ON NOVEL GEOMETRIC STRUCTURES OF LAPLACIAN EIGENFUNCTIONS IN $\mathbb{R}^3$ AND APPLICATIONS TO INVERSE PROBLEMS

XINLIN CAO, HUAIAN DIAO, HONGYU LIU, AND JUN ZOU

Abstract. This is a continuation and an extension of our recent work [5] on the geometric structures of Laplacian eigenfunctions and their applications to inverse scattering problems. We studied in [5] the analytic behaviour of the Laplacian eigenfunctions at a point where two nodal or generalized singular lines intersect. The results reveal an important and intriguing property that the vanishing order of the eigenfunction at the intersecting point is closely related to the rationality of the intersecting angle. In this paper, we continue this development in three dimensions and study the analytic behaviours of the Laplacian eigenfunctions at places where nodal or generalized singular planes intersect. Compared with the two-dimensional case, the geometric situation is much more complicated, so is the corresponding analysis: the intersection of two planes generates an edge corner, whereas the intersection of more than three planes generates a vertex corner. We provide a systematic and comprehensive characterisation of the relations between the analytic behaviours of an eigenfunction at a corner point and the geometric quantities of that corner for all these geometric cases. Moreover, we apply the spectral results to establish some novel unique identifiability results for the geometric inverse problems of recovering the shape as well as the (possible) surface impedance coefficient by the associated scattering far-field measurements.

Keywords Laplacian eigenfunctions, geometric structures, nodal and generalized singular planes, inverse scattering, impedance obstacle, uniqueness, a single far-field pattern

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1. Introduction

In this paper, we consider the geometric structures of Laplacian eigenfunctions and their application to the geometrical inverse scattering problem. The study of the geometric properties of Laplacian eigenfunctions has a rich theory in the literature. As a background introduction and an inspiring source for our study, we mention a few famous examples here. The first one is about the topology of the nodal domains of the Laplacian eigenfunction, which has been an important topic for many years [17, 28]. This includes the celebrated Courant's nodal domain theorem [11]. The second example is the Schiffer's conjecture which states that if a Neumann eigenfunction takes a (non-zero) constant value on the boundary, then the domain must be a ball [28]. The Schiffer's conjecture has a close connection to the Pompeium property in the integral geometry [12, 16, 28] and it has also an interesting implication to invisibility cloaking [20]. The third example is the "hot-spot" conjecture which states that the second Neumann eigenfunction attains its maximum value at a boundary point [3,4,18]. The last example is the eigenfunction concentration/localization and its implication to the quantum ergodicity of the billiard flow [29]. There are many other existing developments on the geometric and analytic properties of the Laplacian eigenfunction and the corresponding study still remains to be an active field. We refer to the introductory section of our recent paper [5] and the related references therein for a relatively more comprehensive discussion of this intriguing topic. The current paper is a continuation as well as a significant further development of our study in [5], where the intersection of two nodal or generalized singular lines is considered. Our results in [5] reveal a novel intriguing property that the vanishing order (analytic quantity) of an eigenfunction at an intersecting point is related to the rationality (geometric quantity) of the corresponding intersecting angle. These spectral results were applied directly in [5] to the inverse obstacle scattering problem and the inverse diffraction grating problem to establish several novel unique identifiability results in determining the polygonal shape/support of an inhomogeneous scattering object as well as the (possible) surface impedance coefficient by a few far-field measurements. We note that determining the shape/support of an unknown scatterer by a minimal/optimal number of far-field measurements constitutes a long-standing open problem in the inverse scattering theory [10]. It is very natural to explore if similar results can be established in the more important three-dimensional case, about the intersections of nodal or generalized singular planes and their implications to the analytic behaviours of the eigenfunctions. But the geometric setup is much more complicated in three dimensions, so is the corresponding analysis: the intersection of two planes produces an edge corner, whereas the intersection of more than three planes produces a vertex corner; see Fig. 1 for a schematic illustration. We aim to derive a comprehensive characterisation of the relationship between the analytic behaviours of an eigenfunction at a corner point and the geometric quantities of that corner. More specifically, at the edge corner case, we can show that the vanishing order of the eigenfunction is related to the rationality of the intersecting angle in a similar manner to the two-dimensional case, whereas at the vertex corner case, the vanishing order of the eigenfunction is proved to be related to the intersecting angle in a more complicated and mysterious manner through the roots of the Legendre polynomials. As an important application, these new spectral results are applied to establish several novel and fundamental unique identifiability results for the geometrical inverse scattering problem of determining an impenetrable obstacle as well as the (possibly) surface impedance by at most a few far-field measurements in the polyhedral setup. The rest of this section is mainly devoted to the introduction of the mathematical setup for our study.

Let Ω be an open set in $\mathbb{R}^3$. Consider $u \in L^2(\Omega)$ and $\lambda \in \mathbb{R}_+$ such that

$$-\Delta u = \lambda u. \tag{1.1}$$

The solution u to (1.1) is referred to as a (generalized) Laplacian eigenfunction. We emphasize that compared with the conventional notion of Laplacian eigenfunctions, we do not prescribe any homogeneous boundary condition for u in (1.1). This means, the spectral results that we shall establish in this paper apply to any function that satisfies (1.1) in the interior of Ω, in particular, including all the conventional Laplacian eigenfunctions with various boundary conditions. We next introduce several critical definitions for our subsequent analysis. In what follows, for Π being a flat plane in $\mathbb{R}^3$, any non-empty open connected subset $\Sigma \Subset \Pi$ is called a *cell* of Π. Let $\widetilde{\Pi} = \Pi_\Sigma$ denote the connected component of $\Pi \cap \Omega$ that contains Σ.

Definition 1.1. Consider a nontrivial eigenfunction u to (1.1). Let $\Sigma \subset \Omega$ be a cell of Π, and $\eta \in \mathbb{C}$ be a constant. If $u|_\Sigma = 0$, Σ is said to be a nodal cell of u in Ω. By analytic continuation, it is seen that $u|_{\widetilde{\Pi}} = 0$, and $\widetilde{\Pi}$ is said to be a nodal plane of u. In a similar manner, in the case $(\partial_\nu u + \eta u)\big|_\Sigma = 0$, where ν is a unit one-sided normal direction of Π and $\eta \in \mathbb{C}$ is a constant, Σ and $\widetilde{\Pi}$ are respectively called the generalized singular cell and plane. In the particular case $\eta \equiv 0$, a generalized singular plane is also called a singular plane. Let $\mathcal{N}_\Omega^\lambda$, $\mathcal{S}_\Omega^\lambda$ and $\mathcal{M}_\Omega^\lambda$, respectively, signify the sets of nodal, singular and generalized singular planes of u in (1.1).

According to Definition 1.1, a nodal/generalized singular plane is actually a cell that is fully extended in Ω. Indeed, by the fact that u is analytic in Ω, we know that if the homogeneous condition is satisfied on a cell, then it is also satisfied on the so-called "plane" in Definition 1.1 by the analytic continuation. In what follows, most of the planes are actually the nodal/generalized singular planes in the sense of Definition 1.1, which should be clear from the context. Moreover, we would like to emphasize that in defining a generalized singular plane, the parameter η can be replaced to be a complex-valued real-analytic function. Indeed, all of the results obtained in this work hold for the case that η is a variable function as mentioned above. However, in order to ease the exposition, we stick to the case that η is a constant till to Section 4, and we shall make more relevant remarks in Section 5.

Let $B_\rho(\mathbf{x})$ denote a ball of radius $\rho \in \mathbb{R}_+$ and centred at $\mathbf{x} \in \mathbb{R}^3$.

Definition 1.2. Let Π_1 and Π_2 be two adjacent faces of a polyhedron $\mathcal{P}$ in Ω. Let $\boldsymbol{l}$ be a connected open portion of the edge formed by Π_1 and Π_2 such that $\boldsymbol{l} \Subset \Omega$. Then any $\mathbf{x} \in \boldsymbol{l}$ is said to be an edge corner point; see Fig. 1 for a schematic illustration. For notational convenience, we also let $\mathcal{E}(\Pi_1, \Pi_2, \boldsymbol{l})$ denote the edge corner as illustrated in Fig. 1.

Definition 1.3. Let $\{\Pi_\ell\}_{\ell=1}^n$ $(n \geq 3)$ be n planes in Ω such that they form a polyhedral cone $\mathcal{K}$ with the vertex $\mathbf{x}_0 \in \Omega$. Let $\rho \in \mathbb{R}_+$ be sufficiently small such that $B_\rho(\mathbf{x}_0) \subset \Omega$, then $\mathcal{K} \cap B_\rho(\mathbf{x}_0)$ is called a vertex corner associated with Π_1, Π_2, ..., Π_n, and denoted by $\mathcal{V}(\{\Pi_\ell\}_{\ell=1}^n, \mathbf{x}_0)$; see Fig. 1 for a schematic illustration.

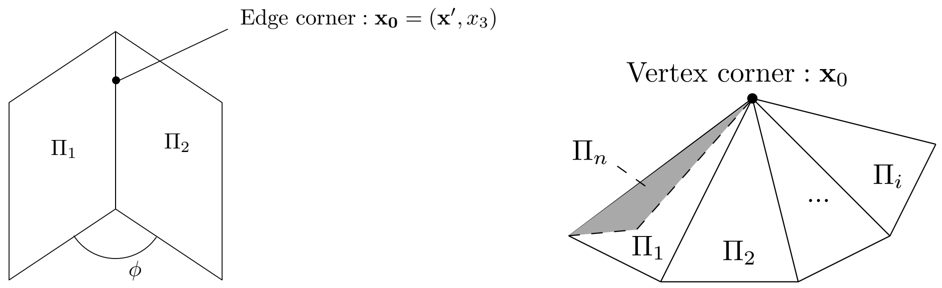

FIGURE 1. Schematic illustrations of edge corner and vertex corner respectively.

It is obvious that a vertex corner $\mathcal{V}(\{\Pi_\ell\}_{\ell=1}^n, \mathbf{x}_0)$ is composed of finite many edge corners, which are intersected by any two adjacent planes. Moreover, a vertex corner must be an edge corner. Definitions 1.1–1.3 describe some geometric notions. Next, we introduce several analytic notions for the Laplacian eigenfunction.

Definition 1.4. Let u be a nontrivial eigenfunction in (1.1). For a given point $\mathbf{x}_0 \in \Omega$, if there exits a number $N \in \mathbb{N} \cup \{0\}$ such that

$$\lim_{\rho \to +0} \frac{1}{\rho^m} \int_{B_\rho(\mathbf{x}_0)} |u(\mathbf{x})| \, \mathrm{d}\mathbf{x} = 0 \quad \text{for} \quad m = 0, 1, \ldots, N+2, \tag{1.2}$$

we say that u vanishes at $\mathbf{x}_0$ up to the order N. The largest possible N such that (1.2) is fulfilled is called the vanishing order of u at $\mathbf{x}_0$, and we write

$$\mathrm{Vani}(u; \mathbf{x}_0) = N.$$

If (1.2) holds for any $N \in \mathbb{N}$, then we say that the vanishing order is infinity.

By the strong UCP, if the vanishing order of u at $\mathbf{x}_0 \in \Omega$ is infinite, we know that $u \equiv 0$ in Ω. Similarly, we can introduce the definition of the vanishing order of u at an edge or vertex corner.

Definition 1.5. Let u be a nontrivial eigenfunction to (1.1). Consider an edge corner $\mathcal{E}(\Pi_1, \Pi_2, \boldsymbol{l}) \Subset \Omega$. For any given $\mathbf{x}_0 \in \boldsymbol{l}$, if

$$\mathrm{Vani}(u; \mathbf{x}_0) = N,$$

we say that u vanishes at $\mathbf{x}_0$ associated with the edge corner $\mathcal{E}(\Pi_1, \Pi_2, \boldsymbol{l}) \Subset \Omega$ up to order N, denoted by

$$\mathrm{Vani}(u; \mathbf{x}_0, \Pi_1, \Pi_2) = N.$$

For a vertex corner $\mathbf{x}_0 \in \Omega$ which is intersected by Π_i, $i = 1, 2, ...n$, the vanishing order of u at $\mathbf{x}_0$ is defined by

$$\mathrm{Vani}(u; \mathbf{x}_0) := \max\big\{ \max_{i=1,2,\ldots n-1} \mathrm{Vani}(u; \mathbf{x}_0, \Pi_i, \Pi_{i+1}), \mathrm{Vani}(u; \mathbf{x}_0, \Pi_n, \Pi_1) \big\}.$$

With the above definitions, we shall investigate in Sections 2, 3 and 4 the detailed vanishing properties of the Laplacian eigenfunctions at places where two or more nodal/singular/generalized singular planes intersect. The remaining part of the paper is organised as follows. In Section 2, we consider the vanishing property of the Laplacian eigenfunction at an edge corner intersected by two planes of three types: nodal planes, singular planes or generalized singular planes. In Section 3, we study the vanishing property at a vertex corner intersected by n planes $(n \geq 3)$, on the basis of Section 2. As a direct consequence of Sections 2 and 3, Section 4 is devoted to the discussion of the irrational intersection as a special case with infinite vanishing order. In Section 5, we remark the extension to the case that η is a variable function instead being a constant. In Section 6, as an important application of our new spectral results, we study an open fundamental mathematical issue in inverse obstacle scattering problems, namely, the unique identifiability results in determining the obstacle as well as the surface impedance by at most two far-field measurements.

2. Vanishing orders at edge corners

In this section, we study the vanishing property of the Laplacian eigenfunction at an edge corner $\mathbf{x}_0 \in \boldsymbol{l}$ associated with $\mathcal{E}(\Pi_1, \Pi_2, \boldsymbol{l})$. The two planes Π_ℓ $(\ell = 1, 2)$ could be either one of the following three types: nodal, singular or generalized singular planes. First, we give a definition of the irrational or rational dihedral angle of two intersecting planes.

Definition 2.1. Let Π_1 and Π_2 be two planes in $\mathbb{R}^3$ that intersect with each other. Let $\phi \in (0, \pi)$ be one of the associated intersecting dihedral angle of Π_1 and Π_2 satisfying

$$\phi = \alpha \cdot \pi, \quad \alpha \in (0, 1).$$

Then, ϕ is said to be an *irrational dihedral angle* if α is an irrational number; and it is said to be a rational dihedral angle of degree q if $\alpha = p/q$ with $p, q \in \mathbb{N}$ and is irreducible.

Since $-\Delta$ is invariant under rigid motions, throughout the rest of this paper, we assume that the edge corner $\mathcal{E}(\Pi_1, \Pi_2, \boldsymbol{l})$ satisfies

$$\boldsymbol{l} = \left\{ \mathbf{x} = (\mathbf{x}', x_3) \in \mathbb{R}^3; \mathbf{x}' = 0, \ x_3 \in (-H, H) \right\} \Subset \Omega, \tag{2.1}$$

where $2H$ is the length of $\boldsymbol{l}$. That is, $\boldsymbol{l}$ coincides with the x_3-axis. We further assume that Π_1 coincides with the (x_1, x_3)-plane while Π_2 possesses a dihedral angle $\alpha\pi$ away from Π_1 in the anti-clockwise direction; see Figure 2 for a schematic illustration. Clearly, we can assume that $\alpha \in (0, 1)$. Moreover, when we consider the vanishing order at an edge corner of $\mathcal{E}(\Pi_1, \Pi_2, \boldsymbol{l})$, we assume throughout this section that the edge corner under consideration is the origin $\mathbf{0} \in \boldsymbol{l}$.

In the next subsection, we first study a relatively simpler case that at least one of the intersecting planes of $\mathcal{E}(\Pi_1, \Pi_2, \boldsymbol{l})$ is a nodal plane. Without loss of generality, we assume $u|_{\Pi_1} \equiv 0$ throughout this subsection.

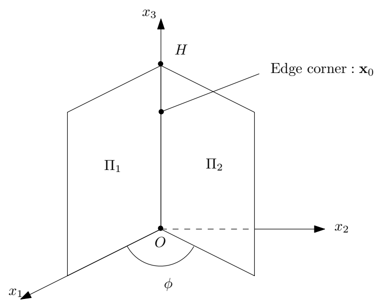

FIGURE 2. Schematic illustration of two intersecting planes with an edge corner and the dihedral angle ϕ.

2.1. **Vanishing orders at an edge corner with at least one plane being nodal.**

We first derive several important auxiliary results for the subsequent analysis, for which we will often use the spherical coordinate of any point $\mathbf{x}$ in $\mathbb{R}^3$:

$$\mathbf{x} = (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r \cos\theta) := (r, \theta, \phi), \ r \geq 0, \ \theta \in [0, \pi), \ \phi \in [0, 2\pi) . \tag{2.2}$$

Then the following proposition is a consequence of direct computings using spherical coordinates.

