\frac{1}{2}}[1 + \log(d/h)]^{\frac{1}{2}} \|\varphi_{\mathrm{F}_b}\|_{0,\mathrm{F}}. \tag{6.33}$$

Proof. For any $v \in H^{1/2}(\Gamma_i)$, let $v_h \in Z_h(\Gamma_i)$ be the $L^2(\Gamma_i)$-projection of v. We see directly from (6.27), (6.23), and (6.1) that

$$\begin{aligned} |\langle \varphi_{\mathrm{F}_b}, v\rangle_{\Gamma_i}| &\le |\langle \varphi_{\mathrm{F}_b}, v - v_h\rangle_{\Gamma_i} + |\langle \varphi_{\mathrm{F}_b}, \mathrm{I}^0_{\partial \mathrm{F}} v_h\rangle_{\Gamma_i}| + |\langle \varphi_{\mathrm{F}_b}, \mathrm{I}^0_{\mathrm{F}} v_h\rangle_{\Gamma_i}| \\ &\le Ch^{\frac{1}{2}}[1 + \log(d/h)]^{\frac{1}{2}} \|\varphi_{\mathrm{F}_b}\|_{0,\mathrm{F}}\, \|v\|_{\frac{1}{2},\Gamma_i} + |\langle \varphi_{\mathrm{F}_b}, \mathrm{I}^0_{\mathrm{F}} v_h\rangle_{\mathrm{F}}|, \end{aligned}$$

where we have used the fact that $\varphi_{\mathrm{F}_b} = 0$ on $\Gamma_i \backslash$F. It remains to show that

$$|\langle \varphi_{\mathrm{F}_b}, \mathrm{I}^0_{\mathrm{F}} v_h\rangle_{\mathrm{F}}| \le Ch^{\frac{1}{2}}[1 + \log(d/h)]^{\frac{1}{2}} \|\varphi_{\mathrm{F}_b}\|_{0,\mathrm{F}}\, \|\mathbf{v}\|_{\frac{1}{2},\Gamma_i}. \tag{6.34}$$

Let F_c denote the union of all triangles that have at least one of their vertices lying on ∂F. We regroup the triangles in F_c such that $\mathrm{F}_c = \cup K$, with each K being one triangle or a union of two triangles and having at least one of its edges lying on ∂F. Then by the definition of φ_{F_b} and the Hölder inequality, we have

$$\begin{aligned} |\langle \varphi_{\mathrm{F}_b}, \mathrm{I}^0_{\mathrm{F}} v_h\rangle_{\mathrm{F}}| = |\langle \varphi_{\mathrm{F}_b}, \mathrm{I}^0_{\mathrm{F}} v_h\rangle_{\mathrm{F}_c}| &= \left|\sum_K \langle \varphi_{\mathrm{F}_b}, \mathrm{I}^0_{\mathrm{F}} v_h\rangle_K\right| \\ &\le \sum_K \|\varphi_{\mathrm{F}_b}\|_{0,K}\, \|\mathrm{I}^0_{\mathrm{F}} v_h\|_{0,K}. \end{aligned} \tag{6.35}$$

As each $K \in \mathrm{F}_c$ has an edge lying on ∂F, $\mathrm{I}^0_{\mathrm{F}} v_h$ vanishes on the edge. Then by Friedrich's inequality we obtain

$$\|\mathrm{I}^0_{\mathrm{F}} v_h\|_{0,K} \le Ch^{\frac{1}{2}} |\mathrm{I}^0_{\mathrm{F}} v_h|_{\frac{1}{2},K}.$$

Plugging this in (6.35) and using the Cauchy–Schwarz inequality, we derive

$$\begin{aligned} |\langle \varphi_{\mathrm{F}_b}, \mathrm{I}^0_{\mathrm{F}} v_h\rangle_{\mathrm{F}}| &\le Ch^{\frac{1}{2}} \left\{\sum_K \|\varphi_{\mathrm{F}_b}\|^2_{0,K}\right\}^{\frac{1}{2}} \left\{\sum_K |\mathrm{I}^0_{\mathrm{F}} v_h|^2_{\frac{1}{2},K}\right\}^{\frac{1}{2}} \\ &= Ch^{\frac{1}{2}} \{\|\varphi_{\mathrm{F}_b}\|^2_{0,\mathrm{F}_c}\}^{\frac{1}{2}} \{|\mathrm{I}^0_{\mathrm{F}} v_h|^2_{\frac{1}{2},\mathrm{F}_c}\}^{\frac{1}{2}} \\ &\le Ch^{\frac{1}{2}} \|\varphi_{\mathrm{F}_b}\|_{0,\mathrm{F}}\ \ |\mathrm{I}^0_{\mathrm{F}} v_h|_{\frac{1}{2},\mathrm{F}}. \end{aligned} \tag{6.36}$$

