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SOME NONOVERLAPPING DOMAIN DECOMPOSITION METHODS*

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Abstract. The purpose of this paper is to give a unified investigation of a class of nonoverlapping domain decomposition methods for solving second-order elliptic problems in two and three dimensions. The methods under scrutiny fall into two major categories: the substructuring–type methods and the Neumann–Neumann-type methods. The basic framework used for analysis is the *parallel subspace correction* method or *additive Schwarz* method, and other technical tools include *local-global* and *global-local* techniques. The analyses for both two- and three-dimensional cases are carried out simultaneously. Some internal relationships between various algorithms are observed and several new variants of the algorithms are also derived.

Key words. nonoverlapping domain decomposition, Schur complement, local-global and global-local techniques, jumps in coefficients, substructuring, Neumann–Neumann, balancing methods

AMS subject classifications. 65N30, 65N55, 65F10

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1. Introduction.

The purpose of this paper is to give an overview of several major nonoverlapping domain decomposition (DD) methods for solving large sparse linear systems of equations which arise from finite element discretizations of second-order self-adjoint elliptic problems defined on a bounded Lipschitz domain $\Omega \subset R^n$ $(n = 2, 3)$. It may be regarded as a sequel of Xu [81], where multigrid methods and overlapping DD methods are presented in a unified framework based on the concept of space decomposition and subspace correction.

Nonoverlapping DD methods refer to methods defined on a decomposition of a domain consisting of a collection of mutually disjoint subdomains. These methods are obviously well suited for parallel computing architectures and they also have some advantages over the overlapping methods (for example, they are efficient for handling elliptic problems with large jumps in coefficients). There is a vast literature on nonoverlapping DD methods, and most theoretical presentations of these methods are quite technical. A primary goal of this paper is to sort out the existing techniques and give a unified presentation on the theoretical aspects of these methods. As a result, some internal connections between various algorithms will be observed by means of elementary technical tools presented in the paper.

The DD methods discussed in this paper are related to the more traditional block Gaussian elimination methods with the blocks given by subdomains. Loosely speaking, these methods are algorithms for preconditioning the Schur complement resulting from the block Gaussian eliminations, thus excluding some other interesting

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nonoverlapping DD methods (for example, the methods based on Lagrangian multiplier techniques [55, 41, 42, 77, 20] and methods based on dual formulation [42]). For clarity, we will focus our discussion only on h-version finite element methods for scalar second-order elliptic equations. Indeed these methods can be extended to other partial differential equations such as elliptic systems (linear elasticity) and plate and shell problems, and they can also be extended to the hp-version finite element methods (treating each high-order element as a subdomain). For discussions of these extensions, we refer to [54, 1, 61, 66, 49] and the references cited therein.

We are not attempting to present a detailed history and literature of the methods concerned. For more thorough references, readers are referred to recent survey papers and books by Chan and Mathew [26], Le Tallec [75], Smith, Bjørstad, and Gropp [73], and the DD proceedings [22, 23, 25, 44, 45, 52, 53]. We also note that [26] and [73] use the matrix representation while [75] and the current survey use the operator representation.

The techniques for analyzing DD methods consist of, roughly speaking, two aspects: algebraic techniques and analytic techniques. These techniques will be summarized in two separate sections. The algebraic techniques will be presented in section 2. This section gives a brief account of some basic facts on conjugate gradient methods and preconditioning techniques. In particular, the framework of the subspace correction method is summarized. Also some very simple techniques, known as *global-local* and *local-global* techniques, will be presented; the global-local technique is for the construction of a subspace preconditioner based on a preconditioner on the whole space, whereas the local-global technique is for constructing a preconditioner on the whole space based on a subspace preconditioner.

These algebraic tools prove to be instrumental in understanding the close relationship among different algorithms. The analytic techniques, on the other hand, will be presented in section 4. This section gives a summary of some basic properties of Sobolev spaces and finite element spaces. In particular, special discussions will be devoted to continuous and discrete harmonic functions, the norm equivalence for finite element functions restricted on the boundary of a domain. Major technical results are included in this section.

The major idea of nonoverlapping DD methods will be introduced in section 3. With a model elliptic boundary value problem, discussions therein lead to two major types of DD methods to be studied in this paper: the substructuring-type methods and the Neumann–Neumann-type methods.

The substructuring methods are presented in section 5. Discussions begin with the simpler two-subdomain case. Then major different methods will be presented for the general multisubdomain case. The main algorithms originated from the papers of Bramble, Pasciak, and Schatz [10, 12], and other types of substructuring methods are presented as natural variants of the methods in [10, 12]. Nevertheless, our presentations are somewhat different from the existing literature and we hope a little easier for the readers to comprehend.

The Neumann–Neumann methods, including the so-called *balancing DD method*, are addressed in section 6. We attempt to present the Neumann–Neumann methods in an intuitively natural way. With the earlier-mentioned global-local and local-global techniques, the balancing method is derived in a simple and straightforward fashion. In particular, the technical estimates of this method are derived as a direct consequence of those for the Neumann–Neumann methods.

Section 7 is devoted to those DD methods derived from other existing methods, such as the hierarchical basis multigrid method, BPX multilevel preconditioner, and

additive overlapping Schwarz methods. Some of these methods are treated again by the global-local and local-global techniques. Section 8 gives a brief discussion of nonoverlapping DD methods based on inexact subdomain solvers. Finally, section 9 discusses the implementation issues for some major algorithms in the paper.

Although we tried hard to make our presentation more coherent and simpler than those in the literature, the paper is unfortunately still quite technical. These methods are indeed technically complicated in nature and their practical efficiency is also difficult to address (and sometimes is actually a matter of controversy). We hope this paper will be of some help to the further development of this type of method.

For convenience, following [81], the symbols $\lesssim, \gtrsim$, and $\eqsim$ will be used in this paper. $x_1 \lesssim y_1, x_2 \gtrsim y_2$, and $x_3 \eqsim y_3$ mean that $x_1 \le C_1 y_1, x_2 \ge c_2 y_2$, and $c_3 x_3 \le y_3 \le C_3 x_3$ for some constants C_1, c_2, c_3, and C_3 that are independent of mesh parameters.

2. Algebraic aspects of preconditioning techniques. All the DD methods discussed in this paper are based on a very important algebraic method: the preconditioned conjugate gradient method (PCG). This section contains some basic facts for the PCG method and techniques for constructing and analyzing preconditioners. The presentation in this section is purely algebraic.

Given a finite-dimensional linear vector space V with an inner product $(\cdot, \cdot)$ and a linear operator A defined on V, consider the following linear equation on V:

$$Au = f. \tag{2.1}$$

We assume that A is symmetric positive definite (SPD) with respect to the inner product $(\cdot, \cdot)$ on V. Next, we give a brief overview of the PCG method, and methods of constructing and analyzing preconditioners for the above linear equation.

For convenience, we will use the following convention: if B_1 and B_2 are two SPD operators on V such that

$$\alpha_1 (B_1 v, v) \le (B_2 v, v) \le \alpha_2 (B_1 v, v) \quad \forall\, v \in V,$$

we then write

$$\alpha_1 B_1 \le B_2 \le \alpha_2 B_1;$$

if furthermore α_1 and α_2 are constants independent of any parameters associated with V, then we write

$$B_1 \eqsim B_2.$$

2.1. PCG method. The well-known conjugate gradient method is the basis of all the preconditioning techniques to be studied in this paper. The PCG method can be viewed as a conjugate gradient method applied to the preconditioned system

$$BAu = Bf. \tag{2.2}$$

Here $B : V \mapsto V$ is another SPD operator and known as a preconditioner for A. Note that BA is symmetric with respect to the inner products $(B^{-1}\cdot, \cdot)$ and $(A\,\cdot, \cdot)$.

Let $\|\cdot\|_A = (A\,\cdot, \cdot)^{1/2}$ be the energy norm and $\kappa(BA)$ be the condition number of BA, i.e., the ratio between the largest and the smallest eigenvalues of BA. It is well known that

$$\|u - u_k\|_A \le 2 \left(\frac{\sqrt{\kappa(BA)} - 1}{\sqrt{\kappa(BA)} + 1} \right)^k \|u - u_0\|_A, \tag{2.3}$$

which implies that the PCG method will generally converge faster with a smaller condition number $\kappa(BA)$.

Observing the formula in the PCG method and the convergence estimate (2.3), we see that the efficiency of a PCG method depends on two main factors: the action of B and the size of $\kappa(BA)$. Hence, a good preconditioner should have the properties that the action of B is relatively easy to compute and that $\kappa(BA)$ is relatively small (at least smaller than $\kappa(A)$).

2.2. Framework of parallel subspace correction. All the DD preconditioners considered in this paper will be interpreted and analyzed within the framework of the *parallel subspace correction method* (PSC) (see Xu [81]) or the *additive Schwarz method* (see Dryja and Widlund [37]).

This framework is based on a sequence of subspaces V_i, $0 \le i \le J$, of V such that

$$V = \sum_{i=0}^{J} V_i.$$

The above *space decomposition* is understood in such a way that for any $v \in V$, there exist $v_i \in V_i$ (not necessarily unique) such that $v = \sum_{i=0}^{J} v_i$.

For each i, we define two orthogonal projections $Q_i, P_i : V \mapsto V_i$ by

$$(Q_i u, v_i) = (u, v_i), \quad (AP_i u, v_i) = (Au, v_i) \quad \forall\ u \in V,\ v_i \in V_i,$$

and the restriction A_i of A on V_i by

$$(A_i u_i, v_i) = (Au_i, v_i) \quad \forall\ u_i, v_i \in V_i.$$

Let t and $*$ denote the adjoint operator with respect to the inner products $(\cdot,\cdot)$ and $(A\cdot,\cdot)$, respectively. Then

$$Q_i = I_i^t \text{ and } P_i = I_i^*.$$

Here $I_i : V_i \mapsto V$ is the inclusion operator. It follows from the definition that

$$A_i P_i = Q_i A \text{ or } A_i I_i^* = I_i^t A.$$

Let $R_i : V_i \mapsto V_i$ be an SPD operator that approximates the inverse of A_i in some sense. The PSC preconditioner for A is formulated as follows:

$$B = \sum_{i=0}^{J} I_i R_i I_i^t. \tag{2.4}$$

Set $T_i = I_i R_i I_i^t A = I_i R_i A_i I_i^*$, $0 \le i \le J$. Note that $T_i : V \mapsto V$ is symmetric with respect to $(A\cdot,\cdot)$ and nonnegative. We obtain from above that

$$BA = \sum_{i=0}^{J} I_i R_i I_i^t A = \sum_{i=0}^{J} T_i.$$

In implementation, the action of B on an element $g \in V$ can be realized by assembling the local contributions w_i: $Bg = \sum_{i=0}^{J} w_i$, where w_i solves the inexact subproblem

$$(R_i^{-1} w_i, v_i) = (g, v_i) \quad \forall\ v_i \in V_i. \tag{2.5}$$

If each subproblem solver is exact, i.e., $R_i = A_i^{-1}$ for each i, then $T_i = I_i I_i^*$, and we obtain the additive Schwarz method

$$BA = \sum_{i=0}^{J} I_i I_i^*.$$

In this case, the subproblem (2.5) becomes

$$(A_i w_i, v_i) = (g, v_i) \quad \forall\, v_i \in V_i.$$

The preconditioner (2.4) can be analyzed by the following theorem.

THEOREM 2.1. *Let K_0 be a positive constant so that, for any $v \in V$, there exists a decomposition $v = \sum_{i=0}^{J} v_i$ such that $v_i \in V_i$ and*

$$\sum_{i=0}^{J} (R_i^{-1} v_i, v_i) \le K_0\, (Av, v). \tag{2.6}$$

Let K_1 be a positive constant given by

$$K_1 = \max_{1 \le j \le J} \sum_{i=1}^{J} \varepsilon_{ij}, \tag{2.7}$$

where, for $1 \le i, j \le J$, $\varepsilon_{ij} = 0$ if $P_i P_j = 0$ (namely, $V_i \perp V_j$), $\varepsilon_{ij} = 1$ otherwise. Then the preconditioner (2.4) *admits the following estimate:*

$$\kappa(BA) \le \omega_1\, K_0\, (1 + K_1),$$

where $\omega_1 = \max_{0 \le i \le J} \lambda_{\max}(R_i A_i)$.

For a proof of the above theorem and a more general theory, we refer to Xu [81]. For related theory, we refer to Dryja and Widlund [38, 35], Bramble et al. [14, 13], and Griebel and Oswald [47].

REMARK 2.2. The parameter ω_1 measures the resolution of R_i for the upper spectrum of A_i, and $\omega_1 = 1$ if exact solvers are used on subspaces (namely $R_i = A_i^{-1}$ for all $0 \le i \le J$). In many DD methods, the boundedness of ω_1 comes as an assumption. The estimate for K_0 often dominates the analysis in DD methods.

REMARK 2.3. In DD methods, K_1 measures the degree of overlapping among the subspaces V_i for $1 \le i \le J$. The space V_i is said to overlap with V_j if V_i is not orthogonal with V_j with respect to $(A\cdot, \cdot)$. Hence K_1 is the largest number of subspaces that a subspace can overlap with (with the exclusion of the subspace V_0). The term "overlapping" comes from the fact that, in DDs, V_i overlaps with V_j if and only if the two geometric subdomains that define these two subspaces overlap with each other. As in all the DD methods considered in this paper, each subdomain overlaps with only a fixed number of other subdomains, hence K_1 is always bounded by a fixed constant.

REMARK 2.4. For other applications such as multigrid methods, the parameter K_1 must be defined in a more precise way than used here. For example, K_1 can be defined as in (2.7), but with ε_{ij} $(1 \le i, j \le J)$ defined by [81]

$$(AT_i u, T_j v) \le \omega_1 \varepsilon_{ij} (AT_i u, u)^{1/2} (AT_j v, v)^{1/2} \quad \forall\, u, v \in V.$$

REMARK 2.5. The PSC method as presented here is similar to the so-called additive Schwarz method. The terminology *additive Schwarz method* was attributed to Dryja and Widlund [37] and it reflects the fact that the method studied in [37] is a variant of an alternating algorithm proposed by Schwarz [68] in 1870. The terminology *parallel subspace correction method* was introduced by Xu [81] to more directly reflect, within the general framework studied in [81], the nature of this type of method (including not only DD methods but other methods such as multigrid methods and even the classic Jacobi and Gauss–Seidel methods as well). The basic idea and the theory of this type of method can be found in Matsokin and Nepomnyaschikh [62]. Some history of the theoretical development of this method can be traced in Matsokin and Nepomnyaschikh [62], Dryja and Widlund [37, 39], Lions [57], Bramble, Pasciak, and Xu [15], Bramble et al. [14, 13], and Xu [81].

2.3. Three special techniques. For constructing preconditioners and investigating the relationship between different DD preconditioners, three special techniques formulated here will prove to be instrumental. The first two techniques (see Xu [84]), known as the global-local and local-global techniques, concern preconditioning techniques using the relationship between a space and its subspace. The third technique, known as the auxiliary space method (see Xu [83]), is on a preconditioning technique using an auxiliary space.

We introduce some notation. Given a subspace $\hat{V}$ of V, as in subsection 2.2, let $\hat{A}$ be the restriction of A on $\hat{V}$ defined by

$$(\hat{A}\hat{u}, \hat{v}) = (A\hat{u}, \hat{v}) \quad \forall\ \hat{u}, \hat{v} \in \hat{V}, \tag{2.8}$$

and $\hat{Q}, \hat{P} : V \mapsto \hat{V}$ be two orthogonal projections with respect to $(\cdot, \cdot)$ and $(A\cdot, \cdot)$, respectively. Let $\hat{P}^t$ be the adjoint of $\hat{P}$ with respect to $(\cdot, \cdot)$.

2.3.1. Global-local technique. This simple technique is for constructing a preconditioner on a subspace $\hat{V}$ of V from a known preconditioner on the space V.

THEOREM 2.6. *Given that B is an SPD operator on V, define $\hat{B} = \hat{P}B\hat{P}^t$. Then $\hat{B}$ is SPD on $\hat{V}$ and on the subspace $\hat{V}$, $\hat{B}\hat{A} = \hat{P}BA$. As a consequence,*

$$\kappa(\hat{B}\hat{A}) \leq \kappa(BA).$$

Proof. By noting $\hat{A}\hat{P} = \hat{Q}A$, we have on the subspace $\hat{V}$ that

$$\hat{B}\hat{A} = \hat{P}B\hat{P}^t\hat{A} = \hat{P}B(\hat{A}\hat{P})^t = \hat{P}B(\hat{Q}A)^t = \hat{P}BA\hat{Q}^t = \hat{P}BA,$$

where we have used the fact that $\hat{Q}^t : \hat{V} \mapsto V$, the adjoint of $\hat{Q}$, is an injection. ∎

2.3.2. Local-global technique. This simple technique is for constructing a preconditioner on the space V by using some known local preconditioner on a subspace $\hat{V}$ of V.

We are given a preconditioner $\hat{B} : \hat{V} \mapsto \hat{V}$ for the operator $\hat{A}$ on $\hat{V}$, which is assumed to be spectrally equivalent to $\hat{A}$ in the sense that there exist two constants α_0 and α_1 such that

$$\alpha_0(\hat{A}\hat{u}, \hat{u}) \leq (\hat{B}\hat{A}\hat{u}, \hat{A}\hat{u}) \leq \alpha_1(\hat{A}\hat{u}, \hat{u}) \quad \forall\ \hat{u} \in \hat{V}. \tag{2.9}$$

Let $\hat{V}^\perp$ be the orthogonal complement of $\hat{V}$ in V with respect to $(A\,\cdot, \cdot)$. Then the following algorithm provides a preconditioner B for A on the space V.

ALGORITHM 2.1. *Given $g \in V$, $u = Bg = u_{_P} + u_{_R}$ is computed as follows:*

1. *Solve the local problem $u_{_P} \in \hat{V}^\perp$ satisfying*

$$(Au_{_P}, v) = (g, v) \quad \forall\ v \in \hat{V}^\perp.$$

2. *Compute $u_{_R}$ by $u_{_R} = \hat{B}\hat{Q}(g - Au_{_P})$.*

THEOREM 2.7. *For Algorithm 2.1 defined above, we have*

$$B = (\hat{P}^\perp + \hat{P}\hat{B}\hat{A}\hat{P})A^{-1},$$

where $\hat{P}^\perp \equiv I - \hat{P} : V \mapsto \hat{V}^\perp$ is an orthogonal projection with respect to $(A\cdot, \cdot)$, and

$$\kappa(BA) \leq \frac{\max(1, \alpha_1)}{\min(1, \alpha_0)}.$$

Proof. By step 1 of Algorithm 2.1, $u_{_P} = \hat{P}^\perp A^{-1}g$. Substituting $u_{_P}$ into $u_{_R}$ in step 2 gives

$$u_{_R} = \hat{B}\hat{Q}(AA^{-1}g - A\hat{P}^\perp A^{-1}g) = \hat{B}\hat{Q}A\hat{P}A^{-1}g.$$

Theorem 2.7 then follows from (2.9) and the expressions of $u_{_P}$ and $u_{_R}$. ∎

REMARK 2.8. We remark that the subproblem in step 1 of Algorithm 2.1 is solved exactly. When $V = \hat{V}$, we set $u_{_P} = 0$, and the preconditioner B degenerates to the original $\hat{B}$ on $\hat{V}$.

2.3.3. Auxiliary and fictitious space methods. The method to be introduced in this section may be viewed as a two-level nonnested multigrid preconditioner or PSC method associated with two nonnested spaces. Assume that these two nonnested linear vector spaces are V and W, with inner products $(\cdot,\cdot)$ and $[\cdot,\cdot]$, respectively.

Let A and S be two given SPD operators on V and W, respectively, with respect to their inner products. Assume that T is a preconditioner for S on W with

$$\lambda_0[w,w]_S \le [TSw,w]_S \le \lambda_1[w,w]_S \quad \forall\ w \in W. \tag{2.10}$$

The technique given here is for constructing a preconditioner B for A on V from the known preconditioner T for S on W. For the purpose, we introduce a prolongation operator $\Pi : W \mapsto V$ and a restriction operator $Q : V \mapsto W$. For a good preconditioner, these two operators are assumed to be bounded in their energy norms, i.e.,

$$\|Qv\|_S^2 \le \beta_0^{-1}\|v\|_A^2, \quad \|\Pi w\|_A^2 \le \beta_1\|w\|_S^2 \quad \forall\ v \in V,\ w \in W \tag{2.11}$$

and the action of ΠQ is not far from the action of the identity, i.e.,

$$\|v - \Pi Q v\|^2 \le \gamma_0^{-1}\rho_A^{-1}\|v\|_A^2 \quad \forall\ v \in V, \tag{2.12}$$

where ρ_A is the spectral radius of A.

Moreover, let R be an SPD operator in V, spectrally equivalent to the scaling of the identity, namely,

$$\alpha_0\rho_A^{-1}(v,v) \le (Rv,v) \le \alpha_1\rho_A^{-1}(v,v) \quad \forall\ v \in V. \tag{2.13}$$

Then we have the following.

THEOREM 2.9. *Under the assumptions* (2.10)–(2.13)*, the SPD operator defined by*

$$B = R + \Pi T \Pi^t$$

can be chosen to be a preconditioner for A, and the condition number of BA is bounded by

$$\kappa(BA) \le (\alpha_1 + \beta_1\lambda_1)((\alpha_0\gamma_0)^{-1} + (\beta_0\lambda_0)^{-1}).$$

In particular, if Q is a right inverse of Π, namely, $\Pi Q = I$, then

$$\kappa((\Pi T \Pi^t)A) \le \frac{\beta_1}{\beta_0}\frac{\lambda_1}{\lambda_0}. \tag{2.14}$$

Theorem 2.9 was presented in Xu [83] to analyze an optimal preconditioner for general unstructured grids by using auxiliary structured grids. The last estimate in the theorem corresponds to the *fictitious space lemma* (see Nepomnyaschikh [63]). In this case, the space W has to be at least as rich as the original space V and hence the construction of Π needs more caution.

3. A model problem and outline. For clarity of presentation, this section is devoted to explaining the basic ideas and motivation of the main algorithms to be discussed in the paper. The style of presentation of this section is informal. Technical details and relevant literature are to be given in subsequent sections.

The following boundary value problem is the model problem to be studied:

$$\begin{aligned} -\nabla \cdot (\rho(x)\nabla U(x)) &= F(x) \quad \text{in } \Omega, \\ U(x) &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.1}$$

Here $\Omega \subset R^n$ is a polygon for $n = 2$ or a polyhedron for $n = 3$, and $\rho(x)$ is piecewise constant in Ω or $\rho(x) \equiv 1$. Without loss of generality, we assume that Ω is of unit diameter.

From a preconditioning point of view, the above model problem is appropriate for the study of a large class of second-order self-adjoint elliptic boundary value problems. Problems with slowly changing variable coefficients may be preconditioned by the Poisson equation, namely, $\rho(x) \equiv 1$ in (3.1); problems with large jumps in coefficients between subdomains may also be preconditioned by the above problem with ρ being piecewise constant.

Let $\mathcal{T}^h = \{\tau_i\}$ be a quasi-uniform triangulation of Ω with the τ_i's being nonoverlapping simplexes of size h, with $h \in (0, 1]$; namely, there exist constants C_0 and C_1 independent of h such that each simplex τ_i contains (resp., is contained in) a ball of radius $C_0 h$ (resp., $C_1 h$). We then define V^h to be the piecewise-linear finite element subspace of $H_0^1(\Omega)$ associated with $\mathcal{T}^h$, as follows:

$$V^h = \{v \in H_0^1(\Omega) : v|_\tau \in \mathcal{P}_1(\tau) \quad \forall\ \tau_i \in \mathcal{T}^h\},$$

where $\mathcal{P}_1$ is the space of linear polynomials.

Then the finite element approximation for (3.1) is to find $u \in V^h$ such that

$$A(u, v) = (F, v) \quad \forall\ v \in V^h, \tag{3.2}$$

where $(\cdot, \cdot)$ is the scalar product in $L^2(\Omega)$, and

$$A(u, v) = \int_\Omega \rho(x)\nabla u \cdot \nabla v\, dx.$$

Later on, we shall also use A to denote the operator on V^h defined by

$$(Au, v) = A(u, v) \quad \forall\ u, v \in V^h,$$

and this operator A may be called the *stiffness* operator.

