ON NODAL AND GENERALIZED SINGULAR STRUCTURES OF LAPLACIAN EIGENFUNCTIONS AND APPLICATIONS TO INVERSE SCATTERING PROBLEMS

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Abstract. In this paper, we present some novel and intriguing findings on the geometric structures of Laplacian eigenfunctions and their deep relationship to the quantitative behaviours of the eigenfunctions in two dimensions. We introduce a new notion of generalized singular lines of the Laplacian eigenfunctions, and carefully study these singular lines and the nodal lines. The studies reveal that the intersecting angle between two of those lines is closely related to the vanishing order of the eigenfunction at the intersecting point. We establish an accurate and comprehensive quantitative characterisation of the relationship. Roughly speaking, the vanishing order is generically infinite if the intersecting angle is *irrational*, and the vanishing order is finite if the intersecting angle is rational. In fact, in the latter case, the vanishing order is the degree of the rationality. The theoretical findings are original and of significant interest in spectral theory. Moreover, they are applied directly to some physical problems of great importance, including the inverse obstacle scattering problem and the inverse diffraction grating problem. It is shown in a certain polygonal setup that one can recover the support of the unknown scatterer as well as the surface impedance parameter by finitely many far-field patterns. Indeed, at most two far-field patterns are sufficient for some important applications. Unique identifiability by finitely many far-field patterns remains to be a highly challenging fundamental mathematical problem in the inverse scattering theory.

Keywords Laplacian eigenfunctions, geometric structures, nodal and generalised singular lines, inverse scattering, impedance obstacle, uniqueness, a single far-field pattern

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1. Introduction

1.1. **Background.** Laplacian eigenvalue problem is arguably the simplest PDE eigenvalue problem, which is stated as finding $u \in L^2(\Omega)$ and $\lambda \in \mathbb{R}_+$ such that

$$-\Delta u = \lambda u, \tag{1.1}$$

where Ω is an open set in $\mathbb{R}^2$, under a certain homogeneous boundary condition on $\partial\Omega$, such as the Dirichlet, Neumann or Robin condition. There is a long and colourful history on the spectral theory of Laplacian eigenvalues and eigenfunctions; see e.g. [5, 18, 21, 29, 30, 40, 44, 52, 64, 66, 67, 69]. It still remains an inspiring source for many technical, practical and computational developments [11, 12, 60–62, 69].

In this paper, we are concerned with the geometric structures of Laplacian eigenfunctions as well as their applications to inverse scattering theory. There is a rich theory on the geometric properties of Laplacian eigenfunctions in the literature; see e.g. the review papers [26, 32, 44]. The celebrated Courant's nodal domain theorem states that the first Dirichlet eigenfunction doest not change sign in Ω and the nth eigenfunction (counting multiplicity) u_n has at most n nodal domains. In particular, a famous conjecture concerning the topology of the 2nd Dirichlet eigenfunction states that in $\mathbb{R}^2$, the nodal line of u_2 divides Ω by intersecting its boundary at exactly two points if Ω is convex (cf. [68]). A large amount of literature has been devoted to this conjecture and significant progresses have been made in various situations [3, 22–25, 31, 33, 36, 42, 46, 54, 59, 63]. The "hot spots" conjecture formulated by J. Rauch in 1974 says that the maximum of the second Neumann eigenfunction is attained at a boundary point. This conjecture was proved to be true for a class of planar domains [6], but the statement may not be correct in general; see several counterexamples [7, 10, 13, 34]. The hot spots conjecture was proved recently for a new class of domains (possibly non-convex and non-Euclidean) [43]. Another famous longstanding problem in spectral theory is the Schiffer conjecture which states that if a Neumann eigenfunction takes constant value on the boundary, then the domain must be a ball. The Schiffer conjecture is closely related to the Pompeium property in integral geometry (cf. [68]) and has also an interesting connection to invisibility cloaking (cf. [47]). In [27], the nodal set of the second Dirichlet Laplacian eigenfunction was proved to be close to a straight line when the eccentricity of a bounded and convex domain $\Omega \subset \mathbb{R}^2$ is large. On the other hand, one may also have some estimate about the size of eigenfunctions, e.g., the size of the first eigenfunction can be estimated uniformly for all convex domains; see [28]. Other geometrical characteristics may also be analyzed, e.g., the volume of a set on which an eigenfunction is positive [58], and lower and upper bounds for the length of the nodal line of an eigenfunction of the Laplace operator in two-dimensional Riemannian manifolds [9, 57].

As we see from the above, the study of the geometric structures of Laplacian eigenfunctions is intriguing and challenging. We shall present some novel findings on the geometric structures of Laplacian eigenfunctions and their deep relationship to the quantitative behaviours of the eigenfunctions in $\mathbb{R}^2$. Roughly speaking, we consider the intersections of certain homogeneous line segments and their implications to the quantitative analytic properties of the underlying eigenfunction. The specific geometric and mathematical setup of our study is heavily motivated by our research into the inverse obstacle scattering problem, which is concerned with the recovery of geometric and physical properties of unknown obstacles from the measurement of the wave pattern due to an impinging field. It is a long-standing and fundamental problem in inverse scattering whether a one-to-one correspondence holds between the geometric shapes of a set of obstacles and their scattering wave patterns due to a single impinging wave field. This is also known as the Schiffer problem in the inverse scattering theory. To tackle this fundamental problem within the polygonal geometry, we suggest in this work a highly novel approach by making full use of the geometric properties of the Laplacian eigenfunctions in the specific setup of the current study. We shall present more relevant discussions in this aspect in the next subsection and Section 8.

1.2. Motivation and discussion of our main findings.

We first introduce three definitions for the descriptions of our main results.

Definition 1.1. For a Laplacian eigenfunction u in (1.1), a line segment $\Gamma_h \subset \Omega$ is called a *nodal line* of u if $u = 0$ on Γ_h, where $h \in \mathbb{R}_+$ signifies the length of the line segment. For a given complex-valued function $\eta \in L^\infty(\Gamma_h)$, if it holds that

$$\partial_\nu u(\mathbf{x}) + \eta(\mathbf{x}) u(\mathbf{x}) = 0, \quad \mathbf{x} \in \Gamma_h, \tag{1.2}$$

then Γ_h is referred to as a *generalized singular line* of u. For the special case that $\eta \equiv 0$ in (1.2), a generalized singular line is also called a *singular line* of u in Ω. We use $\mathcal{N}_\Omega^\lambda$, $\mathcal{S}_\Omega^\lambda$ and $\mathcal{M}_\Omega^\lambda$ to denote the sets of nodal, singular and generalized singular lines, respectively, of an eigenfunction u in (1.1).

According to Definition 1.1, a singular line is obviously a generalized singular line. However, for unambiguity and definiteness, Γ_h in (1.2) is called a generalized singular line only if η is not identically zero, otherwise it is referred to as a singular line. We like to point out that as u is (real) analytic inside Ω, a nodal line or a singular line can be extended by the analytic continuation within Ω (cf. [41]). We are mainly concerned with the local properties of the eigenfunction u around the intersecting point of two lines, and hence the length h of Γ_h does not play an essential role as long as it is positive. We further emphasize that no any specific boundary condition is specified for u on $\partial\Omega$ in Definition 1.1, that is, all our subsequent results hold for a generic Laplacian eigenfunction as long as it satisfies (1.1), therefore applicable to the particular Dirichlet, Neumann or Robin eigenfunction. As mentioned earlier, one of the major motivations of our current study comes from attacking the fundamental Schiffer problem in the inverse scattering theory. The localized feature of our results, which is independent of any specific boundary condition of u on $\partial\Omega$, shall play a key role in our study. This shall become more evident in our subsequent analysis.

Definition 1.2. Let Γ and Γ' be two line segments in Ω that intersect with each other. Denote by $\theta = \angle(\Gamma, \Gamma') \in (0, 2\pi)$ the intersecting angle. Set

$$\theta = \alpha \cdot 2\pi, \quad \alpha \in (0, 1).$$

θ is called an *irrational angle* if α is an irrational number; and it is called a rational angle of degree q if $\alpha = p/q$ with $p, q \in \mathbb{N}$ and irreducible.

Definition 1.3. Let u satisfy (1.1) and be a nontrivial eigenfunction. For a given point $\mathbf{x}_0 \in \Omega$, if there exits a number $N \in \mathbb{N} \cup \{0\}$ such that

$$\lim_{r\to+0} \frac{1}{r^m} \int_{B(\mathbf{x}_0, r)} |u(\mathbf{x})| \,\mathrm{d}\mathbf{x} = 0 \quad \text{for} \quad m = 0, 1, \ldots, N+1, \tag{1.3}$$

where $B(\mathbf{x}_0, r)$ is a disk centered at $\mathbf{x}_0$ with radius $r \in \mathbb{R}_+$, we say that u vanishes at $\mathbf{x}_0$ up to the order N. The largest possible N such that (1.3) is fulfilled is called the vanishing order of u at $\mathbf{x}_0$, and we write

$$\mathrm{Vani}(u; \mathbf{x}_0) = N.$$

If (1.3) holds for any $N \in \mathbb{N}$, then we say that the vanishing order is infinity.

Since u to (1.1) is analytic in Ω, it is straightforward to verify that $\mathrm{Vani}(u; \mathbf{x}_0)$ is actually the lowest order of the nontrivial homogeneous polynomial in the Taylor series expansion of u at $\mathbf{x}_0$. Moreover, by the strong unique continuation principle, we know that if the vanishing order of u is infinity at a given point $\mathbf{x}_0 \in \Omega$, then u is identically zero in Ω.

To elucidate our study, we next consider a simple example which connects the vanishing order of an eigenfunction with the intersecting angle of its nodal lines. Set

$$u(\mathbf{x}) = J_n(\sqrt{\lambda}\, r) \sin n\theta, \quad \mathbf{x} = (x_1, x_2) = r(\cos\theta, \sin\theta) \in \Omega,$$

where J_n is the first-kind Bessel function of order $n \in \mathbb{N}$ (cf. Section 3.4 of [16]). $u(\mathbf{x})$ is a single spherical wave mode and satisfies (1.1), and we can verify that

$$\mathrm{Vani}(u; \mathbf{0}) = n.$$

In particular, it is noted that if one considers u in a central disk B_{r_0} with $\sqrt{\lambda}\, r_0$ being a root of $J_n(t)$ or $J_n'(t)$, then u is actually a Dirichlet or Neumann eigenfunction in $\Omega = B_{r_0}$. However, we are more interested in the nodal lines of u, and it can be easily seen that

$$\Gamma_h^m := \{\mathbf{x} = r(\cos\theta_m, \sin\theta_m); 0 < r < h, \theta_m = \frac{m}{n}\pi\}, \quad m = 0, 1, 2, \ldots, 2n-1. \tag{1.4}$$

The nodal lines in (1.4) all intersect at the origin and the intersecting angle between any two of them is rational of degree n. Clearly, this simple example reveals an intriguing connection between the intersecting angle of two nodal lines and the vanishing order of the eigenfunction at the intersecting point. The aim of the present paper is to establish an accurate and comprehensive characterisation of such a relationship in the most general scenario. Roughly speaking, we shall show that the vanishing order is generically infinity if the intersecting angle is *irrational*, and the vanishing order is finite if the intersecting angle is *rational*. In the latter case, the vanishing order is actually the degree of rationality of the intersecting angle. The result is not only established for the nodal lines, but also for the generalized singular lines. Hence our study uncovers a deep relationship between the nodal and generalized singular structures of the Laplacian eigenfunctions and the quantitative behaviours of the eigenfunctions. To the best of our knowledge, it is the first time in the literature to present a systematic study of such intriguing connections between the vanishing orders of Laplacian eigenfunctions and the intersecting angles of their nodal/generalized singular lines. Hence, these results should be truly original and of significant interest in the spectral theory of Laplacian eigenfunctions, and possibly closely related to the Maxwellian eigenfunctions as well. In fact, it is noted that the Maxwell equations can be transformed to vector-valued Helmholtz equations (cf. [16]). We would also like to comment on the geometric speciality of our study with nodal and singular line segments, which may not always occur for the usual Dirichlet/Neumann/Robin Laplacian eigenfunctions. On the one hand, as noted earlier, we shall not prescribe any boundary condition of u and consider a generic u satisfying (1.1). This allows u to possess much richer structures than those of the specific Dirichlet/Neumann/Robin eigenfunctions. On the other hand, as it will become more evident in Section 8, the aforementioned homogeneous line segments of the general Laplacian eigenfunctions occur naturally when dealing with the fundamental Schiffer problem in inverse scattering within the general polygonal geometry.

In order to establish the aforementioned results, we make essential use of the spherical wave expansion of the eigenfunction, which in combination with the homogeneous conditions on the intersecting lines can yield certain formulae of the Fourier coefficients. In order to trigger off the formulae for us to achieve the desired vanishing order of the eigenfunction, we need to show the vanishing of the first few polynomial terms (basically up to the third order) of the eigenfunction. For this part, we develop a "localized" argument, by making use of some analytical tool from the microlocal analysis in combination with a complex-geometrical-optics (CGO) solution to derive more accurate characterisations of the singularities of the eigenfunction at the intersecting point in the phase space. This involves rather delicate and technical analysis, but it only requires the "local" information of the eigenfunction in a corner region formed by the intersecting lines. This is in sharp contrast to the Fourier expansion, which requires the "global" information of the eigenfunction around the intersecting point. In principle, the arguments that are developed in this work can be used to extend our study to the case with general second order elliptic operators as well as to the case that the nodal or singular lines are lying on the boundary $\partial\Omega$ of the domain. However, we choose in this work to stick to the fundamental case with the Laplacian eigenfunctions and the nodal or generalised singular lines lying within the domain Ω and study the aforementioned technical extensions in our future work.

In addition to their theoretical beauty and profundity, our new spectral findings in this work can be directly applied to some physical problems of great practical importance, including the inverse obstacle scattering problem and the inverse diffraction grating problem. By using the new critical connection between the intersecting angles of the nodal/generalized singular lines and the vanishing order of the eigenfunctions, we establish in a certain polygonal setup that one can recover the support of the unknown scatterer as well as the surface impedance parameter by finitely many far-field patterns. In fact, two far-field patterns are sufficient for some important applications under some mild a-priori knowledge of the underlying obstacles. It is well known that unique identifiability by finitely many far-field patterns remains a highly challenging fundamental mathematical problem in the inverse scattering theory. Using the new spectral findings, we are able not only to establish the unique identifiability results for some open inverse scattering problems, especially for the impedance case, but also to develop a completely new approach that can treat the unique identifiability issue for several inverse scattering problems in a unified manner, especially in terms of general material properties. Most existing analytical theories for the unique identifiability of inverse scattering problems need to handle each special material property very differently. We shall give more background introduction in Section 8 about these practical problems so that we can first focus on the theoretical study of the nodal and singular structures of the Laplacian eigenfunctions.

The rest of the paper is organized as follows. In Sections 2 and 4, we consider the case that the intersecting angle is irrational and show that the vanishing order is infinity. In Sections 3, 5 and 6, we study the case that the intersecting angle is rational and the vanishing order is finite. Section 3 is devoted to the presentation and discussions of the main results, whereas Sections 5 and 6 are concentrated on the corresponding rigorous proofs. Section 7 discusses a generic condition required in Sections 2–6. In Section 8, we establish the unique recovery results for the inverse obstacle problem and the inverse diffraction grating problem by at most two incident waves.

2. Irrational intersection and infinite vanishing order: two intersecting nodal and singular lines

In this section, we consider a relatively simple case that two nodal lines or two singular lines intersect at an irrational angle. We show that in such a case, the vanishing order of the eigenfunction at the intersecting point is generically infinite, and hence it is identically vanishing in Ω.