Proposition 2.2. *Let Π be any of the two planes associated with $\mathcal{E}(\Pi_1, \Pi_2, \boldsymbol{l})$. For any point $\mathbf{x} \in \Pi$, we know that ϕ defined in* (2.2) *is fixed; see Fig. 2. Let ν be the unit normal vector that is perpendicular to Π. Then*

$$\frac{\partial u}{\partial \nu} = \pm \frac{1}{r \sin\theta} \frac{\partial u}{\partial \phi}.$$

Lemma 2.3. *[9, Section 3.3]The solution u to* (1.1) *has the spherical wave expansion in spherical coordinates around the origin:*

$$u(\mathbf{x}) = 4\pi \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \mathrm{i}^n a_n^m j_n(\sqrt{\lambda} r) Y_n^m(\theta, \phi), \tag{2.3}$$

where $j_n(t)$ is the spherical Bessel function of order n, and $Y_n^m(\theta,\phi)$ is the spherical harmonics given by

$$Y_n^m(\theta,\phi) = \sqrt{\frac{2n+1}{4\pi}\frac{(n-|m|)!}{(n+|m|)!}} P_n^{|m|}(\cos\theta) e^{\mathrm{i}m\phi}$$

with $P_n^m(t)$ being the associated Legendre functions.

Lemma 2.4. *[27, Theorem 2.4.4] In the spherical coordinate system, the associated Legendre functions fulfill the following orthogonality condition for any fixed $n \in \mathbb{N}$, and any two integers $m \geq 0$ and $l \leq n$:*

$$\int_{-\pi}^{\pi} \frac{P_n^m(\cos\theta) P_n^l(\cos\theta)}{\sin\theta}\, d\theta = \begin{cases} 0 & \text{if} \quad l \neq m \\ \frac{(n+m)!}{m(n-m)!} & \text{if} \quad l = m \end{cases}.$$

Lemma 2.5. *Suppose that for $t \in (0,h)$, $h \in \mathbb{R}_+$,*

$$\sum_{n=0}^{\infty} \alpha_n j_n(t) = 0, \tag{2.4}$$

where $j_n(t)$ is the n-th spherical Bessel function. Then

$$\alpha_n = 0, \quad n = 0, 1, 2, \cdots. \tag{2.5}$$

Proof. By [9, Section 2.4] we know that

$$j_n(t) := \sum_{p=0}^{\infty} \frac{(-1)^p t^{n+2p}}{2^p p! 1 \cdot 3 \cdots (2n+2p+1)} = \frac{t^n}{(2n+1)!!}\Big(1 + \sum_{p=1}^{\infty} \frac{(-1)^p t^{2p}}{2^p p! N_{l,n}}\Big), \tag{2.6}$$

where $N_{l,n} = (2n+3)\cdot(2n+5)\cdots(2n+2p+1)$. Substituting (2.6) into (2.4) and comparing the coefficient of t^n $(n = 1, 2, \cdots)$, we can deduce (2.5). ☐

We are now in a position to study the general vanishing orders with the help of the spherical wave expansion of the Laplacian eigenfunction u to (1.1) around an intersecting edge corner.

Lemma 2.6. *Let u be a Laplacian eigenfunction to* (1.1)*. Suppose that there exists an edge corner $\mathcal{E}(\Pi_1, \Pi_2, \boldsymbol{l})$ such that*

$$\mathcal{E}(\Pi_1, \Pi_2, \boldsymbol{l}) \Subset \Omega,$$

where Π_1 and Π_2 are from either of $\mathcal{N}_\Omega^\lambda$, $\mathcal{S}_\Omega^\lambda$ and $\mathcal{M}_\Omega^\lambda$. If there exists a sufficiently small $\varepsilon \in \mathbb{R}_+$ such that

$$u|_{B_\varepsilon(\mathbf{0}) \cap \boldsymbol{l}} = 0, \tag{2.7}$$

then it holds for the coefficients in (2.3) *that*

$$a_n^0 = 0, \quad n \in \mathbb{N} \cup \{0\}. \tag{2.8}$$

Proof. Since the line segment $\boldsymbol{l}$ associated with $\mathcal{E}(\Pi_1, \Pi_2, \boldsymbol{l})$ coincides with the x_3-axis, we know $\theta = 0$ or π for $\mathbf{x} \in \boldsymbol{l}$ in the spherical coordinate system (2.2). Combining with Lemma 2.3, we know under the condition (2.7) that

$$u|_{B_\varepsilon(\mathbf{0}) \cap \boldsymbol{l}} = 4\pi \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \mathrm{i}^n a_n^m j_n(\sqrt{\lambda} r) \sqrt{\frac{2n+1}{4\pi}} \sqrt{\frac{(n-|m|)!}{(n+|m|)!}} P_n^{|m|}(\pm 1) e^{\mathrm{i}m\phi} = 0. \tag{2.9}$$

On the other hand, we have that for $m \in \mathbb{N}$ (cf. [2]),

$$P_n^{-m} = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m, \;\; P_n^m(\pm 1) = 0, \;\; P_n^0(+1) = 1, \;\; P_n^0(-1) = (-1)^n. \tag{2.10}$$

Substituting (2.10) into (2.9), it is easy to see that

$$\sum_{n=0}^{\infty} \mathrm{i}^n \sqrt{\frac{2n+1}{4\pi}} a_n^0 j_n(\sqrt{\lambda} r) = 0.$$

By virtue of Lemma 2.5, we readily see

$$\mathrm{i}^n \sqrt{\frac{2n+1}{4\pi}} a_n^0 = 0 \quad \text{for } n = 0, 1, 2, \cdots,$$

which completes the proof of Lemma 2.6. □

First, we consider the case that two nodal planes intersect with each other to yield the edge corner.

Theorem 2.7. *Let u be a Laplacian eigenfunction to* (1.1). *Consider an edge corner $\mathcal{E}(\Pi_1, \Pi_2, \boldsymbol{l}) \Subset \Omega$ where the two planes Π_ℓ, $\ell = 1, 2$ are assumed to be nodal, namely $\Pi_\ell \in \mathcal{N}_\Omega^\lambda$ $(\ell = 1, 2)$. If the corresponding dihedral angle can be written as*

$$\angle(\Pi_1, \Pi_2) = \phi = \alpha \cdot \pi, \quad \alpha \in (0, 1),$$

where α satisfies for an $N \in \mathbb{N}$, $N \geq 3$,

$$\alpha \neq \frac{q}{p}, \quad p = 1, 2, \cdots, N-1, \ q = 1, 2, \cdots, p-1, \tag{2.11}$$

then u vanishes up to order at least N at the edge corner $\mathbf{0}$.

Proof. Since $u|_{\Pi_i} \equiv 0$, $i = 1, 2$, it follows from Lemma 2.3 that

$$u|_{\Pi_1} = 4\pi \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \mathrm{i}^n a_n^m j_n(\sqrt{\lambda} r) \sqrt{\frac{2n+1}{4\pi}} \sqrt{\frac{(n-|m|)!}{(n+|m|)!}} P_n^{|m|}(\cos\theta) = 0, \tag{2.12}$$

$$u|_{\Pi_2} = 4\pi \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \mathrm{i}^n a_n^m j_n(\sqrt{\lambda} r) \sqrt{\frac{2n+1}{4\pi}} \sqrt{\frac{(n-|m|)!}{(n+|m|)!}} P_n^{|m|}(\cos\theta) e^{\mathrm{i}m\alpha\cdot\pi} = 0, \tag{2.13}$$

where $\phi = 0$ on Π_1 and $\phi = \alpha \cdot \pi, \alpha \in (0, 1)$ on Π_2. It is obvious that $u|_{\boldsymbol{l}} = 0$, then we have (2.8) from Lemma 2.6. Thus comparing the coefficient of r and substituting $a_n^0 = 0$ for $n = 0, 1$ into (2.12) and (2.13), we obtain

$$(a_1^1 + a_1^{-1}) P_1^1(\cos\theta) = 0, \quad (a_1^1 e^{\mathrm{i}\alpha\cdot\pi} + a_1^{-1} e^{-\mathrm{i}\alpha\cdot\pi}) P_1^1(\cos\theta) = 0.$$

Since $\theta \in (0, \pi)$ is arbitrary, utilizing the orthogonality condition (Lemma 2.4), we can deduce

$$a_1^1 + a_1^{-1} = 0, \quad a_1^1 e^{\mathrm{i}\alpha\cdot\pi} + a_1^{-1} e^{-\mathrm{i}\alpha\cdot\pi} = 0.$$

Therefore, if $\alpha \neq 0, 1$, we derive that $a_1^{\pm 1} = 0$.

Assume that $a_{n-1}^m = 0$, $m = \pm 1, \pm 2, \cdots, \pm(n-1)$. We next show by induction that $a_n^m = 0$, $m = \pm 1, \pm 2, \cdots, \pm n$. Indeed, comparing the coefficients of r^n, we obtain

$$\sum_{m=-n}^{n} \mathrm{i}^n a_n^m \frac{\sqrt{\lambda}^n}{(2n+1)!!} \sqrt{\frac{2n+1}{4\pi}} \sqrt{\frac{(n-|m|)!}{(n+|m|)!}} P_n^{|m|}(\cos\theta) = 0, \tag{2.14}$$

$$\sum_{m=-n}^{n} \mathrm{i}^n a_n^m \frac{\sqrt{\lambda}^n}{(2n+1)!!} \sqrt{\frac{2n+1}{4\pi}} \sqrt{\frac{(n-|m|)!}{(n+|m|)!}} P_n^{|m|}(\cos\theta) e^{\mathrm{i}m\alpha\cdot\pi} = 0. \tag{2.15}$$

Similarly, substituting $a_n^0 = 0$ into (2.14) and (2.15), noting that θ is arbitrary, and utilizing the orthogonality condition (Lemma 2.4) again, we can derive for $m = 1, 2, \cdots$ that

$$a_n^m + a_n^{-m} = 0, \quad a_n^m e^{\mathrm{i}m\alpha\cdot\pi} + a_n^{-m} e^{-\mathrm{i}m\alpha\cdot\pi} = 0. \tag{2.16}$$

Hence if $\alpha \neq \frac{k}{m}$, $k = 1, 2, \cdots, m-1$, the coefficient matrix fulfills

$$\begin{vmatrix} 1 & 1 \\ e^{\mathrm{i}m\alpha\cdot\pi} & e^{-\mathrm{i}m\alpha\cdot\pi} \end{vmatrix} = -2\mathrm{i}\sin m\alpha\cdot\pi \neq 0,$$

which yields that $a_n^m = 0$ for $m = \pm 1, \pm 2, \cdots, \pm n$, hence completes the proof of Theorem 2.7. $\square$

Remark 2.8. In the proof of Theorem 2.7, we make use of the boundary conditions of u on Π_1 and Π_2 as well as the orthogonality property in Lemma 2.4 to arrive at the homogeneous linear system (2.16), which in turn proves that $a_n^{\pm m} = 0$ provided a certain condition on the dihedral angle $\alpha \cdot \pi$ is fulfilled. This type of argument shall be frequently used in the proofs of Theorems 2.9, 2.11 and 3.1 in what follows.

We now proceed to consider the case that a nodal plane $\Pi_1 \in \mathcal{N}_\Omega^\lambda$ intersects with a generalized singular plane $\Pi_2 \in \mathcal{M}_\Omega^\lambda$.

Theorem 2.9. *Let u be a Laplacian eigenfunction to* (1.1)*. Consider an edge corner $\mathcal{E}(\Pi_1, \Pi_2, \boldsymbol{l}) \Subset \Omega$ such that*

$$\Pi_1 \in \mathcal{N}_\Omega^\lambda, \quad \Pi_2 \in \mathcal{M}_\Omega^\lambda \quad \textit{and} \quad \angle(\Pi_1, \Pi_2) = \phi = \alpha \cdot \pi, \quad \alpha \in (0,1).$$

If for an $N \in \mathbb{N}$, $N \geq 2$, there holds

$$\alpha \neq \frac{2q+1}{2p}, \quad p = 1, 2, \cdots, N-1, \ q = 1, 2, \cdots, p-1,$$

then u vanishes up to order at least N at the edge corner $\mathbf{0}$.

Proof. Since $u|_{\Pi_1} \equiv 0$, it is direct to know that $u|_{\boldsymbol{l}} \equiv 0$, which indicates that $a_n^0 = 0$ for $n = 0, 1, 2, \cdots$ from Lemma 2.6. Furthermore, by Lemma 2.3 we have

$$u|_{\Pi_1} = 4\pi \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \mathrm{i}^n a_n^m j_n(\sqrt{\lambda} r) \sqrt{\frac{2n+1}{4\pi}} \sqrt{\frac{(n-|m|)!}{(n+|m|)!}} P_n^{|m|}(\cos\theta) = 0. \tag{2.17}$$

Combining with Proposition 2.2, we derive the following expression on Π_2:

$$\begin{aligned}
&\frac{\partial u}{\partial \nu} + \eta u\Big|_{\Pi_2} = \frac{1}{r\sin\theta}\frac{\partial u}{\partial \phi} + \eta u\Big|_{\phi=\alpha\cdot\pi} \\
=&\frac{1}{r\sin\theta} 4\pi \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \mathrm{i}^{n+1} m a_n^m j_n(\sqrt{\lambda} r) \sqrt{\frac{2n+1}{4\pi}} \sqrt{\frac{(n-|m|)!}{(n+|m|)!}} P_n^{|m|}(\cos\theta) e^{\mathrm{i} m\alpha\cdot\pi} \\
+&\eta \cdot 4\pi \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \mathrm{i}^n a_n^m j_n(\sqrt{\lambda} r) \sqrt{\frac{2n+1}{4\pi}} \sqrt{\frac{(n-|m|)!}{(n+|m|)!}} P_n^{|m|}(\cos\theta) e^{\mathrm{i} m\alpha\cdot\pi} = 0.
\end{aligned} \tag{2.18}$$

Since $\theta \in (0, \pi)$ and $r > 0$, multiplying $r \sin\theta$ on the both sides of (2.18) we can obtain that

$$\begin{aligned}
&\sum_{n=0}^{\infty} \sum_{m=-n}^{n} \mathrm{i}^{n+1} m a_n^m j_n(\sqrt{\lambda} r) \sqrt{\frac{2n+1}{4\pi}} \sqrt{\frac{(n-|m|)!}{(n+|m|)!}} P_n^{|m|}(\cos\theta) e^{\mathrm{i} m\alpha\cdot\pi} \\
&+ \eta \cdot r\sin\theta \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \mathrm{i}^n a_n^m j_n(\sqrt{\lambda} r) \sqrt{\frac{2n+1}{4\pi}} \sqrt{\frac{(n-|m|)!}{(n+|m|)!}} P_n^{|m|}(\cos\theta) e^{\mathrm{i} m\alpha\cdot\pi} = 0.
\end{aligned} \tag{2.19}$$

Following a similar argument to Theorem 2.7, we may compare the coefficients of r in (2.17) and (2.19) respectively. First for (2.17) we have

$$\sum_{m=-1}^{1} \mathrm{i} a_1^m \frac{\sqrt{\lambda}}{3!!} \sqrt{\frac{3}{4\pi}} \sqrt{\frac{(1-|m|)!}{(1+|m|)!}} P_1^{|m|}(\cos\theta) = 0.$$

Since $a_1^0 = 0$, using Lemma 2.4 we can deduce that

$$a_1^1 + a_1^{-1} = 0. \tag{2.20}$$

Then for (2.19), we have

$$\sum_{m=-1}^{1} m a_1^m \frac{\sqrt{\lambda}}{3!!} \sqrt{\frac{3}{4\pi}} \sqrt{\frac{(1-|m|)!}{(1+|m|)!}} P_1^{m}(\cos\theta) e^{\mathrm{i} m\alpha\cdot\pi} = 0, \tag{2.21}$$

since $a_0^0 = 0$. By the orthogonality condition of P_1^m for arbitrary $\theta \in (0, \pi)$ and the fact that $a_1^0 = 0$ we can simplify (2.21) to get

$$a_1^1 e^{\mathrm{i}\alpha\cdot\pi} - a_1^{-1} e^{-\mathrm{i}\alpha\cdot\pi} = 0.$$

Combining (2.20) with (2.21) we can obtain that if $\alpha \neq \frac{1}{2}$, then $a_1^{\pm 1} = 0$. By induction, we assume that $a_{n-1}^m = 0$, $m = \pm 1, \pm 2, \cdots, \pm(n-1)$. Considering the coefficients of r^n in (2.17), we have

$$\sum_{m=-n}^{n} \mathrm{i}^n a_n^m \frac{\sqrt{\lambda}^n}{(2n+1)!!} \sqrt{\frac{2n+1}{4\pi}} \sqrt{\frac{(n-|m|)!}{(n+|m|)!}} P_n^{|m|}(\cos\theta) = 0,$$

from which we can derive

$$a_n^m + a_n^{-m} = 0 \quad \text{ for } m = 1, 2, \cdots \tag{2.22}$$

by virtue of the fact that $a_n^0 = 0$ and Lemma 2.4. Similarly, for (2.19), we know the coefficients of r^n fulfill that

$$\begin{aligned} &\sum_{m=-n}^{n} \mathrm{i}^{n+1} m a_n^m \frac{\sqrt{\lambda}^n}{(2n+1)!!} \sqrt{\frac{2n+1}{4\pi}} \sqrt{\frac{(n-|m|)!}{(n+|m|)!}} P_n^{|m|}(\cos\theta) e^{\mathrm{i}m\alpha\cdot\pi} \\ &+ \eta \cdot \sin\theta \sum_{m=-(n-1)}^{n-1} \mathrm{i}^{n-1} a_{n-1}^m \frac{\sqrt{\lambda}^{n-1}}{(2n-1)!!} \sqrt{\frac{2n-1}{4\pi}} \sqrt{\frac{(n-1-|m|)!}{(n-1+|m|)!}} P_{n-1}^{|m|}(\cos\theta) e^{\mathrm{i}m\alpha\cdot\pi} = 0. \end{aligned} \tag{2.23}$$

Substituting $a_{n-1}^m = 0$, $m = \pm 1, \pm 2, \cdots, \pm(n-1)$, and $a_n^0 = 0$ into (2.23), utilizing Lemma 2.4 again we derive

$$a_n^m e^{\mathrm{i}m\alpha\cdot\pi} - a_n^{-m} e^{-\mathrm{i}m\alpha\cdot\pi} = 0. \tag{2.24}$$

Therefore, by the virtue of Remark 2.8, we can deduce from (2.22) and (2.24) that if $\alpha \neq \frac{2k+1}{2m}$ ($k = 0, 1, \cdots, m-1$), then $a_n^m = 0$, $m = \pm 1, \pm 2, \cdots, \pm n$, hence completes the proof of Theorem 2.9. □

It is straightforward to verify from the proof of Theorem 2.9 that η can be 0. In such a case, we have the following result.