A DD without overlapping consists of a number of mutually disjoint open subdomains Ω_i such that

$$\bar{\Omega} = \bigcup_{i=1}^{J} \bar{\Omega}_i. \tag{3.3}$$

When the coefficient $\rho(x)$ is piecewise constant, each subdomain Ω_i is chosen in such a way that $\rho(x)$ equals constant ρ_i in Ω_i. We assume that the triangulation $\mathcal{T}^h$ is consistent with (3.3) in the sense that each $\partial\Omega_i$ can be written as a union of boundaries of elements in $\mathcal{T}^h$ (see Fig. 1). Moreover, we assume that all subdomains Ω_i are of size h_0 ($> h$) in the sense that there exist constants C_0 and C_1 independent of h and h_0 such that each Ω_i contains (resp., is contained in) a ball of radius $C_0 h_0$ (resp., $C_1 h_0$).

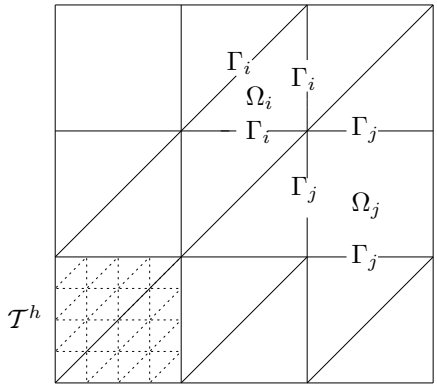

FIG. 1. *Fine mesh $\mathcal{T}^h$ and subdomains $\{\Omega_i\}_{i=1}^J$: some are simplices and some quadrilaterals.*

Corresponding to each subdomain Ω_i, we shall use the notation

$$A_i(u,v) = \int_{\Omega_i} \rho_i \nabla u \cdot \nabla v dx \quad \forall\ u, v \in H^1(\Omega).$$

Obviously $A(u,v) = \sum_{i=1}^J A_i(u,v)$.

Nonoverlapping DD methods discussed in this paper correspond to constructing preconditioners for the system (3.2) by solving certain small problems associated with the subdomains from the DD, possibly plus a small-scale global problem (often called the *coarse* problem). Apparently the first natural step is to solve, for $u_{P,i} \in V_0^h(\Omega_i)$ on each subdomain Ω_i, the following local homogeneous Dirichlet problem:

$$A(u_{P,i}, v) = (F, v) \quad \forall\ v \in V_0^h(\Omega_i),$$

where the local subspace $V_0^h(\Omega_i) \subset V^h$ is defined by (with a slight abuse of notation)

$$V_0^h(\Omega_i) = \{v \in V^h :\ v(x) = 0 \quad \forall\ x \in \Omega \setminus \Omega_i\}.$$

Notice that the computation of $u_{P,i}$ can be carried out on each subdomain concurrently. This is typically how the parallelization is realized in nonoverlapping DD methods.

Let $u_P \in V^h$ be the function that is equal to $u_{P,i}$ on the subdomain Ω_i. Then u_P is clearly a finite element function in the subspace

$$V_P = \{v \in V^h : v(x) = 0 \quad \forall\ x \in \Gamma\}. \tag{3.4}$$

Here $\Gamma = \bigcup_{i=1}^J \Gamma_i$, with $\Gamma_i = \partial\Omega_i \setminus \partial\Omega$, is the interface among all the subdomains $\{\Omega_i\}$ (see Fig. 1).

Obviously, $u_P \in V_P$ is the solution of the following problem:

$$A(u_P, v) = (F, v) \quad \forall\ v \in V_P. \tag{3.5}$$

Of course, u_P is only part of the entire finite element solution u that we are seeking. The remaining part of the solution, $u_H = u - u_P$, lies in the orthogonal complement of V_P in V^h:

$$V_H = \{v \in V^h :\ A(v, \chi) = 0 \quad \forall\ \chi \in V_P\}, \tag{3.6}$$

and u_H obviously satisfies

$$A(u_H, v) = (F, v) - A(u_P, v) \quad \forall\ v \in V^h,$$

or equivalently,

$$A(u_H, v_H) = (F, v) - A(u_P, v) \quad \forall\, v \in V^h. \tag{3.7}$$

Here $v_H \in V_H$ is understood similarly as u_H. We note that

$$A(u, u) = A(u_P, u_P) + A(u_H, u_H). \tag{3.8}$$

The function $u_H \in V_H$ is called a piecewise-discrete harmonic function, since it satisfies, on each subdomain Ω_i,

$$A_i(u_H, \chi) = \rho_i \, (\nabla u_H, \nabla \chi)_{0,\Omega_i} = 0 \quad \forall\, \chi \in V_0^h(\Omega_i).$$

From this, we also know that the value of u_H in Ω is uniquely determined by its value on the interface Γ. Therefore, the previously conducted process actually reduces the global finite element resolution (3.2) to a much smaller problem (3.7) on the interface.

The main concern of this paper is to construct preconditioners for the system (3.7). Note that this system is on the functions defined on the interface Γ, thus it is convenient to consider only those functions on Γ, namely the finite element space $V^h(\Gamma)$. Given $u, v \in V^h(\Gamma)$, let $u_H, v_H \in V_H$ be the discrete harmonic extensions of u, v. The relation between interface functions and their discrete harmonic extensions can be established through the following bilinear form on Γ:

$$S(u, v) = A(u_H, v_H) \quad \forall\, u, v \in V^h(\Gamma), \tag{3.9}$$

which may be analyzed using its equivalent $H^{1/2}$-norm on the boundaries of all subdomains (see (4.13) below):

$$S(v, v) \eqsim \sum_{i=1}^{J} \rho_i |v|^2_{1/2,\partial\Omega_i}. \tag{3.10}$$

We shall also use $S : V^h(\Gamma) \mapsto V^h(\Gamma)$ to denote the *interface* operator induced by

$$\langle Su, v\rangle = S(u, v) \quad \forall\, u, v \in V^h(\Gamma). \tag{3.11}$$

Here and in what follows, $\langle \cdot, \cdot \rangle$ denotes the L^2 scalar product on Γ. Our task is therefore to construct preconditioners for the interface operator S to improve its condition number.

The following lemma shows how the condition number of S depends on the jumps in the coefficients of the model problem, the subdomain size h_0, and the finite element mesh size h.

LEMMA 3.1. *For the interface operator S, we have*

$$\kappa(S) \lesssim \frac{\max_i \rho_i}{\min_i \rho_i} (h_0 h)^{-1}.$$

A proof of this lemma can be found in the appendix.

We end this section with a presentation of the matrix form of the interface operator S. Let $\mathcal{S}$ be the stiffness matrix associated with the bilinear form $S(\cdot, \cdot)$ under the standard nodal basis functions in $V^h(\Gamma)$. One important observation is that $\mathcal{S}$ is a Schur complement of the stiffness matrix $\mathcal{A}$ associated with $A(\cdot, \cdot)$ under the nodal basis functions in V^h. More specifically, if we write the stiffness matrix $\mathcal{A} \in R^{n\times n}$ blockwise,

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{12}^t & \mathcal{A}_{22} \end{pmatrix},$$

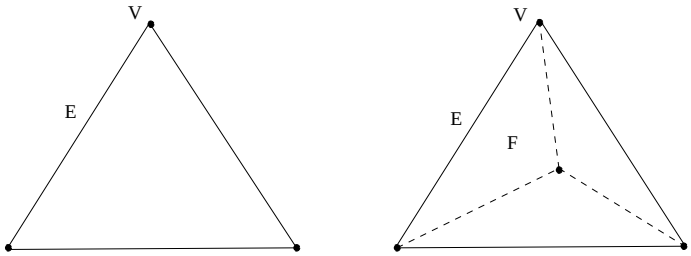

FIG. 2. *Left:* 2*-simplex with its vertices, edges; right:* 3*-simplex with its vertices, edges, faces.*

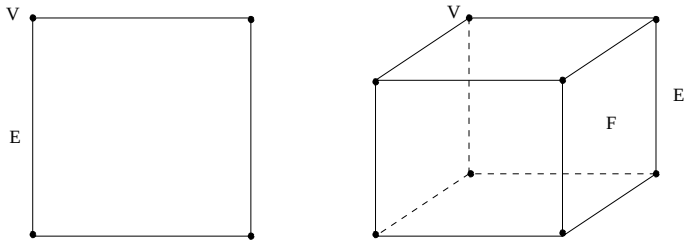

FIG. 3. *Left:* 2*-cube and its vertices and edges; right:* 3*-cube and its vertices, edges, and faces.*

where $\mathcal{A}_{11} \in R^{n_1 \times n_1}$ is the stiffness matrix associated with the nodes in $\Omega \setminus \Gamma$ and $\mathcal{A}_{22} \in R^{n_2 \times n_2}$ the stiffness matrix associated with the nodes on Γ, then

$$\mathcal{S} = \mathcal{A}_{22} - \mathcal{A}_{12}^t \mathcal{A}_{11}^{-1} \mathcal{A}_{12}.$$

It is well known that the condition number of the Schur compliment $\mathcal{S}$ is always less than the one of the matrix $\mathcal{A}$, namely, $\kappa(\mathcal{S}) \leq \kappa(\mathcal{A})$. But using the conditioning estimate of the operator S in Lemma 3.1, we have

$$\kappa(\mathcal{S}) \lesssim \frac{\max_i \rho_i}{\min_i \rho_i} (h_0 h)^{-1}.$$

This indicates that the condition number $\kappa(S)$ deteriorates with respect to the subdomain size h_0, the finite element mesh size h, and the coefficients ρ_i of the model problem. The concern of this paper is to construct preconditioners T such that $\kappa(TS)$ is weakly dependent on h_0 and h and independent of ρ_i, namely, we shall prove the following type of estimate:

$$\kappa(TS) \lesssim \log^\alpha \frac{h_0}{h},$$

which holds uniformly with respect to ρ_i for some $\alpha \geq 0$.

4. Preliminaries of Sobolev spaces and finite element spaces. In this section, we shall discuss some analytical tools for studying DD methods, namely, Sobolev spaces and finite element spaces.

We consider a bounded Lipschitz domain $\Omega \subset R^n$ $(n = 2, 3)$, which plays the role of a general subdomain, e.g., Ω_i in section 3 and in subsequent sections, where we apply the results of this section to DDs. We assume that Ω is a polygonal $(n = 2)$ or polyhedral $(n = 3)$ domain with each edge length of size d, and with boundary $\partial\Omega$ consisting of faces $\{F\}$, edges $\{E\}$, and vertices $\{\mathrm{V}\}$ (e.g., see Fig. 2 and Fig. 3). Most of the inequalities of this section will be given the explicit dependence on the diameter d of Ω. It is implicitly assumed that the constants in certain Sobolev inequalities for a unit-size domain will be uniform for a reasonable class of such domains.

4.1. Some Sobolev spaces. Let $H^1(\Omega)$ be the standard Sobolev space consisting of square integrable functions with square integrable first-order weak derivatives, equipped with the usual seminorm $|\cdot|_{1,\Omega}$ and the scaled full-norm $\|\cdot\|_{1,\Omega}$:

$$|u|_{1,\Omega}^2 = \int_\Omega |\nabla u|^2 dx, \quad \|u\|_{1,\Omega}^2 = d^{-2}\|u\|_{0,\Omega}^2 + |u|_{1,\Omega}^2,$$

where $\|u\|_{0,\Omega} = (\int_\Omega u^2 dx)^{1/2}$, $\nabla u = (\partial_1 u, \ldots, \partial_n u)$, and the derivatives $\partial_i u$, $i = 1, 2, \ldots, n$, are understood in the sense of distributions, and $|\nabla u|$ is the Euclidean norm of ∇u in R^n. We remark that the scaling factor d^{-2} in the norm $\|\cdot\|_{1,\Omega}$ is to make the two terms appearing in the definition have the same scaling with respect to d. $W^{1,\infty}(\Omega)$ will denote the Sobolev space consisting of essentially bounded functions with essentially bounded first-order weak derivatives, equipped with the norm

$$\|u\|_{1,\infty,\Omega} = \max(d^{-1}\|u\|_{0,\infty,\Omega}, \|\nabla u\|_{0,\infty,\Omega}), \quad \|u\|_{0,\infty,\Omega} = \operatorname*{ess\,sup}_{x\in\Omega} |u(x)|.$$

The Sobolev space $H_0^1(\Omega)$, a subspace of $H^1(\Omega)$, is defined to be the closure with respect to the norm $\|\cdot\|_{1,\Omega}$ of $C_0^\infty(\Omega)$ (infinitely differentiable functions with compact support in Ω). In other words, $H_0^1(\Omega)$ consists of functions in $H^1(\Omega)$ that vanish on $\partial\Omega$ (in the sense of trace).

Let $\Sigma \subseteq \partial\Omega$. The space $H^{1/2}(\Sigma)$ can be defined as follows:

$$H^{1/2}(\Sigma) = \{u \in L^2(\Sigma) : |u|_{1/2,\Sigma} < \infty\}$$

equipped with a norm

$$\|u\|_{1/2,\Sigma} = \left(d^{-1}\|u\|_{0,\Sigma}^2 + |u|_{1/2,\Sigma}^2\right)^{1/2},$$

where, with ds denoting the surface element on Σ,

$$|u|_{1/2,\Sigma}^2 = \int_\Sigma\int_\Sigma \frac{(u(x)-u(y))^2}{|x-y|^n} ds(x)ds(y), \quad \|u\|_{0,\Sigma}^2 = \int_\Sigma u^2 ds.$$

For any face F of Ω, the space $H_{00}^{1/2}(F)$ is defined, with $\tilde{v}$ being the zero extension of v into $\partial\Omega \setminus F$, as follows:

$$H_{00}^{1/2}(F) = \{v \in L^2(F) : \ \tilde{v} \in H^{1/2}(\partial\Omega)\}.$$

By a direction calculation (see Grisvard [48]), for any $v \in H_{00}^{1/2}(F)$,

$$|\tilde{v}|_{1/2,\partial\Omega} \eqsim |v|_{H_{00}^{1/2}(F)}^2, \tag{4.1}$$

with

$$|v|_{H_{00}^{1/2}(F)}^2 = \int_F\int_F \frac{(v(x)-v(y))^2}{|x-y|^n} ds(x)ds(y) + \int_F \frac{v^2(x)}{\operatorname{dist}(x,\partial F)} ds(x). \tag{4.2}$$

The space $H_{00}^{1/2}(F)$ is then a Hilbert space with a norm given by

$$\|v\|_{H_{00}^{1/2}(F)}^2 = d^{-1}\|v\|_{0,F}^2 + |v|_{H_{00}^{1/2}(F)}^2.$$

The space $H_{00}^{1/2}$ can be obtained by Hilbert scaling between the spaces L^2 and H_0^1. Let $-\Delta_F : H_0^1(F) \mapsto H^{-1}(F)$ be the Laplacian operator. The following equivalence holds:

$$\|v\|_{H_{00}^{1/2}(F)} \eqsim \|(-\Delta_F)^{1/4} v\|_{0,F} \quad \forall\ v \in H_{00}^{1/2}(F). \tag{4.3}$$

For proof of the above results, we refer to Lions and Magenes [56] for smooth domains and Bramble [9] for Lipschitz domains. By the equivalence between the Hilbert scale and the real method of interpolation (see Lions and Magenes [56]), (4.3) is equivalent to the statement that $H_{00}^{1/2}(F)$ is the interpolated space halfway between the $H_0^1(F)$ and $L^2(F)$ spaces.

Sobolev spaces $H^{1/2}$ and $H_{00}^{1/2}$ are most often used on the boundary of a domain.

We are now in a position to introduce some Sobolev inequalities. The following relations are well known:

$$\inf_{\mu\in R^1} \|u+\mu\|_{1,\Omega} \eqsim |u|_{1,\Omega} \tag{4.4}$$

and

$$\inf_{\mu\in R^1} \|u+\mu\|_{1/2,\partial\Omega} \eqsim |u|_{1/2,\partial\Omega}. \tag{4.5}$$

We have the well-known Poincaré inequality

$$\|u\|_{0,\Omega} \lesssim d\,|u|_{1,\Omega} \quad \forall u \in H_0^1(\Omega) \tag{4.6}$$

and Friedrichs's inequality

$$\|u-\gamma(u)\|_{0,\Omega} \lesssim d\,|u|_{1,\Omega} \quad \forall\ u \in H^1(\Omega), \tag{4.7}$$

where $\gamma(u)$ is either $\gamma_\Omega(u)$, the average of u on Ω, or $\gamma_{\Gamma_0}(u)$, the average of u on $\Gamma_0 \subset \partial\Omega$ with $|\Gamma_0| = \text{meas}(\Gamma_0) > 0$, namely,

$$\gamma_\Omega(u) = \frac{1}{|\Omega|}\int_\Omega u dx, \quad \gamma_{\Gamma_0}(u) = \frac{1}{|\Gamma_0|}\int_{\Gamma_0} u ds.$$

We shall have occasion to use the following elementary inequality (see Grisvard [48]):

$$\|u\|_{0,\partial\Omega} \lesssim \varepsilon^{-1}\|u\|_{0,\Omega} + \varepsilon\,|u|_{1,\Omega} \quad \forall\ u \in H^1(\Omega),\ \varepsilon \in (0,1). \tag{4.8}$$

The following theorem is an important case of the so-called trace theorem (see Nečas [64]).

THEOREM 4.1. *The mapping $u \mapsto u|_{\partial\Omega}$, which is defined for $u \in C^1(\bar\Omega)$, has a unique continuous extension as an operator from $H^1(\Omega)$ onto $H^{1/2}(\partial\Omega)$, namely,*

$$\|v\|_{1/2,\partial\Omega} \lesssim \|v\|_{1,\Omega}, \quad |v|_{1/2,\partial\Omega} \lesssim |v|_{1,\Omega}. \tag{4.9}$$

This operator has a right-continuous inverse, i.e., there is a linear operator $E : H^{1/2}(\partial\Omega) \mapsto H^1(\Omega)$ such that

$$\|Ev\|_{1,G} \lesssim \|v\|_{1/2,\partial\Omega}, \quad |Ev|_{1,G} \lesssim |v|_{1/2,\partial\Omega} \quad \forall\ v \in H^{1/2}(\partial\Omega). \tag{4.10}$$

4.2. Properties of the finite element space. Let $\mathcal{T}^h$ be a quasi-uniform triangulation of the domain Ω and V^h be the corresponding piecewise-linear finite element space on $\mathcal{T}^h$, as defined in section 3. V^h has a natural nodal basis $\{\phi_i\}_{i=1}^{N_h}$ ($N_h = \dim(V^h)$) satisfying

$$\phi_i(x_l) = \delta_{il} \quad \forall\ i,l = 1,\dots,N_h,$$

where $\{x_l : l = 1,\dots,n\} \equiv \mathcal{N}_h$ is the set of all nodal points of the triangulation $\mathcal{T}^h$.

For any subset $G \subseteq \Omega$, we let

$$V^h(G) = \{v|_G : v \in V^h\} \ \text{ and } \ V_0^h(G) = V^h(G) \cap H_0^1(G).$$

In particular, we often write $V_0^h = V_0^h(\Omega)$.

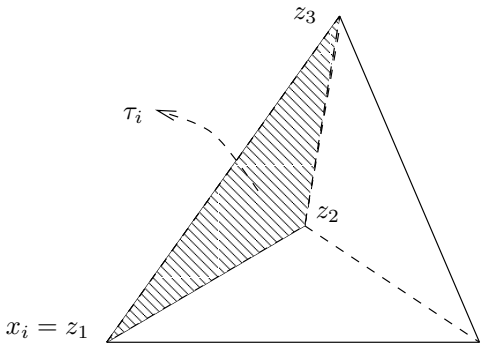

FIG. 4. *Nodal point x_i, 2-simplex τ_i, and vertices z_l $(l = 1, 2, 3)$.*

4.2.1. Interpolation operators. Interpolation operators are the basic tools in finite element analysis. Here we describe two types of such operators.

Standard nodal value interpolant. One of the most commonly used operators in the finite element analysis is the nodal value interpolant $I_h : C(\bar{\Omega}) \mapsto V^h$ defined by

$$(I_h v)(x) = \sum_{i=1}^{N_h} v(x_i)\phi_i(x).$$

Namely, $I_h v$ is the unique finite element function that is equal to v at all of the nodal points $\{x_i\}_{i=1}^{N_h}$.

The properties of the standard nodal value interpolant are discussed in many books, e.g., Ciarlet [29], where most standard finite element convergence theory can be found.

Average nodal value interpolant. The average nodal value interpolant is designed for interpolating functions that are not necessarily continuous.

Given $x_i \in \mathcal{N}_h$ (the set of nodal points in $\mathcal{T}^h$), let τ_i be an $(n-1)$-simplex from the triangulation $\mathcal{T}^h$ with vertices z_l $(l = 1, \ldots, n)$ such that $z_1 = x_i$ (see Fig. 4). The choice of τ_i is not unique in general, but if $x_i \in \partial\Omega$, we take $\tau_i \subset \partial\Omega$. Let $\theta_i \in \mathcal{P}_1(\tau_i)$ be the unique function satisfying

$$(\theta_i, \lambda_l)_{0,\tau_i} = \delta_{l1}, \quad l = 1, \ldots, n,$$

where λ_l is the barycentric coordinate of τ_i (see Ciarlet [29]) with respect to z_l.

Obviously,

$$(\theta_i, v)_{0,\tau_i} = v(x_i) \quad \text{if } v \in \mathcal{P}_1(\tau_i).$$

The average nodal value interpolant Π_h is then defined by

$$(\Pi_h v)(x) = \sum_{i=1}^{N_h} (\theta_i, v)_{0,\tau_i}\phi_i(x). \tag{4.11}$$

LEMMA 4.2. *The operator Π_h defined by* (4.11) *satisfies*

(a) $\Pi_h : H^1(\Omega) \mapsto V^h$ *and* $\Pi_h : H_0^1(\Omega) \mapsto V_0^h$,

(b) $v - \Pi_h v \in H_0^1(\Omega)$ *if* $v|_{\partial\Omega} \in V^h(\partial\Omega)$,

(c) $|v - \Pi_h v|_{t,\Omega} \lesssim h^{s-t}|v|_{s,\Omega} \quad \forall\, v \in H^s(\Omega)$ *$(s = 1, 2, t = 0, 1)$, and*

(d) $|\Pi_h v|_1 \lesssim |v|_{1,\Omega}$ *and* $\|\Pi_h v\|_{1,\Omega} \lesssim \|v\|_{1,\Omega} \quad \forall\, v \in H^1(\Omega)$.

For a proof of the above result we refer to Scott and Zhang [69], where the average nodal value interpolant was proposed. Other similar interpolants can be found in Clément [30].

4.2.2. Discrete harmonic functions. Discrete harmonic functions play a crucial role in nonoverlapping DD methods. Let us first recall, on the continuous level, that a function $u \in H^1(\Omega)$ is called *harmonic* if

$$(\nabla u, \nabla v) = 0 \quad \forall\, v \in H_0^1(\Omega).$$

Thus if v is such that $v - u \in H_0^1(\Omega)$, then $(\nabla u, \nabla(v-u)) = 0$ and

$$(\nabla u, \nabla u) \le (\nabla u, \nabla u) + (\nabla(v-u), \nabla(v-u)) = (\nabla v, \nabla v).$$

This, together with Theorem 4.1, implies

$$|u|_{1,\Omega} = \inf_{v-u \in H_0^1(\Omega)} |v|_{1,\Omega} \eqsim |u|_{1/2,\partial\Omega}. \tag{4.12}$$

A finite element function $u^h \in V^h(\Omega)$ is called *discrete harmonic* if

$$(\nabla u^h, \nabla v) = 0 \quad \forall\, v \in V_0^h(\Omega).$$

Similar to (4.12), the following holds for any discrete harmonic function u^h:

$$|u^h|_{1,\Omega} = \inf_{v-u^h \in V_0^h(\Omega)} |v|_{1,\Omega} \eqsim |u^h|_{1/2,\partial\Omega}. \tag{4.13}$$

In fact, the equality in (4.13) follows again from the orthogonality

$$(\nabla u^h, \nabla(v-u^h)) = 0 \quad \forall\, v \in V^h(\Omega) \text{ such that } v - u^h \in V_0^h(\Omega).$$

To see the equivalence in (4.13), let $U \in H^1(\Omega)$ be the continuous harmonic function such that $U = u^h$ on $\partial\Omega$. Then by (4.12), $|U|_{1,\Omega} \eqsim |u^h|_{1/2,\partial\Omega}$. Let Π_h be as defined in (4.11). It then follows from Lemma 4.2(d) that

$$|u^h|_{1,\Omega} = \inf_{v-u^h \in V_0^h(\Omega)} |v|_{1,\Omega} \lesssim |\Pi_h U|_{1,\Omega} \lesssim |U|_{1,\Omega} \eqsim |u^h|_{1/2,\partial\Omega}.$$

This completes the justification of (4.13).