Theorem 2.1. *Let u be a Laplacian eigenfunction to (1.1). If there are two nodal lines Γ_h^+ and Γ_h^- from $\mathcal{N}_\Omega^\lambda$ such that*

$$\Gamma_h^+ \cap \Gamma_h^- = \mathbf{x}_0 \in \Omega \quad \text{and} \quad \angle(\Gamma_h^+, \Gamma_h^-) = \alpha \cdot 2\pi, \tag{2.1}$$

where $\alpha \in (0,1)$ is irrational. Then the vanishing order of u at $\mathbf{x}_0$ is infinite, namely

$$\mathrm{Vani}(u; \mathbf{x}_0) = +\infty.$$

In order to prove Theorem 2.1, we need some auxiliary results about reflection principles of nodal and singular lines from the following two lemmas. In what follows, for a line segment $\Gamma \subset \mathbb{R}^2$, we define $\mathcal{R}_\Gamma$ to be the (mirror) reflection in $\mathbb{R}^2$ with respect to the line containing Γ.

Lemma 2.2. *Let u be a Laplacian eigenfunction to (1.1). There hold the following reflection principles:*

1. *Let $\Gamma \in \mathcal{N}_\Omega^\lambda$ (resp. $\Gamma \in \mathcal{S}_\Omega^\lambda$) and $\Gamma' \in \mathcal{N}_\Omega^\lambda \cup \mathcal{S}_\Omega^\lambda$. If $\widetilde{\Gamma} = \mathcal{R}_{\Gamma'}(\Gamma) \subset \Omega$, then $\widetilde{\Gamma} \in \mathcal{N}_\Omega^\lambda$ (resp. $\widetilde{\Gamma} \in \mathcal{S}_\Omega^\lambda$);*
2. *Let $\Gamma \in \mathcal{M}_\Omega^\lambda$ with $\partial_\nu u + \eta u = 0$ on Γ and $\Gamma' \in \mathcal{N}_\Omega^\lambda$. If $\widetilde{\Gamma} = \mathcal{R}_{\Gamma'}(\Gamma) \subset \Omega$, then $\widetilde{\Gamma} \in \mathcal{M}_\Omega^\lambda$ satisfies $\partial_{\widetilde{\nu}} u + \widetilde{\eta} u = 0$ on $\widetilde{\Gamma}$, where $\widetilde{\nu} = \mathcal{R}_{\Gamma'}(\nu)$ and $\widetilde{\eta} = \mathcal{R}_{\Gamma'}(\eta)$.*

The reflection principles are rather standard for the Laplacian [50,51]. The first reflection principle in Lemma 2.2 shall be used in the proof of Theorem 2.1, whereas the second one is needed in our subsequent study.

Lemma 2.3. *Let $0 < \alpha_1 < 1$ be an irrational number. Define*

$$\alpha_{n+1} = 1 - \left\lfloor \frac{1}{\alpha_n} \right\rfloor \alpha_n, \quad n = 1, 2, \ldots, \tag{2.2}$$

where $\lfloor\cdot\rfloor$ *is the floor function. Then it holds that*

$$\lim_{n\to\infty}\alpha_n = 0.$$

Proof. We prove this lemma by contradiction. First, by induction, it is easy to see that $\{\alpha_n\}\subset\mathbb{R}\backslash\mathbb{Q}$, where $\mathbb{Q}$ is the set of rational numbers and $\alpha_n>0$. Furthermore, by using

$$0<\frac{1}{\alpha_n}-\left\lfloor\frac{1}{\alpha_n}\right\rfloor<1,$$

we know that the sequence $\{\alpha_n\}$ is bounded below and decreasing. Suppose that

$$\lim_{n\to\infty}\alpha_n=\beta_0>0. \tag{2.3}$$

Since $1/\alpha_n$ is not an integer, we know from [65, p. 15, Eq.(2.1.7)] that the Fourier series expansion of $\lfloor 1/\alpha_n\rfloor$ is given as follows

$$\left\lfloor\frac{1}{\alpha_n}\right\rfloor=\frac{1}{\alpha_n}-\frac{1}{2}+\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\sin(2k\pi/\alpha_n)}{k}. \tag{2.4}$$

Next, we show for a sufficiently small $\epsilon_0\in\mathbb{R}_+:=\{x\in\mathbb{R}|x>0\}$, the Fourier series $\sum_{k=1}^{\infty}k^{-1}\sin(2k\pi x)$ is uniformly convergent with respect to $x\in(\frac{1}{\beta_0}-\epsilon_0,\frac{1}{\beta_0}+\epsilon_0)$. To this end, we first prove by absurdity that $\beta_0\neq\frac{1}{N}$ for any $N\in\mathbb{N}$. It is obvious to see from (2.3) that $\beta_0<1$, since $\{\alpha_n\}$ is decreasing with β_0 as its infimum. Now assume contrarily that $\beta_0=\frac{1}{N}$ for some $N\in\mathbb{N}\backslash\{1\}$ and

$$\lim_{n\to\infty}\frac{1}{\alpha_n}=\frac{1}{\beta_0}=N.$$

Since the sequence $\{\frac{1}{\alpha_n}\}$ is increasing and bounded above by $\frac{1}{N}$, we know that there exists a sufficiently large $n_1\in\mathbb{N}$ such that for any $n\geq n_1$, there holds

$$N-1\leq\frac{1}{\alpha_n}\leq N,$$

Hence,

$$\frac{1}{N}\leq\alpha_n\leq\frac{1}{N-1}\quad\text{for any}\quad n\geq n_1, \tag{2.5}$$

and

$$\left\lfloor\frac{1}{\alpha_n}\right\rfloor=N-1.$$

On the other hand, from definition (2.2), we can also deduce that

$$\frac{1}{N}\leq\alpha_{n+1}=1-\left\lfloor\frac{1}{\alpha_n}\right\rfloor\alpha_n=1-(N-1)\alpha_n\leq\frac{1}{N-1}\quad\text{for } n\geq n_1,$$

which further yields

$$\frac{N-2}{(N-1)^2}\leq\alpha_n\leq\frac{1}{N}\quad\text{for } n\geq n_1. \tag{2.6}$$

Combining (2.5) and (2.6), we can obtain that

$$\alpha_n=\frac{1}{N}\quad\text{for } n\geq n_1,$$

which contradicts to the fact that $\alpha_n\in\mathbb{R}\backslash\mathbb{Q}$ for $n=1,2,\cdots$. Therefore we must have $\beta_0\neq\frac{1}{N}$ for any $N\in\mathbb{N}$.

We proceed to prove the uniform convergence of the Fourier series $\sum_{k=1}^{\infty}k^{-1}\sin(2k\pi x)$ with respect to $x\in(\frac{1}{\beta_0}-\epsilon_0,\frac{1}{\beta_0}+\epsilon_0)$ by Dirichlet's test [19]. Indeed, since $\{\frac{1}{k}\}$ is decreasing with respect to k and $\lim_{k\to\infty}\frac{1}{k}=0$ uniformly in $(\frac{1}{\beta_0}-\epsilon_0,\frac{1}{\beta_0}+\epsilon_0)$, it suffices to show that the partial sum $\sum_{k=1}^{K_0}\sin(2k\pi x)$ is uniformly bounded in $(\frac{1}{\beta_0}-\epsilon_0,\frac{1}{\beta_0}+\epsilon_0)$ for $K_0=1,2,\cdots$. By [20, p. 110, Eq.(7.1.3)], we actually have

$$\left|\sum_{k=1}^{K_0}\sin(2k\pi x)\right|=\left|\frac{\cos\pi x-\cos(2K_0+1)\pi x}{2\sin\pi x}\right|\le\left|\frac{1}{\sin\pi x}\right|\le\frac{2}{|\sin\frac{\pi}{\beta_0}|},\tag{2.7}$$

which readily gives the desired uniform boundedness. The last inequality in (2.7) holds because $\beta_0\neq\frac{1}{N}$ for any $N\in\mathbb{N}$ and hence $\sin\frac{\pi}{\beta_0}\neq 0$. By the continuity, we know there holds that $|\sin\pi x|\ge\frac{1}{2}|\sin\frac{\pi}{\beta_0}|$ for $x\in(\frac{1}{\beta_0}-\epsilon_0,\frac{1}{\beta_0}+\epsilon_0)$.

Taking $x=\alpha_n^{-1}$ in $\sum_{k=1}^{\infty}k^{-1}\sin(2k\pi x)$ and utilizing the uniform convergence result in $\alpha_n\in(\beta_0-\epsilon_0,\beta_0+\epsilon_0)$ for a sufficiently small $\epsilon_0\in\mathbb{R}_+$, we can let $n\to\infty$ on the both sides of (2.4) to derive that

$$\lim_{n\to\infty}\left\lfloor\frac{1}{\alpha_n}\right\rfloor=\lim_{n\to\infty}\left(\frac{1}{\alpha_n}-\frac{1}{2}+\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\sin(2k\pi/\alpha_n)}{k}\right)=\left\lfloor\frac{1}{\beta_0}\right\rfloor.\tag{2.8}$$

By using (2.8) and taking the limits of both sides of (2.2), we further have that

$$\beta_0=1-\left\lfloor\frac{1}{\beta_0}\right\rfloor\beta_0.\tag{2.9}$$

Dividing β_0 on the both sides of (2.9), we finally arrive at a contradiction

$$1=\frac{1}{\beta_0}-\left\lfloor\frac{1}{\beta_0}\right\rfloor,$$

which completes the proof. ☐

We are now ready to present the proof of Theorem 2.1.

Proof of Theorem 2.1. By a rigid motion if necessary, we can assume without loss of generality that $\mathbf{x}_0=\mathbf{0}$ is the origin and Γ_h^- lies in the x_1^+-axis while Γ_h^+ has the angle $2\alpha\pi$ away from Γ_h^- in the anticlockwise direction. Since we are mainly concerned with the local properties, it is assumed that $h\in\mathbb{R}_+$ is sufficiently small such that $\mathcal{R}_{\Gamma_h^+}(\Gamma_h^-)\Subset\Omega$. In what follows, with the help of Lemmas 2.2 and 2.3, we show that there exists a dense set of nodal lines around the origin.

To begin with, we prove that one can assume $\alpha\in(\mathbb{R}\backslash\mathbb{Q})\cap(0,1/3)$ in (2.1). That is, there exist two nodal lines $\Gamma_h^\pm$ satisfying (2.1) with $\alpha\in(0,1/3)$ being irrational. To that end, we first show that the other two separate cases $\alpha\in(\mathbb{R}\backslash\mathbb{Q})\cap(1/3,1/2)$ and $\alpha\in(\mathbb{R}\backslash\mathbb{Q})\cap(1/2,1)$ can be reduced to the case $\alpha\in(\mathbb{R}\backslash\mathbb{Q})\cap(0,1/3)$. We first consider $\alpha\in(\mathbb{R}\backslash\mathbb{Q})\cap(1/3,1/2)$. By Lemma 2.2, we set

$$\Gamma_{1,h}^{(0)}=\mathcal{R}_{\Gamma_h^+}\left(\Gamma_h^-\right)\in\mathcal{N}_\Omega^\lambda,\quad\Gamma_{1,h}^{(1)}=\mathcal{R}_{\Gamma_{1,h}^{(0)}}\left(\Gamma_h^+\right)\in\mathcal{N}_\Omega^\lambda,\tag{2.10}$$

Then it can be directly verified that $\Gamma_{1,h}^{(1)}$ has an angle $\alpha^{(1)}\cdot 2\pi$ away from Γ_h^- in the anticlockwise direction, where

$$\alpha^{(1)}:=1-\left\lfloor\frac{1}{\alpha}\right\rfloor\alpha\in(\mathbb{R}\backslash\mathbb{Q})\cap(0,1/3).$$

Hence, one can replace Γ_h^+ in (2.1) by $\Gamma_{1,h}^{(1)}$ in (2.10) to obtain the desired result. Next, if $\alpha\in(1/2,1)$, we set

$$\Gamma_{1,h}^{(1)}=\mathcal{R}_{\Gamma_h^-}\left(\Gamma_h^+\right)\in\mathcal{N}_\Omega^\lambda.\tag{2.11}$$

Then $\Gamma^{(1)}_{1,h}$ has an angle $2(1-\alpha)\pi$ away from Γ^-_h in the anticlockwise direction. It can be directly verified that

$$\alpha^{(2)} := 1-\alpha \in (\mathbb{R}\backslash\mathbb{Q}) \cap (0, 1/2).$$

If $\alpha^{(2)} \in (0, 1/3)$, then one can replace Γ^+_h in (2.1) by $\Gamma^{(1)}_{1,h}$ in (2.11) to obtain the desired result, whereas if $\alpha^{(2)} \in (1/3, 1/2)$, one can follow the same reflection argument as the previous case (cf. (2.10)) to obtain a nodal line for replacing Γ^+_h in (2.1) satisfying $\alpha \in (\mathbb{R}\backslash\mathbb{Q}) \cap (0, 1/3)$.

Next, starting from (2.1) with $\alpha \in (\mathbb{R}\backslash\mathbb{Q}) \cap (0, 1/3)$, we perform a series of reflections to obtain the dense set of nodal lines around the origin as mentioned at the beginning of the proof; see Fig. 1 for a schematic illustration. We write $\Gamma^{(1)}_{0,h} = \Gamma^-_h$ and $\Gamma^{(1)}_{1,h} = \Gamma^+_h$. By

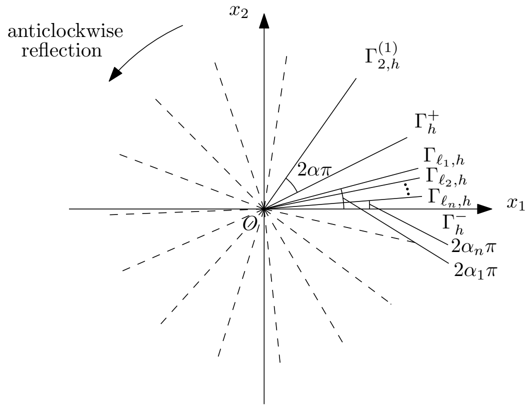

FIGURE 1. Schematic illustration of the reflection argument in the proof of Theorem 2.1.

Lemma 2.2, we see that

$$\Gamma^{(1)}_{2,h} := \mathcal{R}_{\Gamma^{(1)}_{1,h}}\left(\Gamma^{(1)}_{0,h}\right) \in \mathcal{N}^\lambda_\Omega.$$

Since $\alpha \in (0, 1/3)$ we know that

$$\ell_1 := \left\lfloor \frac{1}{\alpha} \right\rfloor \geq 3.$$

By repeating the reflections, one can find $(\ell_1 - 2)$ nodal lines as follows:

$$\Gamma^{(1)}_{m,h} := \mathcal{R}_{\Gamma^{(1)}_{m-1,h}}\left(\Gamma^{(1)}_{m-2,h}\right) \in \mathcal{N}^\lambda_\Omega, \quad m = 3, \ldots, \ell_1.$$

Furthermore, it is easy to verify that

$$\angle(\Gamma^{(1)}_{\ell_1,h}, \Gamma^-_h) = \alpha_1 \cdot 2\pi, \quad \alpha_1 := 1 - \left\lfloor \frac{1}{\alpha} \right\rfloor \alpha < \alpha.$$

Let

$$\Gamma_{\ell_1,h} = \mathcal{R}_{\Gamma^-_h}\left(\Gamma^{(1)}_{\ell_1,h}\right) \in \mathcal{N}^\lambda_\Omega.$$

Then one sees that $\Gamma_{\ell_1,h}$ has an angle $2\alpha_1\pi$ away from Γ^-_h in the anticlockwise direction.