On the other hand, it follows from (6.27) and (6.3) that

$$|\mathrm{I}^0_{\mathrm{F}} v_h|_{\frac{1}{2},\mathrm{F}} = |v_h - \mathrm{I}^0_{\partial \mathrm{F}} v_h|_{\frac{1}{2},\mathrm{F}} \le |v_h|_{\frac{1}{2},\mathrm{F}} + |\mathrm{I}^0_{\partial \mathrm{F}} v_h|_{\frac{1}{2},\mathrm{F}} \le C[1 + \log(d/h)]^{\frac{1}{2}} \|v_h\|_{\frac{1}{2},\Gamma_i}.$$

This, together with (6.36), gives (6.34). ∎

LEMMA 6.9. *Assume that* $\Phi = \mathbf{v}\times\mathbf{n} \in V_h(\Gamma_i)$. *Then*

$$\|\varphi_{\mathrm{F}_b}\|_{0,\mathrm{F}} \le Ch^{-\frac{1}{2}}\|\Phi\|_{*,\mathrm{F}_b}. \tag{6.37}$$

Proof. We have by the definitions of φ_{F_b} that

$$\|\varphi_{\mathrm{F}_b}\|^2_{0,\mathrm{F}} \le C\sum_{e\subset \mathrm{F}_b}\lambda_e^2(\mathbf{v})\|\mathbf{n}_i\cdot\mathbf{curl}\ L_e\|^2_{0,\mathrm{F}}. \tag{6.38}$$

It follows from (2.2) that $\mathbf{curl}\,L_e = c_e\nabla\lambda_1^e\times\nabla\lambda_2^e$, which gives $\|\mathbf{n}_i\cdot\mathbf{curl}\,L_e\|^2_{0,\mathrm{F}} \le Ch^{-2}$. Then we derive from (6.38) that

$$\|\varphi_{\mathrm{F}_b}\|^2_{0,\mathrm{F}} \le Ch^{-2}\sum_{e\subset \mathrm{F}_b}\lambda_e^2(\mathbf{v}).$$

This, together with (6.20), gives the desired results. ∎

LEMMA 6.10. *For any* $\Phi = \mathbf{v}\times\mathbf{n}\in V_h(\Gamma_i)$, *we have*

$$\|\mathrm{div}_\tau(\mathbf{I}^0_{\mathrm{F}}\Phi)\|_{-\frac{1}{2},\Gamma_i} \le C[1+\log(d/h)]\|\mathrm{div}_\tau\Phi\|_{-\frac{1}{2},\Gamma_i} + C[1+\log(d/h)]^{\frac{1}{2}}\|\Phi\|_{*,\mathrm{F}_b}. \tag{6.39}$$

Proof. We use Lemmas 6.7, 6.8, and 6.9 to estimate $\mathrm{div}_\tau(\mathbf{I}^0_{\mathrm{F}}\Phi)$. By Green's formula and the definition of $\mathrm{div}_\tau\Phi$, one can verify (cf. [2]) that

$$\mathrm{div}_\tau\Phi = \mathrm{div}_\tau(\mathbf{v}\times\mathbf{n})|_{\Gamma_i} = -(\mathbf{n}_i\cdot\mathbf{curl}\ \mathbf{v})|_{\Gamma_i} \quad \text{in} \quad H^{-\frac{1}{2}}(\Gamma_i).$$

Thus $\mathrm{div}_\tau\Phi$ is a piecewise constant function on Γ_i. It suffices to prove that

$$\mathrm{div}_\tau(\mathbf{I}^0_{\mathrm{F}_\partial}\Phi) = \mathrm{I}^t_{\mathrm{F}}(\mathrm{div}_\tau\Phi|_{\mathrm{F}}) + \varphi_{\mathrm{F}_b} \quad \text{in} \quad H^{-\frac{1}{2}}(\Gamma_i). \tag{6.40}$$