We now consider the full-weighted H^1 inner product

$$(u,v)_{1,\Omega} = d^{-2}(u,v)_{0,\Omega} + (\nabla u, \nabla v)_{0,\Omega} \quad \forall\, u, v \in H^1(\Omega).$$

A finite element function $u^h \in V^h(\Omega)$ is called *generalized discrete harmonic* if

$$(u^h, v)_{1,\Omega} = 0 \quad \forall\, v \in V_0^h(\Omega). \tag{4.14}$$

Again by Theorem 4.1 and the average nodal value interpolant (see Lemma 4.2(d)), we have

$$\|u^h\|_{1,\Omega} = \inf_{v-u^h \in V_0^h(\Omega)} \|u^h\|_{1,\Omega} \lesssim \|u^h\|_{1/2,\partial\Omega}. \tag{4.15}$$

If, in addition, the mean value of u^h, $\gamma_{\partial\Omega}(u^h)$, on $\partial\Omega$ vanishes or u^h vanishes on one face of Ω, then

$$|u^h|_{1,\Omega} \eqsim |u^h|_{1/2,\partial\Omega}. \tag{4.16}$$

4.2.3. Discrete Sobolev norms. In finite element spaces, certain Sobolev norms can be equivalently expressed in a discrete manner.

Some basic norm equivalence results that hold for any $v \in V^h$ include

$$\|v\|_{0,\Omega}^2 \eqsim h^n \sum_{x_i \in \mathcal{N}_h} v^2(x_i), \tag{4.17}$$

$$|v|_{1,\Omega}^2 \eqsim h^{n-2} \sum_{\tau \in \mathcal{T}_h} \sum_{x_i, x_j \in \tau \cap \mathcal{N}_h} (v(x_i) - v(x_j))^2, \tag{4.18}$$

and

$$|v|_{1/2,F}^2 \eqsim h^{2(n-1)} \sum_{\substack{x_i, x_j \in F \cap \mathcal{N}_h \\ x_i \neq x_j}} \frac{(v(x_i) - v(x_j))^2}{|x_i - x_j|^n}. \tag{4.19}$$

For convenience of subsequent applications, let us now elaborate on the discrete L^2-norms. Let K be the entire boundary $\partial\Omega$ or a face F of $\partial\Omega$, or a union of edges of $\partial\Omega$, or simply Ω itself. We define

$$\langle v, w \rangle_{h,K} = h^\alpha \sum_{x_i \in K \cap \mathcal{N}_h} v(x_i) w(x_i) \quad \forall\, v, w \in V^h(K), \tag{4.20}$$

where $\alpha = \dim(K)$, and the corresponding norm $\|v\|_{h,K} = \langle v, v \rangle_{h,K}^{1/2}$.

We have by the quasi uniformity of $\mathcal{T}^h$ that

$$\|v\|_{h,K} \eqsim \|v\|_{0,K} \quad \forall\, v \in V^h(K). \tag{4.21}$$

Similar to the average value $\gamma_{_K}(v)$ of v on the set K, we define its discrete version by

$$\gamma_{_{h,K}}(v) = \langle v, 1 \rangle_{h,K} / \langle 1, 1 \rangle_{h,K}.$$

It is easy to see that

$$|\gamma_{_K}(v)| \lesssim d^{-\alpha/2} \|v\|_{0,K}, \quad |\gamma_{_{h,K}}(v)| \lesssim d^{-\alpha/2} \|v\|_{0,K}. \tag{4.22}$$

LEMMA 4.3. *Assume that for $n = 3$, Ω is a tetrahedron and u a continuous function on $\partial\Omega$ and linear on each face; for $n = 2$, Ω is a triangle or quadrilateral and u a continuous function on $\partial\Omega$ and linear on each edge. Then*

$$|u|_{1/2,\partial\Omega}^2 \eqsim d^{n-2} \sum_{E_{kl} \subset \partial\Omega} (u(\mathrm{v}_k) - u(\mathrm{v}_l))^2, \tag{4.23}$$

where E_{kl} is the edge with vertices v_k and v_l as its endpoints.

Proof. For $n = 3$, let $\bar{u}$ be the unique linear function on $\bar{\Omega}$ such that $\bar{u}|_{_{\partial\Omega}} = u$. It is easy to verify that a linear function on $\bar{\Omega}$ must be discrete harmonic in Ω, so we have $|u|_{1/2,\partial\Omega} \eqsim |\bar{u}|_{1,\Omega}$ from (4.13). Noting that $\bar{u}$ and u are equal at all vertices of Ω, we immediately see from (4.18) that

$$|u|_{1/2,\partial\Omega}^2 \eqsim |\bar{u}|_{1,\Omega}^2 \eqsim d \sum_{E_{kl} \subset \partial\Omega} (u(\mathrm{v}_k) - u(\mathrm{v}_l))^2.$$

For $n = 2$, taking any edge $E_{kl} \subset \partial\Omega$, the linearity of u on E_{kl} implies

$$\frac{|u(\mathrm{v}_k) - u(\mathrm{v}_l)|}{d} = \frac{|u(x) - u(y)|}{|x - y|} \frac{|\mathrm{v}_k - \mathrm{v}_l|}{d} \lesssim \frac{|u(x) - u(y)|}{|x - y|} \quad \forall\, x, y \in E_{kl}.$$

Integrating both sides of the inequality over E_{kl} and by the definition of $|\cdot|_{1/2,\partial\Omega}$, we obtain

$$(u(\mathrm{v}_k) - u(\mathrm{v}_l))^2 \lesssim \int_{E_{kl}} \int_{E_{kl}} \frac{(u(x)-u(y))^2}{|x-y|^2} ds(x)\, ds(y) \le |u|^2_{1/2,\partial\Omega}.$$

To derive the other direction of (4.23), we consider Ω to be a quadrilateral; the proof for the triangular case is similar. We subdivide the quadrilateral into two triangles by connecting its two opposite vertices, and then extend u naturally onto the whole quadrilateral. Clearly, the resulting function, denoted as $\hat{u}$, is linear on each triangle and is in $H^1(\Omega)$. Hence by the trace Theorem 4.1,

$$|u|^2_{1/2,\partial\Omega} \lesssim |\hat{u}|^2_{1,\Omega} \lesssim \sum_{\mathrm{v}_k,\mathrm{v}_l \in \partial\Omega} (u(\mathrm{v}_k) - u(\mathrm{v}_l))^2 \eqsim \sum_{E_{kl} \subset \partial\Omega} (u(\mathrm{v}_k) - u(\mathrm{v}_l))^2.$$

The estimate (4.23) then follows. □

Similar to (4.3), we have the following equivalence for the $H^{1/2}_{00}$-norm.

$$\|v\|_{H^{1/2}_{00}(F)} \eqsim \langle (-\Delta_{F,h})^{1/2} v, v\rangle_{0,F} \quad \forall\, v \in V^h_0(F), \tag{4.24}$$

where $\Delta_{F,h}$ is the discrete Laplace operator in $V^h_0(F)$. The above relation is equivalent to the following:

$$\|(-\Delta)^{1/4} v\|_{0,F} \eqsim \|(-\Delta_{F,h})^{1/4} v\|_{0,F} \quad \forall\, v \in V^h_0(F),$$

which is well known; see Bank and Dupont [2], Xu [79, 82].

4.2.4. Discrete Sobolev inequalities. Let us first state the following inverse inequalities that hold for all $v \in V^h(\Omega)$:

$$\|v\|_{1,\infty,\Omega} \lesssim h^{-1} \|v\|_{0,\infty,\Omega}, \tag{4.25}$$

$$\|v\|_{0,\infty,\Omega} \lesssim h^{-n/2} \|v\|_{0,\Omega}, \tag{4.26}$$

$$\|v\|_{1,\Omega} \lesssim h^{-1} \|v\|_{0,\Omega}, \tag{4.27}$$

$$\|v\|_{1/2,\partial\Omega} \lesssim h^{-1/2} \|v\|_{0,\partial\Omega}, \tag{4.28}$$

$$\|v\|_{1,\partial\Omega} \lesssim h^{-1/2} \|v\|_{1/2,\partial\Omega}. \tag{4.29}$$

The first three inverse inequalities above are well known, and their proofs can be found in most finite element textbooks, e.g., Ciarlet [29]. Inequalities such as (4.28) and (4.29) and their proofs may be found in Bramble, Pasciak, and Xu [16] and Xu [81]. As an illustration, let us now include a proof for (4.28).

In fact, given $v \in V^h(\partial\Omega)$, if $\hat{v}$ denotes the function in $V^h(\Omega)$ which vanishes at all interior nodes of Ω but equals v at the boundary nodes of Ω, then by (4.9), (4.27), and (4.21),

$$\begin{aligned} \|v\|^2_{1/2,\partial\Omega} &\lesssim d^{-2} \|\hat{v}\|^2_{0,\Omega} + |\hat{v}|^2_{1,\Omega} \lesssim h^{-2} \|\hat{v}\|^2_{0,\Omega} \\ &\lesssim h^{-1} \sum_{x_i \in \partial\Omega \cap \mathcal{N}_h} h^{n-1} v^2(x_i) \eqsim h^{-1} \|v\|^2_{0,\partial\Omega}. \end{aligned}$$

This proves (4.28).

It is known that $H^1(\Omega)$ cannot be embedded into $C^0(\bar{\Omega})$ in two dimensions. But for finite element functions, we have the following well-known result.

LEMMA 4.4. *Let $n = 2$. For any $u \in V^h(\Omega)$,*

$$\|u\|_{0,\infty,\Omega} \lesssim \left(\log \frac{d}{h}\right)^{1/2} \|u\|_{1,\Omega}.$$

Lemma 4.4 can be traced back in Bramble [8]. It is a direct consequence of the following more general inequality (see Xu [79]):

$$\|u\|_{0,\infty,\Omega} \lesssim |\log \varepsilon|^{1/2} \|u\|_{1,\Omega} + \varepsilon d \, \|\nabla u\|_{0,\infty,\Omega} \quad \forall\, u \in W^{1,\infty}(\Omega),\ \varepsilon \in (0, 1/2),$$

taking ε to be an appropriate fraction of h/d and using the inverse inequality (4.25).

The next lemma, which was proved by Dryja [34] using different techniques, is an analogue to Lemma 4.4 on the boundary of Ω.

LEMMA 4.5. *Let $n = 2$. For any $u \in V^h(\partial\Omega)$,*

$$\|u\|_{0,\infty,\partial\Omega} \lesssim \left(\log \frac{d}{h}\right)^{1/2} \|u\|_{1/2,\partial\Omega}.$$

Proof. Let $\tilde{u}$ be the generalized discrete harmonic extension of u in Ω. Then by Lemma 4.4,

$$\|u\|_{0,\infty,\partial\Omega} \lesssim \|\tilde{u}\|_{0,\infty,\Omega} \lesssim \left(\log \frac{d}{h}\right)^{1/2} \|\tilde{u}\|_{1,\Omega},$$

which, combined with (4.15), gives the desired result. ∎

We next present some Poincaré and inverse inequalities on the boundary.

LEMMA 4.6. *Let $n = 2, 3$. For any $v \in V^h(\partial\Omega)$,*

$$d^{-1/2}\|v - \gamma_{\partial\Omega}(v)\|_{0,\partial\Omega} \lesssim |v|_{1/2,\partial\Omega} \lesssim h^{-1/2}\|v - \gamma_{\partial\Omega}(v)\|_{0,\partial\Omega} \tag{4.30}$$

and, in terms of discrete L^2-norms (see (4.20)),

$$d^{-1/2}\|v - \gamma_{h,\partial\Omega}(v)\|_{h,\partial\Omega} \lesssim |v|_{1/2,\partial\Omega} \lesssim h^{-1/2}\|v - \gamma_{h,\partial\Omega} v\|_{h,\partial\Omega}. \tag{4.31}$$

Proof. The minimization property of the average value in the constant space and (4.21) imply (let $K = \partial\Omega$)

$$\begin{aligned}\|v - \gamma_{h,\partial\Omega}(v)\|_{h,\partial\Omega} &\le \|v - \gamma_{\partial\Omega}(v)\|_{h,\partial\Omega} \lesssim \|v - \gamma_{\partial\Omega}(v)\|_{0,\partial\Omega} \\ &\le \|v - \gamma_{h,\partial\Omega}(v)\|_{0,\partial\Omega} \lesssim \|v - \gamma_{h,\partial\Omega}(v)\|_{h,\partial\Omega}.\end{aligned}$$

Hence $\|v-\gamma_{h,\partial\Omega}(v)\|_{h,\partial\Omega} \eqsim \|v-\gamma_{\partial\Omega}(v)\|_{0,\partial\Omega}$. Therefore it suffices to prove either (4.30) or (4.31). Let us show (4.30).

The second inequality in (4.30) follows directly from (4.28), while the first is obtained by (4.9) and (4.16):

$$d^{-1/2}\|v - \gamma_{\partial\Omega}(v)\|_{0,\partial\Omega} \lesssim \|\tilde{v} - \gamma_{\partial\Omega}(v)\|_{1,\Omega} \lesssim |\tilde{v}|_{1,\Omega} \lesssim |v|_{1/2,\partial\Omega},$$

where $\tilde{v}$ is the generalized discrete harmonic extension of v. ∎

4.3. Vertex-edge-face lemmas for finite element functions. The results to be presented below concern the finite element functions restricted in some way on sets of vertices, edges, or faces. These sets will always be understood to be relatively open, namely an edge set does not include its end vertices and a face set does not include its boundary edges. In two dimensions, an edge set is sometimes also referred to as a face set.

Given a subset $K \subset \Omega$, define a restriction operator $I_K^0 : V^h \mapsto V_0^h(K)$ by

$$(I_K^0 v)(x) = \sum_{x_i \in K \cap \mathcal{N}_h} v(x_i)\phi_i(x).$$

Namely, $I_K^0 v$ is obtained by dropping v to be zero on ∂K. Similarly, we define $I_K^0 : V^h(\Gamma) \mapsto V_0^h(K)$ for any subset K on the interface Γ.

We first present a simple result concerning vertex sets.

LEMMA 4.7 (vertex lemma). *If* v *is a vertex of* $\partial\Omega$*, then for any* $u \in V^h(\partial\Omega)$*,*

$$\|I_{\mathrm{V}}^0 u\|_{1/2,\partial\Omega} \lesssim h^{(n-2)/2}|u(\mathrm{V})| \lesssim \left(\log \frac{d}{h}\right)^{1/2} \|u\|_{1/2,\partial\Omega}, \tag{4.32}$$

where $u(\mathrm{V})$ *is the value of* u *at the vertex* v.

Proof. The first part of (4.32) follows from the inverse inequality (4.28) and (4.21):

$$\|I_{\mathrm{V}}^0 u\|_{1/2,\partial\Omega} \lesssim h^{-1/2}\|I_{\mathrm{V}}^0 u\|_{0,\partial\Omega} \eqsim h^{(n-2)/2}|u(\mathrm{V})|.$$

Now we prove the last part of the inequality in (4.32). For $n = 2$, it is a direct consequence of Lemma 4.5. For $n = 3$, let F be a face of $\partial\Omega$ having $\{\mathrm{V}\}$ as one of its vertices. Then by Lemma 4.4 and the inverse inequality (4.29), we deduce that

$$hu^2(\mathrm{V}) \lesssim h \log \frac{d}{h}\|u\|_{1,F}^2 \lesssim \log \frac{d}{h}\|u\|_{1/2,F}^2 \lesssim \log \frac{d}{h}\|u\|_{1/2,\partial\Omega}^2. \qquad \square$$

For the proof of the results concerning edges and faces, the following simple technical result concerning a limiting case of the trace theorem is useful.

LEMMA 4.8. *Let* $\Omega = (0,d)^3$*. Then for any* $u \in V^h(\partial\Omega)$*,*

$$\int_0^d \max_{0\le x\le d} u^2(x,y,0)dy \lesssim \log \frac{d}{h}\|u\|_{1/2,\partial\Omega}^2.$$

Similar results also hold by interchanging the positions of x, y, z*, and for a tetrahedron.*

Proof. Let $\tilde{u}$ be the generalized discrete harmonic extension of $u \in V^h(\partial\Omega)$ into Ω and set $\Delta_s = \Omega \cap \{y = s\}$. By Lemma 4.4, we have

$$\int_0^d \max_{0\le x\le d} |u(x,y,0)|^2 dy \le \log \frac{d}{h}\int_0^d \|\tilde{u}\|_{1,\Delta_y}^2 dy = \log \frac{d}{h}\|\tilde{u}\|_{1,\Omega}^2 \lesssim \log \frac{d}{h}\|u\|_{1/2,\partial\Omega}^2. \qquad \square$$

LEMMA 4.9 (edge lemma). *Assume that* $n = 3$ *and* E *is an edge of* $\partial\Omega$*. Then for any* $u \in V^h(\partial\Omega)$*,*

$$\|I_E^0 u\|_{1/2,\partial\Omega} \lesssim \|u\|_{0,E} \lesssim \left(\log \frac{d}{h}\right)^{1/2} \|u\|_{1/2,\partial\Omega}. \tag{4.33}$$

Proof. The first part of (4.33) follows from the inverse inequality (4.28) and (4.21), namely,

$$\|I_E^0 u\|_{1/2,\partial\Omega} \lesssim h^{-1/2}\|I_E^0 u\|_{0,\partial\Omega} \eqsim \|u\|_{0,E}.$$

To prove the last inequality in (4.33), for simplicity we assume that $\Omega = (0,d)^3$ and the edge is on the z-axis. It then follows from Lemma 4.8 that

$$\|u\|_{0,E}^2 = \int_0^d u^2(0,0,z)\,dz \lesssim \log \frac{d}{h}\|u\|_{1/2,\partial\Omega}^2. \qquad \square$$

LEMMA 4.10 (face lemma). *Let F be a face of $\partial\Omega (n = 3)$ or an edge of $\partial\Omega$ $(n = 2)$. Then for any $u \in V^h(\partial\Omega)$,*

$$\|I_F^0 u\|_{1/2,\partial\Omega} \lesssim \log \frac{d}{h} \|u\|_{1/2,\partial\Omega}.$$

Proof. By definition,

$$\|I_F^0 u\|^2_{1/2,\partial\Omega} \eqsim \|I_F^0 u\|^2_{H_{00}^{1/2}(F)} \eqsim |I_F^0 u|^2_{1/2,F} + \int_F \frac{(I_F^0 u)^2(x)}{\operatorname{dist}(x, \partial F)} ds(x). \tag{4.34}$$

Note that on F, $I_F^0 u = u - I_{\partial F}^0 u$. Thus it follows by definition and Lemma 4.9 that

$$|I_F^0 u|_{1/2,F} \le |u|_{1/2,F} + |I_{\partial F}^0 u|_{1/2,F} \lesssim \left(\log \frac{d}{h}\right)^{1/2} \|u\|_{1/2,\partial\Omega}. \tag{4.35}$$

To estimate the second term in (4.34), without loss of generality, we assume $\Omega = (0, d)^n$. Let us first demonstrate the case when $n = 3$. The proof here follows Bramble, Pasciak, and Schatz [12].

Consider the face on the xy-plane, i.e., $F = \Omega \cap \{z = 0\}$, and the term

$$\begin{aligned}\int_F \frac{(I_F^0 u)^2(x, y, 0)}{x}\, dxdy &= \left(\int_0^d dy \int_0^{x_1} + \int_0^d dy \int_{x_1}^d\right) \frac{(I_F^0 u)^2(x, y, 0)}{x}\, dx \\ &\equiv I_1 + I_2.\end{aligned}$$

Here x_1 is the first node on the x-axis. Since $(I_F^0 u)(0, y, 0) = 0$, we have

$$\begin{aligned} I_1 &\equiv \int_0^d dy \int_0^{x_1} \frac{(I_F^0 u)^2(x, y, 0)}{x}\, dx \\ &\lesssim h^2 \int_0^d \max_{0\le x\le x_1} |\partial_x I_F^0 u(x, y, 0)|^2 dy \\ &\lesssim \int_0^d \max_{0\le x\le d} |u(x, y, 0)|^2 dy \text{ (by inverse inequality (4.25))} \\ &\lesssim \log \frac{d}{h} \|u\|^2_{1/2,\partial\Omega} \text{ (by Lemma 4.8).}\end{aligned}$$

For I_2, by a direct computation, we obtain

$$\begin{aligned} I_2 &\equiv \int_0^d dy \int_{x_1}^d \frac{(I_F^0 u)^2(x, y, 0)}{x}\, dx \\ &\lesssim \log \frac{d}{h} \Big(\int_0^d \max_{0\le x\le d} |u(x, y, 0)|^2 dy \Big) \\ &\lesssim \Big(\log \frac{d}{h}\Big)^2 \|u\|^2_{1/2,\partial\Omega} \text{ (by Lemma 4.8).} \end{aligned} \tag{4.36}$$

Hence I_2 is also bounded as required, which ends the proof for $n = 3$.

Next we prove the result for $n = 2$. Assume $F = \{(x, 0) : 0 \le x \le d\}$ and x_1 is the first node on the x-axis. Let $w(x) = I_F^0 v(x, 0)$. Note $w(0) = 0$. Then by the mean value theorem and inverse inequality,

$$\begin{aligned}\int_0^d \frac{|w(x)|^2}{x} dx &= \int_0^{x_1} \frac{|w(x)|^2}{x} dx + \int_{x_1}^d \frac{|w(x)|^2}{x} dx \\ &\lesssim \left\| \frac{\partial w}{\partial x} \right\|^2_{L^\infty(0,d)} \int_0^{x_1} x dx + \|w\|^2_{L^\infty(0,d)} \int_{x_1}^d \frac{dx}{x} \\ &\lesssim \log \frac{d}{h} \|w\|^2_{L^\infty(0,d)}.\end{aligned}$$

Similarly,

$$\int_0^d \frac{|w(x)|^2}{1-x}dx \lesssim \log\frac{d}{h}\|w\|^2_{L^\infty(0,d)}.$$

Combining the previous two inequalities with Lemma 4.5 completes the proof for $n = 2$. □

The following result concerns a more precise estimate for a special case of the preceding lemma.

LEMMA 4.11. *Let F be a face ($n = 3$) or an edge ($n = 2$) of $\partial\Omega$. Then*

$$\| I_F^0 1\|_{1/2,\partial\Omega} \lesssim d^{(n-2)/2}\Big(\log\frac{d}{h}\Big)^{1/2}.$$

Proof. A direct application of Lemma 4.10 with $u \equiv 1$ gives

$$\| I_F^0 1\|_{1/2,\partial\Omega} \lesssim d^{(n-2)/2}\log\frac{d}{h}.$$

But the extra factor $(\log d/h)^{1/2}$ can be removed from the above estimate by observing the extra logarithmic factor is no longer in (4.36) when $u \equiv 1$. □

We next present two more results concerning edges.

LEMMA 4.12. *Let $n = 2$. For any edge $E \subset \partial\Omega$ and any $v \in V^h(\partial\Omega)$,*

$$\|v - I_{\partial\Omega}v\|_{H_{00}^{1/2}(E)} \lesssim \log\frac{d}{h}|v|_{1/2,\partial\Omega},$$

where $I_{\partial\Omega}v$ equals v on the corner nodes of $\partial\Omega$ and is linear on each edge of $\partial\Omega$.

Proof. For simplicity, we consider only $\Omega = (0,d)^n$ and $E = \{(x,0) : 0 \le x \le d\}$. Let $w = v - I_{\partial\Omega}v$. Following the same lines of proof for the case $n = 2$ in Lemma 4.10, we conclude that

$$\int_0^d \left(\frac{|w(x)|^2}{x} + \frac{|w(x)|^2}{1-x}\right)dx \lesssim \log\frac{d}{h}\|w\|^2_{L^\infty(0,d)}.$$

But by Lemma 4.5,

$$\|w\|_{L^\infty(0,d)} \lesssim \left(\log\frac{d}{h}\right)^{1/2}\|v\|_{1/2,\partial\Omega}.$$

The desired estimate then follows. □

LEMMA 4.13. *Let $n = 3$. For any $u \in V^h(\Omega)$ vanishing on one edge of Ω,*

$$\|u\|_{0,\Omega} \lesssim d\left(\log\frac{d}{h}\right)^{1/2}|u|_{1,\Omega}.$$

Proof. The proof follows [40]. Without loss of generality, assume $\Omega = (0,d)^3$ and that the edge is on the z-axis. Let $\Delta_s = \Omega \cap \{z = s\}$. By the assumption, for $z \in (0,d)$, u vanishes at least at one point in Δ_z. Thus for any constant c,

$$\max_{(x,y)\in\Delta_z}|u(x,y,z)| \le \max_{(x,y)\in\Delta_z}|u(x,y,z)-c| + |c| \le 2\max_{(x,y)\in\Delta_z}|u(x,y,z)-c|.$$

Thus by Lemma 4.4 and Friedrichs's inequality (choosing $c = \gamma_\Omega(u)$),

$$\begin{aligned}\|u\|^2_{0,\Omega} &\le d^2\int_0^d \max_{(x,y)\in\Delta_z}|u(x,y,z)|^2dz \le 4\,d^2\int_0^d \max_{(x,y)\in\Delta_z}|u(x,y,z)-c|^2dz\\ &\le 4d^2\log\frac{d}{h}\int_0^d \|u-c\|^2_{1,\Delta_z}dz\\ &= 4d^2\log\frac{d}{h}\,\|u-c\|^2_{1,\Omega} \lesssim d^2\log\frac{d}{h}\,|u|^2_{1,\Omega}.\end{aligned}$$

□

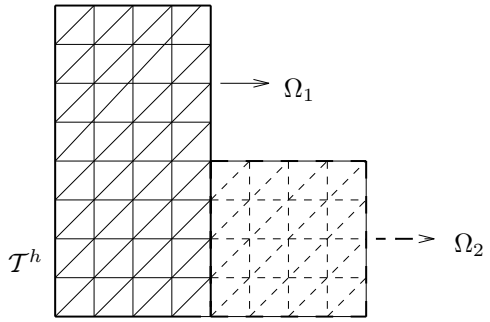

FIG. 5. *Subdomains* Ω_1 *and* Ω_2.