Next, by replacing Γ^+_h with $\Gamma_{\ell_1,h}$ and repeating the above reflection argument, one can find a nodal line $\Gamma_{\ell_2,h}$ such that

$$\Gamma_{\ell_2,h} \in \mathcal{N}^\lambda_\Omega, \quad \ell_2 := \left\lfloor \frac{1}{\alpha_1} \right\rfloor, \quad \angle(\Gamma_{\ell_2,h}, \Gamma^-_h) = \alpha_2 \cdot 2\pi \quad \text{and} \quad \alpha_2 := 1 - \left\lfloor \frac{1}{\alpha_1} \right\rfloor \alpha_1 < \alpha_1.$$

Furthermore, $\Gamma_{\ell_2,h}$ has an angle $2\alpha_2\pi$ away from Γ_h^- in the anticlockwise direction. Clearly, by further repeating this reflection argument, one can find a series of nodal lines $\Gamma_{\ell_n,h}$ such that

$$\Gamma_{\ell_n,h} \in \mathcal{N}_\Omega^\lambda, \quad \ell_n := \left\lfloor \frac{1}{\alpha_{n-1}} \right\rfloor, \quad \angle(\Gamma_{\ell_n,h}, \Gamma_h^-) = \alpha_n \cdot 2\pi, \quad \text{and} \quad \alpha_n := 1 - \left\lfloor \frac{1}{\alpha_{n-1}} \right\rfloor \alpha_{n-1}. \tag{2.12}$$

Moreover, $\Gamma_{\ell_n,h}$ has the angle $2\alpha_n\pi$ away from Γ_h^- in the anticlockwise direction. Then by Lemma 2.3 we have

$$\lim_{n\to\infty} \alpha_n = 0. \tag{2.13}$$

Combining (2.12) and (2.13), we see that $\{\Gamma_{\ell_n,h}\}$ forms a dense set of nodal lines around the origin. Hence, by the continuity of u one readily has that u is identically zero. This completes the proof of Theorem 2.1. □

The next theorem is concerned with the intersection of two singular lines.

Theorem 2.4. *Let u be a Laplacian eigenfunction to (1.1). If there are two singular lines Γ_h^+ and Γ_h^- from $\mathcal{S}_\Omega^\lambda$ such that*

$$\Gamma_h^+ \cap \Gamma_h^- = \mathbf{x}_0 \in \Omega \quad \textit{and} \quad \angle(\Gamma_h^+, \Gamma_h^-) = \alpha \cdot 2\pi,$$

where $\alpha \in (0,1)$ is irrational, then there hold that

$$\begin{aligned} \mathrm{Vani}(u; \mathbf{x}_0) &= 0, && \textit{if } u(\mathbf{x}_0) \neq 0; \\ \mathrm{Vani}(u; \mathbf{x}_0) &= +\infty, && \textit{if } u(\mathbf{x}_0) = 0. \end{aligned}$$

Moreover, if $u(\mathbf{x}_0) \neq 0$, we have the following expansion of u in a neighborhood of $\mathbf{x}_0$:

$$u(\mathbf{x}) = u(\mathbf{x}_0) J_0(\sqrt{\lambda} r), \quad \mathbf{x} = \mathbf{x}_0 + r(\cos\theta, \sin\theta).$$

where $J_0(t)$ is the zeroth Bessel function of the first kind.

In order to prove Theorem 2.4, we need some auxiliary results from the following three lemmas, especially about the spherical wave expansion, for which we refer to [16] for more details. In what follows, $\mathrm{i} := \sqrt{-1}$ is used for the imaginary unit.

Lemma 2.5. *[16, Section 3.4] Suppose that u is an eigenfunction to (1.1), then u has the following spherical wave expansion in polar coordinates around the origin:*

$$u(\mathbf{x}) = \sum_{n=0}^{\infty} \left(a_n e^{\mathrm{i}n\theta} + b_n e^{-\mathrm{i}n\theta} \right) J_n\left(\sqrt{\lambda} r\right), \tag{2.14}$$

where $\mathbf{x} = (x_1, x_2) = r(\cos\theta, \sin\theta) \in \mathbb{R}^2$, λ is the corresponding eigenvalue of (1.1), a_n and b_n are constants, and $J_n(t)$ is the n-th Bessel function of the first kind.

Lemma 2.6. *Let Γ be a line segment that can be parameterized in polar coordinates as $\mathbf{x} \in \Gamma$, where $\mathbf{x} = r(\cos\theta, \sin\theta)$ with $0 \le r < \infty$ and θ fixed. Let ν be the unit normal vector to Γ, then it holds that*

$$\frac{\partial u}{\partial \nu} = \pm \frac{1}{r} \frac{\partial u}{\partial \theta}.$$

Proof. Using the polar coordinates and the chain rule, we have

$$\frac{\partial u}{\partial x_1} = \frac{\partial u}{\partial r} \cos\theta - \frac{\sin\theta}{r} \frac{\partial u}{\partial \theta}, \quad \frac{\partial u}{\partial x_2} = \frac{\partial u}{\partial r} \sin\theta + \frac{\cos\theta}{r} \frac{\partial u}{\partial \theta}.$$

Thus

$$\frac{\partial u}{\partial \nu} = \left(\frac{\partial u}{\partial r}\Big|_\Gamma \cos\theta - \frac{\sin\theta}{r} \frac{\partial u}{\partial \theta}\Big|_\Gamma \right) \cos\varphi + \left(\frac{\partial u}{\partial r}\Big|_\Gamma \sin\theta + \frac{\cos\theta}{r} \frac{\partial u}{\partial \theta}\Big|_\Gamma \right) \sin\varphi, \tag{2.15}$$

where $\nu = (\cos\varphi, \sin\varphi)$ denotes the unit normal vector to Γ . Using the fact $|\varphi - \theta| = \pi/2$, we complete the proof. $\square$

Lemma 2.7. *Suppose that for* $0 < h \ll 1$ *and* $t \in (0, h)$,

$$\sum_{n=0}^{\infty} \alpha_n J_n(t) = 0, \tag{2.16}$$

where $J_n(t)$ *is the* n*-th Bessel function of the first kind. Then*

$$\alpha_n = 0, \quad n = 0, 1, 2, \ldots.$$

Proof. From [16], we know that

$$J_n(t) = \frac{t^n}{2^n n!}\left(1 + \sum_{p=1}^{\infty} \frac{(-1)^p n!}{p!(n+p)!}\left(\frac{t}{2}\right)^{2p}\right). \tag{2.17}$$

Substituting (2.17) into (2.16) and comparing the coefficient of t^n $(n = 0, 1, 2, \ldots)$, we can prove this lemma. $\square$

Now we are in a position to prove Theorem 2.4.

Proof of Theorem 2.4. Without loss of generality, we assume that two singular lines Γ_h^+ and Γ_h^- intersect with each other at the origin. Using the reflection principle and a similar argument to the proof of Theorem 2.1, for any line segment $\Gamma \Subset \Omega = \{\mathbf{x}; \mathbf{x} = r(\cos\beta, \sin\beta),\ 0 \leq r \leq h\}$ pointed out from the origin we can show that

$$\frac{\partial u}{\partial \nu_\Gamma} \equiv 0 \text{ in } \Omega,$$

where ν_Γ is a unit normal vector to Γ. Recalling the expansion (2.14), it is easy to see that

$$\left.\frac{\partial u}{\partial \theta}\right|_\Gamma = \sum_{n=0}^{\infty} \mathrm{i}n\left(a_n e^{\mathrm{i}n\beta} - b_n e^{-\mathrm{i}n\beta}\right) J_n\left(\sqrt{\lambda}r\right) = 0. \tag{2.18}$$

Taking $\beta = 0$ in (2.18), we derive from Lemma 2.7 that

$$\sum_{n=0}^{\infty} \mathrm{i}n\,(a_n - b_n)\, J_n\left(\sqrt{\lambda}r\right) = 0,$$

thus

$$\mathrm{i}n\,(a_n - b_n) = 0, \quad n = 1, 2, \ldots.$$

Moreover, evaluating (2.18) at $\beta = \alpha\pi$ where $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, we can deduce that

$$\mathrm{i}n\left(a_n e^{\mathrm{i}n\alpha\pi} - b_n e^{-\mathrm{i}n\alpha\pi}\right) = 0, \quad n = 1, 2, \ldots.$$

Hence a_n and b_n satisfy

$$\begin{bmatrix} 1 & -1 \\ e^{\mathrm{i}n\alpha\pi} & -e^{-\mathrm{i}n\alpha\pi} \end{bmatrix} \begin{bmatrix} a_n \\ b_n \end{bmatrix} = 0, \quad n = 1, 2, \ldots,$$

which readily implies that $a_n = b_n = 0$ for $n = 1, 2, \cdots$, in view of Lemma 2.7. Therefore, $u(\mathbf{x})$ has the simplified form:

$$u(\mathbf{x}) = (a_0 + b_0) J_0(\sqrt{\lambda}r).$$

Finally, from the assumptions in the theorem we can easily see

$$a_0 + b_0 = u(\mathbf{0}) \neq 0, \quad \text{if } u(\mathbf{0}) \neq 0;$$
$$a_0 + b_0 = u(\mathbf{0}) = 0, \quad \text{if } u(\mathbf{0}) = 0,$$

which complete the proof. □

3. Rational intersection and finite vanishing order

In this section, we consider the general case that two line segments from Definition 1.1 which intersect at a rational angle. Throughout the present section, we let Γ_h^+ and Γ_h^- signify the two line segments which could be either one of the three types: nodal line, singular line or generalized singular line. It is also assumed that for a generalized singular line of the form (1.2), the parameter η is a constant. Nevertheless, we would like to point out that for the case that η is a function in the generalized singular line, we can derive similar conclusions, but through more tedious and subtle calculations. We shall address this point more in Section 5. Let η_1 and η_2 signify the parameters associated with Γ_h^- and Γ_h^+, respectively, if they are generalized singular lines. Set

$$\angle(\Gamma_h^+, \Gamma_h^-) = \alpha \cdot \pi, \quad \alpha \in (0, 2), \tag{3.1}$$

where α is a rational number of the form $\alpha = p/q$ with $p, q \in \mathbb{N}$ and irreducible. Since the Laplace operator $-\Delta$ is invariant under rigid motions, without loss of generality, we assume throughout the rest of this work that

$$\Gamma_h^+ \cap \Gamma_h^- = \mathbf{0} \in \Omega, \tag{3.2}$$

and Γ_h^- coincides with the ${x_1}^+$-axis while Γ_h^+ has the angle $\alpha \cdot \pi$ away from Γ_h^- in the anti-clockwise direction; see Figure 2 for a schematic illustration.

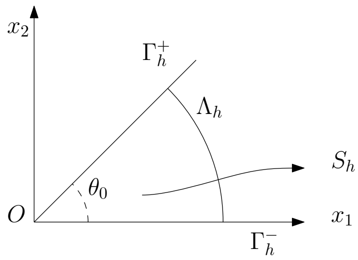

Figure 2. Schematic illustration of the geometry of two intersecting lines with an angle $\theta_0 = \alpha \cdot \pi$ for some $\alpha \in (0, 1) \cap \mathbb{Q}$.

Finally, we mainly deal with the case that the two intersecting line segments Γ_h^+ and Γ_h^- form an angle satisfying

$$\angle(\Gamma_h^+, \Gamma_h^-) = \alpha \cdot \pi, \quad \alpha \in (0, 1), \tag{3.3}$$

and the other case with $1 < \alpha < 2$ can be reduced to the previous case by a straightforward argument. Indeed, for $1 < \alpha < 2$, we know that Γ_h^+ belongs to the half-plane of $x_2 < 0$ (see Fig. 2). Let $\widetilde{\Gamma}_h^+$ be the extended line segment of length h in the half-plane of $x_2 > 0$. Since the Laplacian eigenfunction u is real analytic in the interior of Ω and $\Gamma_h^+ \Subset \Omega$, u is real analytic on Γ_h^+. Moreover, u fulfils a certain homogeneous condition on Γ_h^+. By the analytic continuation (cf. [41]), we know that $\widetilde{\Gamma}_h^+$ fulfils the same homogeneous condition as that on Γ_h^+. That is, $\widetilde{\Gamma}_h^+$ is of the same type of Γ_h^+. Hence, instead of studying the intersection of Γ_h^+ and Γ_h^-, one can study the intersection of $\widetilde{\Gamma}_h^+$ and Γ_h^-, and its relations to the vanishing order of the eigenfunction. Clearly, the angle between $\widetilde{\Gamma}_h^+$ and Γ_h^- satisfies (3.3).

For a clear exposition, the rest of the section is devoted to the presentation and discussion of our main results, and their proofs shall be postponed to Sections 5 and 6. In Section 5, we consider the case where the vanishing of the eigenfunction is up to the third order, whereas in Section 6, we consider the case of general vanishing orders.

Theorem 3.1. *Let u be a Laplacian eigenfunction to (1.1). Suppose that there are two generalized singular lines Γ_h^+ and Γ_h^- from $\mathcal{M}_\Omega^\lambda$ such that (3.2) and (3.3) hold. Assume that $\eta_1 \equiv C_1$ and $\eta_2 \equiv C_2$, where C_1 and C_2 are two constants. Then the Laplacian eigenfunction u vanishes up to the order N at $\mathbf{0}$:*

$$N \geq n, \quad \text{if } u(\mathbf{0}) = 0 \text{ and } \alpha \neq \frac{q}{p},\ p = 1, \ldots, n-1, \tag{3.4}$$

where $n \in \mathbb{N}$, $n \geq 3$ and for a fixed p, $q = 1, 2, \ldots, p-1$.

In Theorem 3.1, we require that $n \geq 3$. That means, we exclude the special case that the intersecting angle is $\pi/2$. Nevertheless, we shall discuss this special case in Remark 3.8 with more details in what follows. In the next two theorems, we consider the case of two intersecting singular and nodal lines, respectively.

Theorem 3.2. *Let u be a Laplacian eigenfunction to (1.1). Suppose that there are two nodal lines Γ_h^+ and Γ_h^- from $\mathcal{N}_\Omega^\lambda$ such that (3.2) and (3.3) hold. Then the Laplacian eigenfunction u vanishes up to the order N at $\mathbf{0}$:*

$$N \geq n, \quad \text{if } \alpha \neq \frac{q}{p},\ p = 1, \ldots, n-1,$$

where $n \in \mathbb{N}$, $n \geq 3$ and for a fixed p, $q = 1, 2, \ldots, p-1$.

Theorem 3.3. *Let u be a Laplacian eigenfunction to (1.1). Suppose that there are two singular lines Γ_h^+ and Γ_h^- from $\mathcal{S}_\Omega^\lambda$ such that (3.2) and (3.3) hold. Then the Laplacian eigenfunction u vanishes up to the order N at $\mathbf{0}$:*

$$N \geq n, \quad \text{if } u(\mathbf{0}) = 0 \text{ and } \alpha \neq \frac{q}{p},\ p = 1, \ldots, n-1,$$

where $n \in \mathbb{N}$, $n \geq 3$ and for a fixed p, $q = 1, 2, \ldots, p-1$.

Example 3.4. Let $\Omega := \{(x_1, x_2) \in \mathbb{R}^2 \mid -2\pi \leq x_1 \leq 2\pi,\ -4\pi \leq x_2 \leq 4\pi\}$ be a rectangle. It is easy to see that

$$u(x_1, x_2) = \sin x_1 \sin 2x_2$$

is an eigenfunction to (1.1) with a homogeneous Dirichlet boundary condition on $\partial\Omega$. The corresponding eigenvalue is $\lambda = 5$. One pair of nodal lines of u in Ω are $\{(x_1, x_2) \mid x_2 = 0, -2\pi + h \leq x_1 \leq 2\pi - h\}$ and $\{(x_1, x_2) \mid x_1 = 0, -4\pi + h \leq x_2 \leq 4\pi - h\}$ for a fixed $0 < h < 2\pi$, which are perpendicular to each other at the origin. Therefore from Theorem 3.2, since $\angle(\Gamma_h^+, \Gamma_h^-) = \pi/2$ which implies that $\alpha \neq 1$, the vanishing order N at the origin is 2. In fact, by the explicit expression of u, we know that the order of the lowest nontrivial homogeneous polynomial of the Taylor expansion of u at the origin is 2, which coincides with the conclusion given by Theorem 3.3.