Corollary 2.10. *Let u be a Laplacian eigenfunction to* (1.1). *Consider an edge corner $\mathcal{E}(\Pi_1, \Pi_2, \boldsymbol{l}) \Subset \Omega$ such that*

$$\Pi_1 \in \mathcal{N}_\Omega^\lambda, \quad \Pi_2 \in \mathcal{S}_\Omega^\lambda \quad \text{and} \quad \angle(\Pi_1, \Pi_2) = \phi = \alpha \cdot \pi, \quad \alpha \in (0, 1).$$

If for an $N \in \mathbb{N}$, $N \geq 2$, there holds

$$\alpha \neq \frac{2q+1}{2p}, \quad p = 1, 2, \cdots, N-1, \ q = 1, 2, \cdots, p-1,$$

then u vanishes up to order at least N at the edge corner $\mathbf{0}$.

2.2. **Vanishing orders at an edge corner intersected by generalized singular planes.** In this subsection, we consider the case that an edge corner $\mathcal{E}(\Pi_1, \Pi_2, \boldsymbol{l})$ is intersected by two generalized singular planes, namely $\Pi_\ell \in \mathcal{M}_\Omega^\lambda$, $\ell = 1, 2$. In what follows, we signify the boundary parameters on Π_ℓ to be η_ℓ, $\ell = 1, 2$. Then we can derive the following three theorems.

Theorem 2.11. *Let u be a Laplacian eigenfunction to* (1.1). *Consider an edge corner $\mathcal{E}(\Pi_1, \Pi_2, \boldsymbol{l}) \Subset \Omega$ with $\Pi_\ell \in \mathcal{M}_\Omega^\lambda, \ell = 1, 2$ and $\angle(\Pi_1, \Pi_2) = \phi = \alpha \cdot \pi$ for $\alpha \in (0, 1)$. If there exists a sufficiently small radius $\varepsilon \in \mathbb{R}_+$ such that*

$$u|_{B_\varepsilon(\mathbf{0}) \cap \boldsymbol{l}} \equiv 0, \tag{2.25}$$

and for an $N \in \mathbb{N}$, $N \geq 3$,

$$\alpha \neq \frac{q}{p}, \quad p = 1, 2, \cdots, N-1, \ q = 1, 2, \cdots, p-1,$$

then u vanishes up to the order at least N at the edge corner $\mathbf{0}$.

Proof. Since $u|_{\Pi_i} = \frac{\partial u}{\partial \nu} + \eta_i u = 0$, $i = 1, 2$, we have by using Proposition 2.2 that

$$\begin{aligned} \frac{\partial u}{\partial \nu} + \eta_1 u\Big|_{\Pi_1} &= -\frac{1}{r\sin\theta}\frac{\partial u}{\partial \phi} + \eta_1 u\Big|_{\phi=0} = 0, \\ \frac{\partial u}{\partial \nu} + \eta_2 u\Big|_{\Pi_2} &= \frac{1}{r\sin\theta}\frac{\partial u}{\partial \phi} + \eta_2 u\Big|_{\phi=\alpha\cdot\pi} = 0, \end{aligned}$$

which can be written more explicitly in spherical coordinate system by Lemma 2.3 as

$$\begin{aligned} &-\sum_{n=0}^{\infty}\sum_{m=-n}^{n} \mathrm{i}^{n+1} m a_n^m j_n(\sqrt{\lambda}r)\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|m|)!}{(n+|m|)!}}P_n^{|m|}(\cos\theta) \\ &+\eta_1 r\sin\theta \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \mathrm{i}^{n} a_n^m j_n(\sqrt{\lambda}r)\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|m|)!}{(n+|m|)!}}P_n^{|m|}(\cos\theta) = 0, \end{aligned} \tag{2.26}$$

and

$$\begin{aligned} &\sum_{n=0}^{\infty}\sum_{m=-n}^{n} \mathrm{i}^{n+1} m a_n^m j_n(\sqrt{\lambda}r)\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|m|)!}{(n+|m|)!}}P_n^{|m|}(\cos\theta)e^{\mathrm{i}m\alpha\cdot\pi} \\ &+\eta_2 r\sin\theta \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \mathrm{i}^{n} a_n^m j_n(\sqrt{\lambda}r)\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|m|)!}{(n+|m|)!}}P_n^{|m|}(\cos\theta)e^{\mathrm{i}m\alpha\cdot\pi} = 0. \end{aligned} \tag{2.27}$$

Under the condition (2.25), we know from from Lemma 2.6 that

$$a_n^0 = 0, \quad \text{for } n = 0, 1, 2, \cdots. \tag{2.28}$$

Comparing the coefficients of r^1 in (2.26) and (2.27) respectively we have

$$\begin{aligned} \sum_{m=-1}^{1} m a_1^m \frac{\sqrt{\lambda}}{3!!}\sqrt{\frac{3}{4\pi}}\sqrt{\frac{(1-|m|)!}{(1+|m|)!}}P_1^{|m|}(\cos\theta) = 0, \\ -\sum_{m=-1}^{1} m a_1^m \frac{\sqrt{\lambda}}{3!!}\sqrt{\frac{3}{4\pi}}\sqrt{\frac{(1-|m|)!}{(1+|m|)!}}P_1^{|m|}(\cos\theta)e^{\mathrm{i}m\alpha\cdot\pi} = 0. \end{aligned}$$

Utilizing the orthogonality condition (Lemma 2.4) and the fact that $a_1^0 = 0$ we can obtain the linear system with respect to $a_1^{\pm 1}$ as

$$a_1^1 - a_1^{-1} = 0, \quad a_1^1 e^{\mathrm{i}\alpha\cdot\pi} - a_1^{-1}e^{-\mathrm{i}\alpha\cdot\pi} = 0.$$

Since $\alpha \in (0, 1)$, which indicates that $\phi \neq 0, \pi$, it is easy to see that $a_1^{\pm 1} = 0$. Using the same argument, by induction, we assume that

$$a_{n-1}^m = 0, \quad m = \pm 1, \pm 2, \cdots, \pm(n-1). \tag{2.29}$$

Then by considering the coefficients of r^n in (2.26) and (2.27) we have

$$\begin{aligned} &-\sum_{m=-n}^{n} \mathrm{i}^{n+1} m a_n^m \frac{\sqrt{\lambda}^n}{(2n+1)!!}\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|m|)!}{(n+|m|)!}}P_n^{|m|}(\cos\theta) \\ &+\eta_1 \sin\theta \sum_{m=-(n-1)}^{n-1} \mathrm{i}^{n-1} a_{n-1}^m \frac{\sqrt{\lambda}^{n-1}}{(2n-1)!!}\sqrt{\frac{2n-1}{4\pi}}\sqrt{\frac{(n-1-|m|)!}{(n-1+|m|)!}}P_{n-1}^{|m|}(\cos\theta) = 0, \end{aligned} \tag{2.30}$$

and

$$\begin{aligned} &\sum_{m=-n}^{n} \mathrm{i}^{n+1} m a_n^m \frac{\sqrt{\lambda}^n}{(2n+1)!!}\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|m|)!}{(n+|m|)!}}P_n^{|m|}(\cos\theta)e^{\mathrm{i}m\alpha\cdot\pi} \\ &+\eta_2 \sin\theta \sum_{m=-(n-1)}^{n-1} \mathrm{i}^{n-1} a_{n-1}^m \frac{\sqrt{\lambda}^{n-1}}{(2n-1)!!}\sqrt{\frac{2n-1}{4\pi}}\sqrt{\frac{(n-1-|m|)!}{(n-1+|m|)!}}P_{n-1}^{|m|}(\cos\theta)e^{\mathrm{i}m\alpha\cdot\pi} = 0. \end{aligned} \tag{2.31}$$

By induction, substituting (2.28) and (2.29) into (2.30) and (2.31), using Lemma 2.4 we can deduce that for $m \in \mathbb{N}_+$,

$$\begin{cases} a_n^m - a_n^{-m} = 0, \\ a_n^m e^{\mathrm{i}m\alpha\cdot\pi} - a_n^{-m} e^{-\mathrm{i}m\alpha\cdot\pi} = 0. \end{cases} \tag{2.32}$$

Hence if $\alpha \neq \frac{k}{m}$, $k = 1, 2, \cdots, m-1$, by virtue of Remark 2.8, we can deduce that $a_n^m = 0$, $m = \pm 1, \pm 2, \cdots, \pm n$, which completes the proof of Theorem 2.11. □

Remark 2.12. It is important and necessary to assume $u \equiv 0$ on $B_\varepsilon(\mathbf{0}) \cap \boldsymbol{l}$ in Theorem 2.11. Otherwise we can not derive the recursive equations with respect to a_n^m from (2.26) and (2.27) to ensure the desired vanishing results.

Remark 2.13. It is straightforward to verify in the proof of Theorem 2.11 that η_1 and/or η_2 can be taken to be zero. That is, Theorem 2.11 also includes the cases that at least one of the two planes Π_ℓ is a singular plane.

3. Vanishing orders at vertex corners

In this section, we study the vanishing property of the Laplacian eigenfunction to (1.1) at a vertex corner $\mathcal{V}(\{\Pi_\ell\}_{\ell=1}^n, \mathbf{x}_0) \Subset \Omega$, where Π_ℓ could be either a nodal plane, a singular plane or a generalized singular plane. It is known that an edge corner $\mathcal{E}(\Pi_1, \Pi_2, \boldsymbol{l})$ can be regarded as part of a vertex corner $\mathcal{V}(\{\Pi_\ell\}_{\ell=1}^n, \mathbf{x}_0)$. In Section 2, we have unveiled that the vanishing order of the eigenfunction u at an edge corner can be determined by the the intersecting dihedral angle of $\mathcal{E}(\Pi_1, \Pi_2, \boldsymbol{l})$ under a generic condition (cf. (2.25)). In this section, we concentrate on the following condition

$$u(\mathbf{x}_0) = 0, \tag{3.1}$$

to study the vanishing property of u at $\mathbf{x}_0$. We should point out that (3.1) is much more relaxed compared with (2.25), and it can be easily fulfilled in certain generic case, e.g., superpositions of two eigenfunctions at the point $\mathbf{x}_0$. In particular, such a condition like (3.1) can be used to show the unique determination of some polyhedral obstacles in $\mathbb{R}^3$ by finitely many measurements in the inverse obstacle scattering problem; see more details in Section 6.

Similar to Section 2, without loss of generality, we assume that the vertex corner $\mathbf{x}_0$ of $\mathcal{V}(\{\Pi_\ell\}_{\ell=1}^n, \mathbf{x}_0)$ coincides with the origin. We first focus on the case that $n = 3$, which implies that the vertex corner $\mathcal{V}(\{\Pi_\ell\}_{\ell=1}^3, \mathbf{x}_0)$ is formed by three planes; see Figure 3 for a schematic illustration. For $n > 3$, the related results can be derived in a similar way; see Theorem 3.6–3.7. It is obvious that $\mathcal{V}(\{\Pi_\ell\}_{\ell=1}^3, \mathbf{x}_0)$ is formed by three edge corners $\mathcal{E}(\Pi_1, \Pi_2, \boldsymbol{l}_1)$, $\mathcal{E}(\Pi_2, \Pi_3, \boldsymbol{l}_2)$ and $\mathcal{E}(\Pi_3, \Pi_1, \boldsymbol{l}_3)$ where $\boldsymbol{l}_1$, $\boldsymbol{l}_2$ and $\boldsymbol{l}_3$ are three line segments of $\Pi_1 \cap \Pi_2$, $\Pi_2 \cap \Pi_3$ and $\Pi_3 \cap \Pi_1$ respectively. Hence, if either of the three planes Π_ℓ is nodal, say Π_3, then one can apply the results in Section 2 to the edge corners $\mathcal{E}(\Pi_2, \Pi_3, \boldsymbol{l}_2)$ and $\mathcal{E}(\Pi_3, \Pi_1, \boldsymbol{l}_3)$ to derive a certain vanishing order at the vertex corder, by regarding it as an edge corner associated with $\mathcal{E}(\Pi_2, \Pi_3, \boldsymbol{l}_2)$ and $\mathcal{E}(\Pi_3, \Pi_1, \boldsymbol{l}_3)$, respectively. Hence, we shall mainly focus on the vanishing order generated through the intersection of the two planes Π_1 and Π_2, both of which are assumed not to be nodal.

Theorem 3.1. *Let u be a Laplacian eigenfunction to* (1.1). *Consider a vertex corner $\mathcal{V}(\{\Pi_\ell\}_{\ell=1}^3, \mathbf{0}) \Subset \Omega$ with $\Pi_\ell \in \mathcal{M}_\Omega^\lambda$, $\ell = 1, 2$, $\angle(\Pi_1, \Pi_2) = \phi = \alpha \cdot \pi$, $\alpha \in (0, 1)$ and $\Pi_3 \in \mathcal{N}_\Omega^\lambda$. Assume that $\Pi_3 = \mathrm{span}\{\vec{a}, \vec{b}\}$, where $\vec{a} = (r, \theta_1, 0) \in \Pi_1 \cap \Pi_3$ and $\vec{b} = (r, \theta_2, \alpha \cdot \pi) \in \Pi_2 \cap \Pi_3$ for $r > 0$, $\alpha \in (0, 1)$, and fixed θ_1 and θ_2 in the spherical coordinate system. If for an $N \in \mathbb{N}$, $N \geq 3$, it holds that*

$$P_p^0(\cos\theta_i) \neq 0, i = 1 \text{ or } 2, \text{ and } \alpha \neq \frac{q}{p}, p = 1, 2, \cdots, N-1, \ q = 1, 2, \cdots, p-1, \tag{3.2}$$

where P_p^0 is the associated Legendre polynomial, then the vanishing order of u at $\mathbf{0}$ generated by the intersection of the two planes Π_1 and Π_2 is at least order N.

Proof. Since Π_1 and Π_2 are two generalized singular planes, we have

$$\frac{\partial u}{\partial \nu} + \eta_1 u\Big|_{\Pi_1} = 0 \quad \text{and} \quad \frac{\partial u}{\partial \nu} + \eta_2 u\Big|_{\Pi_2} = 0. \tag{3.3}$$

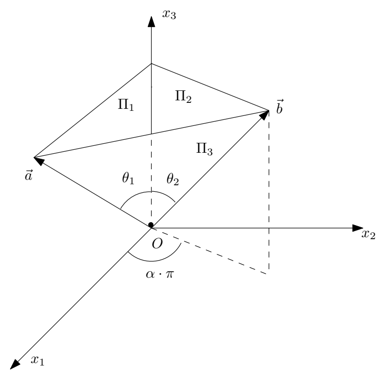

FIGURE 3. Schematic illustration of a vertex corner that is intersected by Π_1, Π_2 and Π_3.