5. Substructuring methods. The methods to be presented in this section, often known as *substructuring* methods, were first studied by Bramble, Pasciak and Schatz [10, 12]. We shall first give an outline of the major ideas behind these methods and then discuss several variants of the methods.

5.1. Outline. For clarity, we shall first examine a simple example using a DD consisting of two subdomains. Assume that $\bar{\Omega} = \bar{\Omega}_1 \bigcup \bar{\Omega}_2$ with $F = \partial\Omega_1 \cap \partial\Omega_2$ (see Fig. 5).

In this case, for the interface bilinear form $S(\cdot,\cdot)$ defined in (3.9) or (3.10), we have, with $\rho_{_F} = (\rho_1 + \rho_2)/2$,

$$S(u,u) \eqsim \rho_{_F}\, |u|^2_{H^{1/2}_{00}(F)} \quad \forall\ u \in V^h(\Gamma).$$

The interface $\Gamma = F$ is either an interval for $n = 2$ or a polygonal domain for $n = 3$. In both cases, by (4.24), the following spectral equivalence holds:

$$S(u,u) \eqsim \rho_{_F} |u|^2_{H^{1/2}_{00}(F)} \eqsim \rho_{_F} \langle (-\Delta_{F,h})^{1/2} u, u\rangle_{0,F}, \tag{5.1}$$

where $-\Delta_{F,h}$ is the discrete Laplacian on the set F. Therefore, the operator S can be preconditioned by

$$T = \rho_{_F}^{-1} (-\Delta_{F,h})^{-1/2}. \tag{5.2}$$

Note that T is a global operator and is generally difficult to evaluate. But when the grid on F is uniform, then this operator can be evaluated efficiently by the fast Fourier transform (FFT) method (cf. Dryja [34], Bramble, Pasciak, and Schatz [10]). In fact, even if the grid on F is not completely uniform, say only quasiuniform, the FFT method may also be employed to obtain a preconditioner for S. Since by (4.19), we may adjust the nodal points on F to make them uniform, the corresponding $H^{1/2}$-norm is still equivalent to the original norm. Therefore, on the algebraic level, we can still use (5.2) on the uniform grid to get an asymptotically well-conditioned preconditioner for S that is associated with the quasi-uniform grid. For two-dimensional second-order elliptic problems with constant coefficients, we can find the exact eigendecomposition for the interface operator S arising from the finite difference discretization of the problems. We refer to Bjørstad and Widlund [6], Chan and Hou [24].

Methods other than FFT for preconditioning S are also possible; see Xu and Zhang [85] or many methods listed in Smith, Bjørstad, and Gropp [73, Chap. 4].

Multiple subdomains. The above procedure extends directly to the decomposition consisting of multiple subdomains without cross points (see Fig. 6). If cross

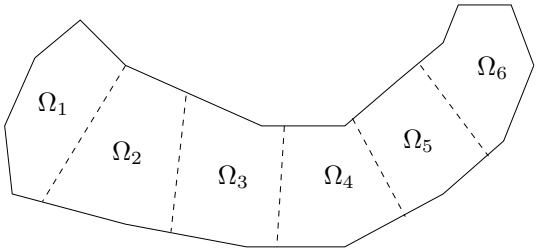

FIG. 6. *Multiple subdomains without cross points.*

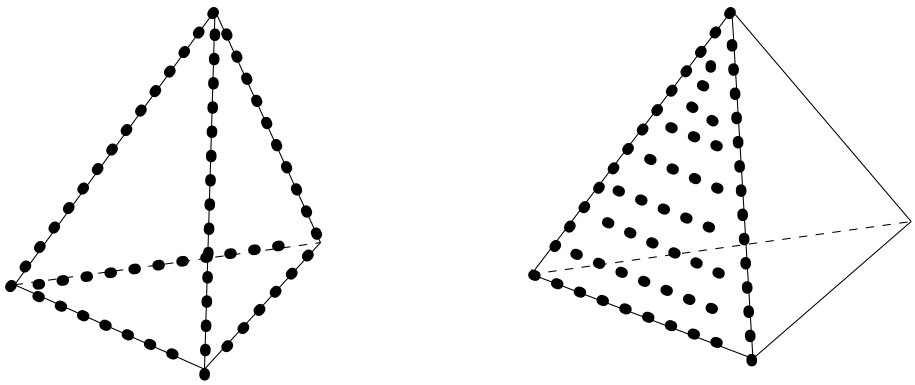

FIG. 7. *Left: nodes on the joint-set $\mathcal{W}$; right: nodes on a face F and its boundary ∂F.*

points are present in the decomposition (e.g., see Fig. 1), however, the situation is much more complicated. For one thing, the interface set Γ is nontrivial and the relation (5.1) is no longer available. In this case, the main idea is to split the interface Γ into small and local substructures on which relation (5.1) may be applied.

The local substructures on the interface Γ we expect to use are obviously edge sets in two dimensions and face sets in three dimensions. For convenience, we shall use the term *joint-set* and the notation $\mathcal{W}$ to represent the set that connects these substructures. The joint-set $\mathcal{W}$ consists of those points belonging to more than two subdomains for $n = 2$ and those edges belonging to more than two subdomains for $n = 3$. For $n = 3$, the joint-set $\mathcal{W}$ looks like a wire-basket (see Fig. 7) and is known as *wire-basket* set in some DD literature.

The "splitting" of Γ is realized by a properly constructed linear operator $\Pi_0 : V^h(\Gamma) \to V^h(\Gamma)$ that is invariant on the joint-set $\mathcal{W}$:

$$(\Pi_0 v)(x) = v(x) \quad \forall\, x \in \mathcal{W}.$$

By means of this operator, we may further split $u_{_H}$ as $u_{_H} = (u_{_H} - \Pi_0 u_{_H}) + \Pi_0 u_{_H}$. Note that $u_{_H} - \Pi_0 u_{_H}$ vanishing on the joint-set $\mathcal{W}$ can be "localized." Although the function $\Pi_0 u_{_H}$ is still global, it is not a major problem if we can force the range of Π_0, often known as a *coarse* space, to be simple and to have relatively small dimensions. If the operator Π_0 has an appropriate boundedness property in the $H^{1/2}$-norm, we then have

$$\sum_{i=1}^{J} \rho_i |u_{_H}|^2_{1/2,\partial\Omega_i} \sim \sum_{i=1}^{J} \rho_i |u_{_H} - \Pi_0 u_{_H}|^2_{1/2,\partial\Omega_i} + \sum_{i=1}^{J} \rho_i |\Pi_0 u_{_H}|^2_{1/2,\partial\Omega_i}$$

$$\sim \sum_{F\subset\Gamma} \rho_F |u_H - \Pi_0 u_H|^2_{H^{1/2}_{00}(F)} + \sum_{i=1}^{J} \rho_i |\Pi_0 u_H|^2_{1/2,\partial\Omega_i}, \tag{5.3}$$

where $F \subset \Gamma$ is any edge for $n = 2$ and any face for $n = 3$.

The expression in (5.3) naturally leads to a preconditioner that depends on local solvers and a small global solver. In fact, the term $|u_H - \Pi_0 u_H|^2_{H^{1/2}_{00}(F)}$ can be replaced by a computationally more efficient quadratic term (similar to (5.1)), which will be discussed below. The resulting preconditioner in such a process is often known as a *substructuring* preconditioner.

As we note, the crucial technical aspect in a substructuring method is the operator Π_0. Such an operator may be known as a *joint-operator*, whereas $I-\Pi_0$ may be known as a *decomposition-operator*. The construction of Π_0 will be discussed in detail below.

The above "splitting" or decomposition process naturally leads to a decomposition of $V^h(\Gamma)$ into a sum of subspaces as follows:

$$V^h(\Gamma) = V_0 + \sum_{F\subset\Gamma} V_0^h(F),$$

with

$$V_0 = \text{range } (\Pi_0), \quad V_0^h(F) = \{v \in V^h(\Gamma);\ \text{supp } v \subset F\}.$$

As a result, the framework of space decomposition and subspace correction (see discussions in subsection 2.2) can also be applied to construct PSC preconditioners.

5.2. Joint-set coarse subspaces. We shall now discuss the construction of coarse subspaces. As mentioned in the previous subsection, the coarse subspace V_0 of $V^h(\Gamma)$ may be viewed as the range of a linear operator Π_0 on $V^h(\Gamma)$, namely,

$$V_0 = \Pi_0(V^h(\Gamma)),\ \text{the range of}\ \ \Pi_0.$$

The effective application of a coarse subspace in DD depends on its approximation property in the L^2-norm and its stability property in the $H^{1/2}$-norm. Namely, it depends on the constants α_0 and α_1 (which usually depend logarithmically on h and h_0) in the following estimates:

$$\|u - \Pi_0 u\|_{L^2_\rho(\Gamma)} \lesssim \alpha_0 h_0^{1/2} |u|_{H^{1/2}_\rho(\Gamma)}, \quad |\Pi_0 u|_{H^{1/2}_\rho(\Gamma)} \lesssim \alpha_1 |u|_{H^{1/2}_\rho(\Gamma)}, \tag{5.4}$$

where

$$|v|^2_{H^{1/2}_\rho(\Gamma)} = \sum_{i=1}^{J} \rho_i |v|^2_{1/2,\Gamma_i}, \quad \|v\|^2_{L^2_\rho(\Gamma)} = \sum_{i=1}^{J} \rho_i \|v\|^2_{0,\Gamma_i} \quad \forall\ v \in V^h(\Gamma).$$

Before discussing the particular construction of the operator Π_0, let us summarize the desired properties of this operator.

(C1) It is invariant on the joint-set $\mathcal{W}$.
(C2) It is invariant on constant functions.
(C3) The bounds α_0 and α_1 in (5.4) are not large.
(C4) Its range is simple and has relatively small dimension.

5.2.1. Two-dimensional case. The most natural and effective coarse subspace of $V^h(\Gamma)$ in two dimensions is the standard coarse subspace V_0 generated from the coarse grid given by the subdomains $\{\Omega_i\}$. This coarse subspace V_0 is defined to be the space consisting of all continuous functions on the interface Γ which are linear on

each edge of all subdomains Ω_i. Obviously, this space V_0 is just the restriction on Γ of the piecewise-linear finite element space corresponding to $\{\Omega_i\}$ in the special case that all Ω_i's are triangles.

Let Π_0 be the nodal value interpolant associated with the coarse space V_0. We have for any $u \in V^h(\Gamma)$ that

$$|\Pi_0 u|^2_{1/2,\partial\Omega_i} \lesssim \log \frac{h_0}{h} \, |u|^2_{1/2,\partial\Omega_i}, \tag{5.5}$$

$$\|u - \Pi_0 u\|^2_{0,\partial\Omega_i} \lesssim h_0 \log^2 \frac{h_0}{h} |u|^2_{1/2,\partial\Omega_i}. \tag{5.6}$$

The inequality (5.5) is a consequence of Lemma 4.3 and Lemma 4.5. The inequality (5.6) is derived from the definition of $\|\cdot\|_{1/2}$, the fact that $u - \Pi_0 u$ vanishes at vertices of $\partial\Omega_i$, and Lemma 4.12:

$$\begin{aligned} h_0^{-1}\|u - \Pi_0 u\|^2_{0,\partial\Omega_i} \lesssim \|u - \Pi_0 u\|^2_{1/2,\partial\Omega_i} &\lesssim \sum_{E_{kl}\subset\partial\Omega_i} \|u - \Pi_0 u\|^2_{H_{00}^{1/2}(E_{kl})} \\ &\lesssim \log^2 \frac{h_0}{h} \, |u|^2_{1/2,\partial\Omega_i}. \end{aligned}$$

Here E_{kl} denotes the edge of $\partial\Omega_i$ connecting two vertices v_k and v_l.

Let S_0 be the restriction of the interface operator S to the coarse subspace V_0. To avoid using the exact solver S_0^{-1}, we can replace S_0 by any of its equivalent forms. Here we consider one such option. It follows from (4.23) that

$$\langle S u_0, u_0\rangle \eqsim \sum_{i=1}^{J} \rho_i \sum_{E_{kl}\subset\partial\Omega_i} (u_0(\mathrm{v}_k) - u_0(\mathrm{v}_l))^2 \quad \forall\, u_0 \in V_0. \tag{5.7}$$

This suggests an inexact coarse solver R_0, called the standard coarse solver, as follows:

$$\langle R_0^{-1} u_0, v_0\rangle = \sum_{i=1}^{J} \rho_i \sum_{E_{kl}\subset\partial\Omega_i} (u_0(\mathrm{v}_k) - u_0(\mathrm{v}_l))\,(v_0(\mathrm{v}_k) - v_0(\mathrm{v}_l)) \quad \forall\, u_0, v_0 \in V_0.$$

5.2.2. Three-dimensional case. The estimates (5.5)–(5.6) indicate that for $n = 2$, the energy of the coarse interpolant $\Pi_0 u$ for any u in $V^h(\Gamma)$, by using only the values of u at vertices of Ω_i, exceeds that of u by at most a factor $\log \frac{h_0}{h}$. However, for $n = 3$, it may result in an $\frac{h_0}{h}$ increase in the energy. In fact, a simple example can show this cannot be improved. Consider a tetrahedral Ω_i and take a nodal basis function $\phi_k \in V^h$ which equals 1 at one vertex v_k of a single Ω_i, but vanishes at all other nodes on $\bar{\Omega}_i$. Let $\Pi_0\phi_k$ be the standard linear interpolation of ϕ_k into Ω_i. Obviously, $\Pi_0\phi_k$ is discrete harmonic in Ω_i. By (4.13) and the linearity of $\Pi_0\phi_k$, we have

$$|\Pi_0\phi_k|^2_{1/2,\partial\Omega_i} \eqsim |\Pi_0\phi_k|^2_{1,\Omega_i} \eqsim h_0^{-2}|\Omega_i| \eqsim \frac{h_0}{h}\, h,$$

while using Lemma 4.7 yields

$$|\phi_k|^2_{1/2,\partial\Omega_i} = |I^0_{\mathrm{V}_k}\phi_k|^2_{1/2,\partial\Omega_i} \lesssim h.$$

This indicates that

$$|\Pi_0\phi_k|^2_{1/2,\partial\Omega_i} \gtrsim \frac{h_0}{h} \, |\phi_k|^2_{1/2,\partial\Omega_i}.$$

Therefore interpolating the value of a function in V^h by exploiting only the vertex values is not effective in three dimensions.

To motivate the construction of an effective Π_0 in three dimensions, recall the first requirement (C1). For any $v \in V^h(\Gamma)$, we define $\Pi_0 v$ to be equal to v at the nodes on the joint-set. In regard to the fourth requirement (C4), it is desirable not to introduce any more degrees of freedom. Hence we need to extend the value on the joint-set inside each face set F. One possible way, as implicitly used in Bramble, Pasciak, and Schatz [11, 12], is to use the discrete harmonic extension. But this approach is computationally a bit too expensive for this purpose. Following Smith [70, 71], we define $\Pi_0 v$ to be a constant value at all interior nodes of each face F, namely the arithmetic average, $\gamma_{h,\partial F}$, of the nodal values of v on ∂F. With this definition, (C2) is clearly satisfied. As we shall see in a moment, (C3) is also satisfied.

In summary, the operator Π_0 is defined, for any $v \in V^h(\Gamma)$ and nodal point x, by

$$(\Pi_0 v)(x) = \begin{cases} v(x) & \text{if } x \in \mathcal{W} \cap \mathcal{N}_h, \\ \gamma_{h,\partial F}(v) & \text{if } x \in F \cap \mathcal{N}_h \quad \text{for each (open) face set } F. \end{cases} \tag{5.8}$$

This interpolant, called the *joint-set interpolant*, satisfies, for any $u \in V^h(\Gamma)$,

$$\|u - \Pi_0 u\|_{0,\partial\Omega_i} \lesssim h_0^{1/2} \log \frac{h_0}{h} |u|_{1/2,\partial\Omega_i}, \quad |\Pi_0 u|_{1/2,\partial\Omega_i} \lesssim \log \frac{h_0}{h} |u|_{1/2,\partial\Omega_i}. \tag{5.9}$$

To see this, we mainly follow Dryja, Smith, and Widlund [36]. Let $u_0 = \Pi_0 u$. Then we can express u_0 on each $\partial\Omega_i$ as

$$u_0 = I^0_{\mathcal{W}_i} u_0 + \sum_{F \subset \partial\Omega_i} \gamma_{h,\partial F}(u_0) I^0_F 1.$$

It follows from the triangle inequality and Lemmas 4.7, 4.9, and 4.11 that

$$|u_0|^2_{1/2,\partial\Omega_i} \lesssim \|u_0\|^2_{h,\mathcal{W}_i} + h_0 \log \frac{h_0}{h} \sum_{F \subset \partial\Omega_i} |\gamma_{h,\partial F}(u_0)|^2 \lesssim \log \frac{h_0}{h} \|u_0\|^2_{h,\mathcal{W}_i}, \tag{5.10}$$

where in the last inequality we have used the bound (4.22). Now the second estimate of (5.9) is a consequence of this and Lemmas 4.7 and 4.9.

Similarly, to show the first estimate of (5.9), we express $u - \Pi_0 u$ as

$$u - \Pi_0 u = \sum_{F \subset \partial\Omega_i} I^0_F (u - \Pi_0 u) = \sum_{F \subset \partial\Omega_i} \big(I^0_F u - \gamma_{h,\partial F}(u) I^0_F 1 \big),$$

and then use the triangle inequality and Lemmas 4.10–4.11 to derive

$$\begin{aligned} & h_0^{-1/2} \|u - \Pi_0 u\|_{0,\partial\Omega_i} \\ & \le \|u - \Pi_0 u\|_{1/2,\partial\Omega_i} \\ & \lesssim \sum_{F \subset \partial\Omega_i} \big(\|I^0_F u\|_{1/2,\partial\Omega_i} + |\gamma_{h,\partial F}(u)| \, \|I^0_F 1\|_{1/2,\partial\Omega_i} \big) \\ & \lesssim \log \frac{h_0}{h} \|u\|_{1/2,\partial\Omega_i}. \end{aligned}$$

The desired estimate follows by noting that $u - \Pi_0 u$ annihilates constant functions.

Now we can state an important property of the *joint-set coarse space* V_0, the range of Π_0 on $V^h(\Gamma)$, in the following lemma.

LEMMA 5.1. *For any $u_0 \in V_0$, we have*

$$\log^{-1} \frac{h_0}{h} |u_0|^2_{1/2,\partial\Omega_i} \lesssim \|u_0 - \gamma_{h,\mathcal{W}_i}(u_0)\|^2_{h,\mathcal{W}_i} \lesssim \log \frac{h_0}{h} |u_0|^2_{1/2,\partial\Omega_i},$$

where $\mathcal{W}_i$ is the joint-set or wire-basket set on $\partial\Omega_i$.

Proof. The first inequality follows by replacing u_0 by $u_0 - \gamma_{h,\mathcal{W}_i}(u_0)$ in (5.10), while the second inequality is an immediate consequence of (4.22) and Lemma 4.9. □

Lemma 5.1 actually defines an inexact coarse solver R_0 by using the discrete norm defined on the joint-set, namely, for all $u_0, v_0 \in V_0$,

$$\langle R_0^{-1} u_0, v_0 \rangle = \log \frac{h_0}{h} \sum_{i=1}^{J} \rho_i \langle u_0 - \gamma_{h,\mathcal{W}_i}(u_0), v_0 - \gamma_{h,\mathcal{W}_i}(v_0) \rangle_{h,\mathcal{W}_i}. \tag{5.11}$$

5.3. Substructuring method I. The algorithms presented in this subsection are based on the coarse spaces defined in subsection 5.2. We first present a basic estimate and then discuss its applications to different variants of substructuring methods.

5.3.1. Basic space decomposition. As mentioned in section 3, the "splitting" process on the interface Γ gives a natural decomposition of the space $V^h(\Gamma)$ as follows:

$$V^h(\Gamma) = V_0 + \sum_{F \subset \Gamma} V_0^h(F), \tag{5.12}$$

where V_0 is the image of the space $V^h(\Gamma)$ under the *joint-operator* Π_0 introduced in subsection 5.2. We shall now discuss the technical details for such a decomposition.

Using Π_0, we have the first decomposition for $u \in V^h(\Gamma)$ as follows:

$$u = u_0 + u_1, \quad \text{with } u_0 = \Pi_0 u \text{ and } u_1 = u - \Pi_0 u.$$

Obviously,

$$S(u,u) \le 2S(u_1,u_1) + 2S(u_0,u_0). \tag{5.13}$$

We now proceed to further decompose the function u_1. Since u_1 vanishes on the joint-set $\mathcal{W}$, we have, by (4.1) and (4.24),

$$\begin{aligned} S(u_1,u_1) &\eqsim \sum_{i=1}^{J} \rho_i \, |u_1|^2_{1/2,\partial\Omega_i} \lesssim \sum_{i=1}^{J} \sum_{F \subset \partial\Omega_i} \rho_i \, |u_{_F}|^2_{H_{00}^{1/2}(F)} \\ &\eqsim \sum_{F \subset \Gamma} \rho_{_F} \, \langle (-\Delta_{F,h})^{1/2} u_{_F}, u_{_F} \rangle_{0,F}, \end{aligned}$$

where $u_{_F} = (u - \Pi_0 u)|_{_F}$. The ρ_F above is the average value of two coefficients associated with two subdomains sharing the common face F.

Using (5.5) and (5.7) for $n = 2$ and using the second inequality in Lemma 5.1 with u_0 replaced by u for $n = 3$, which is true since $u = u_0$ on $\mathcal{W}_i$ yields

$$S(u_0,u_0) \lesssim \gamma(u_0,u_0) \lesssim \begin{cases} \log \dfrac{h_0}{h} S(u,u) & \text{if } n = 2, \\ \log^2 \dfrac{h_0}{h} S(u,u) & \text{if } n = 3, \end{cases} \tag{5.14}$$

where, for $u_0, v_0 \in V_0$,

$$\gamma(u_0,v_0) = \begin{cases} \displaystyle\sum_{i=1}^{J} \rho_i \sum_{E_{kl} \subset \Gamma_i} (u_0(\mathrm{v}_k) - u_0(\mathrm{v}_l))(v_0(\mathrm{v}_k) - v_0(\mathrm{v}_l)) & (n = 2), \\ \displaystyle\log \frac{h_0}{h} \sum_{i=1}^{J} \rho_i \langle u_0 - \gamma_{h,\mathcal{W}_i}(u_0), v_0 - \gamma_{h,\mathcal{W}_i}(v_0) \rangle_{h,\mathcal{W}_i} & (n = 3). \end{cases} \tag{5.15}$$

Combining (5.13) with (5.14) yields

$$S(u,u) \lesssim \sum_{F\subset\Gamma} \rho_{_F} \langle (-\Delta_{F,h})^{1/2} u_{_F}, u_{_F}\rangle_{0,F} + \gamma(u_0,u_0). \tag{5.16}$$

Note that for $n = 3$,

$$u_{_F} = I_F^0(u - \Pi_0 u) = I_F^0(u - \gamma_{\partial F}(u)\,) = I_F^0 u - \gamma_{\partial F}(u) I_F^0 1.$$

Thus we derive by using Lemma 4.10 that

$$\begin{aligned} &\rho_{_F} \langle (-\Delta_{F,h})^{1/2} u_{_F}, u_{_F}\rangle_{0,F} \\ &\eqsim S(u_{_F}, u_{_F}) \qquad (5.17)\\ &\lesssim \log^2 \frac{h_0}{h} (\rho_{j_1} |u|^2_{1/2,\partial\Omega_{j_1}} + \rho_{j_2} |u|^2_{1/2,\partial\Omega_{j_2}}), \qquad (5.18)\end{aligned}$$

where Ω_{j_1} and Ω_{j_2} are two subdomains sharing the face F. Equation (5.18) is also true for $n = 2$ by directly applying Lemma 4.12. Therefore, by noting (5.13)–(5.16) and (5.18), we have obtained the following result.