We now proceed to consider that a nodal line intersects with a generalized singular line. Without loss of generality, we can assume that Γ_h^- is the generalized singular line, while Γ_h^+ is the nodal line.

Theorem 3.5. *Let u be a Laplacian eigenfunction to (1.1). Suppose that a generalized singular line $\Gamma_h^+ \in \mathcal{M}_\Omega^\lambda$ intersects with a nodal line $\Gamma_h^- \in \mathcal{N}_\Omega^\lambda$ at $\mathbf{0}$ with the angle $\angle(\Gamma_h^+, \Gamma_h^-) = \alpha \cdot \pi$. Assume that the boundary parameter $\eta_2 \equiv C_2$ on Γ_h^+ is a constant. Then the Laplacian eigenfunction u vanishes up to the order N at $\mathbf{0}$:*

$$N \geq n, \quad \text{if } \alpha \neq \frac{2q+1}{2p},\ p = 1, \ldots, n-1, \tag{3.5}$$

where $n \in \mathbb{N}$, $n \geq 2$ and for a fixed p, $q = 0, 1, \ldots, p-1$.

Next, we consider the intersection of a singular line and a generalized singular line. Similar to Theorem 3.5, without loss of generality, we can assume that Γ_h^- is the generlized singular line. Indeed, the vanishing order of the eigenfunction in such a case can be obtained from formally taking η_2 on Γ_h^+ to be zero in Theorem 3.1.

Theorem 3.6. *Let u be a Laplacian eigenfunction to (1.1). Suppose that a singular line $\Gamma_h^+ \in \mathcal{S}_\Omega^\lambda$ intersects with a generalized singular line $\Gamma_h^- \in \mathcal{M}_\Omega^\lambda$ at the origin with the angle $\angle(\Gamma_h^+, \Gamma_h^-) = \alpha \cdot \pi$. Assume that the boundary parameter η_1 on Γ_h^- is a non-zero constant, i.e., $\eta_1 \equiv C_1 \neq 0$. Then the Laplacian eigenfunction u vanishes up to the order N at $\mathbf{0}$:*

$$N \geq n, \quad \text{if } u(\mathbf{0}) = 0 \text{ and } \alpha \neq \frac{q}{p},\ p = 1, \ldots, n-1, \tag{3.6}$$

where $n \in \mathbb{N}$, $n \geq 3$ and $q = 1, 2, \ldots, p-1$ for a fixed p.

Using a similar proof to Theorem 3.5, we can find the relationship between the vanishing order of the Laplacian eigenfunction and the intersecting angle between a singular line and a nodal line.

Theorem 3.7. *Let u be a Laplacian eigenfunction to (1.1). Suppose that a singular line $\Gamma_h^+ \in \mathcal{S}_\Omega^\lambda$ intersects with a nodal line $\Gamma_h^- \in \mathcal{N}_\Omega^\lambda$ at the origin with the angle $\angle(\Gamma_h^+, \Gamma_h^-) = \alpha \cdot \pi$. Then the Laplacian eigenfunction u vanishes up to the order N at $\mathbf{0}$:*

$$N \geq n, \quad \text{if } \alpha \neq \frac{2q+1}{2p},\ p = 1, \ldots, n-1, \tag{3.7}$$

where $n \in \mathbb{N}$, $n \geq 2$ and for a fixed p, $q = 0, 1, \ldots, p-1$.

Remark 3.8. As mentioned after Theorem 3.1, we exclude the special case that the intersecting angle between two lines is $\pi/2$. In fact, for Theorems 3.1-3.3 and Theorem 3.6, one may see from their proofs in Section 5 that if $\angle(\Gamma_h^+, \Gamma_h^-) = \pi/2$, then there holds that $\nabla u(\mathbf{0}) = 0$ if $u(\mathbf{0}) = 0$. That means, the eigenfunction is vanishing at least to the second order in such a case. For the other two cases in Theorems 3.5 and 3.7, we can only have that if $\alpha = 1/2$ and $u(\mathbf{0}) = 0$, the eigenfunction is vanishing at least to the first order.

Remark 3.9. It is noted that in Theorems 3.5 and 3.7, we require that $n \geq 2$, whereas in other theorems, we require that $n \geq 3$. In particular, when $n = 2$, $\alpha \neq 1/2$, one can conclude in Theorems 3.5 and 3.7 that the eigenfunction is vanishing at least to the second order. This conclusion is different from Theorems 3.1-3.3 and 3.6, where one has that if $\alpha \neq 1/2$ then the eigenfunction is vanishing at least to the third order.

Remark 3.10. We point out that all the vanishing orders of the eigenfunction in Theorems 3.1 to 3.7 depend only on the intersecting angle of the two homogeneous line segments and are independent of the underlying eigenvalue (cf. (1.1)). This is probably due to the special geometries of our study. We believe when the curved geometries are considered, the underlying eigenvalues should be involved.

4. Irrational intersection and infinite vanishing order: general cases

In this section we consider the irrational intersection, namely α in (3.1) is an irrational number. We show that the eigenfunction is generically vanishing to infinity, namely u is identically zero in Ω. Here, the generic condition is provided by u vanishing or not at the intersecting point. We shall present more discussions on this generic condition in Section 7. By verifying the conditions in (3.4), (3.5),(3.6) and (3.7) for an irrational α, we can readily obtain from Theorems 3.1, 3.5, 3.6 and 3.7 the results in the following four theorems.

Theorem 4.1. *Let u be a Laplacian eigenfunction to (1.1). Suppose that there are two generalized singular lines Γ_h^+ and Γ_h^- from $\mathcal{M}_\Omega^\lambda$ such that (3.2) and (3.3) hold. Assume that $\eta_1 \equiv C_1$ and $\eta_2 \equiv C_2$, where C_1 and C_2 are two constants. If $\angle(\Gamma_h^+, \Gamma_h^-) = \alpha \cdot \pi$ with $\alpha \in (0,2)$ irrational, then there hold that*

$$\mathrm{Vani}(u;\mathbf{0}) = 0, \qquad \text{if } u(\mathbf{0}) \neq 0;$$
$$\mathrm{Vani}(u;\mathbf{0}) = +\infty, \qquad \text{if } u(\mathbf{0}) = 0.$$

Theorem 4.2. *Let u be a Laplacian eigenfunction to (1.1). Suppose that a generalized singular line $\Gamma_h^- \in \mathcal{M}_\Omega^\lambda$ intersects with a nodal line $\Gamma_h^+ \in \mathcal{N}_\Omega^\lambda$ at $\mathbf{0}$ with the angle $\angle(\Gamma_h^+, \Gamma_h^-) = \alpha \cdot \pi$. Assume that the boundary parameter $\eta_1 \equiv C_1$ on Γ_h^- is a constant. If $\alpha \in (0,2)$ is irrational, then there holds*

$$\mathrm{Vani}(u;\mathbf{0}) = +\infty.$$

Theorem 4.3. *Let u be a Laplacian eigenfunction to (1.1). Suppose that a singular line $\Gamma_h^+ \in \mathcal{S}_\Omega^\lambda$ intersects with a generalized singular line $\Gamma_h^- \in \mathcal{M}_\Omega^\lambda$ at $\mathbf{0}$ with the angle $\angle(\Gamma_h^+, \Gamma_h^-) = \alpha \cdot \pi$. Assume that the boundary parameter $\eta_1 \equiv C_1$ on Γ_h^- is a constant. If $\alpha \in (0,2)$ is irrational, then there hold that*

$$\mathrm{Vani}(u;\mathbf{0}) = 0, \qquad \text{if } u(\mathbf{0}) \neq 0;$$
$$\mathrm{Vani}(u;\mathbf{0}) = +\infty, \qquad \text{if } u(\mathbf{0}) = 0.$$

Theorem 4.4. *Let u be a Laplacian eigenfunction to (1.1). Suppose that a singular line $\Gamma_h^- \in \mathcal{S}_\Omega^\lambda$ intersects with a nodal line $\Gamma_h^+ \in \mathcal{N}_\Omega^\lambda$ at $\mathbf{0}$ with the angle $\angle(\Gamma_h^+, \Gamma_h^-) = \alpha \cdot \pi$. If $\alpha \in (0,2)$ is irrational, then there holds*

$$\mathrm{Vani}(u;\mathbf{0}) = +\infty.$$

For readers' convenience, we now summarize in Table 1 all our main results in Sections 3 and 4 about the vanishing order N of the Laplacian eigenfunction u to (1.1) at an intersecting point $\mathbf{x}_0$ of two line segments Γ_h^+ and Γ_h^- which form an angle $\angle(\Gamma_h^+, \Gamma_h^-) = \alpha \cdot \pi$ for $\alpha \in (0,1)$. We recall that the sets of nodal, singular, and generalized singular lines are denoted by $\mathcal{N}_\Omega^\lambda$, $\mathcal{S}_\Omega^\lambda$ and $\mathcal{M}_\Omega^\lambda$, respectively.

	Rational intersection (Finite vanishing order)	Irrational intersection (Infinite vanishing order)
$\Gamma_h^+, \Gamma_h^- \in \mathcal{M}_\Omega^\lambda$	$N \geq n$, if $u(\mathbf{x}_0) = 0$ and $\alpha \neq \frac{q}{p}$, $p = 1, \cdots, n-1$ and $q = 1, \cdots, p-1$	$N = +\infty$, if $u(\mathbf{x}_0) = 0$; $N = 0$, if $u(\mathbf{x}_0) \neq 0$
$\Gamma_h^+, \Gamma_h^- \in \mathcal{N}_\Omega^\lambda$	$N \geq n$, if $\alpha \neq \frac{q}{p}$, $p = 1, \cdots, n-1$ and $q = 1, \cdots, p-1$	$N = +\infty$
$\Gamma_h^+, \Gamma_h^- \in \mathcal{S}_\Omega^\lambda$	$N \geq n$, if $u(\mathbf{x}_0) = 0$ and $\alpha \neq \frac{q}{p}$, $p = 1, \cdots, n-1$ and $q = 1, \cdots, p-1$	$N = +\infty$, if $u(\mathbf{x}_0) = 0$; $N = 0$, if $u(\mathbf{x}_0) \neq 0$
$\Gamma_h^+ \in \mathcal{M}_\Omega^\lambda, \Gamma_h^- \in \mathcal{N}_\Omega^\lambda$	$N \geq n$, if $\alpha \neq \frac{2q+1}{p}$, $p = 1, \cdots, n-1$ and $q = 0, 1, \cdots, p-1$	$N = +\infty$
$\Gamma_h^+ \in \mathcal{S}_\Omega^\lambda, \Gamma_h^- \in \mathcal{N}_\Omega^\lambda$	$N \geq n$, if $\alpha \neq \frac{2q+1}{p}$, $p = 1, \cdots, n-1$ and $q = 0, 1, \cdots, p-1$	$N = +\infty$
$\Gamma_h^+ \in \mathcal{S}_\Omega^\lambda, \Gamma_h^- \in \mathcal{M}_\Omega^\lambda$	$N \geq n$, if $u(\mathbf{x}_0) = 0$ and $\alpha \neq \frac{q}{p}$, $p = 1, \cdots, n-1$ and $q = 1, \cdots, p-1$	$N = +\infty$, if $u(\mathbf{x}_0) = 0$; $N = 0$, if $u(\mathbf{x}_0) \neq 0$

TABLE 1. Vanishing orders of Laplacian eigenfunction at intersecting points

5. Proofs of the theorems in Section 3 up the third order

In this section, we present the proofs of the theorems shown in Section 3, but confined to the case that the vanishing order N is at most 3. We develop a mathematical scheme by making use of tools from microlocal analysis that possesses several remarkable properties. Next, we first introduce the so-called complex-geometrical-optics (CGO) solutions constructed in [8] for the subsequent use. As before, we let (r, θ) denote the polar coordinates in $\mathbb{R}^2$; that is, for $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, one has $r = |\mathbf{x}|$ and $\theta = \arg(x_1 + \mathrm{i}x_2)$. Let B_h be the central disk of radius $h \in \mathbb{R}_+$. Let $\Gamma^\pm$ signify the infinite extension of $\Gamma_h^\pm$ in the half-space $x_2 \geq 0$. Let $\theta_m = 0$ and $\theta_M \in (0, \pi)$ be respectively the polar angles of Γ^- and Γ^+. Consider the open sector

$$W = \Big\{ \mathbf{x} \in \mathbb{R}^2; \mathbf{x} \neq 0,\ \theta_m < \arg(x_1 + \mathrm{i}x_2) < \theta_M \Big\}, \tag{5.1}$$

which is formed by the two half-lines Γ^- and Γ^+. We have the following result.

Lemma 5.1. *[8, Lemma 2.2 and Proposition 2.3] Let*

$$u_0(\mathbf{x}) := \exp\left(\sqrt{r} \left(\cos\left(\frac{\theta}{2} + \pi \right) + \mathrm{i} \sin\left(\frac{\theta}{2} + \pi \right) \right) \right). \tag{5.2}$$

Then $\Delta u_0 = 0$ in $\mathbb{R}^2 \backslash \mathbb{R}^2_{0,-}$, where $\mathbb{R}^2_{0,-} := \{\mathbf{x} \in \mathbb{R}^2 | \mathbf{x} = (x_1, x_2); x_1 \leq 0, x_2 = 0\}$, and $s \mapsto u_0(s\mathbf{x})$ decays exponentially in $\mathbb{R}_+$ whenever $\mathbf{x}$ is in the same domain of harmonicity. Furthermore, it holds for $\alpha, s > 0$ that

$$\int_W |u_0(s\mathbf{x})| |\mathbf{x}|^\alpha \mathrm{d}\mathbf{x} \leq \frac{2(\theta_M - \theta_m)\Gamma(2\alpha + 4)}{\delta_W^{2\alpha+4}} s^{-\alpha-2}, \tag{5.3}$$

where $\delta_W = -\max_{\theta_m < \theta < \theta_M} \cos(\theta/2 + \pi) > 0$, and

$$\int_W u_0(s\mathbf{x}) \mathrm{d}\mathbf{x} = 6\mathrm{i}(e^{-2\theta_M \mathrm{i}} - e^{-2\theta_m \mathrm{i}}) s^{-2}, \tag{5.4}$$

while for $h \in \mathbb{R}_+$,

$$\int_{W \backslash B_h} |u_0(s\mathbf{x})| \mathrm{d}\mathbf{x} \leq \frac{6(\theta_M - \theta_m)}{\delta_W^4} s^{-2} e^{-\delta_W \sqrt{hs}/2}. \tag{5.5}$$

Henceforth, for notational convenience, we set

$$\theta_0 = \alpha \cdot \pi = \angle(\Gamma_h^+, \Gamma_h^-); \tag{5.6}$$

for $\alpha \in (0, 1)$. In order to make use of the CGO solution $u_0(s\mathbf{x})$ given in Lemma 5.1 as a test function to analyze the vanishing order of u at the origin, we consider the following domain (see Fig. 2 for the illustration):

$$S_h = W \cap B_h, \tag{5.7}$$

where $\partial S_h = \Gamma_h^+ \cup \Gamma_h^- \cup \Lambda_h$ and

$$\begin{aligned} &\Gamma_h^+ = \{\mathbf{x} \in \mathbb{R}^2;\ 0 \leq \sqrt{x_1^2 + x_2^2} \leq h,\ \arg(x_1 + \mathrm{i}x_2) = \theta_0\}, \\ &\Gamma_h^- = \{\mathbf{x} \in \mathbb{R}^2;\ 0 \leq \sqrt{x_1^2 + x_2^2} \leq h,\ \arg(x_1 + \mathrm{i}x_2) = 0\}, \\ &\Lambda_h = W \cap \partial B_h. \end{aligned} \tag{5.8}$$

In the definition of the generalized singular line, we recall that the polar angles of the exterior normal vectors of Γ_h^+ (with respect to the domain W) and Γ_h^- are, respectively,

$$\varphi_M = \theta_0 + \frac{\pi}{2}, \quad \varphi_m = -\frac{\pi}{2}. \tag{5.9}$$

In order to investigate the relationship between the vanishing order of u at the origin and the the intersecting angle of the generalized singular lines $\Gamma_h^{\pm}$, we consider the following equations

$$\Delta u + \lambda u = 0 \qquad \text{in} \quad B_h, \tag{5.10a}$$

$$\frac{\partial u}{\partial \nu} + \eta_1 u = 0 \qquad \text{on} \quad \Gamma_h^-, \tag{5.10b}$$

$$\frac{\partial u}{\partial \nu} + \eta_2 u = 0 \qquad \text{on} \quad \Gamma_h^+. \tag{5.10c}$$

Next, we derive several crucial auxiliary results regarding the function u satisfying (5.10a)–(5.10c).