By Proposition 2.2 and Lemma 2.3, we can write (3.3) explicitly as

$$
\begin{aligned}
&-\frac{1}{r\sin\theta}\frac{\partial u}{\partial\phi}+\eta_1 u\Big|_{\phi=0}\\
&=-\frac{1}{r\sin\theta}4\pi\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\mathrm{i}^{n+1}ma_n^m j_n(\sqrt{\lambda}r)\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|m|)!}{(n+|m|)!}}P_n^{|m|}(\cos\theta)\\
&+\eta_1 4\pi\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\mathrm{i}^{n}a_n^m j_n(\sqrt{\lambda}r)\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|m|)!}{(n+|m|)!}}P_n^{|m|}(\cos\theta)=0, \qquad (3.4)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{r\sin\theta}\frac{\partial u}{\partial\phi}+\eta_2 u\Big|_{\phi=\alpha\cdot\pi}\\
&=\frac{1}{r\sin\theta}4\pi\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\mathrm{i}^{n+1}ma_n^m j_n(\sqrt{\lambda}r)\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|m|)!}{(n+|m|)!}}P_n^{|m|}(\cos\theta)e^{\mathrm{i}m\alpha\cdot\pi}\\
&+\eta_2 4\pi\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\mathrm{i}^{n}a_n^m j_n(\sqrt{\lambda}r)\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|m|)!}{(n+|m|)!}}P_n^{|m|}(\cos\theta)e^{\mathrm{i}m\alpha\cdot\pi}=0. \qquad (3.5)
\end{aligned}
$$

Since $\Pi_3=\mathrm{span}\{\vec{a},\vec{b}\}$, where $\vec{a}=(r,\theta_1,0)\in\Pi_1\cap\Pi_3$ and $\vec{b}=(r,\theta_2,\alpha\cdot\pi)\in\Pi_2\cap\Pi_3$ for fixed θ_1, θ_2 and $u|_{\Pi_3}\equiv 0$. It is direct to see $u|_{\vec{a}}=u|_{\vec{b}}=0$, which further indicates that

$$
u|_{\vec{a}}=4\pi\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\mathrm{i}^{n}a_n^m j_n(\sqrt{\lambda}r)\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|m|)!}{(n+|m|)!}}P_n^{|m|}(\cos\theta_1)=0, \qquad (3.6)
$$

and

$$
u|_{\vec{b}}=4\pi\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\mathrm{i}^{n}a_n^m j_n(\sqrt{\lambda}r)\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|m|)!}{(n+|m|)!}}P_n^{|m|}(\cos\theta_2)e^{\mathrm{i}m\alpha\cdot\pi}=0. \qquad (3.7)
$$

Combining with (3.4) and (3.5), it suffices to use (3.6) or (3.7) to study the coefficients of r^n, $n\in\mathbb{N}$. In what follows, without loss of generality, we discuss only (3.6). Since $u|_{\vec{a}}\equiv 0$, the coefficient of r^0 fulfills that

$$
4\pi a_0^0\sqrt{\frac{1}{4\pi}}P_0^0(\cos\theta_1)=0,
$$

where we can know that $a_0^0=0$ since $P_0^0\equiv 1$. Consider the coefficients of r, from (3.4), (3.5) and (3.6), we can respectively see that

$$
\sum_{m=-1}^{1}ma_1^m\frac{\sqrt{\lambda}}{3!!}\sqrt{\frac{3}{4\pi}}\sqrt{\frac{(1-|m|)!}{(1+|m|)!}}P_1^{|m|}(\cos\theta)+\eta_1\sin\theta a_0^0\sqrt{\frac{1}{4\pi}}P_0^0(\cos\theta)=0, \qquad (3.8)
$$

$$\sum_{m=-1}^{1} m a_1^m \frac{\sqrt{\lambda}}{3!!}\sqrt{\frac{3}{4\pi}}\sqrt{\frac{(1-|m|)!}{(1+|m|)!}}P_1^{|m|}(\cos\theta)e^{\mathrm{i}m\alpha\cdot\pi} - \eta_2\sin\theta\sum_{m=-1}^{1} a_0^0\sqrt{\frac{1}{4\pi}}P_0^0(\cos\theta)e^{\mathrm{i}m\alpha\cdot\pi} = 0, \tag{3.9}$$

$$\sum_{m=-1}^{1} \mathrm{i} a_1^m \frac{\sqrt{\lambda}}{3!!}\sqrt{\frac{3}{4\pi}}\sqrt{\frac{(1-|m|)!}{(1+|m|)!}}P_1^{|m|}(\cos\theta_1) = 0. \tag{3.10}$$

Substituting $a_0^0 = 0$ into (3.8) and (3.9), combining with Lemma 2.4, we can directly derive the following linear system with respect to $a_1^{\pm 1}$:

$$a_1^1 - a_1^{-1} = 0, \quad a_1^1 e^{\mathrm{i}\alpha\cdot\pi} - a_1^{-1}e^{-\mathrm{i}\alpha\cdot\pi} = 0. \tag{3.11}$$

Thus we know that $a_1^{\pm 1} = 0$ since $\alpha \in (0,1)$. As a consequence, if $P_1^0(\cos\theta_1) \neq 0$ in (3.10), we can deduce that $a_1^0 = 0$ easily.

By induction, we assume that $a_{n-1}^m = 0$ for $m = 0, \pm 1, \pm 2, \cdots, \pm(n-1)$. Then considering the coefficients of r^n, by (3.4), (3.5) and (3.6), we have

$$\begin{aligned} &-\sum_{m=-n}^{n} \mathrm{i}^{n+1} m a_n^m \frac{\sqrt{\lambda}^n}{(2n+1)!!}\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|m|)!}{(n+|m|)!}}P_n^{|m|}(\cos\theta) \\ &+\eta_1\sin\theta\sum_{m=-(n-1)}^{n-1} \mathrm{i}^{n-1} a_{n-1}^m \frac{\sqrt{\lambda}^{n-1}}{(2n-1)!!}\sqrt{\frac{2n-1}{4\pi}}\sqrt{\frac{(n-1-|m|)!}{(n-1+|m|)!}}P_{n-1}^{|m|}(\cos\theta) = 0, \end{aligned} \tag{3.12}$$

$$\begin{aligned} &\sum_{m=-n}^{n} \mathrm{i}^{n+1} m a_n^m \frac{\sqrt{\lambda}^n}{(2n+1)!!}\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|m|)!}{(n+|m|)!}}P_n^{|m|}(\cos\theta)e^{\mathrm{i}m\alpha\cdot\pi} \\ &+\eta_2\sin\theta\sum_{m=-(n-1)}^{n-1} \mathrm{i}^{n-1} a_{n-1}^m \frac{\sqrt{\lambda}^{n-1}}{(2n-1)!!}\sqrt{\frac{2n-1}{4\pi}}\sqrt{\frac{(n-1-|m|)!}{(n-1+|m|)!}}P_{n-1}^{|m|}(\cos\theta)e^{\mathrm{i}m\alpha\cdot\pi} = 0, \end{aligned} \tag{3.13}$$

and

$$\sum_{m=-n}^{n} \mathrm{i}^{n} a_n^m \frac{\sqrt{\lambda}^n}{(2n+1)!!}\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|m|)!}{(n+|m|)!}}P_n^{|m|}(\cos\theta_1) = 0. \tag{3.14}$$

Utilizing the assumption $a_{n-1}^m = 0$ for $n = 0, \pm 1, \pm 2, \cdots, \pm(n-1)$ in (3.12) and (3.13), we know from the orthogonality condition in Lemma 2.4 that for $m \in \mathbb{N}_+$, a_n^m satisfies

$$a_n^m - a_n^{-m} = 0, \quad a_n^m e^{\mathrm{i}m\alpha\cdot\pi} - a_n^{-m}e^{-\mathrm{i}m\alpha\cdot\pi} = 0. \tag{3.15}$$

Therefore, if $\alpha \neq \frac{k}{m}$, $k = 1, 2, \cdots, m-1$, and by virtue of Remark 2.8 we can derive that $a_n^m = 0$ for $m = \pm 1, \pm 2, \cdots, \pm n$. Now we are in a position to show that $a_n^0 = 0$. Indeed, substituting $a_n^m = 0$, $m = \pm 1, \pm 2, \cdots, \pm n$ into (3.14), we can obtain that if $P_n^0(\cos\theta_1) \neq 0$, then $a_n^0 = 0$, which completes the proof of Theorem 3.1. □

In the above proof of Theorem 3.1, we have analyzed only the condition $u|_{\vec{a}} = 0$ for illustration. For the condition $u|_{\vec{b}} = 0$, we give the discussion in the following remark.

Remark 3.2. In the proof of Theorem 3.1, if we use (3.7) instead of (3.6), combining with (3.4) and (3.5), to consider the coefficient sof r^n, $n \in \mathbb{N}$, then (3.10) becomes

$$\sum_{m=-1}^{1} \mathrm{i} a_1^m \frac{\sqrt{\lambda}}{3!!}\sqrt{\frac{3}{4\pi}}\sqrt{\frac{(1-|m|)!}{(1+|m|)!}}P_1^{|m|}(\cos\theta_2)e^{\mathrm{i}m\alpha\cdot\pi} = 0. \tag{3.16}$$

Since we know $a_1^{\pm 1} = 0$ by (3.8) and (3.9), we can obtain from (3.16) that if $P_1^0(\cos\theta_2) \neq 0$, then $a_1^0 = 0$. By induction, in order to study a_n^0, we replace (3.14) by

$$\sum_{m=-n}^{n} \mathrm{i}^{n} a_n^m \frac{\sqrt{\lambda}^n}{(2n+1)!!}\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|m|)!}{(n+|m|)!}}P_n^{|m|}(\cos\theta_2)e^{\mathrm{i}m\alpha\cdot\pi} = 0. \tag{3.17}$$

Substituting $a_n^m = 0$, $m = \pm 1, \pm 2, \cdots, \pm n$, which is derived from (3.15) to (3.17), we can deduce that if $P_n^0(\cos\theta_2) \neq 0$, then $a_n^0 = 0$.

Hence, from the above discussions we know that it is actually equivalent to consider $u|_{\vec{a}} = 0$ or $u|_{\vec{b}} = 0$ in the proof of Theorem 3.1. Therefore, in our subsequent study, we shall only prove under the condition with respect to $\vec{a}$.

In Theorem 3.1, we have considered the case that $\Pi_3 \in \mathcal{N}_\Omega^\lambda$ is a nodal plane. Next, we study a more complicated case that $\Pi_3 \in \mathcal{M}_\Omega^\lambda$ is a generalized singular plane.

Theorem 3.3. *Let u be a Laplacian eigenfunction to* (1.1). *Consider a vertex corner $\mathcal{V}(\{\Pi_\ell\}_{\ell=1}^3, \mathbf{0}) \Subset \Omega$ with $\Pi_\ell \in \mathcal{M}_\Omega^\lambda$, $\ell = 1,2,3$ and $\angle(\Pi_1, \Pi_2) = \phi = \alpha\cdot\pi$, $\alpha \in (0,1)$. Assume that $\Pi_3 = \operatorname{span}\{\vec{a}, \vec{b}\}$, where $\vec{a} = (r, \theta_1, 0) \in \Pi_1 \cap \Pi_3$ and $\vec{b} = (r, \theta_2, \alpha\cdot\pi) \in \Pi_2 \cap \Pi_3$ for $r > 0$, $\alpha \in (0,1)$, and fixed $\theta_1 \in (0,\pi)$ and $\theta_2 \in (0,\pi)$ in the spherical coordinate system. If for an $N \in \mathbb{N}$, $N \geq 3$, it holds that*

$$u(\mathbf{0}) = 0,\ P_p^1(\cos\theta_i) \neq 0, i = 1 \text{ or } 2,\ \text{and } \alpha \neq \frac{q}{p}, p = 1,2,\cdots,N-1,\ q = 1,2,\cdots,p-1, \tag{3.18}$$

where P_p^1 is the associated Legendre polynomial, then the vanishing order of u at $\mathbf{0}$ generated by the intersection of the two planes Π_1 and Π_2 is at least order N.

Proof. Since Π_i, $i = 1,2,3$, are three generalized singular planes, we have

$$\frac{\partial u}{\partial \nu} + \eta_1 u\Big|_{\Pi_1} = 0, \quad \frac{\partial u}{\partial \nu} + \eta_2 u\Big|_{\Pi_2} = 0 \quad \text{and} \quad \frac{\partial u}{\partial \nu} + \eta_3 u\Big|_{\Pi_3} = 0.$$

From Theorem 3.1, we have already known that u satisfies (3.4) and (3.5) on Π_1 and Π_2 respectively. Besides, by Remark 3.2, we can obtain that

$$\frac{\partial u}{\partial \nu} + \eta_3 u\Big|_{\vec{a}} = 0. \tag{3.19}$$

Since $\Pi_3 = \operatorname{span}\{\vec{a}, \vec{b}\}$, which implies that $\nu = \vec{b} \times \vec{a} = (\sin\theta_2 \sin(\alpha\cdot\pi)\cos\theta_1, \sin\theta_1\cos\theta_2 - \sin\theta_2\cos(\alpha\cdot\pi)\cos\theta_1, -\sin\theta_1\sin\theta_2\sin(\alpha\cdot\pi))^{\mathrm{T}}$, we know that (3.19) can be written as

$$\begin{aligned}\frac{\partial u}{\partial \nu} + \eta_3 u\Big|_{\vec{a}} &= \frac{1}{r}\frac{\partial u}{\partial \theta}\sin\theta_2\sin(\alpha\cdot\pi) + \frac{1}{r\sin\theta_1}\frac{\partial u}{\partial \phi} \\ &\cdot (\sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1\cos\alpha\cdot\pi) + \eta_3 u\Big|_{\theta=\theta_1,\phi=0} = 0.\end{aligned} \tag{3.20}$$

By Lemma 2.3, multiplying $r\sin\theta_1$ on the both sides of (3.20), the equation can be simplified to

$$\begin{aligned}&\sin\theta_1\sin\theta_2\sin(\alpha\cdot\pi)\sum_{n=0}^{\infty}\sum_{m=-n}^{n} \mathrm{i}^n a_n^m j_n(\sqrt{\lambda}r)\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|m|)!}{(n+|m|)!}}\frac{dP_n^{|m|}(\cos\theta)}{d\theta}\Big|_{\theta=\theta_1} \\ &+ (\sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1\cos\alpha\cdot\pi)\sum_{n=0}^{\infty}\sum_{m=-n}^{n} \mathrm{i}^{n+1} m a_n^m j_n(\sqrt{\lambda}r)\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|m|)!}{(n+|m|)!}} \\ &\cdot P_n^{|m|}(\cos\theta_1) + \eta_3\sin\theta_1 r\sum_{n=0}^{\infty}\sum_{m=-n}^{n} \mathrm{i}^n a_n^m j_n(\sqrt{\lambda}r)\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|m|)!}{(n+|m|)!}}P_n^{|m|}(\cos\theta_1) = 0.\end{aligned} \tag{3.21}$$

Since $u(\mathbf{0}) = 0$, we know that $a_0^0 = 0$. Combining (3.4), (3.5) with (3.21), the corresponding coefficients of r respectively fulfil that

$$\sum_{m=-1}^{1} m a_1^m \frac{\sqrt{\lambda}}{3!!}\sqrt{\frac{3}{4\pi}}\sqrt{\frac{(1-|m|)!}{(1+|m|)!}}P_1^{|m|}(\cos\theta) + \eta_1\sin\theta a_0^0\sqrt{\frac{1}{4\pi}}P_0^0(\cos\theta) = 0, \tag{3.22}$$

$$\sum_{m=-1}^{1} m a_1^m \frac{\sqrt{\lambda}}{3!!}\sqrt{\frac{3}{4\pi}}\sqrt{\frac{(1-|m|)!}{(1+|m|)!}}P_1^{|m|}(\cos\theta)e^{\mathrm{i}m\alpha\cdot\pi} - \eta_2\sin\theta\sum_{m=-1}^{1} a_0^0\sqrt{\frac{1}{4\pi}}P_0^0(\cos\theta)e^{\mathrm{i}m\alpha\cdot\pi} = 0, \tag{3.23}$$

and

$$
\begin{aligned}
&\sin\theta_1\sin\theta_2\sin(\alpha\cdot\pi)\sum_{m=-1}^{1}\mathrm{i}a_1^m\frac{\sqrt{\lambda}}{3!!}\sqrt{\frac{3}{4\pi}}\sqrt{\frac{(1-|m|)!}{(1+|m|)!}}\frac{dP_1^{|m|}(\cos\theta)}{d\theta}\Big|_{\theta=\theta_1}\\
&-(\sin\theta_1\cos\theta_2-\sin\theta_2\cos\theta_1\cos(\alpha\cdot\pi))\sum_{m=-1}^{1}ma_1^m\frac{\sqrt{\lambda}}{3!!}\sqrt{\frac{3}{4\pi}}\sqrt{\frac{(1-|m|)!}{(1+|m|)!}}P_1^{|m|}(\cos\theta_1)\\
&+\eta_3\sin\theta_1 a_0^0\sqrt{\frac{1}{4\pi}}P_0^0(\cos\theta_1)=0. \qquad (3.24)
\end{aligned}
$$

Substituting $a_0^0=0$ into (3.22) and (3.23), utilizing the orthogonality condition we can derive

$$
a_1^1-a_1^{-1}=0,\quad a_1^1e^{\mathrm{i}\alpha\cdot\pi}-a_1^{-1}e^{-\mathrm{i}\alpha\cdot\pi}=0, \qquad (3.25)
$$

which yields $a_1^{\pm1}=0$ from the fact that $\alpha\in(0,1)$. In addition, taking $a_0^0=a_1^{\pm1}=0$ in (3.24), we have

$$
\sin\theta_1\sin\theta_2\sin(\alpha\cdot\pi)\mathrm{i}a_1^0\frac{\sqrt{\lambda}}{3!!}\sqrt{\frac{3}{4\pi}}(-P_1^1(\cos\theta_1))=0. \qquad (3.26)
$$

Hence, by the assumptions on θ_1,θ_2 and α, we can obtain that $a_1^0=0$ if $P_1^1(\cos\theta_1)\neq0$.