LEMMA 5.2. *For any $u \in V^h(\Gamma)$,*

$$S(u,u) \lesssim \gamma(u_0,u_0) + \sum_{F\subset\Gamma} \rho_{_F}\, \langle (-\Delta_{F,h})^{1/2} u_{_F}, u_{_F}\rangle_{0,F} \lesssim \log^2 \frac{h_0}{h}\, S(u,u),$$

where $u_0 = \Pi_0 u$, $u_{_F} = (u - \Pi_0 u)|_{_F}$, and $\gamma(u_0, v_0)$ is given in (5.15).

The estimate in the above lemma is the basic technical tool for constructing preconditioners.

5.3.2. Three variants. We shall now discuss three different approaches based on Lemma 5.2.

S-implementation. Within the subspace correction framework discussed in subsection 2.2, one obvious application of the space decomposition result in Lemma 5.2 is to construct a PSC preconditioner. This variant of the substructuring method was considered by Smith [70, 71] for three-dimensional problems; our presentation also includes the two-dimensional case.

The algorithm is again based on the decomposition (5.12). By the subspace correction framework, we only need to choose appropriate subspace solvers. Such choices, however, are quite clear from Lemma 5.2. First of all, the coarse subspace solver R_0 may be given by

$$\langle R_0^{-1} u_0, v_0\rangle = \gamma(u_0, v_0),$$

with $\gamma(\cdot,\cdot)$ given by (5.15). For each face space $V_0^h(F)$, the solver is given by

$$R_F = \rho_F^{-1}\, (-\Delta_{F,h})^{-1/2},$$

where, we recall, ρ_F is the average value of the coefficients ρ_i of the model problem corresponding to subdomains of which F is a face.

The PSC preconditioner defined in (2.4) for S is then given by

$$T = I_0 R_0 I_0^t + \sum_{F\subset\Gamma} I_F R_F I_F^t, \tag{5.19}$$

where $I_0 : V_0 \mapsto V^h(\Gamma)$ and $I_F : V_0^h(F) \mapsto V^h(\Gamma)$ are the natural inclusion operators and t is the adjoint with respect to $\langle\cdot,\cdot\rangle$.

To estimate the condition number $\kappa(TS)$, we apply Theorem 2.1. Lemma 5.2 implies that $K_0 \lesssim \log^2 \frac{h_0}{h}$. Equations (5.14) and (5.17) mean that $\omega_1 \eqsim 1$. By Remark 2.3, it is evident that $K_1 \eqsim 1$. Consequently, we conclude that the preconditioner T given by (5.19) admits the following estimate:

$$\kappa(TS) \lesssim \log^2 \frac{h_0}{h}. \tag{5.20}$$

The action of the preconditioner T given by (5.19) is straightforward; see subsection 9.1.1.

BPS-implementation. Recall that the preconditioner T in (5.19) is for the interface operator S. Using the relation $\langle Su, v\rangle = (Au_{_H}, v_{_H})\ \forall\ u, v \in V^h(\Gamma)$, and the decomposition (3.8), the preconditioner T can be immediately lifted to a preconditioner for the stiffness operator A on the whole domain Ω by replacing the term $A(u_{_H}, v_{_H})$ in (3.8) by the counterpart from the interface preconditioner, namely

$$(\hat{A}u, v) = A(u_P, v_P) + \gamma(u_0, v_0) + \sum_{F\subset\Gamma} \rho_{_F}\, \langle(-\Delta_{F,h})^{1/2} u_{_F}, v_{_F}\rangle_{0,F}, \tag{5.21}$$

where $u_0 = \Pi_0 u$ and $u_{_F} = (u - \Pi_0 u)|_{_F}$, and $v_{_F}$ is defined similarly. This type of preconditioner was derived in the original paper of Bramble, Pasciak, and Schatz [10, 12]. It follows then from Lemma 5.2 that

$$\kappa(\hat{A}^{-1}A) \lesssim \log^2 \frac{h_0}{h}. \tag{5.22}$$

Remark 5.3. This simple lifting technique from an interface preconditioner to a stiffness preconditioner can be applied to all interface preconditioners discussed in the paper.

DW-implementation. We now describe an interesting variant of the substructuring method. This variant was considered by Dryja and Widlund [39] (mainly for two dimensions), but our presentation and especially our analysis are quite different. This variant is based on the following space decomposition:

$$V^h = \tilde{V}_0 + \sum_{ij} V_{ij}, \tag{5.23}$$

where the space $\tilde{V}_0$ is the discrete harmonic extension of the coarse space V_0, and i, j are taken for all subdomains Ω_i and Ω_j which share a common face, and

$$V_{ij} = V_0^h(\Omega_{ij}),$$

with $\Omega_{ij} = \Omega_i \cup \Omega_j \cup F_{ij}$, where F_{ij} is the common face of Ω_i and Ω_j.

The subspaces in the decomposition (5.23) obviously have a great deal of overlapping. Such overlapping makes it possible to use inexact solvers $R_{ij} : V_{ij} \to V_{ij}$ for the operator $A_{ij} : V_{ij} \to V_{ij}$ associated with $A(\cdot,\cdot)$. By the definition of V_0, on the other hand, an inexact solver on $\tilde{V}_0$ is hard to come by because a harmonic extension means an exact solver. Nevertheless, in a special case where $n = 2$ and all subdomains Ω_i are triangles, $\tilde{V}_0$ is just the space of piecewise-linear functions, and an inexact solver can be used.

In any case, we assume $R_0 : \tilde{V}_0 \to \tilde{V}_0$ and $R_{ij} : V_{ij} \to V_{ij}$ are SPD operators satisfying

$$R_0 \eqsim A_0^{-1}, \quad R_{ij} \eqsim A_{ij}^{-1},$$

and consider the following preconditioner B for A:

$$B = I_0 R_0 I_0^t + \sum_{ij} I_{ij} R_{ij} I_{ij}^t. \tag{5.24}$$

We claim that

$$\kappa(BA) \lesssim \log^2 \frac{h_0}{h}. \tag{5.25}$$

To see this, we use Theorem 2.1. Again, the corresponding K_1 and ω_1 are all uniformly bounded. Next we show $K_0 \lesssim \log^2 \frac{h_0}{h}$.

Given $v \in V^h$, let v_Γ be its restriction on Γ and $v_0 = \widetilde{\Pi_0 v_\Gamma}$ (harmonic extension), where Π_0 is the joint-set interpolant (see subsection 5.2.1 for $n = 2$ and subsection 5.2.2 for $n = 3$). We write

$$v - v_0 = w_{_P} + w_{_H},$$

with $w_{_P} \in V_P$ (see (3.4)) and $w_{_H} = \sum_{ij} w_{_{H,ij}} \in V_H$ (see (3.6)) such that $w_{_{H,ij}} \in V_H$, $w_{_{H,ij}} = w_{_H}$ on F_{ij}, $w_{_{H,ij}} = 0$ on $\partial\Omega_i \bigcup \partial\Omega_j \setminus F_{ij}$, and $\mathrm{supp}(w_{_{H,ij}}) \subset \bar{\Omega}_{ij}$. Given Ω_k, let m_k be the number of F_{ij} that belong to $\partial\Omega_k$, and $w_{P,k}$, having support on Ω_k, be the restriction of $w_{_P}$ on Ω_k. Set

$$v_{ij} = w_{_{P,i}}/m_i + w_{_{P,j}}/m_j + w_{_{H,ij}}.$$

Obviously

$$v = v_0 + \sum_{ij} v_{ij}.$$

Then, showing $K_0 \lesssim \log^2 \frac{h_0}{h}$ is equivalent to proving

$$A(v_0, v_0) + \sum_{ij} A(v_{ij}, v_{ij}) \lesssim \log^2 \frac{h_0}{h} A(v, v),$$

which follows immediately if we can show

$$A(v_0, v_0) + \sum_{ij} A(w_{_{H,ij}}, w_{_{H,ij}}) \lesssim \log^2 \frac{h_0}{h} A(v, v), \tag{5.26}$$

since by the triangle inequality and (3.8) we have

$$\sum_{ij} A(v_{ij}, v_{ij}) \lesssim \sum_i A(w_{_{P,i}}, w_{_{P,i}}) + \sum_{ij} A(w_{_{H,ij}}, w_{_{H,ij}})$$

and

$$\sum_i A(w_{_{P,i}}, w_{_{P,i}}) = A(w_{_P}, w_{_P}) \le A(v - v_0, v - v_0) \lesssim A(v, v) + A(v_0, v_0).$$

To see (5.26), it follows from (5.5) and (5.9) that

$$A(v_0, v_0) \lesssim \log^2 \frac{h_0}{h} A(v, v).$$

Furthermore,

$$\begin{aligned} A(w_{_{H,ij}}, w_{_{H,ij}}) &\lesssim \rho_{_{F_{ij}}} |w_{_{H,ij}}|^2_{H_{00}^{1/2}(F_{ij})} = \rho_{_{F_{ij}}} |(I - \Pi_0) v_\Gamma|^2_{H_{00}^{1/2}(F_{ij})} \\ &\lesssim \log^2 \frac{h_0}{h} \big(\rho_i |v|^2_{1,\Omega_i} + \rho_j |v|^2_{1,\Omega_j}\big). \end{aligned}$$

The desired estimate then follows.

5.4. Substructuring method II. We next present another substructuring technique which is based on Bramble, Pasciak, and Schatz [12]. In fact this approach is a rather general one which could lead to the methods studied in the previous subsection.

The basic technical result is the following lemma.

LEMMA 5.4. *For any $v \in V^h(\partial\Omega_i)$, we have*

$$\|v\|^2_{1/2,\partial\Omega_i} \lesssim G_i(v,v) \lesssim \log^2 \frac{h_0}{h} \|v\|^2_{1/2,\partial\Omega_i}, \tag{5.27}$$

where

$$G_i(v,v) = h^{n-2} \sum_{x_j \in \mathcal{W}_i \cap \mathcal{N}_h} v^2(x_j) + \sum_{F \subset \partial\Omega_i} \langle (-\Delta_{F,h})^{1/2} I_F^0 v, I_F^0 v \rangle_{0,F}.$$

Proof. Note that $v = I^0_{\mathcal{W}_i} v + \sum_{F\subset\partial\Omega_i} I^0_F v$ for $v \in V^h(\partial\Omega_i)$. By Lemmas 4.7–4.10, we deduce that, for all $v \in V^h(\partial\Omega_i)$,

$$\begin{aligned} \|v\|^2_{1/2,\partial\Omega_i} &\lesssim \|I^0_{\mathcal{W}_i} v\|^2_{1/2,\partial\Omega_i} + \sum_{F\subset\partial\Omega_i} \|I^0_F v\|^2_{1/2,\partial\Omega_i} \\ &\lesssim \log^2 \frac{h_0}{h} \|v\|^2_{1/2,\partial\Omega_i}. \end{aligned}$$

The desired result then follows easily. ∎

Based on the above relation, we shall choose, for each i, an appropriate operator $\mu_i : V^h(\partial\Omega_i) \mapsto V^h(\partial\Omega_i)$ such that the operator $\hat{S} : V^h(\Gamma) \mapsto V^h(\Gamma)$ defined by

$$\langle \hat{S}u, u\rangle = \sum_{i=1}^{J} \rho_i \, G_i(u - \mu_i(u), u - \mu_i(u)) \tag{5.28}$$

provides a good preconditioner for S.

First approach. The first choice of μ_i was made by Bramble, Pasciak, and Schatz [12] as follows:

$$G_i(v - \mu_i(v), 1) = 0. \tag{5.29}$$

This leads to

$$\begin{aligned} &\log^{-2} \frac{h_0}{h} G_i(u - \mu_i(u), u - \mu_i(u)) \\ &\le \log^{-2} \frac{h_0}{h} G_i(u - \gamma_{\partial\Omega_i}(u), u - \gamma_{\partial\Omega_i}(u)) \ (\mu_i\text{'s minimizing}) \\ &\lesssim \|u - \gamma_{\partial\Omega_i}(u)\|^2_{1/2,\partial\Omega_i} \ (\text{by (5.27)}) \\ &\lesssim |u|^2_{1/2,\partial\Omega_i} \ (\text{by (4.5)}) \\ &\lesssim G_i(u - \mu_i(u), u - \mu_i(u)) \ (\text{from (5.27)}). \end{aligned}$$

Namely, for all $v \in V^h(\partial\Omega_i)$,

$$|v|^2_{1/2,\partial\Omega_i} \lesssim G_i(v - \mu_i(v), v - \mu_i(v)) \lesssim \log^2 \frac{h_0}{h} |v|^2_{1/2,\partial\Omega_i}. \tag{5.30}$$

Consequently, the operator $\hat{S}$ given by (5.28) with μ_i given by (5.29) satisfies

$$\log^{-2} \frac{h_0}{h} \langle \hat{S}u, u\rangle \lesssim \langle Su, u\rangle \lesssim \langle \hat{S}u, u\rangle. \tag{5.31}$$

This last relation suggests the following preconditioner for the stiffness operator A:

$$(\hat{A}u, v) = A(u_P, v_P) + \sum_{i=1}^{J} \rho_i \, G_i(u - \mu_i(u), v - \mu_i(v)) \quad \forall\, u, v \in V^h. \tag{5.32}$$

Clearly, from above we have

$$\kappa(\hat{A}^{-1}A) \lesssim \log^2 \frac{h_0}{h}.$$

Second approach. It is possible to make other choices of μ_i to relate some known preconditioners and also to obtain new preconditioners.

Our first criterion for choosing μ_i is based on the following estimate:

$$\alpha_0 |v|^2_{1/2,\partial\Omega_i} \le \|v - \mu_i(v)\|^2_{1/2,\partial\Omega_i} \le \alpha_1 |v|^2_{1/2,\partial\Omega_i} \tag{5.33}$$

in such a way that α_1/α_0 is relatively small.

To this end, the most natural choice would be the average on $\partial\Omega$ given by

$$\mu_i(v) = \gamma_{\partial\Omega_i}(v) = \frac{1}{|\partial\Omega_i|} \int_{\partial\Omega_i} v. \tag{5.34}$$

In this case, (5.33) holds with $\alpha_0, \alpha_1 \eqsim 1$ and thus the corresponding $\hat{S}$ also satisfies (5.31).

Another choice we would like to mention is the arithmetic average in the joint-set $\mathcal{W}$:

$$\mu_i(v) = \gamma_{h,\mathcal{W}_i}(v). \tag{5.35}$$

The corresponding algorithms would be closely related to the substructuring preconditioners discussed in the previous sections.

If we choose μ_i to be equal to the special interpolant Π_0 defined in the previous subsections, then the preconditioners from there can be recovered.

REMARK 5.5. If each boundary subdomain shares at least a common face with the boundary of the original domain Ω, then we can choose all the constants μ_i corresponding to these boundary subdomains Ω_i to be zero, with the rest of μ_i defined still by (5.29) or (5.34). All results of the section are still valid because the crucial inequality (5.27) is satisfied for the boundary subdomains with the full norm $\|\cdot\|_{1/2,\partial\Omega_i}$ replaced by the seminorm $|\cdot|_{1/2,\partial\Omega_i}$. Consequently, we need to solve a coarse system with one unknown μ_i for each interior subdomain; see section 9.

6. Neumann–Neumann methods. We shall now discuss a class of techniques for preconditioning the interface operator S based on local Neumann problems.

6.1. Basic ideas. Again, let us first look at the two-subdomain case (see Fig. 5). We observe that $S = S_1 + S_2$, where $S_i : V^h(\Gamma) \to V^h(\Gamma)$ for $i = 1, 2$ is defined by

$$\langle S_i u, v\rangle = A_{\Omega_i}(u_H, v_H),$$

where u_H and v_H are the discrete harmonic extensions of u and v, respectively. Note that, by (4.13), we have for any $u \in V^h(\Gamma)$,

$$\langle S_i u, u\rangle_{0,\Gamma} \eqsim |u|^2_{1/2,\partial\Omega_i} \eqsim A_{\Omega_i}(u_H, u_H) \eqsim \|u\|^2_{H_{00}^{1/2}(\Gamma)} \eqsim \langle Su, u\rangle.$$

This means that either S_1^{-1} or S_2^{-1} is an optimal preconditioner for the interface operator S. As a result, their summation

$$T_h = S_1^{-1} + S_2^{-1} \tag{6.1}$$

is also an optimal preconditioner.

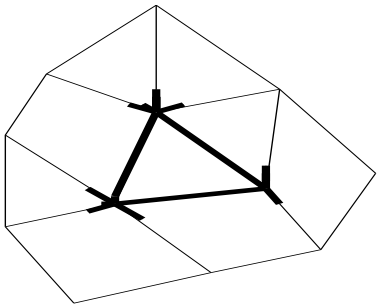

FIG. 8. *A typical subspace V_i related to Ω_i with supporting set denoted by the bold line.*

The action of T in (6.1) amounts to solving two local Neumann problems in Ω_i for $i = 1, 2$ (i.e., problems with Neumann boundary conditions on the interface). For this reason, this type of preconditioning technique is known as a Neumann–Neumann method.

The use of the summation of S_i^{-1} as in (6.1) would give a balance of two subdomains, and it also generalizes to multiple subdomains in a more natural way. Its advantage is clearer, as we shall see later, when applied to problems with jumps in coefficients.

We shall now discuss the generalization of the above procedure to multiple subdomains. A natural attempt is to define S_i on each subspace $V^h(\partial\Omega_i)$ as

$$\langle S_i u, v\rangle_{0,\Gamma_i} = A_i(u_H, v_H) = \rho_i(\nabla u_H, \nabla v_H)_{0,\Omega_i}, \tag{6.2}$$

and then, as in the two-subdomain case, take the sum of their inverses. But there are at least two difficulties in this approach. First of all, $V^h(\partial\Omega_i)$ is not a subspace of $V^h(\Gamma)$; secondly, not all S_i are invertible. To overcome the first difficulty, we introduce a subspace $V_i \subset V^h(\Gamma)$ that in some sense is closest to $V^h(\partial\Omega_i)$, namely, the smallest subspace of $V^h(\Gamma)$ that contains the degrees of freedom of $V^h(\partial\Omega_i)$ (see Fig. 8). Let $\Theta_i : V^h(\partial\Omega_i) \to V_i$ be an appropriate linear operator that links these two spaces. Using the operator Θ_i, a solver on V_i can be obtained from a solver on $V^h(\partial\Omega_i)$. But as mentioned above, the most natural operator S_i is not always invertible. One way to overcome this difficulty is to use a modified operator $\hat{S}_i$ obtained from S_i by adding a lower-order term. By (2.14), $\Theta_i \hat{S}_i^{-1} \Theta_i^t$ should be a good candidate as a preconditioner in the subspace V_i. With an appropriate coarse subspace V_0 and a solver R_0, we obtain a preconditioner in the form of

$$T = I_0 R_0 I_0^t + \sum_{j=1}^{J} I_i \Theta_i \hat{S}_i^{-1} \Theta_i^t I_i^t.$$

As mentioned earlier, such a type of preconditioner is sometimes called a Neumann–Neumann method.

Another effective approach of avoiding the possible singularity of S_i is to remove the null part of S_i in $V^h(\Gamma_i)$ by first solving a proper coarse space. This kind of idea was used by Farhat and Roux [41, 42] for the so-called FETI method, and also by Mandel [58] for the so-called balancing DD method (see subsection 6.3). The balancing DD method will be viewed by the local-global approach discussed in subsection 2.3.

For simplicity of exposition, throughout this section, we shall implicitly assume that each boundary subdomain (from the DD) shares a common face with $\partial\Omega$. We shall make special remarks for more general cases.

6.2. Neumann–Neumann methods. In this subsection, we discuss the methods based on modifying the local operator S_i. The presentation here mainly follows Dryja and Widlund [40].

The modification, first proposed in [40], is based on the following scaled full H^1 inner product:

$$(u,v)_{1,\Omega_i} = (\nabla u, \nabla v)_{0,\Omega_i} + h_0^{-2}(u,v)_{0,\Omega_i} \quad \forall\ u,v \in H^1(\Omega_i).$$

Correspondingly, a nonsingular operator $\check{S}_i : V^h(\Gamma_i) \mapsto V^h(\Gamma_i)$ can be defined as follows:

$$\langle \check{S}_i u, v\rangle_{0,\Gamma_i} = \rho_i(\check{u},\check{v})_{1,\Omega_i} \quad \forall\ u,v \in V^h(\Gamma_i). \tag{6.3}$$

Here $\check{u}$ denotes the generalized discrete harmonic extension of u on each Ω_i (see (4.14)).

Note that

$$\langle \check{S}_i v, v\rangle_{0,\Gamma_i} \eqsim \rho_i \|v\|^2_{1/2,\Gamma_i}.$$

The motivation of the above modified operator is as follows. In the space $V^h(\Gamma_i)/R^1$, S_i is nonsingular and spectrally equivalent to $\check{S}_i$.

We now proceed to discuss how to define a solver (or preconditioner) on the subspace V_i based on $\check{S}_i$ on $V^h(\Gamma_i)$. In view of Theorem 2.9, $V^h(\Gamma_i)$ can be viewed as an auxiliary space. The auxiliary space preconditioner is realized by an isomorphic "link" $\Theta_i : V^h(\Gamma_i) \to V_i$ which is to be specified later. The link in the other direction between the two spaces is established by the adjoint $\Theta_i^t : V_i \mapsto V^h(\Gamma_i)$ given by

$$\langle \Theta_i^t u_i, v_i\rangle_{0,\Gamma_i} = \langle u_i, \Theta_i v_i\rangle \quad \forall\ u_i \in V_i,\ v_i \in V^h(\Gamma_i).$$

A subspace solver or a preconditioner on each V_i can then be defined by

$$R_i = \Theta_i \check{S}_i^{-1} \Theta_i^t. \tag{6.4}$$

With a proper choice of subspace V_0 and a solver R_0, we obtain the corresponding PSC preconditioner

$$T = I_0 R_0 I_0^t + \sum_{i=1}^{J} I_i \Theta_i \check{S}_i^{-1} \Theta_i^t I_i^t. \tag{6.5}$$

To motivate the optimal choice of Θ_i, we first show the following result.