Recall that S_h is defined in (5.7). Since u_0 is only smooth in $S_h \backslash B_\varepsilon (0 < \varepsilon < h)$, we cannot use Green's formula for the Laplacian eigenfunction u and the CGO solution u_0 in S_h directly. Instead, we may overcome this difficulty by taking a limit of the volume integral with the integrand $u_0 \Delta u$ over $S_h \backslash B_\varepsilon$ and carefully investigate the boundary integrals on $\partial(S_h \backslash B_\varepsilon)$. We only present the result in the following proposition and its proof is similar to the argument of Lemma 3.2 in [8].

Lemma 5.2. *The CGO solution $u_0(sx)$ defined in (5.2) is harmonic in $S_h \backslash \mathbf{0}$ and decays exponentially as $s \to \infty$ for $0 < \theta < \theta_0$, where θ_0 is the intersecting angle of Γ_h^+ and Γ_h^-. Moreover, for the Laplacian eigenfunction u to (1.1), the Green's formula holds*

$$\int_{S_h} (u_0(s\mathbf{x})\Delta u - u \Delta u_0(s\mathbf{x}))\, \mathrm{d}\mathbf{x} = I_1^+ + I_1^- + I_2, \tag{5.11}$$

where

$$\begin{aligned} I_1^+ &= \int_{\Gamma_h^+} \left(u_0(s\mathbf{x}) \frac{\partial u}{\partial \nu} - u(\mathbf{x}) \frac{\partial u_0(s\mathbf{x})}{\partial \nu} \right) \mathrm{d}\sigma, \\ I_1^- &= \int_{\Gamma_h^-} \left(u_0(s\mathbf{x}) \frac{\partial u}{\partial \nu} - u(\mathbf{x}) \frac{\partial u_0(s\mathbf{x})}{\partial \nu} \right) \mathrm{d}\sigma, \\ I_2 &= \int_{\Lambda_h} \left(u_0(s\mathbf{x}) \frac{\partial u}{\partial \nu} - u(\mathbf{x}) \frac{\partial u_0(s\mathbf{x})}{\partial \nu} \right) \mathrm{d}\sigma. \end{aligned} \tag{5.12}$$

From Lemma 2.6, by direct calculations, we have the following proposition regarding the exterior normal derivative of the CGO solution $u_0(s\mathbf{x})$ on any straight line.

Proposition 5.3. *For any straight line Γ, where $\mathbf{x} = r(\cos\theta, \sin\theta) \in \Gamma$, let the exterior unit normal vector to Γ be $\nu = (\cos\varphi, \sin\varphi)$. Then the CGO solution $u_0(s\mathbf{x})$ given in Lemma 5.1 fulfills*

$$\frac{\partial u_0(s\mathbf{x})}{\partial \nu}\Big|_{\Gamma} = \beta(\theta) e^{\sqrt{sr}\zeta(\theta)} \sqrt{\frac{s}{r}}, \tag{5.13}$$

where $\zeta(\theta)$ and $\beta(\theta)$ are given by

$$\zeta(\theta) = e^{\mathrm{i}(\theta/2 + \pi)} = -e^{\mathrm{i}\theta/2},\ \beta(\theta) = \frac{1}{2}\sin(\varphi - \theta)\tilde{\zeta}(\theta),\ \tilde{\zeta}(\theta) = -\mathrm{i}e^{\mathrm{i}\theta/2}. \tag{5.14}$$

By induction and straightforward calculations, we can derive the explicit formulas of the following integrals in Lemma 5.4, which is essential in showing the relationship between the vanishing order of u and the intersecting angle of the generalized singular lines $\Gamma_h^{\pm}$. The detailed proof of Lemma 5.4 is omitted.

Lemma 5.4. *For a given $\zeta(\theta)\in\mathbb{C}$ and $\ell=0,1,2,\ldots,$ it holds that*

$$\int_0^h r^\ell e^{\sqrt{sr}\zeta(\theta)}\mathrm{d}r=\frac{2}{s^{\ell+1}}\Big\{\frac{(2\ell+1)!}{\zeta(\theta)^{2\ell+2}}+e^{\sqrt{sh}\zeta(\theta)}\sum_{j=0}^{2\ell+1}\frac{(-1)^j(2\ell+1)!}{(2\ell+1-j)!\zeta(\theta)^{j+1}}(sh)^{(2\ell+1-j)/2}\Big\},$$

$$\int_0^h r^\ell e^{\sqrt{sr}\zeta(\theta)}\sqrt{\frac{s}{r}}\mathrm{d}r=\frac{2}{s^{\ell}}\Big\{-\frac{(2\ell)!}{\zeta(\theta)^{2\ell+1}}+e^{\sqrt{sh}\zeta(\theta)}\sum_{j=0}^{2\ell}\frac{(-1)^j(2\ell)!}{(2\ell-j)!\zeta(\theta)^{j+1}}(sh)^{(2\ell-j)/2}\Big\}.$$

Furthermore, the following asymptotic expansions are true for $\Re(\zeta(\theta))<0$ and $s\to\infty$:

$$\begin{aligned}\int_0^h r^\ell e^{\sqrt{sr}\zeta(\theta)}\mathrm{d}r&=\frac{2}{s^{\ell+1}}\cdot\frac{(2\ell+1)!}{\zeta(\theta)^{2\ell+2}}+\mathcal{O}\left(s^{-1/2}e^{\sqrt{sh}\zeta(\theta)}\right),\\ \int_0^h r^\ell e^{\sqrt{sr}\zeta(\theta)}\sqrt{\frac{s}{r}}\mathrm{d}r&=-\frac{2}{s^{\ell}}\cdot\frac{(2\ell)!}{\zeta(\theta)^{2\ell+1}}+\mathcal{O}\left(e^{\sqrt{sh}\zeta(\theta)}\right).\end{aligned}\tag{5.15}$$

In Lemmas 5.5 and 5.6 below, we will investigate the asymptotic behaviors of the integrals associated with u and the CGO solution $u_0(s\mathbf{x})$ (or their corresponding exterior normal derivatives) with respect to the positive parameter s as $s\to\infty$.

Lemma 5.5. *Recall that Γ_h^- and Γ_h^+ are defined in (5.8). Denote*

$$I_{11}^+=\int_{\Gamma_h^+}u(\mathbf{x})\frac{\partial u_0(s\mathbf{x})}{\partial\nu}\mathrm{d}\sigma,\quad I_{11}^-=\int_{\Gamma_h^-}u(\mathbf{x})\frac{\partial u_0(s\mathbf{x})}{\partial\nu}\mathrm{d}\sigma.\tag{5.16}$$

Then the following asymptotic expansions hold with respect to s as $s\to\infty$:

$$\begin{aligned}I_{11}^+&=-\frac{2\beta(\theta_0)}{\zeta(\theta_0)}u(\mathbf{0})-\frac{1}{s}\cdot\frac{4\beta(\theta_0)}{\zeta(\theta_0)^3}c_1(\theta_0)-\frac{1}{s^2}\cdot\frac{48\beta(\theta_0)}{\zeta(\theta_0)^5}c_2(\theta_0)+\mathcal{O}(s^{-3}),\\ I_{11}^-&=-\frac{2\beta(0)}{\zeta(0)}u(\mathbf{0})-\frac{1}{s}\cdot\frac{4\beta(0)}{\zeta(0)^3}c_1(0)-\frac{1}{s^2}\cdot\frac{48\beta(0)}{\zeta(0)^5}c_2(0)+\mathcal{O}(s^{-3}),\end{aligned}\tag{5.17}$$

where

$$\begin{aligned}c_1(\theta)&=\frac{\partial u}{\partial x_1}\Big|_{\mathbf{x}=\mathbf{0}}\cos\theta+\frac{\partial u}{\partial x_2}\Big|_{\mathbf{x}=\mathbf{0}}\sin\theta,\\ c_2(\theta)&=\frac{1}{2}\left(\frac{\partial^2u}{\partial x_1^2}\Big|_{\mathbf{x}=\mathbf{0}}\cos^2\theta+\frac{\partial^2u}{\partial x_1x_2}\Big|_{\mathbf{x}=\mathbf{0}}\sin2\theta+\frac{\partial^2u}{\partial x_2^2}\Big|_{\mathbf{x}=\mathbf{0}}\sin^2\theta\right).\end{aligned}\tag{5.18}$$

Proof. It is easy to see that the exterior unit normal vector to Γ_h^+ is

$$\nu=(\cos\varphi_M,\sin\varphi_M),\quad\varphi_M=\theta_0+\frac{\pi}{2}.\tag{5.19}$$

From Proposition 5.3, on Γ_h^+ we obtain that

$$\frac{\partial u_0(s\mathbf{x})}{\partial\nu}\Big|_{\Gamma_h^+}=\beta(\theta_0)e^{\sqrt{sr}\zeta(\theta_0)}\sqrt{\frac{s}{r}},\tag{5.20}$$

where $\zeta(\theta_0)=-e^{\mathrm{i}\theta_0/2}$, and

$$\beta(\theta_0)=\frac{1}{2}\sin(\varphi_M-\theta_0)\tilde{\zeta}(\theta_0)=-\frac{\mathrm{i}e^{\mathrm{i}\theta_0/2}}{2},\quad\tilde{\zeta}(\theta_0)=-\mathrm{i}e^{\mathrm{i}\theta_0/2}.\tag{5.21}$$

Noting the analyticity of the Laplacian eigenfunction u to (1.1) in Ω, we have the expansion near a neighborhood of the origin:

$$u(\mathbf{x})=\sum_{\alpha\in\mathbb{N}_0^2,\,|\alpha|\geq0}\frac{(\partial^\alpha u)(\mathbf{0})}{\alpha!}\mathbf{x}^\alpha,\tag{5.22}$$

where $\mathbb{N}_0^2 = \{(\alpha_1, \alpha_2) \mid \alpha_j \in \mathbb{N} \cup \{0\}, \quad j = 1, 2\}$. Then substituting (5.20) and (5.22) into I_{11}^+, we deduce that

$$
\begin{aligned}
I_{11}^+ = & \int_{\Gamma_h^+} u(\mathbf{x}) \frac{\partial u_0(s\mathbf{x})}{\partial \nu} \mathrm{d}\sigma = u(\mathbf{0}) \beta(\theta_0) \int_0^h e^{\sqrt{sr}\zeta(\theta_0)} \sqrt{\frac{s}{r}} \mathrm{d}r \\
& + \left(\frac{\partial u}{\partial x_1}\Big|_{\mathbf{x}=\mathbf{0}} \cos\theta_0 + \frac{\partial u}{\partial x_2}\Big|_{\mathbf{x}=\mathbf{0}} \sin\theta_0 \right) \beta(\theta_0) \int_0^h r e^{\sqrt{sr}\zeta(\theta_0)} \sqrt{\frac{s}{r}} \mathrm{d}r \\
& + \frac{1}{2} \left(\frac{\partial^2 u}{\partial x_1^2}\Big|_{\mathbf{x}=\mathbf{0}} \cos^2\theta_0 + 2 \frac{\partial^2 u}{\partial x_1 x_2}\Big|_{\mathbf{x}=\mathbf{0}} \sin\theta_0 \cos\theta_0 + \frac{\partial^2 u}{\partial x_2^2}\Big|_{\mathbf{x}=\mathbf{0}} \sin^2\theta_0 \right) \\
& \times \beta(\theta_0) \int_0^h r^2 e^{\sqrt{sr}\zeta(\theta_0)} \sqrt{\frac{s}{r}} \mathrm{d}r + r_{I_{11}^+},
\end{aligned} \tag{5.23}
$$

where

$$
r_{I_{11}^+} = \sum_{\alpha \in \mathbb{N}_0^2, |\alpha| \geq 3} \frac{(\partial^\alpha u)(\mathbf{0})}{\alpha!} \int_{\Gamma_h^+} \mathbf{x}^\alpha \frac{\partial u_0(s\mathbf{x})}{\partial \nu} \mathrm{d}\sigma.
$$

But this term can be estimated as $s \to \infty$ by means of (5.15),

$$
\left| r_{I_{11}^+} \right| \leq |\beta(\theta_0)| \int_0^h r^3 \cdot e^{\sqrt{sr}\Re(\zeta(\theta_0))} \sqrt{\frac{s}{r}} \mathrm{d}r \sum_{\alpha \in \mathbb{N}_0^2, |\alpha| \geq 3} h^{|\alpha|-3} \left| \frac{(\partial^\alpha u)(\mathbf{0})}{\alpha!} \right| = \mathcal{O}(s^{-3}),
$$

where we have used the fact that $\Re(\zeta(\theta_0)) = -\cos(\theta_0/2) < 0$ for $\theta_0 \in (0, \pi)$. Now the asymptotic expansion (5.17) for I_{11}^+ follows directly by substituting (5.15) into (5.23). Similar argument also applies to the asymptotic expansion of I_{11}^- and the detail is omitted.