Proving by induction, we assume that $a_{n-1}^m=0$ for $m=0,\pm1,\pm2,\cdots\pm(n-1)$. Then considering the coefficients of r^n in (3.4), (3.5) and (3.21) accordingly, we know that there hold (3.12), (3.13) and also

$$
\begin{aligned}
&\sin\theta_1\sin\theta_2\sin(\alpha\cdot\pi)\sum_{m=-n}^{n}\mathrm{i}^na_n^m\frac{\sqrt{\lambda}^n}{(2n+1)!!}\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|m|)!}{(n+|m|)!}}\frac{dP_n^{|m|}(\cos\theta)}{d\theta}\Big|_{\theta=\theta_1}\\
&+(\sin\theta_1\cos\theta_2-\sin\theta_2\cos\theta_1\cos\alpha\cdot\pi)\sum_{m=-n}^{n}\mathrm{i}^{n+1}ma_n^m\frac{\sqrt{\lambda}^n}{(2n+1)!!}\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|m|)!}{(n+|m|)!}}\\
&\cdot P_n^{|m|}(\cos\theta_1)+\eta_3\sin\theta_1\sum_{m=-(n-1)}^{n-1}\mathrm{i}^{n-1}a_{n-1}^m\frac{\sqrt{\lambda}^{n-1}}{(2n-1)!!}\sqrt{\frac{2n-1}{4\pi}}\sqrt{\frac{(n-1-|m|)!}{(n-1+|m|)!}}P_{n-1}^{|m|}(\cos\theta_1)=0.
\end{aligned}
$$

Using the assumption that $a_{n-1}^m=0$, $m=0,\pm1,\pm2,\cdots\pm(n-1)$ in (3.12) and (3.13), similar to Theorem 3.3, we can obtain that if $\alpha\neq\frac{k}{m}$, $k=1,2,\cdots,m-1$, then $a_n^m=0$ for $m=\pm1,\pm2,\cdots\pm n$. Therefore, we can deduce from the last relation above that

$$
\sin\theta_1\sin\theta_2\sin(\alpha\cdot\pi)\mathrm{i}^na_n^0\frac{\sqrt{\lambda}^n}{(2n+1)!!}\sqrt{\frac{2n+1}{4\pi}}(-P_n^1(\cos\theta_1))=0, \qquad (3.27)
$$

which indicates that $a_n^0=0$ if $P_n^1(\cos\theta_1)\neq0$, hence completes the proof of Theorem 3.3. □

Remark 3.4. Following a similar argument in Theorem 3.3, if we take into account the condition $\frac{\partial u}{\partial\nu}+\eta_3u\Big|_{\vec b}\equiv0$ on Π_3, then we can derive similar results with respect to θ_2 instead of θ_1.

Remark 3.5. By direct verifications in the proof of Theorem 3.3, one can show that either of the boundary parameters η_ℓ, $\ell=1,2,3$, can be taken to be zero. That means, the generalized singular planes in Theorem 3.1 can be replaced by singular planes, and the vanishing results still hold.

In Theorems 3.1 and 3.3, we have considered the vanishing properties at a vertex corner that is intersected by three planes ($n=3$). In fact, the similar arguments work for the case that $n>3$, in which the third plane no longer intersects with Π_1 or Π_2. Without loss of generality, we denote the third plane to be discussed by $\Pi_j=\mathrm{span}\{\overrightarrow{OA_j},\overrightarrow{OA_{j+1}}\}$, where $3\leq j\leq n$ and if $j=n$, we assume that $A_{n+1}:=A_1$. Let Π_1 coincide with the (x_1,x_3)-plane, Π_2 possesses a dihedral angle $\alpha\cdot\pi$ away from Π_1 in the anti-clockwise direction and $\overrightarrow{OA_2}$ lies on the x_3-axis; see Figure 4 for a schematic illustration.

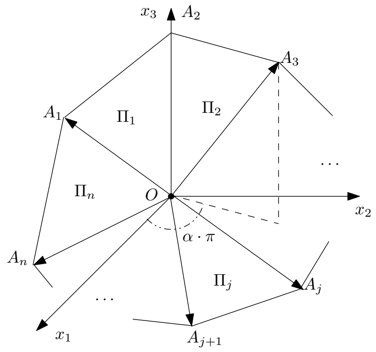

FIGURE 4. Schematic illustration of a vertex corner that is intersected by Π_1, Π_2, $\cdots$, Π_n with $n > 3$.

Theorem 3.6. *Let u be a Laplacian eigenfunction to (1.1). Consider a vertex corner $\mathcal{V}(\{\Pi_\ell\}_{\ell=1}^n, \mathbf{0}) \in \Omega$ as described above with $\Pi_\ell \in \mathcal{M}_\Omega^\lambda$, $\ell = 1, 2$, $\angle(\Pi_1, \Pi_2) = \phi = \alpha \cdot \pi$, $\alpha \in (0, 1)$, and $\Pi_j \in \mathcal{N}_\Omega^\lambda$. Assume that $\Pi_j = \mathrm{span}\{\overrightarrow{OA_j}, \overrightarrow{OA_{j+1}}\}$, where $\overrightarrow{OA_j} = (r, \theta_j, \phi_j)$ and $\overrightarrow{OA_{j+1}} = (r, \theta_{j+1}, \phi_{j+1})$ for $r > 0$, $\theta_j, \theta_{j+1} \in (0, \pi)$, and $\phi_j, \phi_{j+1} \in (0, 2\pi)$ such that $0 < \phi_{j+1} - \phi_j < \pi$ in the spherical coordinate system. If for an $N \in \mathbb{N}$, $N \geq 3$, it holds that*

$$P_p^0(\cos\theta_\tau) \neq 0, \tau = j \text{ or } j+1, \text{ and } \alpha \neq \frac{q}{p}, p = 1, 2, \cdots, N-1, \ q = 1, 2, \cdots, p-1, \tag{3.28}$$

where P_p^0 is the associated Legendre polynomial, then the vanishing order of u at $\mathbf{0}$ generated by the intersection of the two planes Π_1 and Π_2 is at least order N.

Proof. Since Π_1 and Π_2 are two generalized singular planes , we can derive (3.4) and (3.5) immediately. Considering Π_j, we know that $u|_{\Pi_j} = 0$, which indicates that $u|_{\overrightarrow{OA_j}} \equiv 0$ and $u|_{\overrightarrow{OA_{j+1}}} \equiv 0$. By Remark 3.2, it suffices to analyze $u|_{\overrightarrow{OA_j}} \equiv 0$ as follows:

$$u|_{\overrightarrow{OA_j}} = 4\pi \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \mathrm{i}^n a_n^m j_n(\sqrt{\lambda} r) \sqrt{\frac{2n+1}{4\pi}} \sqrt{\frac{(n-|m|)!}{(n+|m|)!}} P_n^{|m|}(\cos\theta_j) e^{\mathrm{i}m\phi_j} = 0. \tag{3.29}$$

Taking $m = n = 0$ in (3.29) we have $4\pi a_0^0 \sqrt{\frac{1}{4\pi}} P_0^0(\cos\theta_j) = 0$, where we can derive $a_0^0 = 0$ since $P_0^0 \equiv 1$. Thus from (3.4), (3.5) and (3.29), we know that the coefficients of r satisfies (3.11) and thus $a_1^{\pm 1} = 0$. Moreover, we have

$$\sum_{m=-1}^{1} \mathrm{i} a_1^m \frac{\sqrt{\lambda}}{3!!} \sqrt{\frac{3}{4\pi}} \sqrt{\frac{(1-|m|)!}{(1+|m|)!}} P_1^{|m|}(\cos\theta_j) e^{\mathrm{i}m\phi_j} = 0, \tag{3.30}$$

which can be further simplified as $a_1^0 P_1^0(\cos\theta_j) = 0$ after substituting $a_1^{\pm 1} = 0$ into (3.30). Hence, it is easy to see that $a_1^0 = 0$ if $P_1^0(\cos\theta_j) \neq 0$.

By induction, we assume that $a_{n-1}^m = 0$ for $m = 0, \pm 1, \pm 2, \cdots \pm (n-1)$. Considering the coefficients of r^n, we can obtain (3.12) and (3.13) which induce (3.15) as well as the equation

$$\sum_{m=-n}^{n} \mathrm{i}^n a_n^m \frac{\sqrt{\lambda}^n}{(2n+1)!!} \sqrt{\frac{2n+1}{4\pi}} \sqrt{\frac{(n-|m|)!}{(n+|m|)!}} P_n^{|m|}(\cos\theta_j) e^{\mathrm{i}m\phi_j} = 0. \tag{3.31}$$

Since we have already known that if $\alpha \neq \frac{k}{m}$, $k = 1, 2, \cdots - m - 1$, then $a_n^m = 0$ for $m = \pm 1, \pm 2, \cdots, \pm n$ from (3.15). Substituting this result into (3.31), we can deduce $a_n^0 P_n^0(\cos\theta_j) = 0$. Therefore, we know that $a_n^0 = 0$ if $P_n^0(\cos\theta_j) \neq 0$. Similarly, if we utilize the condition $u|_{\overrightarrow{OA_{j+1}}} \equiv 0$, then the same argument and results work for θ_{j+1}, which completes our proof. $\square$

We proceed to consider the case that Π_j is a generalized singular plane instead of a nodal plane as in Theorem 3.6.

Theorem 3.7. *Let u be a Laplacian eigenfunction to* (1.1). *Consider a vertex corner $\mathcal{V}(\{\Pi_\ell\}_{\ell=1}^n, \mathbf{0}) \Subset \Omega$ with $\Pi_\ell \in \mathcal{M}_\Omega^\lambda$, $\ell = 1, 2$, $\angle(\Pi_1, \Pi_2) = \phi = \alpha \cdot \pi$, $\alpha \in (0,1)$, and $\Pi_j \in \mathcal{M}_\Omega^\lambda$. Assume that $\Pi_j = \mathrm{span}\{\overrightarrow{OA_j}, \overrightarrow{OA_{j+1}}\}$, where $\overrightarrow{OA_j} = (r, \theta_j, \phi_j)$ and $\overrightarrow{OA_{j+1}} = (r, \theta_{j+1}, \phi_{j+1})$ for $r > 0$, $\theta_j, \theta_{j+1} \in (0,\pi)$, and $\phi_j, \phi_{j+1} \in (0, 2\pi)$ such that $0 < \phi_{j+1} - \phi_j < \pi$ in the spherical coordinate system. If for an $N \in \mathbb{N}$, $N \geq 3$, there holds*

$$u(\mathbf{0}) = 0, \quad P_p^1(\cos\theta_\tau) \neq 0, \tau = j \text{ or } j+1, \text{ and } \alpha \neq \frac{q}{p}, \tag{3.32}$$

where $p = 1, 2, \cdots, N-1$, $q = 1, 2, \cdots, p-1$ and P_p^1 is the associated Legendre polynomial, then the vanishing order of u at $\mathbf{0}$ generated by the intersection of the two planes Π_1 and Π_2 is at least order N.

Proof. From Theorem 3.3 and the fact that Π_1 and Π_2 are two generalized singular planes, we know u fulfils (3.4) and (3.5). Now consider Π_j, there holds $\frac{\partial u}{\partial \nu} + \eta_j u = 0$ on Π_j. Since $\Pi_j = \mathrm{span}\{\overrightarrow{OA_j}, \overrightarrow{OA_{j+1}}\}$, we have $\frac{\partial u}{\partial \nu} + \eta_j u\Big|_{\overrightarrow{OA_j}} = 0$ and

$$\nu = \overrightarrow{OA_j} \times \overrightarrow{OA_{j+1}} = \begin{pmatrix} \sin\theta_j \sin\phi_j \cos\theta_{j+1} - \sin\theta_{j+1}\sin\phi_{j+1}\cos\theta_j \\ -\sin\theta_j\cos\phi_j\cos\theta_{j+1} + \sin\theta_{j+1}\cos\phi_{j+1}\cos\theta_j \\ \sin\theta_j\cos\phi_j\sin\theta_{j+1}\sin\phi_{j+1} - \sin\theta_{j+1}\cos\phi_{j+1}\sin\theta_j\sin\phi_j \end{pmatrix}.$$

Combining with Lemma 2.3, we can obtain by direct computations that

$$\begin{aligned} &\frac{\partial u}{\partial \nu} + \eta_j u\Big|_{\overrightarrow{OA_j}} = \frac{1}{r\sin\theta_j}\frac{\partial u}{\partial \phi}(\sin\theta_{j+1}\cos\theta_j\cos(\phi_j - \phi_{j+1}) - \sin\theta_j\cos\theta_{j+1}) \\ &+ \frac{1}{r}\frac{\partial u}{\partial \theta}\sin\theta_{j+1}\sin(\phi_j - \phi_{j+1}) + \eta_j u\Big|_{\theta=\theta_j, \phi=\phi_j} = 0. \end{aligned} \tag{3.33}$$

Since $\theta_j \in (0,\pi)$, multiplying $r\sin\theta_j$ on the both sides of (3.33), we can deduce that

$$\begin{aligned} &(\sin\theta_{j+1}\cos\theta_j\cos(\phi_j - \phi_{j+1}) - \sin\theta_j\cos\theta_{j+1}) \sum_{n=0}^\infty \sum_{m=-n}^n \mathrm{i}^{n+1} m a_n^m j_n(\sqrt{\lambda} r)\sqrt{\frac{2n+1}{4\pi}} \\ &\cdot \sqrt{\frac{(n-|m|)!}{(n+|m|)!}} P_n^{|m|}(\cos\theta_j) e^{\mathrm{i}m\phi_j} + \sin\theta_j\sin\theta_{j+1}\sin(\phi_j - \phi_{j+1}) \sum_{n=0}^\infty\sum_{m=-n}^n \mathrm{i}^n a_n^m j_n(\sqrt{\lambda} r) \\ &\cdot \sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|m|)!}{(n+|m|)!}} \frac{dP_n^{|m|}(\cos\theta)}{d\theta}\Big|_{\theta=\theta_j} e^{\mathrm{i}m\phi_j} + \eta_j \sin\theta_j r \sum_{n=0}^\infty\sum_{m=-n}^n \mathrm{i}^n a_n^m j_n(\sqrt{\lambda} r) \\ &\cdot \sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|m|)!}{(n+|m|)!}} P_n^{|m|}(\cos\theta_j) e^{\mathrm{i}m\phi_j} = 0. \end{aligned} \tag{3.34}$$

Since $u(\mathbf{0}) = 0$, we have $a_0^0 = 0$. Considering the coefficients with respect to r in (3.4), (3.5) and (3.34), we know that $a_1^{\pm 1}$ fulfills (3.11) which induces that $a_1^{\pm 1} = 0$ since $\alpha \in (0,1)$. Moreover, it is easy to see from (3.34) that

$$\sin\theta_j\sin\theta_{j+1}\sin(\phi_j - \phi_{j+1})\mathrm{i} a_1^0 \frac{\sqrt{\lambda}}{3!!}\sqrt{\frac{3}{4\pi}}(-P_1^1(\cos\theta_j)) = 0.$$

Since $\theta_j, \theta_{j+1} \in (0,\pi)$ and $0 < \phi_j - \phi_{j+1} < \pi$, we know $a_1^0 = 0$ if $P_1^1(\cos\theta_j) \neq 0$.