LEMMA 6.1. *For any $v_i \in V_i$,*

$$\langle S v_i, v_i\rangle \lesssim \log^2 \frac{h_0}{h}\, \rho_i^{-2} \langle \check{S}_i I_h(\nu_\rho v_i), I_h(\nu_\rho v_i)\rangle_{0,\Gamma_i}, \tag{6.6}$$

where $I_h v \in V^h(\Gamma_i)$ and $I_h v(x_i) = v(x_i)$ at any node $x_i \in \Gamma_i$, and ν_ρ is a weighted counting function given by

$$\nu_\rho(x) = \sum_{j=1}^{J} \rho_j \chi_{\partial\Omega_i}(x) \quad \forall\ x \in \Omega, \tag{6.7}$$

and the estimate (6.6) *is uniform with respect to the jumps in coefficients.*

Proof. We now prove (6.6). Note that

$$\langle S v_i, v_i\rangle \eqsim \sum_m \rho_m |v_i|^2_{1/2,\partial\Omega_m}, \tag{6.8}$$

where the summation is over all subdomains Ω_m which share either a face, an edge, or a vertex with Ω_i. We can split v_i into

$$v_i = \sum_{F\subset\Gamma_i} I_F^0 v_i + \sum_{E\subset\Gamma_i} I_E^0 v_i + \sum_{V_k\in\Gamma_m} I_{V_k}^0 v_i. \tag{6.9}$$

Given $F \subset \Gamma_i$ is a face for $n = 3$ or an edge for $n = 2$, let $\mathcal{N}(F) = \{i : \quad F \subset \Gamma_i\}$. Then

$$\begin{aligned}
&\sum_m \rho_m |I_F^0 v_i|^2_{1/2,\partial\Omega_m}\\
&\eqsim \sum_{m\in\mathcal{N}(F)} \rho_m \|I_F^0 v_i\|^2_{H_{00}^{1/2}(F)} \text{ (by definition)}\\
&\lesssim \sum_{m\in\mathcal{N}(F)} \frac{\rho_m\rho_i}{\rho_F^2}\Big(\rho_i^{-1}\|I_F^0(\nu_\rho v_i)\|^2_{H_{00}^{1/2}(F)}\Big)\\
&\lesssim \log^2\frac{h_0}{h}\,\rho_i^{-1}\,\|I_h(\nu_\rho v_i)\|^2_{1/2,\partial\Omega_i} \text{ (Lemma 4.10)}\\
&\eqsim \log^2\frac{h_0}{h}\,\rho_i^{-2}\,\langle \check{S}_i I_h(\nu_\rho v_i), I_h(\nu_\rho v_i)\rangle_{0,\Gamma_i}.
\end{aligned}$$

Applying a similar argument to the second and third terms in (6.9) with Lemma 4.7, we obtain, for any given feasible E and V_k, that

$$\begin{aligned}
&\sum_m \rho_m\big(|I_E^0 v_i|^2_{1/2,\partial\Omega_m} + |I_{V_k}^0 v_i|^2_{1/2,\partial\Omega_m}\big)\\
&\lesssim \log\frac{h_0}{h}\,\rho_i^{-2}\langle \check{S}_i I_h(\nu_\rho v_i), I_h(\nu_\rho v_i)\rangle_{0,\Gamma_i}.
\end{aligned}$$

The desired estimate then follows easily. □

Based on (6.6), the isomorphism $\Theta_i : V^h(\Gamma_i) \to V_i$ should be defined in such a way that

$$\rho_i^{-2}\langle \check{S}_i I_h(\nu_\rho u_i), I_h(\nu_\rho u_i)\rangle_{0,\partial\Omega_i} \eqsim \langle R_i^{-1}u_i, u_i\rangle_{0,\Gamma_i} = \langle \check{S}_i\Theta_i^{-1}u_i, \Theta_i^{-1}u_i\rangle_{0,\Gamma_i} \quad \forall\ u_i \in V_i,$$

which amounts to the following definition:

$$\Theta_i u_i = \rho_i\, I_{\Gamma_i}^0(\nu_\rho^{-1}u_i) \quad \forall\ u_i \in V^h(\Gamma_i). \tag{6.10}$$

With such a choice of Θ_i, the corresponding R_i given by (6.4) then satisfy

$$\langle Su_i, u_i\rangle \lesssim \log^2\frac{h_0}{h}\langle R_i^{-1}u_i, u_i\rangle_{0,\Gamma_i} \quad \forall\ u_i \in V_i.$$

The above estimate means that

$$\rho(R_iS_i) \lesssim \log^2\frac{h_0}{h}, \quad 1 \le i \le J, \tag{6.11}$$

where $S_i : V_i \to V_i$ is the restriction of S on V_i, namely,

$$\langle S_iu_i, v_i\rangle = \langle Su_i, v_i\rangle \quad \forall\ u_i, v_i \in V_i.$$

We now proceed to discuss the choice of the coarse space V_0 and the estimate of the corresponding decomposition constant K_0. We first note that the definition of Θ_i

naturally extends to the whole $V^h(\Gamma)$ and the following partition of unit holds on the space $V^h(\Gamma)$:

$$\sum_{i=1}^{J} \Theta_i = I.$$

Given any $u \in V^h(\Gamma)$, set $u_0 = \sum_{i=1}^{J} \Theta_i \mu_i(u)$ and $u_i = \Theta_i(u - \mu_i(u)) \in V_i$, where $\mu_i(u)$ is a constant such that

$$h_0^{-1/2} \|u - \mu_i(u)\|_{0,\partial\Omega_i} \lesssim |u|_{1/2,\partial\Omega_i},$$

which implies

$$\|u - \mu_i(u)\|_{1/2,\partial\Omega_i} \lesssim |u - \mu_i(u)|_{1/2,\partial\Omega_i}. \tag{6.12}$$

Obviously, $u = \sum_{i=0}^{J} u_i$. Since $R_i = \Theta_i \check{S}_i^{-1} \Theta_i^t$, it follows from (6.12) that

$$\begin{aligned}\sum_{i=1}^{J} \langle R_i^{-1} u_i, u_i \rangle_{0,\Gamma_i} &= \sum_{i=1}^{J} \langle \check{S}_i(u - \mu_i(u)), u - \mu_i(u) \rangle_{0,\Gamma_i} \\ &\eqsim \sum_{i=1}^{J} \rho_i \|u - \mu_i(u)\|^2_{1/2,\partial\Omega_i} \lesssim \sum_{i=1}^{J} \rho_i |u|^2_{1/2,\partial\Omega_i} \eqsim \langle Su, u \rangle.\end{aligned}$$

Note that the presence of the constant $\mu_i(u)$ is crucial in this derivation.

The above argument leads to a natural choice of the coarse space, called the *weighted coarse space*, as follows:

$$V_0 = \operatorname{span}\{\Theta_i \mu_i : \ \mu_i \in R^1 \text{ for all } i \text{ such that } \partial\Omega_i \cap \partial\Omega = \emptyset\}. \tag{6.13}$$

Note that, in view of (6.12) and the Poincaré inequality, $\mu_i(u)$ can be taken to be zero if $\partial\Omega_i \cap \partial\Omega \neq \emptyset$ in the above choice of u_0. Then we see that u_0 belongs to the above space V_0. If S_0 is the restriction of S on V_0, we choose

$$R_0 = \log^2 \frac{h_0}{h} S_0^{-1}.$$

By (6.11), we have

$$\omega_1 = \max_{0 \le i \le J} \rho(R_i S_i) \lesssim \log^2 \frac{h_0}{h}. \tag{6.14}$$

It remains to prove that $K_0 \lesssim 1$. Using the fact that $u_0 = u - \sum_{i=1}^{J} u_i$, $K_1 \lesssim 1$, and $\omega_1 \lesssim \log^2 \frac{h_0}{h}$, we deduce that

$$\begin{aligned}\langle R_0^{-1} u_0, u_0 \rangle &= \left(\log \frac{h_0}{h} \right)^{-2} \langle Su_0, u_0 \rangle \\ &\lesssim \left(\log \frac{h_0}{h} \right)^{-2} \left\{ \langle Su, u \rangle + \sum_{i=1}^{J} \langle Su_i, u_i \rangle \right\} \\ &\lesssim \langle Su, u \rangle + \sum_{i=1}^{J} \langle R_i^{-1} u_i, u_i \rangle \\ &\lesssim \langle Su, u \rangle.\end{aligned}$$

We conclude that $K_0 \lesssim 1$. In summary, by Theorem 2.1, we have the following result.

THEOREM 6.2. *With the aforementioned choice of weighted coarse space V_0 and the coarse solver R_0, the preconditioner T given by* (6.5) *satisfies*

$$\kappa(TS) \lesssim \log^2 \frac{h_0}{h}.$$

REMARK 6.3. If for some i, $\partial\Omega_i \cap \partial\Omega$ is a vertex (for $n = 2$) or an edge (for $n = 3$), the estimates for K_0 are slightly different. In this case, the following estimate holds by Lemma 4.13:

$$\|v\|_{0,\Omega_i} \lesssim h_0 \Big(\log \frac{h_0}{h} \Big)^{1/2} |v|_{1,\Omega_i}.$$

As a result, the corresponding preconditioner admits the following estimate:

$$\kappa(TS) \lesssim \log^3 \frac{h_0}{h}.$$

The use of the standard coarse space. In this subsection, we shall investigate the possibility of using the standard coarse space, which obviously has a much simpler structure than the weighted space used in the previous subsection. But it turns out that such a choice of coarse space would not in general, for $n = 3$, give rise to a preconditioner that is uniform with respect to jumps in the coefficients ρ_i. Therefore, for simplicity, we shall assume that $\rho_i = 1$ for all i in the discussion below. Define

$$\Theta_i u_i = I^0_{\Gamma_i}(\nu^{-1} u_i) \quad \forall\ u_i \in V^h(\Gamma_i), \tag{6.15}$$

where ν is an "unweighted" counting function as in (6.7).

Let $V_0(\Gamma)$ be a space consisting of all continuous functions which are piecewise linear on each edge of the interface Γ and R_0 be the exact coarse solver S_0^{-1}. Then the coarse component $w_0 = I_0 R_0 I_0^t g \in V_0(\Gamma)$ in the action of T given by (6.5) with $g \in V$ can be obtained by solving

$$S(w_0, \phi) = \langle g, \phi \rangle \quad \forall\ \phi \in V_0(\Gamma). \tag{6.16}$$

We remark that for a general subdomain decomposition $\{\Omega_i\}_{i=1}^J$ in $n = 2$ (i.e., both triangles and quadrilaterals are allowed) and for a tetrahedral subdomain decomposition in $n = 3$, the above coarse solver R_0 can be replaced by the following standard solver (see Lemma 4.3):

$$\langle R_0^{-1} u_0, v_0 \rangle = h_0^{n-2} \sum_{i=1}^{J} \sum_{E_{kl} \subset \partial\Omega_i} \big(u_0(\mathrm{v}_k) - u_0(\mathrm{v}_l)\big)\big(v_0(\mathrm{v}_k) - v_0(\mathrm{v}_l)\big) \quad \forall\ u_0, v_0 \in V_0(\Gamma). \tag{6.17}$$

Clearly, this coarse solver R_0 is much simpler and less expensive compared with the weighted coarse space solver S_0^{-1}. For the latter, forming its coefficient matrix with entries $\langle S\phi_i, \phi_j \rangle$, $\{\phi_i\}$ being the basis of the weighted coarse space V_0, is expensive.

For the preconditioner T in (6.5) with Θ_i given by (6.15) and the standard coarse space, we have the following theorem.

THEOREM 6.4.

$$\kappa(TS) \lesssim \log^2 \frac{h_0}{h}.$$

Proof. By Theorem 2.1, we need to estimate K_0 and ω_1. It is clear from (6.6) that we still have $\omega_1 \lesssim \log^2 \frac{h_0}{h}$.

Now we estimate K_0. For any $u \in V^h(\Gamma)$, to define a partition of u, we first use the discrete harmonic extension of u, denoted by $\tilde{u}$, and its projection into V_0 under the L^2-projection $Q_0 = I_0^t$ to define the coarse component u_0, i.e., $u_0 = (I_0^t \tilde{u})|_\Gamma$. Then other local components u_i are defined by $u_i = \Theta_i w$ with $w = u - u_0$.

Obviously, $u_i \in V_i$ for $0 \le i \le J$ and $u = \sum_{i=0}^J u_i$, because $\sum_{i=1}^J \Theta_i w = w$. By definition of R_i and the L^2-approximation of Q_0, we derive

$$\begin{aligned}\sum_{i=1}^J \langle R_i^{-1} u_i, u_i\rangle_{0,\Gamma_i} &\lesssim \sum_{i=1}^J \|u - u_0\|^2_{1/2,\partial\Omega_i} \\ &\lesssim \sum_{i=1}^J (\tilde{u} - Q_0\tilde{u}, \tilde{u} - Q_0\tilde{u})_{1,\Omega_i} \\ &\lesssim |\tilde{u}|^2_{1,\Omega} \lesssim \sum_{i=1}^J |u|^2_{1/2,\partial\Omega_i}.\end{aligned}$$

Again by (4.13) and the stability of Q_0,

$$\begin{aligned}\langle R_0^{-1} u_0, u_0\rangle = \langle S u_0, u_0\rangle &\le \sum_{i=1}^J A_i(\tilde{u}_0, \tilde{u}_0) \le \sum_{i=1}^J A_i(Q_0\tilde{u}, Q_0\tilde{u}) \\ &\lesssim A(\tilde{u}, \tilde{u}) \le \langle Su, u\rangle. \qquad (6.18)\end{aligned}$$

Combining the above two estimates then yields that $K_0 \lesssim 1$, which proves Theorem 6.4. □

REMARK 6.5. For $n = 2$, the standard coarse space may be used for jump coefficients (see (5.5) and (5.6)). Natural injection may be used for the substitute of the operator Θ_i (i.e., taking $\nu = 1$ in the definition of Θ_i), but then $K_0 \lesssim \log^2 \frac{h_0}{h}$. In both cases, we have

$$\kappa(TS) \lesssim \log^4 \frac{h_0}{h}.$$

6.3. Balancing DD method. The balancing DD method provides another approach to treating the singularity of S_i on $V^h(\partial\Omega_i)$. Rather than modifying the expression of the operator S_i itself as was done in the previous subsection, S_i can be made nonsingular by removing its null space. The idea is first to solve the equation on the weighted coarse subspace V_0 defined by (6.13) so that the resulting residual is orthogonal to the null space of S_i, and then to apply the Neumann–Neumann-type algorithm (with an unmodified S_i) to the residual equation.

We shall now discuss the details. When Ω_i does not intersect with $\partial\Omega$, the null space $\mathrm{null}(S_i)$ is a one-dimensional space consisting of constant functions; otherwise $\mathrm{null}(S_i) = \{0\}$. Consider the following subspace:

$$\hat{V}(\Gamma_i) = \{v_i \in V^h(\Gamma_i) : \ \gamma_{\Gamma_i}(v_i) = 0 \quad \text{if} \quad \mathrm{null}(S_i) \ne \{0\}\}, \qquad (6.19)$$

or $\hat{V}(\Gamma_i) = V^h(\Gamma_i)$ otherwise. Then $S_i : \hat{V}(\Gamma_i) \mapsto \hat{V}(\Gamma_i)$ is a nonsingular operator and its inverse will be denoted by S_i^+.

Note that, as a direct consequence of the Poincaré or Friedrichs's inequality,

$$h_0^{-1/2}\|v\|_{0,\Gamma_i} \lesssim |v|_{1/2,\Gamma_i} \quad \forall\, v \in \hat{V}(\Gamma_i),$$

or, equivalently,

$$\|v\|_{1/2,\Gamma_i} \eqsim |v|_{1/2,\Gamma_i} \quad \forall\, v \in \hat{V}(\Gamma_i). \qquad (6.20)$$

The above equivalence implies that, for $\check{S}_i$ defined by (6.3), the following holds:

$$\langle \check{S}_i v, v\rangle_{0,\Gamma_i} \eqsim \langle S_i v, v\rangle_{0,\Gamma_i} \quad \forall\ v \in \hat{V}(\Gamma_i).$$

Consequently,

$$\langle \check{S}_i^{-1} v, v\rangle_{0,\Gamma_i} \eqsim \langle S_i^{+} v, v\rangle_{0,\Gamma_i} \quad \forall\ v \in \hat{V}(\Gamma_i). \tag{6.21}$$

With the basic relation (6.21), we now proceed to introduce the balancing method by using the global-local and local-global techniques described in subsection 2.3.

Let V_0 be the weighted coarse subspace defined by (6.13). Consider the subspace $\hat{V} = V_0^{\perp}$, the complement of V_0 with respect to the inner product $\langle S\cdot, \cdot\rangle$. We first use the global-local technique to derive from the preconditioner (6.5) a subspace preconditioner for the restriction of S on the subspace $\hat{V}$ as follows:

$$\hat{T} = P_0^{\perp} \left(\sum_{i=1}^{J} I_i \Theta_i \check{S}_i^{-1} \Theta_i^t I_i^t \right) (P_0^{\perp})^t, \tag{6.22}$$

where $P_0^{\perp} = I - P_0$, with $P_0 : V^h(\Gamma) \mapsto V_0$ being the orthogonal projection with respect to $\langle S\cdot, \cdot\rangle$.

Note that, since $\Theta_i 1 \in V_0$,

$$\langle I_i \Theta_i^t \hat{I}_i^t (P_0^{\perp})^t v, 1\rangle_{0,\Gamma_i} = \langle v, (I - P_0)\Theta_i 1\rangle = 0.$$

This means that $I_i \Theta_i^t \hat{I}_i^t (P_0^{\perp})^t v \in \hat{V}(\Gamma_i)$. It then follows from (6.21) that the operator $\hat{T}$ defined by (6.22) satisfies

$$\hat{T} \eqsim \tilde{T} \quad \text{on} \quad \hat{V}, \tag{6.23}$$

where

$$\tilde{T} = P_0^{\perp} \left(\sum_{i=1}^{J} I_i \Theta_i S_i^{+} \Theta_i^t I_i^t \right) (P_0^{\perp})^t. \tag{6.24}$$

With the above preconditioner $\tilde{T}$ for the (local) subspace $\hat{V}$, we can then use the local-global technique in subsection 2.3 to obtain a preconditioner T_b for the operator S on the (global) space $V^h(\Gamma)$, which is given by

$$T_b S = P_0 + P_0^{\perp} \tilde{T} \hat{S} P_0^{\perp} = P_0 + P_0^{\perp} \left(\sum_{i=1}^{J} I_i \Theta_i S_i^{+} \Theta_i^t I_i^t \right) (P_0^{\perp})^t \hat{S} P_0^{\perp}.$$

Namely,

$$T_b S = P_0 + (I - P_0) \left(\sum_{i=1}^{J} I_i \Theta_i S_i^{+} \Theta_i^t I_i^t \right) S (I - P_0). \tag{6.25}$$

Combining Theorem 2.7 with (6.23), we obtain the following theorem.

THEOREM 6.6. *Assume that for each subdomain Ω_i, $\partial\Omega_i \cap \partial\Omega$ is either empty or a face ($n = 3$) or an edge ($n = 2$) of Ω_i. Then the balancing preconditioner T given by* (6.25) *satisfies*

$$\kappa(T_b S) \lesssim \log^2 \frac{h_0}{h}.$$

Additional bibliography remarks. The methods studied in this section, often known as Neumann–Neumann-type algorithms, can be traced back to the work by Dinh,

Glowinski, and Périaux [33] and Glowinski and Wheeler [46]. Thereafter there are a few extensions in the theory and algorithms. We refer to Bourgat et al. [7], Roeck and Le Tallec [67], Le Tallec, Roeck, and Vidrascu [76], Mandel and Brezina [58, 59], and Dryja and Widlund [40]. For extension of the approach for mixed finite element framework by Glowinski and Wheeler [46] to the many-subdomain case, see Cowsar and Wheeler [32].

The weighted coarse subspace originated from Bramble, Pasciak, and Schatz [11, 12] for solving their coarse problem, and was used by Dryja and Widlund [40] and Mandel [58] for the Neumann–Neumannn-type methods. In [40], the use of standard coarse subspaces was also considered for elliptic problems with uniformly bounded coefficients. Here we gave a unified presentation for both two and three dimensions with the case of large jumps in coefficients included. In particular, we added the case of using the zero extensions E_i in local solvers R_i instead of weighted operators Θ_i.

The balancing DD method was proposed by Mandel [58], and later its convergence proof was improved by Mandel and Brezina [59, 60]. The balancing method is in some sense a dual method of the FETI algorithm of Farhat and Roux [41, 42, 43]. Cowsar, Mandel, and Wheeler [31] extended the balancing DD method to mixed finite elements.

7. Some other interface preconditioners. In this section, we shall discuss several other preconditioners that do not fall into the categories of the algorithms studied above. The first two examples are multilevel preconditioners, namely the hierarchical basis and BPX preconditioner on the interface, and another example is the overlapping additive Schwarz method on the interface.

7.1. Multilevel preconditioners. The multilevel preconditioners given here on the interface will be derived from the known multilevel preconditioners defined on the whole domain. To do this, we shall again apply the global-local technique in subsection 2.3.

In view of subsection 2.3, we choose the space $V = V^h$ and the subspace $\hat{V}$ of V^h to be the space of discrete harmonic functions, i.e., $\hat{V} = V_H$. Recall that $\hat{P} : V^h \mapsto V_H$ is an orthogonal projection with respect to $A(\cdot, \cdot)$. If we use the orthogonal decomposition $u = u_{_P} + u_{_H}$ (see section 3), with $u_{_P} \in V_0^h(\Omega_i)$ on each subdomain Ω_i and $u_{_H} \in V_H$ being discrete harmonic, then we see that

$$A(\hat{P}u, v_{_H}) = A(u, v_{_H}) = A(u_{_H}, v_{_H}) \quad \forall\ u \in V^h,\ v_{_H} \in V_H.$$

This means that $\hat{P}u$, for any $u \in V^h$, equals the discrete harmonic extension $u_{_H}$ of the restriction of u on the interface Γ.

Let $\hat{A}$ be the discrete harmonic operator, namely the restriction (see (2.8)) of the stiffness operator A on the harmonic subspace $V_H = \hat{V}$, and $\hat{B}$ a preconditioner on $\hat{V}$ for the harmonic operator $\hat{A}$. Then we can define a preconditioner T for the interface operator S by

$$T = \hat{R}\hat{B}\hat{R}^t, \tag{7.26}$$

where $\hat{R} : \hat{V} \mapsto V^h(\Gamma)$ is the restriction operator defined by $\hat{R}u_{_H} = u$ for any $u_{_H} \in \hat{V}$, and its adjoint $\hat{R}^t$ by

$$(\hat{R}^t u, v_{_H}) = \langle u, \hat{R}v_{_H} \rangle.$$

We have immediately that

$$\kappa(\hat{B}\hat{A}) = \kappa(TS), \tag{7.27}$$

since it is easy to see that $\hat{A} = \hat{R}^t S \hat{R}$ and

$$\begin{aligned} \langle TSu, Su\rangle &= \langle \hat{R}\hat{B}\hat{R}^t Su, Su\rangle = \langle \hat{B}\hat{R}^t S\hat{R}\tilde{u}, \hat{R}^t S\hat{R}\tilde{u}\rangle \\ &= \langle \hat{B}\hat{A}\tilde{u}, \hat{A}\tilde{u}\rangle. \end{aligned} \tag{7.28}$$

Now we are in a position to introduce multilevel methods defined on the whole domain.

Given a sequence of nested piecewise-linear finite element subspaces

$$V_1 \subset V_2 \subset \cdots \subset V_J = V^h$$

corresponding to a sequence of nested quasi-uniform triangulations $\{\mathcal{T}^k\}_{k=1}^J$ of Ω with mesh-size parameters

$$h_k = \max_{\tau\in\mathcal{T}^k} \operatorname{diam}(\tau) \eqsim 2^{-k},$$

let $\{\phi_i^k\}_{i=1}^{n_k}$ be the set of nodal basis functions of V_k $(1 \le k \le J)$ corresponding to the set $\mathcal{N}_k$ of interior nodal points of the triangulation $\mathcal{T}^k$.

Hierarchical basis preconditioner. Let V_k and $\mathcal{N}_k$, $1 \le k \le J$, be defined as above. The hierarchical basis preconditioner B_{hb} for the stiffness operator A can be written as (see Xu [79, 81] and Yserentant [86])

$$B_{hb}v = \sum_{k=1}^{J} \sum_{x_i^k \in \mathcal{N}_k \setminus \mathcal{N}_{k-1}} \frac{(v, \phi_i^k)}{A(\phi_i^k, \phi_i^k)}\, \phi_i^k \quad \forall\ v \in V^h$$

and

$$\kappa(B_{hb}A) \lesssim \gamma_n(h),$$

with

$$\gamma_n(h) = \begin{cases} 1 & \text{if } n = 1, \\ |\log h|^2 & \text{if } n = 2, \\ h^{-1} & \text{if } n = 3. \end{cases} \tag{7.29}$$

Then, by Theorem 2.6, the corresponding preconditioner $\hat{B}$ for the operator $\hat{A}$ is

$$\hat{B}\tilde{v} = \sum_{k=1}^{J} \sum_{x_i^k \in \Gamma \bigcap (\mathcal{N}_k \setminus \mathcal{N}_{k-1})} \frac{(\tilde{v}, \tilde{\phi}_i^k)}{A(\phi_i^k, \phi_i^k)}\, \tilde{\phi}_i^k \quad \forall\ \tilde{v} \in V_H.$$

Note that only basis functions on the interface Γ appear above, since the discrete harmonic extensions of all interior basis functions vanish.

For the interface preconditioner T defined as in (7.26), by (7.27) and Theorem 2.6, we have

$$\kappa(TS) = \kappa(\hat{B}\hat{A}) \le \kappa(B_{hb}A) \lesssim \gamma_n(h).$$

Interface overlapping BPX multilevel preconditioner. A simple version of the BPX preconditioner B for the stiffness operator A can be written as (see Bramble, Pasciak, and Xu [15] and Xu [79, 81])

$$B_{bpx}v = \sum_{k=1}^{J} \sum_{x_i^k \in \mathcal{N}_k} \frac{(v, \phi_i^k)}{A(\phi_i^k, \phi_i^k)} \phi_i^k \quad \forall\ v \in V^h.$$

It has been proved that (see [18, 80, 81, 65])

$$\kappa(B_{bpx}A) \lesssim \min(\alpha(\rho), \beta_n(h)).$$

Here $\alpha(\rho)$ is a constant dependent on ρ but independent of h, while $\beta_n(h)$ is a constant independent of ρ but dependent on h. The bound $\beta_n(h)$ depends on the distribution of ρ and subdomains and its worst bound is, as in the hierarchical basis method,

$$\beta_n(h) \le \gamma_n(h),$$

with $\gamma_n(h)$ given by (7.29).