The proof is complete. □

Lemma 5.6. *Recall that Γ_h^- and Γ_h^+ are defined in (5.8). Suppose that u satisfies the boundary conditions (5.10b) and (5.10c) on Γ_h^- and Γ_h^+, respectively. Moreover, assume that $\eta_1 \in C^\gamma(\Gamma_h^-)$ and $\eta_2 \in C^\gamma(\Gamma_h^+)$ for $\gamma \in (0, 1]$, and*

$$
I_{12}^+ = -\int_{\Gamma_h^+} u_0(s\mathbf{x}) \frac{\partial u}{\partial \nu} \mathrm{d}\sigma, \quad I_{12}^- = -\int_{\Gamma_h^-} u_0(s\mathbf{x}) \frac{\partial u}{\partial \nu} \mathrm{d}\sigma. \tag{5.24}
$$

Then the following asymptotic expansions hold for $I_{12}^\pm$ with respect to s as $s \to \infty$:

$$
\begin{aligned}
I_{12}^+ &= \frac{2}{s} \cdot \frac{\eta_2(\mathbf{0}) u(\mathbf{0})}{\zeta(\theta_0)^2} + \frac{12}{s^2} \cdot \frac{\eta_2(\mathbf{0}) c_1(\theta_0)}{\zeta(\theta_0)^4} + u(\mathbf{0}) \cdot \mathcal{O}(s^{-1-\gamma}) + \mathcal{O}(s^{-2-\gamma}), \\
I_{12}^- &= \frac{2}{s} \cdot \frac{\eta_1(\mathbf{0}) u(\mathbf{0})}{\zeta(0)^2} + \frac{12}{s^2} \cdot \frac{\eta_1(\mathbf{0}) c_1(0)}{\zeta(0)^4} + u(\mathbf{0}) \cdot \mathcal{O}(s^{-1-\gamma}) + \mathcal{O}(s^{-2-\gamma}).
\end{aligned} \tag{5.25}
$$

Proof. Since η_1 and η_2 are of C^γ-smooth, we have

$$
\eta_i(x) = \eta_i(\mathbf{0}) + \delta\eta_i(x), \quad |\delta\eta_i| \leq \|\eta_i\|_{C^\gamma} \cdot |x|^\gamma. \tag{5.26}
$$

Then using (5.15), (5.22) and (5.26), we can deduce in polar coordinates on Γ_h^+ that

$$
\begin{aligned}
I_{12}^+ = \int_{\Gamma_h^+} u_0(s\mathbf{x}) \eta_2 u \mathrm{d}\sigma = & \, \eta_2(\mathbf{0}) u(\mathbf{0}) \int_0^h e^{\sqrt{sr}\zeta(\theta_0)} \mathrm{d}r + u(\mathbf{0}) \int_0^h \delta\eta_2 e^{\sqrt{sr}\zeta(\theta_0)} \mathrm{d}r \\
& + \eta_2(\mathbf{0}) c_1(\theta_0) \int_0^h r e^{\sqrt{sr}\zeta(\theta_0)} \mathrm{d}r + r_{I_{12}^+},
\end{aligned} \tag{5.27}
$$

where

$$r_{I_{12}^+} = \eta_2(\mathbf{0}) \sum_{\alpha\in\mathbb{N}_0^2,\,|\alpha|\geq 2} \frac{(\partial^\alpha u)(\mathbf{0})}{\alpha!} \int_{\Gamma_h^+} u_0(s\mathbf{x})\mathbf{x}^\alpha \mathrm{d}\sigma + \sum_{\alpha\in\mathbb{N}_0^2,\,|\alpha|\geq 1} \frac{(\partial^\alpha u)(\mathbf{0})}{\alpha!} \int_{\Gamma_h^+} u_0(s\mathbf{x})\mathbf{x}^\alpha \delta\eta_2 \mathrm{d}\sigma.$$

For this term, it follows from (5.15) that

$$\begin{aligned} \left| r_{I_{12}^+} \right| &\leq |\eta_2(\mathbf{0})| \sum_{\alpha\in\mathbb{N}_0^2,\,|\alpha|\geq 2} h^{|\alpha|-2} \left| \frac{(\partial^\alpha u)(\mathbf{0})}{\alpha!} \right| \int_0^h r^2 e^{\sqrt{sr}\Re(\zeta(\theta_0-\pi))} \mathrm{d}r \\ &+ \|\eta_2\|_{C^\gamma} \sum_{\alpha\in\mathbb{N}_0^2,\,|\alpha|\geq 1} h^{|\alpha|-1} \left| \frac{(\partial^\alpha u)(\mathbf{0})}{\alpha!} \right| \int_0^h r^{1+\gamma} e^{\sqrt{sr}\Re(\zeta(\theta_0-\pi))} \mathrm{d}r \\ &= \mathcal{O}(s^{-2-\gamma}). \end{aligned} \tag{5.28}$$

Using this and (5.15) again, we can derive (5.25) from (5.27). Similar derivation can be done for the asymptotic expansion of I_{12}^-. □

The following lemma is about the exterior normal derivative of $\partial_\nu u$ with respect to any singular line of a Laplacian eigenfunction u.

Lemma 5.7. *Suppose $\Gamma := \{\mathbf{x} \in \mathbb{R}^2 \mid \mathbf{x} = r(\cos\theta, \sin\theta), r > 0\}$ (θ is fixed) is a singular line of the Laplacian eigenfunction u, and φ is the polar angle of the unit normal vector to Γ, then*

$$\cos\varphi\cos\theta \frac{\partial^2 u}{\partial x_1^2}\Big|_{\mathbf{x}=\mathbf{0}} + \sin\varphi\sin\theta \frac{\partial^2 u}{\partial x_2^2}\Big|_{\mathbf{x}=\mathbf{0}} + \sin(\varphi+\theta)\frac{\partial^2 u}{\partial x_1 \partial x_2}\Big|_{\mathbf{x}=\mathbf{0}} = 0\,. \tag{5.29}$$

Proof. Recalling the definition of a singular line, one has

$$\nabla\left(\frac{\partial u}{\partial \nu}\right) \cdot (\cos\theta, \sin\theta)^\top = 0. \tag{5.30}$$

Then we can derive (5.29) by evaluating (5.30) in more detail at $\mathbf{x} = \mathbf{0}$. □

For the subsequent analysis, we also need the following lemma.

Lemma 5.8. *Suppose that u has the expansion (5.22). For any straight line segment $\Gamma_h := \{x \in \mathbb{R}^2 \mid \mathbf{x} = r(\cos\theta, \sin\theta), 0 \leq r \leq h \ll 1\}$ (with θ fixed) satisfying $\Re(\zeta(\theta)) < 0$ where $\zeta(\theta)$ is defined in (5.14), we assume that $\nu = (\cos\varphi, \sin\varphi)$ is the exterior unit normal vector to Γ_h. Then it holds as $s \to \infty$ that*

$$\begin{aligned} I_{12} = \int_{\Gamma_h} u_0(s\mathbf{x})\frac{\partial u}{\partial \nu}\mathrm{d}\sigma &= \frac{2}{s}\cdot\left(\frac{\partial u}{\partial x_1}\Big|_{\mathbf{x}=\mathbf{0}}\cos\varphi + \frac{\partial u}{\partial x_2}\Big|_{\mathbf{x}=\mathbf{0}}\sin\varphi\right)\cdot\frac{1}{\zeta(\theta)^2} \\ &+ \frac{12}{s^2}\cdot\left(\frac{\partial^2 u}{\partial x_1^2}\Big|_{\mathbf{x}=\mathbf{0}}\cos\varphi\cos\theta + \frac{\partial^2 u}{\partial x_2^2}\Big|_{\mathbf{x}=\mathbf{0}}\sin\varphi\sin\theta + \frac{\partial^2 u}{\partial x_1\partial x_2}\Big|_{\mathbf{x}=\mathbf{0}}\sin(\varphi+\theta)\right)\cdot\frac{1}{\zeta(\theta)^4} \\ &+ \mathcal{O}(s^{-3})\,. \end{aligned} \tag{5.31}$$

Proof. Using the polar coordinate on Γ_h, it is easy to see that

$$\begin{aligned} \frac{\partial u}{\partial \nu} &= \frac{\partial u}{\partial x_1}\Big|_{\mathbf{x}=\mathbf{0}}\cos\varphi + \frac{\partial u}{\partial x_2}\Big|_{\mathbf{x}=\mathbf{0}}\sin\varphi + r\left(\frac{\partial^2 u}{\partial x_1^2}\Big|_{\mathbf{x}=\mathbf{0}}\cos\varphi\cos\theta + \frac{\partial^2 u}{\partial x_2^2}\Big|_{\mathbf{x}=\mathbf{0}}\sin\varphi\sin\theta\right. \\ &\left. + \frac{\partial^2 u}{\partial x_1\partial x_2}\Big|_{\mathbf{x}=\mathbf{0}}\sin(\varphi+\theta)\right) + R(u, r, \varphi, \theta) \end{aligned} \tag{5.32}$$

where

$$|R(u,r,\varphi,\theta)| \leq r^2 \sum_{\alpha\in\mathbb{N}_0^2,\,|\alpha|\geq 3} h^{|\alpha|-3} \left| \frac{(\partial^\alpha u)(\mathbf{0})}{(\alpha-1)!} \right|.$$

Therefore we can further write

$$\begin{aligned} I_{12} &= \int_{\Gamma_h} u_0(s\mathbf{x}) \frac{\partial u}{\partial \nu} \mathrm{d}\sigma = \left(\frac{\partial u}{\partial x_1}\Big|_{\mathbf{x}=\mathbf{0}} \cos\varphi + \frac{\partial u}{\partial x_2}\Big|_{\mathbf{x}=\mathbf{0}} \sin\varphi \right) \int_0^h e^{\sqrt{sr}\zeta(\theta)} \mathrm{d}r \\ &+ \left(\frac{\partial^2 u}{\partial x_1^2}\Big|_{\mathbf{x}=\mathbf{0}} \cos\varphi\cos\theta + \frac{\partial^2 u}{\partial x_2^2}\Big|_{\mathbf{x}=\mathbf{0}} \sin\varphi\sin\theta + \frac{\partial^2 u}{\partial x_1 \partial x_2}\Big|_{\mathbf{x}=\mathbf{0}} \sin(\varphi+\theta) \right) \\ &\times \int_0^h r e^{\sqrt{sr}\zeta(\theta)} \mathrm{d}r + \int_0^h R(u,r,\varphi,\theta) e^{\sqrt{sr}\zeta(\theta)} \mathrm{d}r. \end{aligned} \tag{5.33}$$

Then the desired result follows from the estimate as $s \to \infty$ by using (5.15):

$$\left| \int_0^h R(u,r,\varphi,\theta) e^{\sqrt{sr}\zeta(\theta)} \mathrm{d}r \right| \leq \int_0^h r^2 e^{\sqrt{sr}\zeta(\theta)} \mathrm{d}r \cdot \sum_{\alpha\in\mathbb{N}_0^2,\,|\alpha|\geq 3} h^{|\alpha|-2} \left| \frac{(\partial^\alpha u)(\mathbf{0})}{(\alpha-1)!} \right| = \mathcal{O}(s^{-3}).$$

□

Suppose that Γ_h is a nodal line of u. Using the polar coordinate and evaluating (5.22) on Γ_h, we can prove the following lemma.

Lemma 5.9. *Suppose that u is a Laplacian eigenfunction to (1.1) and $u = 0$ on Γ_h, where $\Gamma_h := \{\mathbf{x} \in \mathbb{R}^2;\ x = r(\cos\theta, \sin\theta), 0 \leq r \leq h\}$ (with θ fixed) is a line segment. Then the functions $c_1(\theta)$ and $c_2(\theta)$ defined in (5.18) are both identically zero.*

Now we are in a position to present the proof of the theorems in Section 3 in the specific case that the vanishing order N is up to 3. Before that, we make two important remarks.

First, throughout the present section, if a generalized singular line of the form (1.2) is involved, the parameter η can be a C^1 function other than a constant. Indeed, our argument in the present section can deal with this more general case. In principle, we believe that the theorems in Section 3 can also be extended to the more general case that η is a function other than a constant. However, when dealing with higher vanishing orders, the corresponding analysis becomes radically more tedious and complicated. Hence, in the next section for the general vanishing order case, we shall stick to the case that η is a constant.

Second, in proving the vanishing order of the eigenfunction (up to 3), we shall only make use of the eigenfunction confined in S_h. This is achieved by means of of the auxiliary results established in Lemmas 5.2–5.8. It is emphasized that this is in sharp contrast to the argument in the next section (for the higher vanishing orders), which applies the spherical wave expansion of the eigenfunction in B_h. Hence, our argument in the present section is "localized". This "localization" property enables one to consider, e.g., the quantitative behaviours of the Dirichlet eigenfunction in Ω up to the boundary $\partial\Omega$. Moreover, combining with our first remark above, the argument can also be used to study the quantitative behaviours of eigenfunctions associated with a general second order elliptic operator other than the Laplacian. We shall investigate these interesting extensions in our future work.

We first deal with Theorem 3.1. According to our discussion above, we actually prove the following more general theorem.

Theorem 5.10. *Let u be a Laplacian eigenfunction to (1.1). Suppose that there are two generalized singular lines Γ_h^+ and Γ_h^- from $\mathcal{M}_\Omega^\lambda$ such that (3.2) and (3.3) hold. Assume that*

$\eta_1 \in C^1(\Gamma_h^-)$ and $\eta_2 \in C^1(\Gamma_h^+)$. If the conditions

$$u(\mathbf{0}) = 0 \ \text{and} \ \alpha \neq \frac{1}{2} \tag{5.34}$$

are satisfied, the Laplacian eigenfunction u vanishes up to the order 3 at $\mathbf{0}$.

Proof. Recall Figure 2. Evaluating (5.10b) and (5.10c) at $\mathbf{0}$, using $u(\mathbf{0}) = 0$ we derive

$$\nabla u\Big|_{\mathbf{x}=\mathbf{0}} \cdot (\cos\varphi_m, \sin\varphi_m) = -\eta_1(\mathbf{0})u(\mathbf{0}) = 0, \quad \nabla u\Big|_{\mathbf{x}=\mathbf{0}} \cdot (\cos\varphi_M, \sin\varphi_M) = -\eta_2(\mathbf{0})u(\mathbf{0}) = 0.$$

Since $\theta_0 \in (0, \pi)$, we know that Γ_h^- and Γ_h^+ are non-collinear. Therefore it is easy to see

$$\nabla u(\mathbf{0}) = 0. \tag{5.35}$$

Noting that u is the Laplacian eigenfunction satisfying (5.10a), we derive the integral equality from (5.11):

$$-\lambda \int_{S_h} u_0(s\mathbf{x})u(\mathbf{x})\mathrm{d}\mathbf{x} = \int_{S_h} (u_0(s\mathbf{x})\Delta u - u\Delta u_0(s\mathbf{x}))\,\mathrm{d}\mathbf{x} = I_1^+ + I_1^- + I_2, \tag{5.36}$$

where S_h is defined in (5.7), and $I_1^\pm$ and I_2 are defined in (5.12).

Since $u \in H^2(B_h)$, which can be embedded into $C^\gamma(B_h)(0 < \gamma < 1)$, we know that

$$u(\mathbf{x}) = u(\mathbf{0}) + \delta u(\mathbf{x}), \quad |\delta u(\mathbf{x})| \leq \|u\|_{C^\gamma} |\mathbf{x}|^\gamma, \tag{5.37}$$

from which it follows that

$$\int_{S_h} u_0(s\mathbf{x})u(\mathbf{x})\mathrm{d}\mathbf{x} = u(\mathbf{0})\int_{S_h} u_0(s\mathbf{x})\mathrm{d}\mathbf{x} + I_3 = u(\mathbf{0})\left(\int_W u_0(s\mathbf{x})\mathrm{d}\mathbf{x} - I_4\right) + I_3 \tag{5.38}$$

where

$$I_3 = \int_{S_h} u_0(s\mathbf{x})\delta u(\mathbf{x})\mathrm{d}\mathbf{x}, \quad I_4 = \int_{W\backslash B_h} u_0(s\mathbf{x})\mathrm{d}\mathbf{x}.$$

Substituting (5.38) into (5.36) and combining with (5.4), we derive that

$$-6\lambda\mathrm{i}(e^{-2\mathrm{i}\theta_0} - 1)s^{-2}u(\mathbf{0}) = I_1^+ + I_1^- + I_2 + \lambda I_3 - u(\mathbf{0})\lambda I_4. \tag{5.39}$$

Using $u(\mathbf{0}) = 0$, we further obtain

$$0 = I_1^+ + I_1^- + I_2 + \lambda I_3. \tag{5.40}$$

Since $u \in H^2(B_h)$, we have from [8, Page 6263] and (5.3) that for some $c' > 0$,

$$|I_2| \leq Ce^{-c'\sqrt{s}}, \quad |I_3| \leq \mathcal{O}(s^{-\gamma-2}) \quad \text{as } s \to \infty\,. \tag{5.41}$$

Recalling the definitions of $I_1^\pm, I_2$, $I_{11}^\pm$ and $I_{12}^\pm$ given by (5.12), (5.16) and (5.24), respectively, it is easy to see that

$$I_1^+ = -(I_{11}^+ + I_{12}^+), \quad I_1^- = -(I_{11}^- + I_{12}^-). \tag{5.42}$$