Similarly, we assume that $a_{n-1}^m = 0$, $m = 0, \pm1, \pm2, \cdots, \pm(n-1)$. Then combining with Theorem 3.3, we know that a_n^m satisfies (3.12), (3.13) and

$$
\begin{aligned}
&(\sin\theta_{j+1}\cos\theta_j\cos(\phi_j-\phi_{j+1})-\sin\theta_j\cos\theta_{j+1})\sum_{m=-n}^{n}\mathrm{i}^{n+1}ma_n^m\frac{\sqrt{\lambda}^n}{(2n+1)!!}\sqrt{\frac{2n+1}{4\pi}}\\
&\cdot\sqrt{\frac{(n+|m|)!}{(n-|m|)!}}P_n^{|m|}(\cos\theta_j)e^{\mathrm{i}m\phi_j}+\sin\theta_j\sin\theta_{j+1}\sin(\phi_j-\phi_{j+1})\sum_{m=-n}^{n}\mathrm{i}^na_n^m\frac{\sqrt{\lambda}^n}{(2n+1)!!}\\
&\cdot\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|m|)!}{(n+|m|)!}}\frac{dP_n^{|m|}(\cos\theta)}{d\theta}\Big|_{\theta=\theta_j}e^{\mathrm{i}m\phi_j}+\eta_j\sin\theta_j\sum_{m=-(n-1)}^{n-1}\mathrm{i}^{n-1}a_{n-1}^m\frac{\sqrt{\lambda}^{n-1}}{(2n-1)!!}\\
&\cdot\sqrt{\frac{2n-1}{4\pi}}\sqrt{\frac{(n-1-|m|)!}{(n-1+|m|)!}}P_{n-1}^{|m|}(\cos\theta_j)e^{\mathrm{i}m\phi_j}=0.
\end{aligned}
\tag{3.35}
$$

In (3.12) and (3.13), utilizing the assumption $a_{n-1}^m = 0$ for $m = 0, \pm1, \pm2, \cdots \pm(n-1)$, we know that if $\alpha \neq \frac{k}{m}$, $k = 1, 2, \cdots, m$, then $a_n^m = 0$, $\pm1, \pm2, \cdots, \pm n$. Hence (3.35) can be simplified to

$$
\sin\theta_j\sin\theta_{j+1}\sin(\phi_j-\phi_{j+1})\mathrm{i}^na_n^0\frac{\sqrt{\lambda}^n}{(2n+1)!!}\sqrt{\frac{2n+1}{4\pi}}(-P_n^1(\cos\theta_j))=0.
$$

Since $\theta_j, \theta_{j+1} \in (0, \pi)$ and $0 < \phi_j - \phi_{j+1} < \pi$, we can derive that $a_n^0 = 0$ if $P_n^1(\cos\theta_j) \neq 0$. The same results work for θ_{j+1} if we take into account that $\frac{\partial u}{\partial \nu} + \eta_j u\Big|_{\overrightarrow{OA_{j+1}}} = 0$. This completes the proof of Theorem 3.7. □

Remark 3.8. Similarly to Remark 3.5, one can have by direct verifications that the vanishing results in Theorem 3.7 still hold if any of the generalized singular planes involved is replaced by a singular plane.

4. Irrational intersections and infinite vanishing orders

From the results derived in Sections 2 and 3, one can identify that the vanishing order of the eigenfunction u at an edge or a vertex corner relies on the degree of the dihedral angle of the underlying corner. In the following two definitions, we first introduce the irrational and rational edge or vertex corner. Then, based on the results in Sections 2 and 3, we show that the vanishing order of the eigenfunction at an irrational edge or vertex corner is generically infinity and hence it vanishes identically in Ω.

Definition 4.1. Let $\mathcal{E}(\Pi_1, \Pi_2, \boldsymbol{l})$ be an edge corner defined in Definition 1.2 and the corresponding dihedral angle of Π_1 and Π_2 is denoted by $\phi = \alpha \cdot \pi$, $\alpha \in (0, 1)$. If ϕ is an irrational dihedral angle, namely, α is an irrational number, then $\mathcal{E}(\Pi_1, \Pi_2, \boldsymbol{l})$ is said to be an *irrational* edge corner. Otherwise it is said to be a *rational* edge corner. For a rational edge corner $\mathcal{E}(\Pi_1, \Pi_2, \boldsymbol{l})$, the dihedral angle between Π_1 and Π_2 is called the *rational degree* of $\mathcal{E}(\Pi_1, \Pi_2, \boldsymbol{l})$.

Definition 4.2. Let $\mathcal{V}(\{\Pi_\ell\}_{\ell=1}^n, \mathbf{x}_0)$ be a vertex corner defined in Definition 1.3, where $n \in \mathbb{N}$ and $n \geq 3$. It is clear that $\mathcal{V}(\{\Pi_\ell\}_{\ell=1}^n, \mathbf{x}_0)$ is composed of the following n edge corners

$$
\mathcal{E}_\ell := \mathcal{E}(\Pi_\ell, \Pi_{\ell+1}, \boldsymbol{l}_\ell), \quad \mathcal{E}_n := \mathcal{E}(\Pi_n, \Pi_1, \boldsymbol{l}_n), \quad \Pi_{n+1} := \Pi_1, \quad \ell = 1, 2, \ldots, n-1,
$$

where $\boldsymbol{l}_\ell$ is the line segment of $\Pi_\ell \cap \Pi_{\ell+1}$ and $\boldsymbol{l}_n$ is a line segment of $\Pi_n \cap \Pi_1$, respectively. Denote

$$
\begin{aligned}
I_{\mathsf{IR}} &= \{\ell \in \mathbb{N} \mid 1 \leq \ell \leq n, \quad \mathcal{E}_\ell \text{ is an irrational edge corner}\},\\
I_{\mathsf{R}} &= \{\ell \in \mathbb{N} \mid 1 \leq \ell \leq n, \quad \mathcal{E}_\ell \text{ is a rational edge corner}\}.
\end{aligned}
\tag{4.1}
$$

If $\#I_{\mathsf{IR}} \geq 1$, then $\mathcal{V}(\{\Pi_\ell\}_{\ell=1}^n, \mathbf{x}_0)$ is said to be an *irrational* vertex corner. If $\#I_{\mathsf{IR}} \equiv 0$, then $\mathcal{V}(\{\Pi_\ell\}_{\ell=1}^n, \mathbf{x}_0)$ is said to be a *rational* vertex corner. For a rational vertex corner $\mathcal{V}(\{\Pi_\ell\}_{\ell=1}^n, \mathbf{x}_0)$ composed of edge corners $\mathcal{E}_\ell := \mathcal{E}(\Pi_\ell, \Pi_{\ell+1}, \boldsymbol{l}_\ell)$, the largest degree of $\mathcal{E}_\ell\,(\ell = 1, \ldots, n)$ is referred to as the *rational degree* of $\mathcal{V}(\{\Pi_\ell\}_{\ell=1}^n, \mathbf{x}_0)$.

When an irrational edge corner $\mathcal{E}(\Pi_1,\Pi_2,\boldsymbol{l})$ is intersected by two nodal planes of u, we can derive the following result from Theorem 2.7.

Theorem 4.3. *Let u be a Laplacian eigenfunction to (1.1). Suppose that $\mathcal{E}(\Pi_1,\Pi_2,\boldsymbol{l})\Subset\Omega$ is an irrational edge corner with $\Pi_1,\Pi_2\in\mathcal{N}_\Omega^\lambda$. Then it holds that*

$$\mathrm{Vani}(u;\mathbf{0},\Pi_1,\Pi_2)=+\infty,\quad \mathbf{0}\in\boldsymbol{l}.$$

If the intersecting two planes of the irrational edge corner are either of the three types: a nodal plane, a singular plane or a generalized singular plane, namely for the general case, we have the irrational intersection results as shown below.

Theorem 4.4. *Let u be a Laplacian eigenfunction to (1.1). Suppose that $\mathcal{E}(\Pi_1,\Pi_2,\boldsymbol{l})\Subset\Omega$ is an irrational edge corner with $\Pi_1\in\mathcal{N}_\Omega^\lambda$ and $\Pi_2\in\mathcal{M}_\Omega^\lambda$. Then it holds that*

$$\mathrm{Vani}(u;\mathbf{0},\Pi_1,\Pi_2)=+\infty,\quad \mathbf{0}\in\boldsymbol{l}.$$

The same result can be derived for the case that $\eta\equiv 0$, which indicates that Π_2 is a singular plane. The detailed discussion can be found in Theorem 2.9.

The next theorem is concerned with the intersection of two generalized singular planes, which is a direct corollary of Theorem 2.11.

Theorem 4.5. *Let u be a Laplacian eigenfunction to (1.1). Suppose that $\mathcal{E}(\Pi_1,\Pi_2,\boldsymbol{l})\Subset\Omega$ is an irrational edge corner with $\Pi_\ell\in\mathcal{M}_\Omega^\lambda\,(\ell=1,2)$. If there exists a sufficiently small $\varepsilon>0$ such that*

$$u|_{B_\varepsilon(\mathbf{0})\cap\boldsymbol{l}}\equiv 0, \tag{4.2}$$

then it holds that

$$\mathrm{Vani}(u;\mathbf{0},\Pi_1,\Pi_2)=+\infty,\quad \mathbf{0}\in\boldsymbol{l}.$$

If $\eta_1=0$ or $\eta_2=0$, which indicates that either Π_1 or Π_2 becomes a singular plane, we can deduce the same vanishing property as Theorem 4.5. Moreover, if $\eta_1=\eta_2=0$, for the intersection of two singular planes, we can further obtain the explicit form of u as below.

Theorem 4.6. *Let u be a Laplacian eigenfunction to (1.1). Suppose that $\mathcal{E}(\Pi_1,\Pi_2,\boldsymbol{l})\Subset\Omega$ is an irrational edge corner and $\Pi_\ell\in\mathcal{S}_\Omega^\lambda\,(\ell=1,2)$. If (4.2) is satisfied, then it holds that*

$$\mathrm{Vani}(u;\mathbf{0},\Pi_1,\Pi_2)=+\infty,\quad \mathbf{0}\in\boldsymbol{l}. \tag{4.3}$$

Moreover, if $u|_{B_\varepsilon(\mathbf{0})\cap\boldsymbol{l}}\not\equiv 0$, then we have the following expansion of u in a neighborhood of the edge corner $\mathbf{0}$ in the polar coordinate system:

$$u(\mathbf{x})=4\pi\sum_{n=0}^{\infty}\mathrm{i}^n a_n^0 j_n(\sqrt{\lambda}r)Y_n^0(\theta,\phi), \tag{4.4}$$

where $Y_n^0(\theta,\phi)$ is the spherical harmonics and $j_n(t)$ is the n-th Bessel function.

Proof. By Theorem 2.11 and Remark 2.13, it is easy to verify that (4.3) holds under the generic condition (4.2). However, if (4.2) fails to be fulfilled, then we can not derive $a_n^0=0$ for $n=0,1,2,\cdots$.

Since $\left.\frac{\partial u}{\partial\nu}\right|_{\Pi_\ell}\equiv 0$, $\ell=1,2$, we can obtain by direct computation that

$$-\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\mathrm{i}^{n+1}ma_n^m j_n(\sqrt{\lambda}r)\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|m|)!}{(n+|m|)!}}P_n^{|m|}(\cos\theta)=0, \tag{4.5}$$

and

$$\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\mathrm{i}^{n+1}ma_n^m j_n(\sqrt{\lambda}r)\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|m|)!}{(n+|m|)!}}P_n^{|m|}(\cos\theta)e^{\mathrm{i}m\alpha\cdot\pi}=0, \tag{4.6}$$

on Π_1 and Π_2 respectively. By comparing the coefficients of r in (4.5) and (4.6), with the help of the orthogonality condition, we can still obtain that $a_1^{\pm1}=0$ since $\alpha\in(0,1)$ for the dihedral angle $\phi = \alpha \cdot \pi$. By induction, following a similar argument to the proof of Theorem 2.11, we can deduce that

$$-\sum_{m=-n}^{n} \mathrm{i}^{n+1} m a_n^m \frac{\sqrt{\lambda}^n}{(2n+1)!!}\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|m|)!}{(n+|m|)!}}P_n^{|m|}(\cos\theta) = 0, \tag{4.7}$$

and

$$\sum_{m=-n}^{n} \mathrm{i}^{n+1} m a_n^m \frac{\sqrt{\lambda}^n}{(2n+1)!!}\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|m|)!}{(n+|m|)!}}P_n^{|m|}(\cos\theta)e^{\mathrm{i}m\alpha\cdot\pi} = 0. \tag{4.8}$$

Therefore by Lemma 2.4, we see that $a_n^m = 0$ $(m = \pm 1, \pm 2, \cdots, \pm n)$ since the corresponding dihedral angle is irrational. Hence, we are able to obtain the explicit expression (4.4) around the edge corner $\mathbf{0}$. ☐

Based on the irrational intersection at an edge corner by two planes, we next consider the corresponding properties at a vertex corner which is intersected by n planes where $n \geq 3$.

Using Theorem 3.6 and Remark 3.8, we have the following results for an irrational vertex corner.

Theorem 4.7. *Let u be a Laplacian eigenfunction to* (1.1). *Consider an irrational vertex corner $\mathcal{V}(\{\Pi_\ell\}_{\ell=1}^n, \mathbf{0}) \Subset \Omega$, where the intersecting n planes $\Pi_1, \Pi_2, \cdots, \Pi_n$ could be either of the three types: a nodal plane, a singular plane or a generalized singular plane, $n \in \mathbb{N}$ and $n \geq 3$. Assume that for $i = 1, 2, \cdots, n$, $\Pi_i = \mathrm{span}\{\overrightarrow{OA_i}, \overrightarrow{OA_{i+1}}\}$, where $\overrightarrow{OA_i} = (r, \theta_i, \phi_i)$, $\overrightarrow{OA_{i+1}} = (r, \theta_{i+1}, \phi_{i+1})$ for $r > 0$, $\theta_i, \theta_{i+1} \in (0, \pi)$ and $\phi_i, \phi_{i+1} \in (0, 2\pi)$ such that $0 < \phi_{i+1} - \phi_i < \pi$ in the spherical coordinate system. Particularly when $i = n$, we denote $\Pi_{n+1} := \Pi_1$. Recall that I_{R} and I_{IR} are defined in* (4.1). *If one of the following conditions is fulfilled that*

(1) there exists an index $\ell_0 \in I_{\mathsf{IR}}$ such that $\Pi_{\ell_0} \in \mathcal{N}_\Omega^\lambda$ or $\Pi_{\ell_0+1} \in \mathcal{N}_\Omega^\lambda$;
(2) for any $\ell \in I_{\mathsf{IR}}$, if $\Pi_\ell, \Pi_{\ell+1} \in \{\mathcal{S}_\Omega^\lambda \cup \mathcal{M}_\Omega^\lambda\}$, $u(\mathbf{0}) = 0$ and for a fixed $\ell_0 \in I_{\mathsf{IR}}$ there exists an index $j \in \{1, \ldots, n\}$ such that the corresponding plane $\Pi_j = \mathrm{span}\{\overrightarrow{OA_j}, \overrightarrow{OA_{j+1}}\}$ satisfies $P_p^0(\cos\theta_\tau) \neq 0$ and $P_p^1(\cos\theta_\tau) \neq 0$ for all $p \in \mathbb{N}$, $\tau = j, j+1$, where P_p^0 and P_p^1 are the associated Legendre polynomials;

then there holds that $\mathrm{Vani}(u; \mathbf{0}) = +\infty$.