Then, by Theorem 2.6, the corresponding preconditioner $\hat{B}$ for the operator $\hat{A}$ is

$$\hat{B}_{bpx}\tilde{v} = \hat{P}B\hat{P}^t\tilde{v} = \sum_{k=1}^{J}\sum_{x_i^k\in\Gamma}\frac{(\tilde{v},\tilde{\phi}_i^k)}{A(\phi_i^k,\phi_i^k)}\,\tilde{\phi}_i^k \quad \forall\ \tilde{v}\in V_H,$$

where $\tilde{v}$ is the discrete harmonic extension of $v \in V^h(\Gamma)$. For the corresponding interface preconditioner T defined as in (7.26), by (7.27) with Theorem 2.6, we have

$$\kappa(TS) = \kappa(\hat{B}_{bpx}\hat{A}_{bpx}) \le \kappa(BA) \lesssim \min(\alpha(\rho), \beta_n(h)).$$

REMARK 7.1. The hierarchical basis method was proposed by Yserentant [86] and studied later by many authors, e.g., Bank, Dupont, and Yserentant [3], Yserentant [87], Bank and Xu [4], and Xu [79, 81]. The interface hierarchical basis preconditioner was considered by Smith and Widlund [74] and Haase, Langer, and Meyer [50].

The BPX preconditioner was proposed by Bramble, Pasciak, and Xu [15] and Xu [79]. The interface BPX preconditioner was studied by Xu [79], Tong, Chan, and Kuo [78], and Bramble, Pasciak, and Xu [17].

7.2. Interface overlapping additive Schwarz method. In this section, we consider an interface additive Schwarz algorithm. The method can be viewed as a variant of the standard overlapping additive Schwarz algorithm on the interface Γ.

We first decompose the domain Ω into q_0 nonoverlapping subregions $\tilde{\Omega}_i^0$ with diameters of size h_0. Note that $\{\tilde{\Omega}_i^0\}$ may be different from the existing subdomains $\{\Omega_i\}_{i=1}^J$. We consider only the case that the original subdomains $\{\Omega_i\}_{i=1}^J$ are a simplicial triangulation of Ω (see Remark 7.3 for more general cases). We then extend each $\tilde{\Omega}_i^0$ to a larger subregion $\tilde{\Omega}_i$ with $\operatorname{dist}(\partial\tilde{\Omega}_i \cap \Omega, \partial\tilde{\Omega}_i^0) \eqsim h_0$. Let

$$\Gamma^i = \Gamma \cap \tilde{\Omega}_i, \quad 1 \le i \le q_0.$$

Without loss of generality, we assume that each Γ^i has a positive measure, and has a nonempty intersection with at most n_0 subregions from $\{\Gamma^j\}_{j=1}^{q_0}$. Note that $\{\Gamma^j\}_{j=1}^{q_0}$ forms an overlapping decomposition of Γ, namely,

$$\Gamma = \bigcup_{j=1}^{q_0}\Gamma^j. \tag{7.30}$$

Let $V_0(\Gamma)$ be the standard coarse subspace defined in subsection 5.2.1, i.e., the restriction on Γ of the piecewise-linear finite element space corresponding to the simplicial triangulation $\{\Omega_i\}_{i=1}^J$, and $V_i(\Gamma) \subset H_0^1(\Gamma_i)$ be the restriction of the space $V_0^h(\tilde{\Omega}_i)$ on the interface Γ for $0 \le i \le q_0$. We then define the additive Schwarz preconditioner T for the interface operator S as follows:

$$T = I_iR_0I_0^t + \sum_{i=1}^{q_0} I_iS_i^{-1}I_i^t,$$

where S_i is the restriction of S on $V_i(\Gamma)$, and the coarse solver $R_0 : V_0(\Gamma) \mapsto V_0(\Gamma)$ is the standard coarse solver defined in subsection 5.2.1, i.e.,

$$\langle R_0^{-1} u_0, v_0 \rangle = h_0^{n-1} \sum_{i=1}^{p} \rho_i \sum_{E_{kl} \subset \Gamma_i} (u_0(\mathrm{v}_k) - u_0(\mathrm{v}_l))\,(v_0(\mathrm{v}_k) - v_0(\mathrm{v}_l)) \quad \forall\, u_0, v_0 \in V_0(\Gamma).$$

We have the following result.

THEOREM 7.2. *For the additive Schwarz preconditioner T defined above,*

$$\kappa(TS) \lesssim r(\rho),$$

where $r(\rho)$ is the coefficient ratio, i.e., $\max_{1\le i\le J} \rho_i / \min_{1\le i\le J} \rho_i$.

Proof. By Theorem 2.1, we need to estimate the three parameters K_0, K_1, and ω_1. We readily note that $K_1 \le n_0$, since each Γ^i has a nonempty intersection with at most n_0 subregions from $\{\Gamma^j\}$.

For the estimate of ω_1, it suffices, by definition, to estimate $\lambda_{\max}(R_0 S_0)$ for the coarse subspace, because all local solvers are exact. This is easily obtained for $n = 2$ from (5.7), and for $n = 3$ from Lemma 4.3, which shows for any $u_0 \in V_0(\Gamma)$,

$$\langle S u_0, u_0 \rangle \eqsim \langle R_0^{-1} u_0, u_0 \rangle,$$

which shows $\omega_1 \le 1$.

Finally, we estimate K_0 associated with the partition of

$$V^h(\Gamma) = V_0(\Gamma) + \sum_{i=1}^{q_0} V_i(\Gamma).$$

For any $u \in V^h(\Gamma)$, it is known (see Chan and Zou [28] and Cai [21]) that there exists a partition for the discrete harmonic extension $\tilde{u}$ of u: $\tilde{u} = \sum_{i=0}^{q_0} u_i$ such that $u_i \in V_0^h(\tilde{\Omega}_i)$ and $\sum_{i=0}^{q_0} \|u_i\|_{1,\Omega}^2 \lesssim \|u\|_{1,\Omega}^2$. This together with the Poincaré inequality implies that

$$\sum_{i=0}^{q_0} A(u_i, u_i) \lesssim r(\rho)\, A(u, u). \tag{7.31}$$

Obviously, the restrictions w_i of u_i on the interface Γ, i.e., $w_i = u_i|_\Gamma$, belong to $V_i(\Gamma)$ and $u = \sum_{i=0}^{q_0} w_i$ by definition of $V_0^h(\tilde{\Omega}_i)$ and the choice of u_i. Now we show that

$$\sum_{i=0}^{q_0} \langle S w_i, w_i \rangle \lesssim r(\rho)\, \langle S u, u \rangle. \tag{7.32}$$

Using the fact that the discrete harmonic function $\tilde{w}_i$ equals the nonharmonic function u_i for $0 \le i \le q_0$ on the interface, we have

$$\langle S w_i, w_i \rangle = A(\tilde{w}_i, \tilde{w}_i) \le A(u_i, u_i),$$

which together with (7.31) gives (7.32), so $K_0 \lesssim r(\rho)$. This completes the proof of Theorem 7.2. ∎

REMARK 7.3. If we replace the standard interface coarse space $V_0(\Gamma)$ used above by the joint-set coarse space defined in subection 5.2.2, the condition number grows at a logarithmic rate of $\frac{h_0}{h}$ but is independent of the jumps in coefficients. In this case, the original subdomains do not need to form a triangulation of Ω.

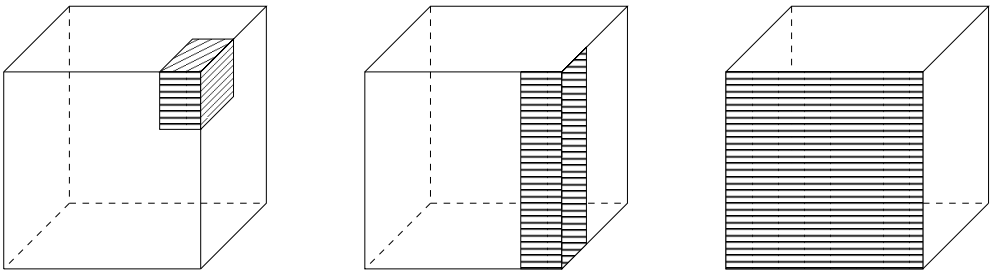

FIG. 9. *Vertex-, edge-, and face-related overlapping subregions on the interface* Γ.

REMARK 7.4. Note that subdomains $\{\tilde{\Omega}_i\}$ are not necessarily a triangulation of Ω. Each $\tilde{\Omega}_i$ can be of arbitrary shape but $\partial\tilde{\Omega}_i$ must be a union of boundaries of elements in $\mathcal{T}^h$.

REMARK 7.5. If we choose some special subdomains $\{\tilde{\Omega}_i\}$ such that the interface subregions Γ^i are vertex-, edge-, and face-related overlapping subregions of the interface Γ (see Fig. 9), then the algorithm discussed in this subsection reduces to the *vertex algorithm* studied by Matsokin and Nepomnyaschikh [62] and Smith [70, 72].

8. Methods with inexact subdomain solvers. All the nonoverlapping DD methods studied so far require exact subdomain solvers. Such a requirement can severely degrade the efficiency of the methods. We shall briefly discuss how such a requirement can be removed.

Assume that $E : V^h(\Gamma) \to V$ is a linear extension operator satisfying

$$A(Ev, Ev) \lesssim S(v, v) \quad \forall\, v \in V^h(\Gamma). \tag{8.33}$$

For any $v \in V$, let $v_\Gamma = v|_\Gamma$. We write

$$v = (v - Ev_\Gamma) + Ev_\Gamma.$$

Note that $v - Ev_\Gamma \in V_P$. By (8.33), we then obtain a "stable" space decomposition

$$V = V_P + V_E \quad \text{with} \quad V_E = \text{range of } E.$$

By the auxiliary or fictitious space lemma (see subsection 2.3.3), if T is a good preconditioner of S, then ETE^t is a good preconditioner for the operator associated with the space V_E. Therefore, if B_P is a good preconditioner for the operator associated with the space V_P, by Theorem 2.1, we obtain a preconditioner for the original operator A as follows:

$$B = I_P B_P I_P^t + I_E ETE^t I_E^t. \tag{8.34}$$

What becomes crucial is the extension operator E. The easiest mathematically is the discrete harmonic extension, but this extension requires exact subdomain solvers. The algorithms discussed earlier correspond to such types of extension.

We now give a simple example of extension operators by using the arithmetic average. For works related to this kind of idea, we refer to Bjørstad, Dryja, and Vainikko [5] and Bramble, Pasciak, and Vassilev [19].

Let G be any subdomain Ω_j and $v \in V^h(\partial G)$. We can define Ev by extending v to be a constant (the arithmetic average of the nodal values on $\partial\Omega$) inside each G, namely, for any interior nodal point $x \in G \cap \mathcal{N}_h$,

$$Ev(x) = \frac{1}{m} \sum_{y \in \partial G \cap \mathcal{N}_h} v(y),$$

where m_i is the number of nodes on ∂G.

Of course, we should not expect such an extension operator to have a good bound in terms of the subdomain size h_0 and the finite element size h. Nevertheless, the following estimate can be easily proved (for $n = 2$):

$$|Ev|^2_{1,G} \lesssim \frac{h_0}{h}|v|^2_{1/2,\partial G}.$$

Consequently, the resulting preconditioner T using this simple extension would in general admit the following type of estimate:

$$\kappa(TS) \lesssim \frac{h_0}{h}.$$

Despite such a deterioration, this preconditioner may still be useful in practice because of its simplicity.

Nonharmonic and yet nearly bounded extensions are also possible to obtain, but they are in general rather technical; we refer to Nepomnyaschikh [63] (and the references cited therein). The extension operator can be more easily constructed if the finite element space has a nested multilevel structure; see Haase et al. [51].

9. Implementation issues. In this section, we shall give a brief discussion of the implementation issues on the main algorithms studied in the paper. We will focus only on the preconditioning part in the PCG method (see section 2.1), i.e., the action of preconditioners.

9.1. Substructuring algorithms. For the substructuring methods studied in section 5, we shall present the S-, BPS- and DW-implementations of substructuring method I and also substructuring method II.

Before the presentation, we first discuss the possible solver for the coarse problem

$$\gamma(u_0, v_0) = f(v_0) \quad \forall\ v_0 \in V_0 \tag{9.1}$$

for a given function f on V_0. The bilinear form $\gamma(u_0, v_0)$ and the joint-set coarse space V_0 above are defined in (5.15) and subsection 5.2, respectively. In two dimensions, the problem becomes

$$\sum_{i=1}^{J} \rho_i \sum_{E_{kl}\subset\Gamma_i} (u_0(\mathrm{v}_k) - u_0(\mathrm{v}_l))(v_0(\mathrm{v}_k) - v_0(\mathrm{v}_l)) = f(v_0) \quad \forall\ v_0 \in V_0, \tag{9.2}$$

which involves all vertices of the subdomains as unknown variables, and can be solved using a standard method, say, simply using a direct method.

In three dimensions, the problem (9.1) becomes

$$\log\frac{h_0}{h}\sum_{i=1}^{J} \rho_i \langle u_0 - \gamma_{h,\mathcal{W}_i}(u_0), v_0 - \gamma_{h,\mathcal{W}_i}(v_0)\rangle_{h,\mathcal{W}_i} = f(v_0) \quad \forall\ v_0 \in V_0. \tag{9.3}$$

We next discuss a solver for this algebraic system. By symmetry, (9.3) is equivalent to the minimization problem

$$\min_{w_0\in V_0(\Gamma)} \frac{1}{2}\log\frac{h_0}{h}\sum_{i=1}^{J} \rho_i \min_{\lambda_i\in R^1} \|w_0 - \lambda_i\|^2_{h,\mathcal{W}_i} - f(w_0), \tag{9.4}$$

or

$$\min_{w_0\in V_0(\Gamma),\Lambda\in R^J} \frac{1}{2}\log\frac{h_0}{h}\sum_{i=1}^{J} \rho_i \|w_0 - \lambda_i\|^2_{h,\mathcal{W}_i} - f(w_0),$$

where $\Lambda = (\lambda_1, \ldots, \lambda_J)$.

The Euler equation for the above problem is

$$\sum_{i=1}^{J} \rho_i \langle u_0 - \lambda_i,\ v_0 \rangle_{h,\mathcal{W}_i} = f(v_0) \quad \forall\ v_0 \in V_0,$$

where we have used the fact that

$$\langle u_0 - \lambda_i, 1 \rangle_{h,\mathcal{W}_i} = 0.$$

With certain ordering of the nodes $\{x_k\}$ on the joint-set, for each i, let $\mathcal{I}_i$ be a matrix with zero off-diagonal entries whose kth diagonal entry equals 1 if $x_k \in \mathcal{W}_i$ and 0 otherwise. Let μ be the vector with components $u_0(x_k)$ for all joint-set nodes x_k. The aforementioned Euler equation can be written as

$$\sum_{i=1}^{J} \rho_i \mathcal{I}_i (\mu - \lambda_i e) = b \tag{9.5}$$

for some appropriate vector b, and

$$e^t \mathcal{I}_i (\mu - \lambda_i e) = 0, \quad 1 \le i \le J. \tag{9.6}$$

Let $\mathcal{D} = \sum_{i=1}^{J} \rho_i \mathcal{I}_i$, which is obviously a nonsingular diagonal matrix. By (9.5) we have

$$\mu = \mathcal{D}^{-1} \sum_{i=1}^{J} \rho_i \mathcal{I}_i \lambda_i e + \mathcal{D}^{-1} b. \tag{9.7}$$

Substituting (9.7) into (9.6) yields

$$e^t \mathcal{I}_i \left(\mathcal{D}^{-1} \sum_{j=1}^{J} \rho_j \mathcal{I}_j \lambda_j e - \lambda_i e \right) = -e^t \mathcal{I}_i \mathcal{D}^{-1} b, \quad 1 \le i \le J. \tag{9.8}$$

This is a global problem with one unknown λ_i for each subdomain Ω_i. Solving the equation (9.8), we get λ_i, and then we can calculate μ from (9.7).

The above construction can be found in [71] and [73], and is based on an earlier work by Mandel [61], where it was recognized that the μ_i are minimizing. The minimizing property is the key to extension to problems like elasticity, where the coarse problem consists of rigid body motions in each subdomain; there, we simply take the minimum over a subspace of dimension larger than one.

9.1.1. Substructuring I: S-implementation. For the preconditioner T defined in (5.19), its action Tg for any $g \in V^h(\Gamma)$ satisfies

$$u = Tg = I_0 R_0 I_0^t g + \sum_{F \subset \Gamma} I_F R_F I_F^t g \equiv u_0 + \sum_{F \subset \Gamma} u_{_F}.$$

By definition, $u_0 = I_0 R_0 I_0^t g$ solves

$$\gamma(u_0, v) = \langle g, v \rangle \quad \forall\ v \in V_0,$$

while $u_{_F}$ solves

$$\rho_F \langle (-\Delta_{F,h})^{1/2} u_{_F}, v_{_F} \rangle_{0,F} = \langle g, v_{_F} \rangle.$$

We then have the following algorithm.

ALGORITHM 9.1 (substructuring algorithm I). *For any given $g \in V^h(\Gamma)$, $u = Tg = u_0 + \sum_F u_{_F}$ can be obtained as follows:*

1. *Solve for* $u_F \in V_0^h(F)$ *on each face* $F \subset \Gamma$*:*

$$\rho_F \langle (-\Delta_{F,h})^{1/2} u_F, v_F \rangle_{0,F} = \langle g, v_F \rangle \quad \forall\ v_F \in V_0^h(F).$$

2. *Solve for* $u_0 \in V_0$ *the coarse problem*

$$\gamma(u_0, v) = \langle g, v \rangle \quad \forall\ v \in V_0.$$

Here the coarse problem in step 2 can be solved as shown at the beginning of subsection 9.1.

9.1.2. Substructuring I: BPS-implementation. For the stiffness preconditioner $\hat{A}$ defined by (5.21), its action $u = \hat{A}^{-1} g$ for any given g in V^h solves the equation

$$A(u_P, v_P) + \gamma(u_0, v_0) + \sum_{F \subset \Gamma} \rho_F \langle (-\Delta_{F,h})^{1/2} u_F, v_F \rangle_{0,F} = (g, v) \quad \forall\ v \in V^h,$$

where $u_0 = \Pi_0 u$ and $u_F = (u - \Pi_0 u)_F$. This yields immediately the following algorithm.

ALGORITHM 9.2 (substructuring algorithm II). *For given* $g \in V^h$*, let* $u = \hat{A}^{-1} g$*. Then* $u = u_P + u_H$ *can be obtained as follows:*

1. *Solve for* $u_P \in V_0^h(\Omega_i)$ *on each subdomain* Ω_i*:*

$$A(u_P, v) = (g, v) \quad \forall\ v \in V_0^h(\Omega_i).$$

2. *Solve for* $u_F \in V_0^h(F)$ *on each face* $F \subset \Gamma$*:*

$$\rho_F \langle (-\Delta_{F,h})^{1/2} u_F, v_F \rangle_{0,F} = (g, \hat{v}) - A(u_P, \hat{v}) \quad \forall\ v \in V_0^h(F).$$

3. *Solve for* $u_0 \in V_0$ *the coarse problem*

$$\gamma(u_0, v) = (g, \hat{v}) - A(u_P, \hat{v}) \quad \forall\ v \in V_0.$$

4. *Solve for* $u_H \in V^h(\Omega_i)$ *on each subdomain* Ω_i*:*

$$A(u_H, v) = 0 \quad \forall\ v \in V_0^h(\Omega_i),$$

with u_H *specified by* $u_0 + u_F$ *on each face* F *of* $\partial\Omega_i$*.*

REMARK 9.1. The function $\hat{v}$ in steps 2–3 above can be any extension of v in V^h because of the introduction of the correcting term $A(u_P, \cdot)$. Mathematically, the BPS stiffness preconditioner $\hat{A}$ is equivalent to the interface preconditioner T of the S-implementation in subsection 9.1.1. But from the algorithmic point of view there is a difference between the two. Considering each PCG iteration for both algorithms, BPS needs to compute the action $\hat{A}^{-1} A g$ for given $g \in V^h$ (Algorithm 9.2 is only for computing $\hat{A}^{-1} \tilde{g} \equiv \hat{A}^{-1}(Ag)$), while the S-implementation needs to compute the action TSg for given $g \in V^h(\Gamma)$ (Algorithm 9.1 is just for computing $T\tilde{g} \equiv T(Sg)$). Step 1 of Algorithm 9.2 is the same as the action of the interface operator S on a given g, i.e., the first computation for TSg. The second and third steps of Algorithm 9.2 amount to the first and second steps of Algorithm 9.1. In summary, the first three steps in the BPS-algorithm are equivalent to the action TSg. This indicates that the BPS-algorithm is a bit more expensive, since it requires the discrete harmonic extension at each iteration (step 4), while the S-implementation requires the extension only after the termination of the PCG iteration (transforming the interface solution to the global domain Ω).

9.1.3. Substructuring I: DW-implementation. We present only the two-dimensional version in view of the practical popularity and numerical efficiency. Let T be the preconditioner defined in (5.24) as B. Then its action Tg for a given $g \in V^h$ can be written as

$$u = Tg = I_0 R_0 I_0^t g + \sum_{ij} I_{ij} R_{ij} I_{ij}^t g \equiv u_0 + \sum_{ij} u_{ij}.$$

Considering the inexact coarse solver R_0 discussed in subsection 5.2.1 and the exact local solvers $R_{ij} = A_{ij}^{-1}$ (can be replaced by any equivalent inexact solver), we obtain the following algorithm.

ALGORITHM 9.3 (substructuring algorithm III). *For given $g \in V^h$, $u = Tg = u_0 + \sum_{ij} u_{ij}$ can be obtained as follows:*

1. *Solve for $u_{ij} \in V_0^h(\Omega_{ij})$ corresponding to each face $F_{ij} \subset \Gamma$:*

$$A(u_{ij}, v) = (g, v) \quad \forall\ v \in V_0^h(\Omega_{ij}).$$

2. *Solve for $u_0 \in V_0$ the coarse problem* (9.2) *with $f(v) = (g, v)$.*

9.1.4. Substructuring method II. Let $\hat{A}$ be the preconditioner defined in (5.32) with μ_i satisfying (5.29). Following [12], we now discuss how to implement the action of $u = \hat{A}^{-1} g$ for any given function $g \in V^h$. Clearly, u solves $\hat{A}u = g$. Then by definition of $\hat{A}$ and the property (5.29), we know $u = u_{_P} + u_{_H}$ solves

$$A(u_{_P}, v_{_P}) + \sum_{j=1}^{J} \rho_j G_j(u - \mu_j, v) = (g, v) \quad \forall\ v \in V^h.$$

As before, $u_{_P}$ can be obtained by solving a homogeneous Dirichlet problem independently on each subdomain. Once $u_{_P}$ is known, then the value of u on the interface Γ, equal to $u_{_H}$, satisfies

$$\sum_{j=1}^{J} \rho_j G_j(u - \mu_j, v) = F(\hat{v}), \quad v \in V_\Gamma, \tag{9.9}$$

with $F(\hat{v}) = (g, \hat{v}) - A(u_{_P}, \hat{v})$ and $\hat{v}$ being any extension of v into $V^h(\Omega_i)$ for $1 \le i \le J$. Once the interface value of u is known, $u_{_H}$ can be obtained by the discrete harmonic extension into each subdomain Ω_i, i.e., solving a local Dirichlet problem. Now we show how to solve (9.9) for the interface value of u.

Let us introduce some notation. For each local interface Γ_i, V_i denotes a subspace of $V^h(\Gamma)$ with functions vanishing at nodes in $\Gamma \setminus \Gamma_i$. For any point and face on the interface Γ, we define a corresponding index subset of subdomains:

$$\mathcal{N}(x) = \{i;\ \ x \in \partial\Omega_i\}, \quad \mathcal{N}(F) = \{i;\ \ F \subset \partial\Omega_i\}. \tag{9.10}$$

Based on the index subset $\mathcal{N}(x)$, we construct one basis function per subdomain, denoted by ϕ_i $(1 \le i \le J)$, such that $\phi_i \in V^h(\Gamma)$ and

$$\phi_i(x_k) = \begin{cases} \nu_\rho^{-1}(x_k) & \text{for any node } x_k \in \Gamma_i, \\ 0 & \text{for any node } x_k \in \Gamma \setminus \Gamma_i, \end{cases} \tag{9.11}$$

where $\nu_\rho(x)$ is the weighted counting function given by (6.7).