Since η_1 and η_2 are C^1 functions on the boundary $\Gamma_h^\pm$, they fulfill the requirement of Lemma 5.6. Using (5.35) and recalling $c_1(\theta_0)$ and $c_1(0)$ given in (5.18), we know that

$$c_1(\theta_0) = c_1(0) = 0. \tag{5.43}$$

Substituting $u(\mathbf{0}) = 0$ and (5.43) into (5.17) and (5.25) yields

$$\begin{aligned} &I_{11}^+ = -\frac{1}{s^2} \cdot \frac{48\beta(\theta_0)}{\zeta(\theta_0)^5} c_2(\theta_0) + \mathcal{O}(s^{-3}), \quad I_{11}^- = -\frac{1}{s^2} \cdot \frac{48\beta(0)}{\zeta(0)^5} c_2(0) + \mathcal{O}(s^{-3}), \\ &I_{12}^+ = \mathcal{O}(s^{-2-\gamma}), \quad I_{12}^- = \mathcal{O}(s^{-2-\gamma}). \end{aligned} \tag{5.44}$$

But by means of (5.42), we deduce by substituting (5.44) into (5.40) that

$$\frac{1}{s^2}\cdot\frac{48\beta(\theta_0)}{\zeta(\theta_0)^5}c_2(\theta_0)+\frac{1}{s^2}\cdot\frac{48\beta(0)}{\zeta(0)^5}c_2(0)=-\lambda I_3-I_2+\mathcal{O}(s^{-2-\gamma}),\tag{5.45}$$

where $c_2(\theta)$ is defined in (5.18). Multiplying s^2 on the both sides of (5.45), combining with (5.41), and letting $s\to\infty$, we can obtain that

$$\frac{\beta(\theta_0)}{\zeta(\theta_0)^5}c_2(\theta_0)+\frac{\beta(0)}{\zeta(0)^5}c_2(0)=0.\tag{5.46}$$

Using the eigen-equation, $-\Delta u=\lambda u$, we can see

$$\frac{\partial^2 u}{\partial x_1^2}\Big|_{\mathbf{x}=\mathbf{0}}+\frac{\partial^2 u}{\partial x_2^2}\Big|_{\mathbf{x}=\mathbf{0}}=-\lambda u(\mathbf{0})=0.\tag{5.47}$$

Substituting this equation into the expression of $c_2(\theta)$, we derive that

$$c_2(0)=\frac{1}{2}\frac{\partial^2 u}{\partial x_1^2}\Big|_{\mathbf{x}=\mathbf{0}},\quad c_2(\theta_0)=\frac{1}{2}\left(\frac{\partial^2 u}{\partial x_1^2}\Big|_{\mathbf{x}=\mathbf{0}}\cos 2\theta_0+\frac{\partial^2 u}{\partial x_1 x_2}\Big|_{\mathbf{x}=\mathbf{0}}\sin 2\theta_0\right).\tag{5.48}$$

From (5.14), it is easy to see that

$$\frac{\beta(\theta_0)}{\zeta(\theta_0)^5}=\frac{\mathrm{i}e^{-2\mathrm{i}\theta_0}}{2},\quad \frac{\beta(0)}{\zeta(0)^5}=-\frac{\mathrm{i}}{2}\tag{5.49}$$

Then substituting (5.48) and (5.49) into (5.46), we can deduce that

$$\left(1-e^{-2\mathrm{i}\theta_0}\cos 2\theta_0\right)\frac{\partial^2 u}{\partial x_1^2}\Big|_{\mathbf{x}=\mathbf{0}}-e^{-2\mathrm{i}\theta_0}\sin 2\theta_0\frac{\partial^2 u}{\partial x_1 x_2}\Big|_{\mathbf{x}=\mathbf{0}}=0.\tag{5.50}$$

Recalling that

$$f(x):=\frac{\partial u}{\partial \nu}+\eta_2 u\equiv 0\qquad \text{on }\ \Gamma_h^+,$$

we know the directional derivative of f with respect to the direction $(\cos\theta_0,\sin\theta_0)$ satisfies

$$\nabla\left(\frac{\partial u}{\partial\nu}+\eta_2 u\right)\cdot(\cos\theta_0,\sin\theta_0)^\top=0.\tag{5.51}$$

Since $\eta_1\in C^1(\Gamma_h^-)$ and $\eta_2\in C^1(\Gamma_h^+)$, we can use the fact that

$$\nabla\left(\frac{\partial u}{\partial\nu}+\eta_2 u\right)=\begin{bmatrix}\frac{\partial^2 u}{\partial x_1^2}\cos\varphi_M+\frac{\partial^2 u}{\partial x_1\partial x_2}\sin\varphi_M+\frac{\partial\eta_2}{\partial x_1}u+\eta_2\frac{\partial u}{\partial x_1}\\ \frac{\partial^2 u}{\partial x_1\partial x_2}\cos\varphi_M+\frac{\partial^2 u}{\partial x_2^2}\sin\varphi_M+\frac{\partial\eta_2}{\partial x_2}u+\eta_2\frac{\partial u}{\partial x_2}\end{bmatrix},$$

where $\varphi_M=\theta_0+\frac{\pi}{2}$, and evaluate (5.51) at $\mathbf{x}=\mathbf{0}$, then derive by using $u(\mathbf{0})=0$ and (5.35) that

$$\cos\varphi_M\cos\theta_0\frac{\partial^2 u}{\partial x_1^2}\Big|_{\mathbf{x}=\mathbf{0}}+\sin\varphi_M\sin\theta_0\frac{\partial^2 u}{\partial x_2^2}\Big|_{\mathbf{x}=\mathbf{0}}+\sin(\varphi_M+\theta_0)\frac{\partial^2 u}{\partial x_1\partial x_2}\Big|_{\mathbf{x}=\mathbf{0}}=0.\tag{5.52}$$

Substituting (5.47) into (5.52), together with $\varphi_M=\theta_0+\frac{\pi}{2}$ we can further obtain that

$$\sin 2\theta_0\frac{\partial^2 u}{\partial x_1^2}\Big|_{\mathbf{x}=\mathbf{0}}-\cos 2\theta_0\frac{\partial^2 u}{\partial x_1\partial x_2}\Big|_{\mathbf{x}=\mathbf{0}}=0.\tag{5.53}$$

Now combining (5.50) with (5.53), we can get a system of linear equations with respect to $\frac{\partial^2 u}{\partial x_1^2}(\mathbf{0})$ and $\frac{\partial^2 u}{\partial x_1\partial x_2}(\mathbf{0})$, with the determinant of its coefficient matrix given by

$$\begin{vmatrix}1-e^{-2\mathrm{i}\theta_0}\cos 2\theta_0 & -e^{-2\mathrm{i}\theta_0}\sin 2\theta_0\\ \sin 2\theta_0 & -\cos 2\theta_0\end{vmatrix}=-\cos 2\theta_0+e^{-2\mathrm{i}\theta_0}=-\mathrm{i}\sin 2\theta_0\neq 0$$

since $\theta_0 \neq \pi/2$. Therefore, together with (5.47) we can conclude that

$$\frac{\partial^2 u}{\partial x_1^2}\Big|_{x=0} = \frac{\partial^2 u}{\partial x_1 \partial x_2}\Big|_{\mathbf{x}=\mathbf{0}} = \frac{\partial^2 u}{\partial x_2^2}\Big|_{\mathbf{x}=\mathbf{0}} = 0.$$

Since the order of the lowest nontrivial homogeneous polynomial in Taylor expansion (5.22) around the origin is larger than 2, its vanishing order is at least up to 3. This completes the proof. □

We next deal with Theorem 3.5, but under a more general situation with $\eta_2 \in C^1(\Gamma_h^+)$.

Theorem 5.11. *Let u be a Laplacian eigenfunction to (1.1). Suppose that a generalized singular line $\Gamma_h^+ \in \mathcal{M}_\Omega^\lambda$ intersects with a nodal line $\Gamma_h^- \in \mathcal{N}_\Omega^\lambda$ such that (3.2) and (3.3) hold, and $\eta_2 \in C^1(\Gamma_h^+)$. If the following condition is fulfilled*

$$\alpha \neq \frac{1}{4}, \frac{1}{2} \text{ and } \frac{3}{4} \tag{5.54}$$

then the Laplacian eigenfunction u vanishes up to the order 3 at $\mathbf{0}$.

Proof. Since $u = 0$ on Γ_h^-, we know from (5.47) that

$$\frac{\partial u}{\partial x_1}\Big|_{\mathbf{x}=\mathbf{0}} = \frac{\partial^2 u}{\partial x_1^2}\Big|_{\mathbf{x}=\mathbf{0}} = \frac{\partial^2 u}{\partial x_2^2}\Big|_{\mathbf{x}=\mathbf{0}} = 0. \tag{5.55}$$

Further, we derive by evaluating

$$\frac{\partial u}{\partial \nu} + \eta_2 u = 0$$

on Γ_h^+ at $\mathbf{x} = \mathbf{0}$ that

$$\frac{\partial u}{\partial x_1}\Big|_{\mathbf{x}=\mathbf{0}} \cos\varphi_M + \frac{\partial u}{\partial x_2}\Big|_{\mathbf{x}=\mathbf{0}} \sin\varphi_M = -\eta_2(\mathbf{0})u(\mathbf{0}) = 0, \tag{5.56}$$

where $\varphi_M = \theta_0 + \pi/2$. Then substituting (5.55) into (5.56), it is easy to see that

$$\cos\theta_0 \cdot \frac{\partial u}{\partial x_2}\Big|_{\mathbf{x}=\mathbf{0}} = 0.$$

Hence if $\theta_0 \neq \pi/2$, we have

$$\frac{\partial u}{\partial x_2}\Big|_{\mathbf{x}=\mathbf{0}} = 0. \tag{5.57}$$

Now recalling that we have the boundary condition (5.10c) on Γ_h^+, with $\eta_2 \in C^1(\Gamma_h^+)$, and the fact that $u = 0$ on Γ_h^-, we can establish the following integral equality by the same argument for deriving (5.40)

$$0 = I_1^+ - I_{12}^- + I_2 + \lambda I_3, \tag{5.58}$$

where $I_1^+ = -(I_{11}^+ + I_{12}^+)$, I_2, I_{12}^- and I_3 are defined in (5.16), (5.12), (5.24) and (5.38) , respectively. For the term I_{11}^+, it follows from (5.17) that

$$I_{11}^+ = -\frac{2\beta(\theta_0)}{\zeta(\theta_0)}u(\mathbf{0}) - \frac{1}{s} \cdot \frac{4\beta(\theta_0)}{\zeta(\theta_0)^3}c_1(\theta_0) - \frac{1}{s^2} \cdot \frac{48\beta(\theta_0)}{\zeta(\theta_0)^5}c_2(\theta_0) + \mathcal{O}(s^{-3})$$

which can be further reduced to

$$I_{11}^+ = -\frac{1}{s^2} \cdot \frac{24\beta(\theta_0)}{\zeta(\theta_0)^5} \cdot \frac{\partial^2 u}{\partial x_1 x_2}\Big|_{\mathbf{x}=\mathbf{0}} \sin 2\theta_0 + \mathcal{O}(s^{-3}) \tag{5.59}$$

by (5.55) and (5.57).

To estimate the term I_{12}^{+}, we recall that η_2 has the expansion (5.26). We have $c_1(\theta_0)=0$ from (5.55) and (5.57). Then using $u(\mathbf{0})=c_1(\theta_0)=0$, we deduce from (5.25) that

$$I_{12}^{+}=\mathcal{O}(s^{-2-\gamma}). \tag{5.60}$$

Next we estimate I_{12}^{-}. It follows from (5.31), combining with (5.55) and (5.57), that

$$I_{12}^{-}=-\int_{\Gamma_h^-}u_0(s\mathbf{x})\frac{\partial u}{\partial\nu}\mathrm{d}\sigma=\frac{12}{s^2}\cdot\frac{\partial^2 u}{\partial x_1\partial x_2}\Big|_{\mathbf{x}=\mathbf{0}}\cdot\frac{1}{\zeta(0)^4}-\mathcal{O}(s^{-3}). \tag{5.61}$$

Substituting (5.59)-(5.61) into (5.58), we can get that

$$\frac{1}{s^2}\cdot\frac{24\beta(\theta_0)}{\zeta(\theta_0)^5}\cdot\frac{\partial^2 u}{\partial x_1 x_2}\Big|_{\mathbf{x}=\mathbf{0}}\sin 2\theta_0-\frac{12}{s^2}\cdot\frac{\partial^2 u}{\partial x_1\partial x_2}\Big|_{\mathbf{x}=\mathbf{0}}\cdot\frac{1}{\zeta(0)^4}-\mathcal{O}\left(s^{-2-\gamma}\right)=-(I_2+\lambda I_3).$$

Multiplying s^2 on the both sides of the above equality, we deduce from (5.41) that

$$\left(\frac{2\beta(\theta_0)}{\zeta(\theta_0)^5}\cdot\sin 2\theta_0-\frac{1}{\zeta(0)^4}\right)\cdot\frac{\partial^2 u}{\partial x_1 x_2}\Big|_{\mathbf{x}=\mathbf{0}}=0$$

as $s\to\infty$. But we see from (5.49) that

$$\frac{2\beta(\theta_0)}{\zeta(\theta_0)^5}\cdot\sin 2\theta_0-\frac{1}{\zeta(0)^4}=\mathrm{i}e^{-2\mathrm{i}\theta_0}\sin 2\theta_0-1=-\cos 2\theta_0 e^{-2\mathrm{i}\theta_0}\neq 0$$

if $\theta_0\neq\pi/4$ and $\theta_0\neq 3\pi/4$. Hence we obtain that

$$\frac{\partial^2 u}{\partial x_1\partial x_2}\Big|_{\mathbf{x}=\mathbf{0}}=0,$$

which completes the proof. □

5.1. **Proof of Theorem 3.2.** Using $u=0$ on $\Gamma_h^{\pm}$, we have

$$\nabla u\Big|_{\mathbf{x}=\mathbf{0}}\cdot(1,0)^\top=\nabla u\Big|_{\mathbf{x}=\mathbf{0}}\cdot(\cos\theta_0,\sin\theta_0)^\top=0.$$

This implies

$$\nabla u\Big|_{\mathbf{x}=\mathbf{0}}=0. \tag{5.62}$$

Now we recall that u has the expansion (5.22), then we can derive on Γ_h^- by using (5.62) and polar coordinates that

$$\sum_{\substack{\alpha\in\mathbb{N}_0^2,\,|\alpha|\geq 2\\ \alpha=(\alpha_1,\alpha_2)}}\frac{(\partial^\alpha u)(\mathbf{0})}{\alpha!}r^{|\alpha|}\cos^{\alpha_1}(0)\sin^{\alpha_2}(0)\Big|_{x\in\Gamma_h^-}\equiv 0,\quad 0\leq r\leq h\,,$$

from which it is not difficult to see that

$$c_2(0)=\frac{1}{2}\frac{\partial^2 u}{\partial x_1^2}\Big|_{\mathbf{x}=\mathbf{0}}\cdot\cos^2 0=0\,. \tag{5.63}$$

This can be used, along with (5.47), to deduce that

$$\frac{\partial^2 u}{\partial x_2^2}\Big|_{\mathbf{x}=\mathbf{0}}=0\,. \tag{5.64}$$