5. Vanishing at edge and vertex corners involving generalized singular planes with variable parameters

In Sections 2–4, whenever a generalized singular plane Π is concerned, the parameter η (cf. Definition 1.1) was assumed to be a constant. In this section, we remark that with some straightforward modifications, all the results derived in Sections 2–4 equally hold for the case that η is a (variable) analytic function on Π. To that end, we make the following crucial observation. In the sequel, for an analytic function f on Π with the following series representation,

$$f(x) = \sum_{\ell=0}^{\infty} a_\ell(\theta, \phi) r^\ell, \quad \mathbf{x} = (r, \theta, \phi) \in \Pi, \tag{5.1}$$

we define $\deg_{\mathrm{low}}(f) = N$ if $a_\ell = 0$ for $\ell = 0, 1, \ldots, N$ while $a_{N+1} \neq 0$. Next, we first assume that η is an analytic function of the form (5.1) on Π. Recall that u has the expansion (2.3) and assume that

$$a_n^m = 0 \quad \text{for} \quad n = 0, 1, \ldots, N \text{ and } m = \pm n, \pm(n-1), \ldots, \pm 1, 0. \tag{5.2}$$

Then by straightforward calculations, one has

$$\begin{aligned}
&\frac{\partial u}{\partial \nu}+\eta u\\
=&\frac{1}{r\sin\theta}4\pi\sum_{n=N}^{\infty}\sum_{m=-n}^{n}\mathrm{i}^{n+1}ma_n^m j_n(\sqrt{\lambda}r)\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|m|)!}{(n+|m|)!}}P_n^{|m|}(\cos\theta)e^{\mathrm{i}m\alpha\cdot\pi}\\
+&\sum_{\ell=0}^{\infty}a_\ell(\theta,\phi)r^\ell\cdot 4\pi\sum_{n=N}^{\infty}\sum_{m=-n}^{n}\mathrm{i}^n a_n^m j_n(\sqrt{\lambda}r)\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|m|)!}{(n+|m|)!}}P_n^{|m|}(\cos\theta)e^{\mathrm{i}m\alpha\cdot\pi}.
\end{aligned} \tag{5.3}$$

Using (2.6) and (5.3), it is straightforward to verify that $\deg_{\rm low}(\eta u)\geq N$, while the leading-order term for $\partial u/\partial\nu+\eta u$ is the r^{N-1}-term, completely determined by $\partial u/\partial \nu$. With such an observation, it is straightforward to show that all the results in Sections 2–4 hold for the case that η is of the form (5.1) on Π. For the general case that η is analytic on Π, since all of our mathematical arguments can actually be localized around the corner points, one can complete the proofs by using the fact that η has the series expansion (5.1) locally around the corner points.

6. Unique identifiability for inverse obstacle problems

In this section, we apply the results we have obtained in previous sections about the vanishing properties of an eigenfunction at a vertex corner to study a fundamental mathematical issue in inverse scattering problems, namely the unique identifiability of the inverse problem recovering the shape of some unknown objects by certain wave probing data. The inverse obstacle problem arises from many applications, such as those using radar, sonar and geophysical explorations.

Let $\Omega\subset\mathbb{R}^3$ be a bounded Lipschitz domain such that $\mathbb{R}^3\backslash\bar{\Omega}$ is connected. Let u^i be an incident field, and it is assumed in the subsequent analysis to be a plane wave of the form

$$u^i:=u^i(\mathbf{x};k,\mathbf{d})=e^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}},\quad \mathbf{x}\in\mathbb{R}^3,$$

where $k\in\mathbb{R}_+$ signifies the wavenumber and $\mathbf{d}\in\mathbb{S}^2$ denotes the incident direction. Physically speaking, u^i is the detecting wave field and Ω denotes an impenetrable obstacle which interrupts the propagation of the incident wave and generates the corresponding scattered wave field u^s. Define $u:=u^i+u^s$ to be the total wave field, then the forward scattering problem of this process can be described by the following system

$$\begin{cases}\Delta u+k^2u=0 & \text{in}\quad \mathbb{R}^3\backslash\overline{\Omega},\\ u=u^i+u^s & \text{in}\quad \mathbb{R}^3,\\ \mathscr{B}(u)=0 & \text{on}\quad \partial\Omega,\\ \lim\limits_{r\to\infty} r\left(\dfrac{\partial u^s}{\partial r}-\mathrm{i}ku^s\right)=0,\end{cases} \tag{6.1}$$

where the last equation is the Sommerfeld radiation condition that holds uniformly in $\hat{\mathbf{x}}:=\mathbf{x}/|\mathbf{x}|\in\mathbb{S}^2$. If $\mathscr{B}(u):=u$, the boundary condition is of Dirichlet type and Ω is said to be a sound-soft obstacle; if $\mathscr{B}(u):=\partial_\nu u$, the boundary condition is of Neumann type and Ω is said to be a sound-hard obstacle; if $\mathscr{B}(u):=\partial_\nu u+\eta u$, Ω becomes an impedance obstacle with Robin type boundary condition where ν denotes the exterior unit normal vector to $\partial\Omega$ and $\eta\in L^\infty(\partial\Omega)$ signifies the corresponding impedance boundary parameter. For unification of the notation, we write all these three types of boundary conditions as

$$\mathscr{B}(u):=\partial_\nu u+\eta u=0\quad\text{on }\partial\Omega, \tag{6.2}$$

where the cases that $\eta=\infty$ and $\eta=0$ stand for the Dirichlet and Neumann boundary conditions respectively.

The forward scattering problem (6.1) has been studied in [9, 25] and there exists a unique solution $u \in H^1_{loc}(\mathbb{R}^3\backslash\overline{\Omega})$ fulfilling the following expansion:

$$u^s(\mathbf{x}; k, \mathbf{d}) = \frac{e^{ikr}}{r} u_\infty(\hat{\mathbf{x}}; k, \mathbf{d}) + \mathcal{O}\left(\frac{1}{r^2}\right) \quad \text{as } r \to \infty, \tag{6.3}$$

where u_∞ is known as the associated far-field pattern or the scattering amplitude. The asymptotic form (6.3) holds uniformly with respect to all directions $\hat{\mathbf{x}} := \mathbf{x}/|\mathbf{x}| \in \mathbb{S}^2$.

The inverse obstacle scattering problem corresponding to (6.1) is to recover Ω (and η as well in the impedance case) by the knowledge of the far-field pattern $u_\infty(\hat{\mathbf{x}}; k, \mathbf{d})$. By introducing an operator $\mathcal{F}$ which sends the obstacle to the corresponding far-field pattern, defined by the forward scattering system (6.1), the aforementioned inverse problem can be formulated as

$$\mathcal{F}(\Omega, \eta) = u_\infty(\hat{\mathbf{x}}; k, \mathbf{d}). \tag{6.4}$$

It can be directly verified that the inverse problem (6.4) is nonlinear. The problem is also known as the Schiffer problem in the inverse scattering theory, which has a long and colorful history since 1960 by M. Schiffer's pioneering work [19]. It constitutes an open problem whether one can establish the one-to-one correspondence for (6.4) by a single far-field pattern or a finite number of far-field patterns (namely with a fixed pair of k and $\mathbf{d}$ or a finite number of the pairs k and d). We refer to a recent survey paper [10] by Colton and Kress for more discussions about the historical developments of this fundamental problem.

Some significant progresses have been made recently about the Schiffer problem when the unknown obstacles are of general polyhedral type in $\mathbb{R}^n$, $n \geq 2$. Uniqueness and stability results can be found in [1, 6–8, 13–15, 21–24] by using a finite number of far-field patterns. Particularly, the unique determination for impedance-type obstacles was studied in [24] for a partial solution to this fundamental problem. Very recently in [5], we have developed a completely new method that is applicable for sound-soft, sound-hard and also impedance type obstacles to provide a solution to the inverse obstacle problem in the two-dimensional space. We have shown that in a rather general scenario one can determine the convex hull of an impedance obstacle as well as its boundary parameter by at most two far-field patterns by utilizing this new approach. In this section, we apply the spectral results established in the previous sections to study this fundamental issue, namely to recover the obstacle and its surface impedance in $\mathbb{R}^3$. We shall first obtain some local uniqueness results for the inverse problem since the method developed here is completely local. Nevertheless, if the underlying obstacles are further assumed to be convex, the local uniqueness results imply the global uniqueness results. Moreover, we would like to point out that in deriving those unique determination results, we need to introduce certain restrictive conditions on the underlying polyhedral obstacles. One of such conditions is contained in the following admissibility definition.

Definition 6.1. Let $\Omega \subset \mathbb{R}^3$ be an open polyhedron associated with the generalized impedance boundary condition (6.2). Then Ω is said to be an *admissible polyhedral obstacle* if the following conditions are fulfilled:

- On each face of $\partial\Omega$, the surface impedance η is either a constant (possibly zero) or ∞.
- For any vertex of Ω that is intersected by n planes: $\Pi_1, \Pi_2, \cdots, \Pi_n$, $n \geq 3$, there exists a plane $\Pi_j := \{\overrightarrow{OA_j}, \overrightarrow{OA_{j+1}}\}$, where O denotes the vertex locating at the origin. $\overrightarrow{OA_\tau} = (r, \theta_\tau, \phi_\tau)$, for $r > 0$, $\theta_\tau \in (0, \pi)$ and $\phi_\tau \in (0, 2\pi)$ in spherical coordinate system such that $P_n^0(\cos\theta_\tau) \neq 0$ and $P_n^1(\cos\theta_\tau) \neq 0$, where $\tau = j, j+1$, $n \in \mathbb{N}$, P_n^0 and P_n^1 are the associated Legendre polynomials.

Remark 6.2. By noting the fact that $P_n^0(1) \equiv 1$, $P_n^1(1) \equiv 0$ when $\theta = 0$ for all $n \in \mathbb{N}$, and the continuity of the associated Legendre polynomials, one easily knows that there exists $\delta_0 > 0$ such that for any $\epsilon > 0$ and $\theta \in (0, \delta_0)$, $P_n^0(\cos\theta) \in (1-\epsilon, 1)$ and $P_n^1(\cos\theta) \in (0, \epsilon)$, which in turn imply the existence of θ_τ in Definition 6.1. Therefore, the definition of the aforementioned admissible polyhedral obstacle is well defined.

Remark 6.3. In view of Section 5, in Definition 6.1, the surface impedance η can be relaxed to that it is either an analytic function (possibly zero) or ∞; see also Remark 6.12 in what follows.

Throughout this section, we signify an admissible polyhedral obstacle as (Ω, η). Then we define the rational and irrational obstacle in $\mathbb{R}^3$ based on Definition 4.2 for the rational and irrational vertex corner of Ω.

Definition 6.4. Let (Ω, η) be an admissible polyhedral obstacle. If there exists a rational vertex corner, then it is said to be a *rational obstacle*. If all the vertex corners of Ω are irrational, then it is called an *irrational obstacle*. The smallest degree of the rational corner of Ω is referred to as the *rational degree* of Ω.

Definition 6.5. Ω is said to be an admissible complex polyhedral obstacle if it consists of finitely many admissible polyhedral obstacles. That is,

$$(\Omega, \eta) = \bigcup_{j=1}^{l} (\Omega_j, \eta_j),$$

where $l \in \mathbb{N}$ and each (Ω_j, η_j) is an admissible polyhedral obstacle. Here, we define

$$\eta = \sum_{j=1}^{l} \eta_j \chi_{\partial\Omega_j}.$$

Moreover, Ω is said to be irrational if all of its component polyhedral obstacles are irrational, otherwise it is said to be rational. For the latter case, the smallest degree among all the degrees of its rational components is defined to be the degree of the complex obstacle Ω.

Next, we give the unique determination result for an admissible complex irrational polyhedral obstacle by at most two far-field patterns.

Theorem 6.6. *Consider a fixed $k \in \mathbb{R}_+$, and two distinct incident directions $\mathbf{d}_1$ and $\mathbf{d}_2$ from $\mathbb{S}^2$. Let (Ω, η) and $(\widetilde{\Omega}, \widetilde{\eta})$ be two admissible complex irrational obstacles, with u_∞ and $\widetilde{u}_\infty$ being their corresponding far-field patterns and $\mathbf{G}$ being the unbounded connected component of $\mathbb{R}^3 \backslash \overline{(\Omega \cup \widetilde{\Omega})}$. If u_∞ and $\widetilde{u}_\infty$ are the same in the sense that*

$$u_\infty(\hat{\mathbf{x}}; k, \mathbf{d}_\ell) = \widetilde{u}_\infty(\hat{\mathbf{x}}; k, \mathbf{d}_\ell), \quad \textit{for} \ \ \ell = 1, 2 \ \textit{and all} \ \hat{\mathbf{x}} \in \mathbb{S}^2, \tag{6.5}$$

then $(\partial\Omega \backslash \partial\overline{\widetilde{\Omega}}) \bigcup (\partial\widetilde{\Omega} \backslash \partial\overline{\Omega})$ cannot possess a vertex corner on $\partial\mathbf{G}$. Moreover,

$$\eta = \widetilde{\eta} \quad \textit{on} \quad \partial\Omega \cap \partial\widetilde{\Omega}. \tag{6.6}$$

Proof. We prove the theorem by contradiction. Assume that $(\partial\Omega \backslash \partial\overline{\widetilde{\Omega}}) \bigcup (\partial\widetilde{\Omega} \backslash \partial\overline{\Omega})$ has a vertex corner $\mathbf{x}_c$ on $\partial\mathbf{G}$. Then, $\mathbf{x}_c$ is either located at Ω or $\widetilde{\Omega}$. Without loss of generality, we assume that $\mathbf{x}_c$ is a 3D vertex corner of $\widetilde{\Omega}$, which also indicates that $\mathbf{x}_c$ lies outside Ω. Let $h \in \mathbb{R}_+$ be sufficiently small such that $B_h(x_c) \Subset \mathbb{R}^2 \backslash \overline{\Omega}$, then we can suppose for $n \geq 3$ that

$$B_h(\mathbf{x}_c) \cap \partial\widetilde{\Omega} = \Pi_i, \quad i = 1, 2, \cdots, n,$$

where Π_i are the n planes lying on the n faces of $\widetilde{\Omega}$ that intersect at $\mathbf{x}_c$.

Recall that $\mathbf{G}$ is the unbounded connected component of $\mathbb{R}^3 \backslash \overline{(\Omega \cup \widetilde{\Omega})}$. By (6.5) and the Rellich theorem (cf. [9]), we know that

$$u(\mathbf{x}; k, \mathbf{d}_\ell) = \widetilde{u}(\mathbf{x}; k, \mathbf{d}_\ell), \quad x \in \mathbf{G}, \ \ell = 1, 2. \tag{6.7}$$

Since $\Pi_i \subset \partial\mathbf{G}$, $i = 1, 2, \cdots, n$, combining (6.7) with the generalized boundary condition (6.2) on $\partial\widetilde{\Omega}$, it is easy to obtain for $n \geq 3$ that

$$\partial_\nu u + \widetilde{\eta} u = \partial_\nu \widetilde{u} + \widetilde{\eta}\widetilde{u} = 0 \quad \text{on} \quad \Pi_i, \quad i = 1, 2, \cdots, n.$$

Furthermore, since $B_h(x_c) \Subset \mathbb{R}^2 \backslash \overline{\Omega}$, we have $-\Delta u = k^2 u$ in $B_h(\mathbf{x}_c)$. We divide our remaining proof into two separate cases.