Using these basis functions ϕ_i, we can write

$$\sum_{j=1}^{J} \rho_j G_j(u-\mu_j,\phi_i) = \sum_{j=1}^{J} \rho_j G_j(u,\phi_i) - \sum_{j=1}^{J} \rho_j \mu_j G_j(1,\phi_i)$$

$$\equiv I_1 + I_2. \tag{9.12}$$

By the definition of $G_i(\cdot,\cdot)$ and ϕ_i, I_2 and I_1 can be rewritten as

$$\begin{aligned} I_2 &= \sum_{j=1}^{J} h\rho_j\mu_j \sum_{x_k\in\mathcal{W}_j} \phi_i(x_k) + \sum_{j=1}^{J} \rho_j\mu_j \sum_{F\subset\partial\Omega_j} \langle(-\Delta_{F,h})^{1/2} I_F^0 1, I_F^0\phi_i\rangle_{0,F} \\ &= h \sum_{x_k\in\mathcal{W}_i} \sum_{j\in\mathcal{N}(x_k)} \frac{\rho_j}{\rho(x_k)}\mu_j + \sum_{F\subset\partial\Omega_i} \sum_{j\in\mathcal{N}(F)} l_0(F)\frac{\rho_j}{\rho_F}\mu_j, \end{aligned}$$

where $l_0(F) = \langle(-\Delta_{F,h})^{1/2} I_F^0 1, I_F^0 1\rangle_{0,F}$, and

$$\begin{aligned} I_1 &= h\sum_{j=1}^{J} \rho_j \sum_{x_k\in\mathcal{W}_j} u(x_k)\phi_i(x_k) + \sum_{j=1}^{J} \rho_j \sum_{F\subset\partial\Omega_j} \langle(-\Delta_{F,h})^{1/2} I_F^0 u, I_F^0\phi_i\rangle_{0,F} \\ &= h \sum_{x_k\in\mathcal{W}_i} \sum_{j\in\mathcal{N}(x_k)} \rho_j u(x_k)\phi_i(x_k) + \rho_F \sum_{F\subset\partial\Omega_i} \langle(-\Delta_{F,h})^{1/2} I_F^0 u, I_F^0\phi_i\rangle_{0,F} \\ &= h \sum_{x_k\in\mathcal{W}_i} u(x_k) + \sum_{F\subset\partial\Omega_i} \langle(-\Delta_{F,h})^{1/2} I_F^0 u, I_F^0 1\rangle_{0,F} \\ &= G_i(u,1) = \mu_i\, G_i(1,1). \end{aligned}$$

Substituting I_1 and I_2 into (9.9) gives, for $1 \le i \le J$,

$$\mu_i G_i(1,1) - h \sum_{x_k\in\mathcal{W}_i} \sum_{j\in\mathcal{N}(x_k)} \frac{\rho_j}{\rho(x_k)}\,\mu_j - \sum_{F\subset\partial\Omega_i} \sum_{j\in\mathcal{N}(F)} l_0(F)\frac{\rho_j}{\rho_F}\,\mu_j = F(\hat{\phi}_i). \tag{9.13}$$

Solving this $J\times J$ system for J unknowns μ_j, we obtain μ_j.

Once μ_j is known, it follows from (9.9) that

$$\sum_{j=1}^{J} \rho_j G_j(u,v) = F(\hat{v}) + \sum_{j=1}^{J} \rho_j\mu_j G_j(1,v) \equiv \hat{F}(v) \quad \forall\ v\in V^h(\Gamma).$$

This is equivalent to, tracing back to the expressions of I_1 above,

$$h \sum_{x_k\in\mathcal{W}_i} \rho(x_k)u(x_k)v(x_k) + \rho_F \sum_{F\subset\partial\Omega_i} \langle(-\Delta_{F,h})^{1/2} I_F^0 u, I_F^0 v\rangle_{0,F} = \hat{F}(v).$$

Taking $v\in V_0^h(F)$ in the above equation, we can get the value $u_{_F} = I_F^0 u$ of u on the face F by solving

$$\rho_F\langle(-\Delta_{F,h})^{1/2} u_{_F}, v\rangle_{0,F} = \hat{F}(v) \quad \forall\ v\in V_0^h(F). \tag{9.14}$$

Then taking v to be the standard nodal basis θ_k at one node $x_k\in\mathcal{W}$, we obtain

$$h\,\rho(x_k)u(x_k) = \hat{F}(\theta_k), \tag{9.15}$$

which gives directly the nodal value of $u(x_k)$ for any node x_k on the joint-set $\mathcal{W}$.

In summary, the action of $u = \hat{A}^{-1}g$ for any given g in V^h can be done as in the next algorithm.

ALGORITHM 9.4 (substructuring algorithm IV). *The components $u_{_P}$ and $u_{_H}$ of $u = u_{_P} + u_{_H}$ can be computed as follows:*

1. *Solve for* $u_{_P} \in V_0^h(\Omega_i)$ *on each subdomain* Ω_i*:*

$$A(u_{_P}, v) = (g, v) \quad \forall\ v \in V_0^h(\Omega_i).$$

2. *Solve the coarse problem* (9.13) *for the average values* μ_j *for* $1 \le j \le J$.
3. *Solve for* $u_{_F} \in V_0^h(F)$ *on each face* $F \subset \Gamma$*:*

$$\rho_{_F} \langle (-\Delta_{F,h})^{1/2} u_{_F}, v \rangle_{0,F} = \hat{F}(v) \quad \forall\ v \in V_0^h(F).$$

4. *Calculate* $u_{\mathcal{W}}$ *belonging to* $V^h(\Gamma)$ *and* $u_{\mathcal{W}}(x_k) = u(x_k)$ *for any* $x_k \in \mathcal{W}$*, from the relation* (9.15).
5. *Solve for* $u_{_H} \in V^h(\Omega_i)$ *on each subdomain* Ω_i*:*

$$A(u_{_H}, v) = 0 \quad \forall\ v \in V_0^h(\Omega_i),$$

with $u_{_H}$ *specified on the boundary* $\partial\Omega_i$ *by* $\sum_F u_{_F} + u_{\mathcal{W}}$.

9.2. Neumman–Neumann algorithms. For this type of method, we shall discuss two examples.

9.2.1. Neumman–Neumann algorithm with weighted coarse space. Let T be the preconditioner defined by (6.5) with the weighted coarse space V_0 specified by (6.13). For a given function $g \in V^h(\Gamma)$, the action Tg satisfies

$$u = Tg = I_0 R_0 I_0^t g + \sum_{i=1}^{J} I_i \Theta_i \check{S}_i^{-1} \Theta_i^t I_i^t g \equiv \sum_{i=0}^{J} u_i. \tag{9.16}$$

For $i \neq 0$, let w_i be a function such that $\Theta_i w_i = u_i$. By definition of $\check{S}_i$ in (6.3), $u_i = \Theta_i w_i$ can be obtained by solving the local Neumann problem for $w_i = \check{w}_i|_{_\Gamma}$:

$$\rho_i(\check{w}_i, v_i)_{1,\Omega_i} = \langle g, \Theta_i v_i \rangle \quad \forall\ v_i \in V^h(\Omega_i).$$

Since the coarse solver is the scaled exact solver $R_0 = \log^2 \frac{h_0}{h} S_0^{-1}$, u_0 solves the coarse problem

$$\langle S u_0, v_0 \rangle = \log^2 \frac{h_0}{h} \langle g, v_0 \rangle \quad \forall\ v_0 \in V_0. \tag{9.17}$$

To solve this system, we have to form the coefficient matrix $\mathcal{S} = (s_{ij})$ with entries $s_{ij} = \langle S\phi_i, \phi_j \rangle$. Here ϕ_i are the basis functions of the weighted coarse space V_0. Each calculation of $S\phi_i$ is equivalent to solving a local Dirichlet problem on each immediate adjacent subdomain to Ω_i.

We note that the above exact coarse solver is expensive. Here we want to present another coarse solver which is almost spectrally equivalent to the above exact solver, but with much less cost. To define this new solver, using the bilinear form $G_i(\cdot,\cdot)$ defined in subsection 5.4, we know for any $u_0 \in V_0$ that

$$\langle S u_0, u_0 \rangle \lesssim \sum_{i=1}^{J} \rho_i\, G_i(u_0 - \mu_i(u_0), u_0 - \mu_i(u_0)) \lesssim \log^2 \frac{h_0}{h} \langle S u_0, u_0 \rangle.$$

Here $\mu_i = 0$ for all boundary subdomains Ω_i (see Remark 5.4), but otherwise define μ_i by

$$G_i(u_0 - \mu_i(u_0), 1) = 0.$$

The above equivalence suggests that we can define the coarse solver R_0 by

$$\langle R_0^{-1}u_0, v_0\rangle = \log^{-2}\frac{h_0}{h}\sum_{i=1}^{J}\rho_i\, G_i(u_0-\mu_i(u_0), v_0-\mu_i(v_0)) \quad \forall\ u_0, v_0 \in V_0.$$

Then u_0 in (9.16) can be calculated by solving

$$\sum_{i=1}^{J}\rho_i\, G_i(u_0-\mu_i(u_0), v_0) = \log^2\frac{h_0}{h}\langle g, v_0\rangle. \tag{9.18}$$

The unknowns $\mu_j(u_0)$, one per interior subdomain, solve the coarse problem

$$\mu_i G_i(1,1) - \sum_{x_k\in\mathcal{W}_i}\sum_{j\in\mathcal{N}(x_k)}\frac{h\rho_j}{\rho(x_k)}\,\mu_j - \sum_{F\subset\partial\Omega_i}\sum_{j\in\mathcal{N}(F)} l_0(F)\frac{\rho_j}{\rho_F}\,\mu_j = \log^2\frac{h_0}{h}\langle g, v_0\rangle.$$

Once the μ_i are known, then u_0 is obtained by solving

$$\sum_{i=1}^{J}\rho_i\, G_i(u_0, v_0) = \log^2\frac{h_0}{h}\langle g, v_0\rangle + \sum_{i=1}^{J}\mu_i\,\rho_i\, G_i(1, v_0) \quad \forall\ v_0 \in V_0.$$

For this system, we can form the coefficient matrix by using the special basis functions ϕ_k in V_0 and calculate the entries $G_i(\phi_k, \phi_m)$.

From above, we have the following algorithm.

ALGORITHM 9.5 (Neumann–Neumann algorithm). *For given $g \in V^h(\Gamma)$, $u = Tg = \sum_{i=0}^{J} u_i$ can be obtained as follows:*

1. *Solve for $u_i \in V^h(\Omega_i)$ on each subdomain Ω_i ($i \neq 0$):*

$$\rho_i(u_i, v_i)_{1,\Omega_i} = \langle g, \Theta_i v_i\rangle \quad \forall\ v_i \in V^h(\Omega_i).$$

2. *Solve for the average values μ_i the global problem*

$$\mu_i G_i(1,1) - \sum_{x_k\in\mathcal{W}_i}\sum_{j\in\mathcal{N}(x_k)}\frac{h\rho_j}{\rho(x_k)}\,\mu_j - \sum_{F\subset\partial\Omega_i}\sum_{j\in\mathcal{N}(F)} l_0(F)\frac{\rho_j}{\rho_F}\,\mu_j = \log^2\frac{h_0}{h}\langle g, v_0\rangle.$$

3. *Solve for u_0 the coarse problem*

$$\sum_{i=1}^{J}\rho_i\, G_i(u_0, v_0) = \log^2\frac{h_0}{h}\langle g, v_0\rangle + \sum_{i=1}^{J}\mu_i\,\rho_i\, G_i(1, v_0) \quad \forall\ v_0 \in V_0.$$

REMARK 9.2. If the standard coarse space is used instead of the weighted coarse space, then step 2 and step 3 are replaced by the following step:

2′. *Solve for $u_0 \in V_0$ the coarse problem*

$$h_0^{n-2}\sum_{i=1}^{J}\sum_{E_{kl}\subset\partial\Omega_i}(u_0(\mathrm{v}_k)-u_0(\mathrm{v}_l))\,(v_0(\mathrm{v}_k)-v_0(\mathrm{v}_l)) = \langle g, v_0\rangle \quad \forall\ v_0 \in V_0.$$

9.2.2. Balancing DD algorithm. Let T_b be the preconditioner defined by (6.25) for the interface operator S. The action $u = T_b g$, for any given $g \in V^h(\Gamma)$, can be written as

$$\begin{aligned} u = T_b g &= P_0 S^{-1} g + (I-P_0)\left(\sum_{i=1}^{J} I_i\Theta_i S_i^{+}\Theta_i^t I_i^t\right) S(I-P_0)S^{-1}g \\ &= \left(u_0 - P_0\sum_{i=1}^{J}\Theta_i\check{u}_i\right) + \sum_{i=1}^{J}\Theta_i\check{u}_i, \end{aligned}$$

where $u_0 = P_0 S^{-1} g$ and, for $i = 1, 2, \ldots, J$,

$$\check{u}_i = (S_i^+ \Theta_i^t I_i^t) S (I - P_0) S^{-1} g.$$

By definition of P_0, u_0 solves the coarse problem

$$S(u_0, v_0) = \langle g, v_0 \rangle \quad \forall\ v_0 \in V_0. \tag{9.19}$$

Let $r_0 = g - S u_0$ be the residual. Then $\check{u}_i$ solves, by definition of S_i^+,

$$S(\check{u}_i, v_i) = \langle r_0, \Theta_i v_i \rangle \quad \forall\ v_i \in \hat{V}_i(\Gamma),$$

or equivalently,

$$A(\check{u}_i, v_i) = \langle r_0, \Theta_i v_i \rangle \quad \forall\ v_i \in \hat{V}^h(\Omega_i),$$

with $\hat{V}^h(\Omega_i) = \{v \in V^h(\Omega_i) : \gamma_{\Gamma_i}(v) = 0\}$. Using these $\check{u}_i$, we can obtain $w_0 \equiv (u_0 - P_0 \sum_{i=1}^J \Theta_i \check{u}_i)$ by solving the residual equation

$$S(w_0, v_0) = \left\langle g - S \sum_{i=1}^J \Theta_i \check{u}_i, v_0 \right\rangle \quad \forall\ v_0 \in V_0.$$

Thus we have the action $u = T_b g = w_0 + \sum_{i=1}^J \Theta_i \check{u}_i$. We therefore come to the following algorithm.

ALGORITHM 9.6 (balancing DD algorithm). *For any $g \in V^h(\Gamma)$, the action $u = T_b g = w_0 + \sum_{i=1}^J \Theta_i \check{u}_i$ can be obtained as follows:*

1. *Balance the original residual by solving*

$$S(u_0, v_0) = \langle g, v_0 \rangle \quad \forall\ v_0 \in V_0.$$

   Calculate the residual $r_0 = g - S u_0$.
2. *Compute $\check{u}_i \in \hat{V}^h(\Omega_i)$ for $i \neq 0$ concurrently:*

$$A(\check{u}_i, v_i) = \langle r_0, \Theta_i v_i \rangle \quad \forall\ v_i \in \hat{V}^h(\Omega_i).$$

3. *Balance the updated residual $r_0 := g - S \sum_{i=1}^J \Theta_i \check{u}_i$ by solving*

$$S(w_0, v_0) = \langle r_0, v_0 \rangle \quad \forall\ v_0 \in V_0.$$

4. *Compute the action $u = w_0 + \sum_{i=1}^J \Theta_i \check{u}_i$.*

REMARK 9.3. As for the coarse problem (9.17) in subsection 9.2.1, we can replace the coarse problems in step 1 and step 3 above by the more effective coarse solvers proposed in subection 9.2.1.

9.3. Some other interface algorithms. This section will address the implementation of the interface versions of the hierarchical basis, BPX multilevel, and additive Schwarz preconditioners.

9.3.1. Hierarchical basis algorithm. Let T be the interface hierarchical basis preconditioner defined in (7.26), i.e., $T = \hat{R} \hat{B} \hat{R}^t$ with $\hat{B}$ given by

$$\hat{B}\tilde{v} = \sum_{k=1}^J \sum_{x_i^k \in \Gamma \bigcap (\mathcal{N}_k \setminus \mathcal{N}_{k-1})} \frac{(\tilde{v}, \tilde{\phi}_i^k)}{A(\phi_i^k, \phi_i^k)} \tilde{\phi}_i^k \quad \forall\ \tilde{v} \in V_H.$$

For any given $g \in V^h(\Gamma)$, we have $Tg = \hat{R}\hat{B}\hat{R}^t g$. By definition of $\hat{R}$ and its adjoint $\hat{R}^t$,

$$(\hat{R}^t g, \tilde{\phi}_i^k) = \langle g, \hat{R}\tilde{\phi}_i^k \rangle = \langle g, \phi_i^k \rangle.$$

Substituting this into the expression of $\hat{B}\hat{R}^t g$ gives

$$\hat{B}\hat{R}^t g = \sum_{k=1}^{J} \sum_{x_i^k \in \Gamma \bigcap (\mathcal{N}_k \setminus \mathcal{N}_{k-1})} \frac{\langle g, \phi_i^k \rangle}{A(\phi_i^k, \phi_i^k)} \tilde{\phi}_i^k.$$

This with the definition of $\hat{R}$ implies

$$Tg = \hat{R}\hat{B}\hat{R}^t g = \sum_{k=1}^{J} \sum_{x_i^k \in \Gamma \bigcap (\mathcal{N}_k \setminus \mathcal{N}_{k-1})} \frac{\langle g, \phi_i^k \rangle}{A(\phi_i^k, \phi_i^k)} \phi_i^k.$$

This yields the following algorithm.

ALGORITHM 9.7 (hierarchical basis algorithm). *Given $g \in V^h(\Gamma)$:*

1. *For each hierarchical basis ϕ_i^k at node $x_i^k \in \Gamma \bigcap (\mathcal{N}_k \setminus \mathcal{N}_{k-1})$, $1 \le k \le J$, compute*

$$\alpha_i^k = \frac{\langle g,\ \phi_i^k \rangle}{A(\phi_i^k, \phi_i^k)}.$$

2. *Sum up $\alpha_i^k \phi_i^k$ over all hierarchical basis functions on the interface Γ; the summand gives the action Tg.*

9.3.2. Interface BPX algorithm. Let T be the interface overlapping BPX multilevel preconditioner defined in (7.26), i.e., $T = \hat{R}\hat{B}_{bpx}\hat{R}^t$ with $\hat{B}$ given by

$$\hat{B}_{bpx}\tilde{v} = \hat{P}B\hat{P}^t\tilde{v} = \sum_{k=1}^{J} \sum_{x_i^k \in \Gamma} \frac{(\tilde{v}, \tilde{\phi}_i^k)}{A(\phi_i^k, \phi_i^k)} \tilde{\phi}_i^k \quad \forall\ \tilde{v} \in V_H.$$

The same derivation as in subsection 9.3.1 gives

$$Tg = \hat{R}\hat{B}_{bpx}\hat{R}^t g = \sum_{k=1}^{J} \sum_{x_i^k \in \Gamma} \frac{\langle g, \phi_i^k \rangle}{A(\phi_i^k, \phi_i^k)} \phi_i^k.$$

Then we have the following algorithm.

ALGORITHM 9.8 (BPX multilevel algorithm). *Given $g \in V^h(\Gamma)$:*

1. *For each interface basis ϕ_i^k at node $x_i^k \in \Gamma$, $1 \le k \le J$, compute*

$$\alpha_i^k = \frac{\langle g,\ \phi_i^k \rangle}{A(\phi_i^k, \phi_i^k)}.$$

2. *Sum up $\alpha_i^k \phi_i^k$ over all interface basis functions at all levels; the summand gives the action Tg.*

9.3.3. Interface overlapping additive Schwarz algorithms. We consider only the most practical case, i.e., the interface subregions $\{\Gamma^i\}$ in subsection 7.2 are edge-, vertex-, and face-related overlapping subregions (see Fig. 9), denoted as $\{\Gamma^{E_i}\}$, $\{\Gamma^{V_j}\}$, and $\{\Gamma^{F_k}\}$ respectively. More accurately, each Γ^{E_i} is a region consisting of the edge E_i and an overlap onto the adjacent faces to a distance of order h_0 from

E_i. Each Γ^{V_j} is a region consisting of the vertex v_j and an overlap onto the adjacent faces and edges to a distance of order h_0 from v_j. But each Γ^{F_k} is the face F_k itself. The corresponding subspaces to each Γ^{E_i}, Γ^{V_j}, and Γ^{F_k} are $V_0^h(\Gamma^{E_i})$, $V_0^h(\Gamma^{\mathrm{V}_j})$, and $V_0^h(\Gamma^{F_k})$, respectively. Then the interface additive Schwarz preconditioner defined in subsection 7.2 becomes

$$T = I_0 R_0 I_0^t + \sum_i I_{E_i} S_{E_i}^{-1} I_{E_i}^t + \sum_j I_{\mathrm{V}_j}^t S_{\mathrm{V}_j}^{-1} I_{\mathrm{V}_j}^t + \sum_k I_{F_k} S_{F_k}^{-1} I_{F_k}^t,$$

where S_{E_i} is the restriction of the interface operator S on $V_0^h(\Gamma^{E_i})$; S_{V_j} and V_{F_k} are defined similarly.

Using (4.24), we can replace the face operator S_{F_k} by the discrete harmonic operator $(-\Delta_{F,h})^{1/2}$; this yields immediately the following algorithm.

ALGORITHM 9.9 (interface overlapping additive Schwarz algorithm). *For any $g \in V^h(\Gamma)$, the action $Tg = u_0 + \sum_i u_{E_i} + \sum_j u_{\mathrm{V}_j} + \sum_k u_{F_k}$ can be obtained as follows:*

1. *Solve for $u_0 \in V_0(\Gamma)$ the coarse problem*

$$h_0^{n-1} \sum_{i=1}^{J} \rho_i \sum_{E_{kl} \subset \Gamma_i} (u_0(\mathrm{v}_k) - u_0(\mathrm{v}_l))\,(v_0(\mathrm{v}_k) - v_0(\mathrm{v}_l)) = \langle g, v_0 \rangle \quad \forall\ v_0 \in V_0(\Gamma).$$

2. *Solve for $u_{E_i} \in V_0^h(\Gamma^{E_i})$ for each edge E_i the edge problem*

$$S(u_{E_i}, v) = \langle g, v \rangle \quad \forall\ v \in V_0^h(\Gamma^{E_i}).$$

3. *Solve for $u_{\mathrm{V}_j} \in V_0^h(\Gamma^{\mathrm{V}_j})$ for each vertex V_j the vertex problem*

$$S(u_{\mathrm{V}_j}, v) = \langle g, v \rangle \quad \forall\ v \in V_0^h(\Gamma^{\mathrm{V}_j}).$$

4. *Solve for $u_{F_k} \in V_0^h(\Gamma^{F_k})$ on each face F_k the face problem*

$$\rho_F \langle (-\Delta_{F,h})^{1/2} u_{F_k}, v \rangle_{0,F} = \langle g, v \rangle \quad \forall\ v \in V_0^h(\Gamma^{F_k}).$$

REMARK 9.4. Step 1 to step 4 in the above interface additive Schwarz algorithm are completely independent and can be implemented in parallel. In practical implementations, the coefficient matrices corresponding to the edge and vertex subproblems in steps 2–3 need not be formed explicitly. Instead we may use some tools, e.g., probing techniques or Fourier approximations, to find approximations of these coefficient blocks of the Schur complement; see Chan, Mathew, and Shao [27, 26]. Another way is to equivalently extend the subproblem on each edge E_i (resp., vertex v_j) to a homogeneous Dirichlet problem defined on a subregion which includes the edge subregion Γ^{E_i} (resp., vertex subregion Γ^{V_j}) in its interior; see Dryja, Smith, and Widlund [36]. Each extended subregion can be made much smaller than the union of all subdomains which have Γ^{E_i} (resp., Γ^{V_j}) as part of their boundaries.

Appendix: A proof of Lemma 3.1. Given $u \in V^h(\Gamma)$, let $\tilde{u}$ be the generalized discrete harmonic extension to all Ω_i. By definition, the Poincaré inequality, and (4.13) we have

$$\begin{aligned}\left(\min_i \rho_i\right)^{-1} \langle Su, u \rangle &\gtrsim \|\tilde{u}\|_{1,\Omega}^2 \gtrsim h_0^2 \sum_{i=1}^{J} \|\tilde{u}\|_{1,\Omega_i}^2 \\ &\gtrsim h_0^2 \sum_{i=1}^{J} \|\tilde{u}\|_{1/2,\partial\Omega_i}^2 \geq h_0 \sum_{i=1}^{J} \|u\|_{0,\partial\Omega_i}^2 = h_0 \langle u, u \rangle.\end{aligned}$$

This implies that $\lambda_{min}(S) \gtrsim h_0 \min_i \rho_i$. On the other hand, by (3.10) and (4.28),

$$\Big(\max_i \rho_i\Big)^{-1} \langle Su, u\rangle \lesssim \sum_{i=1}^{J} |u|^2_{1/2,\partial\Omega_i} \lesssim h^{-1} \sum_{i=1}^{J} \|u\|^2_{0,\partial\Omega_i} = h^{-1}\langle u, u\rangle.$$

This shows that $\lambda_{max}(S) \lesssim h^{-1} \max_i \rho_i$. The desired estimate then follows and the proof of Lemma 3.1 is completed.

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