As $\mathrm{div}_\tau(\mathbf{I}^0_{\mathrm{F}_\partial}\Phi) = 0$ on $\Gamma_i\backslash\bar{\mathrm{F}}$, the inequality (6.40) is valid in $\Gamma_i\backslash\bar{\mathrm{F}}$. However, on the face $\bar{\mathrm{F}}$, we have by (2.4) and (2.5) that

$$\mathrm{div}_\tau\Phi = \sum_{e\subset\overline{\mathrm{F}}}\lambda_e(\mathbf{v})\mathrm{div}_\tau(L_e\times\mathbf{n}_i)\,, \quad \mathrm{div}_\tau(\mathbf{I}^0_{\mathrm{F}_\partial}\Phi) = \sum_{e\subset\mathrm{F}_\partial}\lambda_e(\mathbf{v})\mathrm{div}_\tau(L_e\times\mathbf{n}_i)\,.$$

Hence

$$\mathrm{div}_\tau\Phi - \mathrm{div}_\tau(\mathbf{I}^0_{\mathrm{F}_\partial}\Phi) = \sum_{e\subset\mathrm{F}_b}\lambda_e(\mathbf{v})\mathrm{div}_\tau(L_e\times\mathbf{n}_i) \quad \text{on} \quad \bar{\mathrm{F}}. \tag{6.41}$$

Noting that (see (2.10) in [2])

$$\mathrm{div}_\tau(L_e\times\mathbf{n}_i)|_{\Gamma_i} = -(\mathbf{n}_i\cdot\mathbf{curl}\ L_e)|_{\Gamma_i} \quad \text{in} \quad H^{-\frac{1}{2}}(\Gamma_i),$$

we see that (6.40) holds also on $\bar{\mathrm{F}}$, using (6.41) and (6.32). ∎

The following result can be proved in an analogous way as Lemma 6.6.

LEMMA 6.11. *For any* $\Phi\in V_h(\Gamma_i)$ *and any face* F *of* Γ_i, *we have*

$$\|\mathbf{I}^0_{\mathrm{F}_b}\Phi\|_{-\frac{1}{2},\mathrm{F}} \le Ch^{\frac{1}{2}}[1+\log(d/h)]^{\frac{1}{2}}\|\mathbf{I}^0_{\mathrm{F}_b}\Phi\|_{0,\mathrm{F}}. \tag{6.42}$$

Below, we start to prove Lemmas 4.7, 4.8, and 4.9. Lemma 4.7 is a direct consequence of Lemmas 4.5, 6.5, and 6.10, and it indicates that the norm $\|\mathbf{I}^0_{\mathrm{F}}\Phi\|_{\mathcal{X}_{\Gamma_i}}$ cannot be bounded only by $\|\Phi\|_{\mathcal{X}_{\Gamma_i}}$ (compare to the estimate (6.2)).

Proof of Lemma 4.8. Using (6.40) and the relations

$$\mathbf{I}^0_{\Delta_i}\Phi = \sum_{\mathrm{F}\subset\Gamma_i} \mathbf{I}^t_{\mathrm{F}}(\mathbf{I}^0_{\mathrm{F}_b}\Phi)|_{\mathrm{F}}, \quad \mathbf{I}^t_{\mathrm{F}}(\mathbf{I}^0_{\mathrm{F}_b}\Phi)|_{\mathrm{F}} = \mathbf{I}^t_{\mathrm{F}}\Phi - \mathbf{I}^0_{\mathrm{F}_\partial}\Phi$$

and the facts that $\mathbf{I}^t_{\mathrm{F}}\Phi)|_{\Gamma_i\backslash\bar{\mathrm{F}}} = \mathbf{0}$ but $(\mathbf{I}^0_{\mathrm{F}_b}\Phi)|_{\Gamma_i\backslash\bar{\mathrm{F}}} \neq \mathbf{0}$, we can write