Case 1. Suppose that either $u(\mathbf{x}_c; k, \mathbf{d}_1)$ or $u(\mathbf{x}_c; k, \mathbf{d}_2)$ is zero. Without loss of generality, we assume that $u(\mathbf{x}_c; k, \mathbf{d}_1) = 0$. By the assumption of the theorem that $\widetilde{\Omega}$ is an admissible irrational obstacle, we know that $\mathbf{x}_c$ is an irrational vertex corner of $\widetilde{\Omega}$, which also implies that there exist Π_{i_0} and Π_{i_0+1} such that the corresponding intersecting dihedral angle is irrational. Hence, by our results in Sections 3 and 4, we can immediately derive that

$$u(\mathbf{x}; k, \mathbf{d}_1) = 0 \quad \text{in} \quad B_h(\mathbf{x}_c),$$

which in turn yields by the analytic continuation that

$$u(\mathbf{x}; k, \mathbf{d}_1) = 0 \quad \text{in} \quad \mathbb{R}^3 \backslash \overline{\Omega}. \tag{6.8}$$

In particular, one has from (6.8) that

$$\lim_{|\mathbf{x}| \to \infty} |u(\mathbf{x}; k, \mathbf{d}_1)| = 0. \tag{6.9}$$

But this contradicts to the fact that follows from (6.3):

$$\lim_{|\mathbf{x}| \to \infty} |u(\mathbf{x}; k, \mathbf{d}_1)| = \lim_{|\mathbf{x}| \to \infty} \left| e^{\mathrm{i} k \mathbf{x} \cdot \mathbf{d}_1} + u^s(\mathbf{x}; k, \mathbf{d}_1) \right| = 1. \tag{6.10}$$

Case 2. Suppose that both $u(\mathbf{x}_c; k, \mathbf{d}_1) \neq 0$ and $u(\mathbf{x}_c; k, \mathbf{d}_2) \neq 0$. Set

$$\alpha_1 = u(\mathbf{x}_c; k, \mathbf{d}_2) \quad \text{and} \quad \alpha_2 = -u(\mathbf{x}_c; k, \mathbf{d}_1), \tag{6.11}$$

and

$$v(\mathbf{x}) = \alpha_1 u(\mathbf{x}; k, \mathbf{d}_1) + \alpha_2 u(\mathbf{x}; k, \mathbf{d}_2) \quad \forall x \in B_h(\mathbf{x}_c). \tag{6.12}$$

It is easy to verify for $n \geq 3$ that v fulfills

$$-\Delta v = k^2 v \quad \text{in} \quad B_h(\mathbf{x}_c) \quad \text{and} \quad \partial_\nu v + \widetilde{\eta} v = 0 \quad \text{on} \quad \Pi_i, \ i = 1, 2, \cdots, n. \tag{6.13}$$

Moreover, by the choice of α_1 and α_2 in (6.11), it is obvious to see that $v(\mathbf{x}_c) = 0$. Hence, by our results in Sections 3 and 4, we deduce that

$$v = 0 \quad \text{in} \quad B_h(\mathbf{x}_c),$$

and thus

$$\alpha_1 u(\mathbf{x}; k, \mathbf{d}_1) + \alpha_2 u(\mathbf{x}; k, \mathbf{d}_2) = 0 \quad \text{in} \quad \mathbb{R}^3 \backslash \overline{\Omega} \tag{6.14}$$

by the analytic continuation. However, since $\mathbf{d}_1$ and $\mathbf{d}_2$ are distinct, we know from [9, Chapter 5] that $u(\mathbf{x}; k, \mathbf{d}_1)$ and $u(\mathbf{x}; k, \mathbf{d}_2)$ are linearly independent in $\mathbb{R}^3 \backslash \overline{\Omega}$. Therefore, from (6.14) we can obtain that

$$\alpha_1 = \alpha_2 = 0, \tag{6.15}$$

which contracts to the assumption at the beginning that both α_1 and α_2 are nonzero.

It remains to prove (6.6), and we do it by contradiction. Let $\mathcal{E} \subset \partial\Omega \cap \partial\widetilde{\Omega}$ be an open subset such that $\eta \neq \widetilde{\eta}$ on $\mathcal{E}$. By taking a smaller subset of $\mathcal{E}$ if necessary, we can assume that η (respectively $\widetilde{\eta}$) is either a fixed constant or ∞ on $\mathcal{E}$. Clearly, one has $u = \widetilde{u}$ in $\mathbb{R}^3 \backslash \overline{(\Omega \cup \widetilde{\Omega})}$. Hence it holds that

$$\partial_\nu u + \eta u = 0, \quad \partial_\nu \widetilde{u} + \widetilde{\eta} \widetilde{u} = 0, \quad u = \widetilde{u}, \quad \partial_\nu u = \partial_\nu \widetilde{u} \quad \text{on} \quad \mathcal{E}. \tag{6.16}$$

Combining with the assumption that $\eta \neq \widetilde{\eta}$ on $\mathcal{E}$, we can deduce by direct computing that

$$u = \partial_\nu u = 0 \quad \text{on} \quad \mathcal{E}, \tag{6.17}$$

which in turn yields by the Homogren's uniqueness result (cf. [23]) that $u = 0$ in $\mathbb{R}^3 \backslash \Omega$. Therefore, we arrive at the same contradiction as that in (6.9), leading to the conclusion (6.6). □

Theorem 6.6 presents a local uniqueness result by showing the lack of an irrational vertex corner. If the underlying admissible complex irrational obstacles are convex, we can obtain a global unique identifiability result by two far-field patterns as follows.

Corollary 6.7. *Consider a fixed $k \in \mathbb{R}_+$, and two distinct incident directions $\mathbf{d}_1$ and $\mathbf{d}_2$ from $\mathbb{S}^2$. Let (Ω, η) and $(\widetilde{\Omega}, \widetilde{\eta})$ be two convex admissible complex irrational obstacles, with u_∞ and $\widetilde{u}_\infty$ being their corresponding far-field patterns and $\mathbf{G}$ being the unbounded connected component of $\mathbb{R}^3 \backslash \overline{(\Omega \cup \widetilde{\Omega})}$. If u_∞ and $\widetilde{u}_\infty$ are the same in the sense that*

$$u_\infty(\hat{\mathbf{x}}; k, \mathbf{d}_\ell) = \widetilde{u}_\infty(\hat{\mathbf{x}}; k, \mathbf{d}_\ell), \quad \text{for } \ell = 1, 2 \text{ and all } \hat{\mathbf{x}} \in \mathbb{S}^2, \tag{6.18}$$

then

$$\Omega = \widetilde{\Omega}, \quad \eta = \widetilde{\eta}.$$

Proof. Since Ω and $\widetilde{\Omega}$ are both convex, due to Krein-Milman theorem [26], we know that a convex polyhedron is fully determined by the set of its vertices. Now we prove by absurdity. If $\Omega \neq \widetilde{\Omega}$, it is obvious that there exists a vertex corner $\mathbf{x}_c$ on $\partial \mathbf{G}$, where $\mathbf{G}$ is the unbounded connected component of $\mathbb{R}^3 \backslash \overline{(\Omega \cup \widetilde{\Omega})}$. Following the similar argument in Theorem 6.6, by the condition (6.18), we can arrive at the contradictions (6.10) or (6.15). Therefore, we can prove that $\Omega = \widetilde{\Omega}$. The conclusion $\eta = \widetilde{\eta}$ can be obtained by using the similar argument in Theorem 6.6. □

We proceed to consider the unique determination of rational obstacles. By Definitions 4.2 and 6.4, we know that a rational obstacle contains at least one rational vertex corner. Recalling the results in Sections 2 and 3, for a fixed rational vertex corner $\mathbf{x}_c$ which is intersected by Π_i, where $\Pi_i = \text{span}\{\overrightarrow{OA_i}, \overrightarrow{OA_{i+1}}\}$ with the n dihedral angles $\angle(\Pi_i, \Pi_{i+1}) = \alpha_i \cdot \pi$, $i = 1, 2, \cdots, n$ $(n \geq 3)$, it is direct to verify that the eigenfunction u to (1.1) of the form

$$u(\mathbf{x}) = 4\pi \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \mathrm{i}^n a_n^m j_n(\sqrt{\lambda} r) \sqrt{\frac{2n+1}{4\pi} \frac{(n-|m|)!}{(n+|m|)!}} P_n^{|m|}(\cos\theta) e^{\mathrm{i}m\phi}$$

satisfies that $a_0^0 = 0$ if $u(\mathbf{x}_c) = 0$, where $\mathbf{x} = (x_1, x_2, x_3) = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \in \mathbb{R}^3$, λ is the corresponding eigenvalue, $P_n^m(t)$ denotes the associated Legendre function and $j_n(t)$ signifies the n-th spherical Bessel function. Since $\alpha_i \in (0, 1)$ for any $i = 1, 2, \cdots, n$, one can immediately obtain that $a_1^{\pm 1} = 0$; see Theorems 2.9, 3.1 and 3.3 for detailed discussions. Moreover, if we denote

$$\overrightarrow{OA_i} = (r, \theta_i, \phi_i) \text{ for } r > 0, \theta_i \in (0, \pi) \text{ and } \phi_i \in (0, 2\pi) \tag{6.19}$$

in the spherical coordinate system, then there always holds that $P_1^1(\cos\theta_i) = -\sin\theta_i \neq 0$. However, since $P_1^0(\cos\theta_i) = \cos\theta_i$, we know that $P_1^0(\cos\theta_i) \neq 0$ is only true for $\theta_i \neq \frac{\pi}{2}$, and thus $a_1^0 = 0$. That is, the eigenfunction u vanishes at least to the second order when $\theta_i \neq \frac{\pi}{2}$, and u vanishes at least to the first order otherwise.

Let Ω be a polyhedron in $\mathbb{R}^3$ and $\mathbf{x}_c$ be a vertex corner of Ω. Then we introduce for $r \in \mathbb{R}_+$ that $\Omega_r(\mathbf{x}_c) = B_r(\mathbf{x}_c) \cap \mathbb{R}^3 \backslash \overline{\Omega}$, and define for any $f \in L^2_{loc}(\mathbb{R}^3 \backslash \overline{\Omega})$ that

$$\mathcal{L}(f)(\mathbf{x}_c) := \lim_{r \to +0} \frac{1}{|\Omega_r(\mathbf{x}_c)|} \int_{\Omega_r(\mathbf{x}_c)} f(\mathbf{x}) \, \mathrm{d}\mathbf{x}$$

if the limit exists. It is easy to see that if $f(\mathbf{x})$ is continuous in $\overline{\Omega_{\epsilon_0}(\mathbf{x}_c)}$ for a sufficiently small $\epsilon_0 \in \mathbb{R}_+$, then $\mathcal{L}(f)(\mathbf{x}_c) = f(\mathbf{x}_c)$.

Now we are ready to study the unique determination of rational obstacles.

Theorem 6.8. *Consider a fixed $k \in \mathbb{R}_+$, and two distinct incident directions $\mathbf{d}_1$ and $\mathbf{d}_2$ from $\mathbb{S}^2$. Let (Ω, η) and $(\widetilde{\Omega}, \widetilde{\eta})$ be two admissible complex rational obstacles of degree $p \geq 3$, with $u_\ell(\mathbf{x}) := u(\mathbf{x}; k, \mathbf{d}_\ell)$ and $\widetilde{u}_\ell := \widetilde{u}(\mathbf{x}; k, \mathbf{d}_\ell)$ being their corresponding total wave fields associated with the incident field $e^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}_\ell}$, and $u_\infty(\hat{\mathbf{x}}; k, \mathbf{d}_\ell)$ and $\widetilde{u}_\infty(\hat{\mathbf{x}}; , k, \mathbf{d}_\ell)$ being their corresponding far-field patterns for $\ell = 1, 2$. We further write $\mathbf{G}$ for the unbounded connected component of $\mathbb{R}^3 \backslash \overline{(\Omega \cup \widetilde{\Omega})}$. Then the set $(\partial\Omega \backslash \partial\overline{\widetilde{\Omega}}) \cup (\partial\widetilde{\Omega} \backslash \partial\overline{\Omega})$ can not possess a vertex corner on $\partial\mathbf{G}$, if the following conditions are satisfied:*

$$u_{\ell,\infty}(\hat{\mathbf{x}}; k, \mathbf{d}_\ell) = \widetilde{u}_{\ell,\infty}(\hat{\mathbf{x}}; k, \mathbf{d}_\ell), \quad \hat{\mathbf{x}} \in \mathbb{S}^2, \ell = 1, 2, \tag{6.20}$$

$$\mathcal{L}\left(u_2 \cdot \nabla u_1 - u_1 \cdot \nabla u_2\right)(\mathbf{x}_c) \neq 0 \quad \text{and} \quad \mathcal{L}\left(\widetilde{u}_2 \cdot \nabla \widetilde{u}_1 - \widetilde{u}_1 \cdot \nabla \widetilde{u}_2\right)(\mathbf{x}_c) \neq 0 \tag{6.21}$$

for all vertices $\mathbf{x}_c$ *of* Ω. *Moreover,*

$$\eta = \widetilde{\eta} \quad on \quad \partial\Omega \cap \partial\widetilde{\Omega}. \tag{6.22}$$

Proof. We prove the theorem by contradiction. Assume that (6.20) holds but $(\partial\Omega \backslash \partial\overline{\widetilde{\Omega}}) \cup (\partial\widetilde{\Omega} \backslash \partial\overline{\Omega})$ has a vertex corner $\mathbf{x}_c$ on $\partial\mathbf{G}$. Without loss of generality, we still assume that $\mathbf{x}_c$ is a vertex corner of $\widetilde{\Omega}$. In what follows, we adopt the same notations as those introduced in the proof of Theorem 6.6.

By following a similar argument to the proof of Theorem 6.6, one can show that there exist n pieces of planes $\Pi_i \subset \partial\mathbf{G}$ intersecting at $\mathbf{x}_c$, such that $\partial_\nu u + \widetilde{\eta} u = 0$ on Π_i, $i = 1, 2, \cdots, n$. Using the fact that $u = \widetilde{u}$ near $\mathbf{x}_c$, we derive by Rellich Lemma and the condition (6.21) on $(\widetilde{\Omega}, \widetilde{\eta})$ that

$$u(\mathbf{x}_c; k, \mathbf{d}_2) \cdot \nabla u(\mathbf{x}_c; k, \mathbf{d}_1) - u(\mathbf{x}_c; k, \mathbf{d}_1) \cdot \nabla u(\mathbf{x}_c; k, \mathbf{d}_2) \neq 0. \tag{6.23}$$

Clearly, this implies that $\alpha_1 := u(\mathbf{x}_c; k, \mathbf{d}_2)$ and $\alpha_2 = -u(\mathbf{x}_c; k, \mathbf{d}_1)$ cannot be identically zero. Let v be the same combination as introduced in (6.12), then we can directly verify that v fulfills (6.13) and

$$v(\mathbf{x}_c) = 0 \quad \text{and} \quad \nabla v(\mathbf{x}_c) \neq 0. \tag{6.24}$$

Noting that $\widetilde{\Omega}$ is a rational obstacle of degree $p \geq 3$, we know that Π_i, $i = 1, 2, \cdots, n$, intersect either at an irrational vertex corner or a rational vertex corner of degree $p \geq 3$. In either case, we see by our results in Sections 2, 3 and 4 that v vanishes at least to second order at $\mathbf{x}_c$ if $\theta_i \neq \frac{\pi}{2}$ in (6.19) for $i = 1, 2, \cdots, n$. Hence, there holds that $\nabla v(\mathbf{x}_c) = 0$, which is a contradiction to (6.24). Following a similar argument in the proof of Theorem 6.6, if $\eta \neq \widetilde{\eta}$, one can directly verify that (6.16) and (6.17) still hold. Therefore, (6.22) can be derived directly by using Homogren's uniqueness principle. ☐

Remark 6.9. In the proof of Theorem 6.8, we may illustrate the vanishing order of u by the normal derivatives in Taylor expansion. That is, the conditions that $v(\mathbf{x}_c) = \nabla v(\mathbf{x}_c) = 0$ imply that v vanishes at $\mathbf{x}_c$ at least up to the second order. Indeed, this is equivalent to the vanishing condition that $a_0^0 = a_1^{\pm 1} = a_1^0 = 0$ for the coefficients of u in the spherical wave expansion, which follows readily from the main theorems of Sections 2 and 3 under the condition that $\theta_i \neq \frac{\pi}{2}$ for $i = 1, 2, \cdots, n$.

Remark 6.10. The uniqueness results and the corresponding argument in Theorems 6.6 and 6.8 are "localized" around the corner $\mathbf{x}_c$ based on the spectral results in Sections 2 and 3. This provides a novel and very effective analytical approach to study inverse scattering problems. Similar to Corollary 6.7, if the underlying admissible complex rational obstacle is convex, then it can be uniquely determined by two far-field patterns under the same generic conditions as those in Theorem 6.8.

Remark 6.11. We would like to point out that the condition (6.21) can be fulfilled under certain generic conditions on Ω. For instance, if the obstacle Ω is sufficiently small compared with the wavelength, namely $k \cdot \text{diam}(\Omega) \ll 1$, then it is known from the physical point of view that the scattered wave field due to the obstacle is of a much smaller magnitude than the incident field, and thus the incident plane wave dominates in the total wave field $u = u^i + u^s$. Under this circumstance, (6.21) can be verified directly. Condition (6.21) can hold in more general scenarios, but we shall not explore this technical issue further in the present paper.

Remark 6.12. We would like to point out that by using the results in Section 5, Theorems 6.6 and 6.8 equally hold for the case that the surface impedance η is a (variable) analytic function; see also Remark 6.3.

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Department of Mathematics, Hong Kong Baptist University, Kowloon, Hong Kong, China.
E-mail address: xlcao.math@foxmail.com

School of Mathematics and Statistics, Northeast Normal University, Changchun, Jilin 130024, China.
E-mail address: hadiao@nenu.edu.cn

Department of Mathematics, City University of Hong Kong, Kowloon Tong, Hong Kong, China.
E-mail address: hongyu.liuip@gmail.com, hongyliu@cityu.edu.hk

Department of Mathematics, The Chinese University of Hong Kong, Hong Kong, China
E-mail address: zou@math.cuhk.edu.hk