$$\begin{aligned}\mathrm{div}_\tau(\mathbf{I}^0_{\Delta_i}\Phi) &= \mathrm{div}_\tau\left(\sum_{\mathrm{F}\subset\Gamma_i}\mathbf{I}^t_{\mathrm{F}}\Phi - \sum_{\mathrm{F}\subset\Gamma_i}\mathbf{I}^0_{\mathrm{F}}\Phi\right) = \mathrm{div}_\tau\left(\Phi - \sum_{\mathrm{F}\subset\Gamma_i}\mathbf{I}^0_{\mathrm{F}_\partial}\Phi\right)\\ &= \mathrm{div}_\tau\Phi - \sum_{\mathrm{F}\subset\Gamma_i}\mathrm{div}_\tau(\mathbf{I}^0_{\mathrm{F}_\partial}\Phi) = \sum_{\mathrm{F}\subset\Gamma_i}(\mathbf{I}^t_{\mathrm{F}}\mathrm{div}_\tau(\Phi)|_{\mathrm{F}} - \mathrm{div}_\tau(\mathbf{I}^0_{\mathrm{F}_\partial}\Phi))\\ &= \sum_{\mathrm{F}\subset\Gamma_i}\varphi_{\mathrm{F}_b}.\end{aligned}$$

This leads to

$$\|\mathbf{I}^0_{\Delta_i}\Phi\|_{-\frac{1}{2},\Gamma_i} \le \sum_{\mathrm{F}\subset\Gamma_i}\|\mathbf{I}^0_{\mathrm{F}_b}\Phi\|_{-\frac{1}{2},\mathrm{F}}\,, \quad \|\mathrm{div}_\tau(\mathbf{I}^0_{\Delta_i}\Phi)\|_{-\frac{1}{2},\Gamma_i} \le \sum_{\mathrm{F}\subset\Gamma_i}\|\varphi_{\mathrm{F}_b}\|_{-\frac{1}{2},\Gamma_i}.$$

Using these two estimates, together with Lemmas 6.11 and 6.8, we have

$$\|\mathbf{I}^0_{\Delta_i}\Phi\|_{\mathcal{X}_{\Gamma_i}} \le Ch^{\frac{1}{2}}[1+\log(d/h)]^{\frac{1}{2}}\sum_{\mathrm{F}\subset\Gamma_i}(d^{-1}\|\mathbf{I}^0_{\mathrm{F}_b}\Phi\|_{0,\mathrm{F}} + \|\varphi_{\mathrm{F}_b}\|_{0,\mathrm{F}}). \tag{6.43}$$

Substituting (6.17) and (6.37) into (6.43), we obtain the desired result. □

Proof of Lemma 4.9. By Lemma 6.10 we have

$$\begin{aligned}&\|\mathrm{div}_\tau[\mathbf{I}^0_{\mathrm{F}_\partial}(\Upsilon_{\partial\mathrm{F}}(\Pi_0\mathbf{v})\times\mathbf{n})]\|^2_{-\frac{1}{2},\Gamma_k}\\ &\le C([1+\log(d/h)]^2\|\mathrm{div}_\tau[\Upsilon_{\partial\mathrm{F}}(\Pi_0\mathbf{v})\times\mathbf{n}|_{\Gamma_k}]\|^2_{-\frac{1}{2},\Gamma_k}\\ &\quad+[1+\log(d/h)]\|\Upsilon_{\partial\mathrm{F}}(\Pi_0\mathbf{v})\times\mathbf{n}\|^2_{*,\mathrm{F}_b}).\end{aligned} \tag{6.44}$$

It is easy to see that

$$\|\Upsilon_{\partial\mathrm{F}}(\Pi_0\mathbf{v})\times\mathbf{n}_k\|^2_{*,\mathrm{F}_b} = \|\Upsilon_{\partial\mathrm{F}}(\Pi_0\mathbf{v})\times\mathbf{n}_k\|^2_{*,\mathrm{F}_b} \le C\|(\Pi_0\mathbf{v})\times\mathbf{n}\|^2_{*,\mathrm{F}_b}. \tag{6.45}$$

Since $\Upsilon_{\partial\mathrm{F}}(\Pi_0\mathbf{v})$ is a constant vector, we have

$$\mathrm{div}_\tau(\Upsilon_{\partial\mathrm{F}}(\Pi_0\mathbf{v})\times\mathbf{n}|_{\Gamma_k}) = 0 \quad \text{in} \quad H^{-\frac{1}{2}}(\Gamma_k).$$

Hence

$$\|\mathrm{div}_\tau(\Upsilon_{\partial\mathrm{F}}(\Pi_0\mathbf{v})\times\mathbf{n}|_{\Gamma_k})\|_{-\frac{1}{2},\Gamma_k} = 0.$$

Substituting (6.45) and the above inequality into (6.44) yields

$$\|\mathrm{div}_\tau[\mathbf{I}^0_{\mathrm{F}_\partial}(\Upsilon_{\partial\mathrm{F}}(\Pi_0\mathbf{v})\times\mathbf{n})]\|^2_{-\frac{1}{2},\Gamma_k} \le C[1+\log(d/h)]\|(\Pi_0\mathbf{v})\times\mathbf{n}\|^2_{*,\mathrm{F}_b}. \tag{6.46}$$

On the other hand, it follows from Lemmas 6.11 and 6.4 that

$$\begin{aligned}&d^{-1}\|\mathbf{I}^0_{\mathrm{F}_\partial}(\Upsilon_{\partial\mathrm{F}}(\Pi_0\mathbf{v})\times\mathbf{n})\|_{-\frac{1}{2},\Gamma_k} = d^{-1}\|\mathbf{I}^0_{\mathrm{F}_\partial}(\Upsilon_{\partial\mathrm{F}}(\Pi_0\mathbf{v})\times\mathbf{n})\|_{-\frac{1}{2},\mathrm{F}}\\ &\le C(d^{-1}\|\Upsilon_{\partial\mathrm{F}}(\Pi_0\mathbf{v})\times\mathbf{n}\|_{-\frac{1}{2},\mathrm{F}} + d^{-1}h[1+\log(d/h)]^{\frac{1}{2}}\|\Upsilon_{\partial\mathrm{F}}(\Pi_0\mathbf{v})\times\mathbf{n}\|_{*,\mathrm{F}_b}).\end{aligned} \tag{6.47}$$

However, for any $\Psi \in (H^{\frac{1}{2}}(\mathrm{F}))^3$, we have

$$\begin{aligned} d^{-1}|\langle \Upsilon_{\partial \mathrm{F}}(\Pi_0 \mathbf{v}) \times \mathbf{n}, \Psi\rangle_{\mathrm{F}}| &\le d^{-1}\|\Upsilon_{\partial \mathrm{F}}(\Pi_0 \mathbf{v}) \times \mathbf{n}\|_{0,\mathrm{F}}\,\|\Psi\|_{0,\mathrm{F}} \\ &\le C d^{-\frac{1}{2}}\|\Upsilon_{\partial \mathrm{F}}(\Pi_0 \mathbf{v} \times \mathbf{n})\|_{0,\mathrm{F}}\,\|\Psi\|_{\frac{1}{2},\mathrm{F}} \\ &\le C d^{\frac{1}{2}}|\Upsilon_{\partial \mathrm{F}}(\Pi_0 \mathbf{v} \times \mathbf{n})|\,\|\Psi\|_{\frac{1}{2},\mathrm{F}} \\ &\le C\|(\Pi_0 \mathbf{v}) \times \mathbf{n}\|_{0,\partial \mathrm{F}}\,\|\Psi\|_{\frac{1}{2},\mathrm{F}} \\ &\le C\|(\Pi_0 \mathbf{v}) \times \mathbf{n}\|_{*,\mathrm{F}_b}\,\|\Psi\|_{\frac{1}{2},\mathrm{F}}, \end{aligned}$$

which implies

$$d^{-1}\|\Upsilon_{\partial \mathrm{F}}(\Pi_0 \mathbf{v}) \times \mathbf{n}\|_{-\frac{1}{2},\mathrm{F}} \le C\|(\Pi_0 \mathbf{v}) \times \mathbf{n}\|_{*,\mathrm{F}_b}.$$

Plugging this and (6.45) in (6.47) leads to

$$d^{-1}\|\mathbf{I}^0_{\mathrm{F}_\partial}(\Upsilon_{\partial \mathrm{F}}(\Pi_0 \mathbf{v}) \times \mathbf{n})\|_{-\frac{1}{2},\Gamma_k} \le C[1 + \log(d/h)]^{\frac{1}{2}}\|(\Pi_0 \mathbf{v}) \times \mathbf{n}\|_{*,\mathrm{F}_b},$$

which, together with Lemmas 6.4 and 6.9, gives the desired result. ∎

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