By means of the fact $u=0$ on $\Gamma_h^{\pm}$ again, we have the integral identitys

$$0=-I_{12}^{+}-I_{12}^{-}+I_2+\lambda I_3, \tag{5.65}$$

where I_2, $I_{12}^{\pm}$ and I_3 are defined in (5.12), (5.24) and (5.38), respectively. By Lemma 5.8, together with (5.62)-(5.64) it is easy to see that

$$
\begin{aligned}
I_{12}^{+} &= \frac{12}{s^2}\cdot \frac{\partial^2 u}{\partial x_1 \partial x_2}\Big|_{\mathbf{x}=\mathbf{0}} \sin(\varphi_M+\theta_0)\cdot \frac{1}{\zeta(\theta_0)^4} + \mathcal{O}(s^{-3}),\\
I_{12}^{-} &= \frac{12}{s^2}\cdot \frac{\partial^2 u}{\partial x_1 \partial x_2}\Big|_{\mathbf{x}=\mathbf{0}} \sin(\varphi_m)\cdot \frac{1}{\zeta(0)^4} + \mathcal{O}(s^{-3}),
\end{aligned}
\tag{5.66}
$$

where $\varphi_m = -\pi/2$ and $\varphi_M = \theta_0 + \pi/2$ are the arguments of the exterior unit normal vectors to Γ_h^- and Γ_h^+, respectively. From (5.14), we have

$$
\zeta(\theta_0)^4 = e^{2\mathrm{i}\theta_0}, \quad \zeta(0)^4 = 1.
$$

Substituting (5.66) into (5.65), we derive that

$$
\frac{12}{s^2}\left(1-\cos 2\theta_0 e^{-2\mathrm{i}\theta_0}\right)\frac{\partial^2 u}{\partial x_1 \partial x_2}\Big|_{\mathbf{x}=\mathbf{0}} + I_2 + \lambda I_3 - \mathcal{O}(s^{-3}) = 0. \tag{5.67}
$$

Now multiplying s^2 on the both sides of (5.67), noting (5.41), we have

$$
\left(1-\cos 2\theta_0 e^{-2\mathrm{i}\theta_0}\right)\frac{\partial^2 u}{\partial x_1 \partial x_2}\Big|_{\mathbf{x}=\mathbf{0}} = 0
$$

from which we can deduce that

$$
\frac{\partial^2 u}{\partial x_1 \partial x_2}\Big|_{\mathbf{x}=\mathbf{0}} = 0
$$

if $\theta_0 \neq \pi/2$. This completes our proof. □

5.2. **Proof of Theorem 3.7.** Since u satisfies the boundary condition $u=0$ on Γ_h^-, we know that

$$
u(\mathbf{0}) = 0, \quad \nabla u\Big|_{\mathbf{x}=\mathbf{0}}\cdot (1,0)^\top = 0 \Longrightarrow \frac{\partial u}{\partial x_1}\Big|_{\mathbf{x}=\mathbf{0}} = 0.
$$

Recalling that $\varphi_M = \theta_0 + \pi/2$ is the argument of the exterior unit normal vector of Γ_h^+ and $\partial_\nu u = 0$ on Γ_h^+, we easily see

$$
\frac{\partial u}{\partial x_1}\Big|_{\mathbf{x}=\mathbf{0}}(-\sin\theta_0) + \frac{\partial u}{\partial x_2}\Big|_{\mathbf{x}=\mathbf{0}}\cos\theta_0 = 0.
$$

Therefore if $\theta_0 \neq \pi/2$, we know

$$
\frac{\partial u}{\partial x_2}\Big|_{\mathbf{x}=\mathbf{0}} = 0.
$$

Furthermore, since Γ_h^- is a nodal line, we know by Lemma 5.9 that

$$
\frac{\partial^2 u}{\partial x_1^2}\Big|_{\mathbf{x}=\mathbf{0}} = 0.
$$

Substituting $u(\mathbf{0}) = 0$ into (5.47), we have

$$
\frac{\partial^2 u}{\partial x_2^2}\Big|_{\mathbf{x}=\mathbf{0}} = -\frac{\partial^2 u}{\partial x_1^2}\Big|_{\mathbf{x}=\mathbf{0}} = 0.
$$

Then using the fact that Γ_h^+ is a singular line of u, we derive from Lemma 5.7 that

$$
\sin(\varphi_M+\theta_0)\cdot \frac{\partial^2 u}{\partial x_1 \partial x_2}\Big|_{x=0} = \cos 2\theta_0 \cdot \frac{\partial^2 u}{\partial x_1 \partial x_2}\Big|_{\mathbf{x}=\mathbf{0}} = 0.
$$

Hence if $\theta_0 \neq \pi/4$ and $\theta_0 \neq 3\pi/4$, we can prove

$$
\frac{\partial^2 u}{\partial x_1 \partial x_2}\Big|_{\mathbf{x}=\mathbf{0}} = 0,
$$

which means that the expansion (5.22) of u has at least nontrivial homogeneous polynomial with the order of three, hence completes our proof. □

Remark 5.12. In the subsection 5.2 above, we may also use the CGO solution as the test function to study the vanishing property of the second order partial derivatives of u at the origin with respect to θ_0, which can lead to the same conclusion.

6. Proofs of the theorems in Section 3 for general cases

In this section, detailed proofs of the theorems for general vanishing order in Section 3 are presented, by using the spherical wave expansion of the Laplacian eigenfunction u near the intersecting point between two line segments.

From (2.14) and (2.15), we have the following lemma regarding the exterior normal derivative of u on $\Gamma_h^\pm$ by using the spherical wave expansion (2.14) of u.

Lemma 6.1. *Under the polar coordinate, we have the following expansion of the normal derivative of u given by (2.14) on $\Gamma_h^\pm$ around the origin*

$$\begin{aligned}\frac{\partial u}{\partial \nu}\Big|_{\Gamma_h^+} &= \frac{1}{r}\frac{\partial u}{\partial \theta}\Big|_{\theta=\theta_0} = \frac{1}{r}\sum_{n=0}^{\infty} \mathrm{i}n\left(a_n e^{\mathrm{i}n\theta_0} - b_n e^{-\mathrm{i}n\theta_0}\right) J_n\left(\sqrt{\lambda}r\right),\\ \frac{\partial u}{\partial \nu}\Big|_{\Gamma_h^-} &= -\frac{1}{r}\frac{\partial u}{\partial \theta}\Big|_{\theta=0} = -\frac{1}{r}\sum_{n=0}^{\infty} \mathrm{i}n\left(a_n - b_n\right) J_n\left(\sqrt{\lambda}r\right).\end{aligned} \tag{6.1}$$

In Section 5, we proved the vanishing order up to three of the Laplacian eigenfunction at the origin. In the next lemma, we clarify the relationship between the coefficients a_n, b_n in (2.14) and the vanishing order of u at the origin.

Lemma 6.2. *Suppose that u has the spherical wave expansion (2.14) around the origin, and the coefficients a_n, b_n in (2.14) satisfy*

$$a_0+b_0=0,\ \ a_n=b_n=0,\ n=1,2,\ldots,N-1,\ \ N\geq 2, \tag{6.2}$$

if and only if the vanishing order of u at the origin is at least N.

Proof. Substituting (6.2) into (2.14) yields

$$u(x)=\sum_{n=N}^{\infty}\left(a_n e^{\mathrm{i}n\theta}+b_n e^{-\mathrm{i}n\theta}\right) J_n\left(\sqrt{\lambda}r\right). \tag{6.3}$$

The lemma is readily proved by noting from (2.17) that the power of the lowest order in (6.3) with respect to r is N. □

Recalling the definition of the generalized singular line of the Laplacian eigenfunction u, and using the spherical wave expansion (2.14) of u and Lemma 6.1, we can deduce some equations for the undetermined coefficients $\{a_n, b_n\}$ in (2.14). These equations will be used in the proof of Theorem 3.1.

Lemma 6.3. *Let $\Gamma_h^\pm$ be two generalized singular lines of u with the boundary parameters $\eta_1\equiv C_1$ and $\eta_2\equiv C_2$ defined on Γ_h^- and Γ_h^+, respectively, where C_1 and C_2 are two constants. Suppose that $\Gamma_h^\pm$ intersect with each other at the origin and (6.2) is fulfilled. Then the following equations hold for the coefficients $\{a_N, b_N\}$ and $\{a_{N+1}, b_{N+1}\}$ in (2.14):*

$$a_N e^{\mathrm{i}N\theta_0}-b_N e^{-\mathrm{i}N\theta_0}=0,\ \ a_N-b_N=0, \tag{6.4}$$

$$2C_2(a_N e^{\mathrm{i}N\theta_0}+b_N e^{-\mathrm{i}N\theta_0})+\mathrm{i}\sqrt{\lambda}(a_{N+1}e^{\mathrm{i}(N+1)\theta_0}-b_{N+1}e^{-\mathrm{i}(N+1)\theta_0})=0, \tag{6.5}$$

$$2C_1(a_N+b_N)-\mathrm{i}\sqrt{\lambda}(a_{N+1}-b_{N+1})=0. \tag{6.6}$$

Proof. Substituting (2.14) and (6.1) into

$$\frac{\partial u}{\partial \nu} + \eta_2 u = 0 \text{ on } \Gamma_h^+ \quad \text{and} \quad \frac{\partial u}{\partial \nu} + \eta_1 u = 0 \text{ on } \Gamma_h^-,$$

and combining with the following identity (cf. [1])

$$J_n(\sqrt{\lambda}r) = \frac{\sqrt{\lambda}r}{2n}\left(J_{n-1}(\sqrt{\lambda}r) + J_{n+1}(\sqrt{\lambda}r)\right), \quad n \in \mathbb{N},$$

we can obtain

$$\begin{aligned}&\sum_{n=1}^{\infty} \frac{\mathrm{i}\sqrt{\lambda}}{2}(a_n e^{\mathrm{i}n\theta_0} - b_n e^{-\mathrm{i}n\theta_0})\left(J_{n-1}(\sqrt{\lambda}r) + J_{n+1}(\sqrt{\lambda}r)\right)\\ &\quad + C_2\sum_{n=0}^{\infty}(a_n e^{\mathrm{i}n\theta_0} + b_n e^{-\mathrm{i}n\theta_0})J_n(\sqrt{\lambda}r) = 0,\end{aligned} \tag{6.7}$$

and

$$\sum_{n=1}^{\infty} \frac{\mathrm{i}\sqrt{\lambda}}{2}(a_n - b_n)\left(J_{n-1}(\sqrt{\lambda}r) + J_{n+1}(\sqrt{\lambda}r)\right) - C_1\sum_{n=0}^{\infty}(a_n + b_n)J_n(\sqrt{\lambda}r) = 0. \tag{6.8}$$

Using (6.2) and comparing the coefficients of r^{N-1} and r^N on both sides of (6.7), one can derive by straightforward calculations the first equation in (6.4) and (6.5). Similarly, we can derive the second equation in (6.4) and (6.6) on Γ_h^- by using (6.2) and (6.8). □

Similar to Lemma 6.3, we can obtain the equations for $\{a_n, b_n\}$ by using (2.14) and the boundary conditions on $\Gamma_h^\pm$ for the singular lines in Lemmas 6.4.

Lemma 6.4. *Suppose that $\Gamma_h^\pm$ are two singular lines of u which intersect with each other at the origin and (6.2) is fulfilled. Then the following equations hold for the coefficients $\{a_N, b_N\}$ in (2.14):*

$$a_N e^{\mathrm{i}N\theta_0} - b_N e^{-\mathrm{i}N\theta_0} = 0, \tag{6.9}$$

$$a_N - b_N = 0. \tag{6.10}$$

Proof. Using (6.1) and Lemma 2.7, together with (6.2), we can derive (6.9) and (6.10). □

In particular, for two intersecting nodal lines $\Gamma_h^\pm$, we can derive the equations for $\{a_n, b_n\}$, $n = 0, 1, \cdots$, by Lemma 2.7 as follows.

Lemma 6.5. *Suppose that $\Gamma_h^\pm$ are two nodal lines of u which intersect with each other at the origin, then the following equations hold for the coefficients $\{a_n, b_n\}$ in (2.14), $n = 0, 1, \ldots$:*

$$a_n e^{\mathrm{i}n\theta_0} + b_n e^{-\mathrm{i}n\theta_0} = 0,$$

$$a_n + b_n = 0.$$

Proof. Substituting (2.14) into $u = 0$ on $\Gamma_h^\pm$, we can obtain that

$$0 = \sum_{n=0}^{\infty}\left(a_n e^{\mathrm{i}n\theta_0} + b_n e^{-\mathrm{i}n\theta_0}\right) J_n(\sqrt{\lambda}r), \quad 0 = \sum_{n=0}^{\infty}(a_n + b_n)\, J_n(\sqrt{\lambda}r).$$

Then the desired results follow directly from Lemma 2.7. □

In the rest of this section, we provide detailed proofs of theorems for general vanishing orders in Section 3.

6.1. **Proof of Theorem 3.1.** According to Theorem 5.10, we know that Theorem 3.1 holds at least for $N = 3$. Hence, by virtue of Lemma 6.2, we see that (6.2) holds for $N = 3$. Now we apply the mathematical induction, and assume that (6.2) holds for any $N \geq 3$ and $N \in \mathbb{N}$. Then using (6.4) we can directly derive by virtue of (3.4) that

$$\begin{vmatrix} e^{2\mathrm{i}N\theta_0} & -1 \\ 1 & -1 \end{vmatrix} = 1 - e^{2\mathrm{i}N\theta_0} \neq 0$$

for $\theta_0 \neq \frac{m\pi}{N}$ $(m = 0, 1, \ldots, N-1)$. This implies that $a_N = b_N = 0$, and completes the proof of Theorem 3.1. □

6.2. **Proof of Theorem 3.2.** Since $u = 0$ on $\Gamma_h^{\pm}$, by Lemma 6.5 we know

$$\begin{vmatrix} e^{\mathrm{i}n\theta_0} & e^{-\mathrm{i}n\theta_0} \\ 1 & 1 \end{vmatrix} = e^{-\mathrm{i}n\theta_0}(e^{2\mathrm{i}n\theta_0} - 1) \neq 0$$

if $\theta_0 \neq \frac{m\pi}{n}$ $(m = 0, 1, \ldots, n-1)$. This readily implies that $a_n = b_n = 0$, $n = 1, 2, \ldots$. □

6.3. **Proof of Theorem 3.3.** In combination with Lemma 6.4, Theorem 3.3 can be proved by following a completely similar argument to the one for Theorem 3.1 in Subsection 6.1 by formally taking $\eta_1 \equiv 0$ and $\eta_2 \equiv 0$.

6.4. **Proof of Theorem 3.5.** From Theorem 5.11, we know that Theorem 3.5 holds at least for $N = 3$. Therefore, by virtue Lemma 6.2, one can conclude that (6.2) holds for $N = 3$. Now we apply the mathematical induction, and assume that (6.2) holds for any $N \geq 3$ and $N \in \mathbb{N}$. Since $u = 0$ on Γ_h^-, from Lemma 6.5, one has

$$a_N + b_N = 0. \tag{6.11}$$

Since $\frac{\partial u}{\partial \nu} + \eta_2 u = 0$ on Γ_h^+, by virtue of (6.4), it holds that

$$a_N e^{2\mathrm{i}N\theta_0} - b_N = 0. \tag{6.12}$$

In view of (6.11) and (6.12), if $\theta_0 \neq \frac{(2m+1)\pi}{2N}$ $(m = 0, 1, \ldots, N-1)$, then

$$\begin{vmatrix} e^{2\mathrm{i}N\theta_0} & -1 \\ 1 & 1 \end{vmatrix} = e^{2\mathrm{i}N\theta_0} + 1 \neq 0,$$

which readily shows that $a_N = b_N = 0$. The proof is complete. □

7. Discussions about the condition $u(\mathbf{0}) = 0$

We recall a critical condition $u(\mathbf{0}) = 0$ that was used in Theorems 3.1, 3.3 and 3.6 in Section 3. It is also noted that in other three theorems of the same section, the condition that $u(\mathbf{0}) = 0$ is always fulfilled because one of the two line segments is a nodal line there. In this section, by illustrating with several specific examples, we show that the condition $u(\mathbf{0}) = 0$ can be fulfilled in certain scenarios in Theorems 3.1, 3.3 and 3.6, if one imposes certain generic conditions on the boundary parameters C_i, the intersecting angle $\alpha \cdot \pi$ and the eigenvalue λ.

It is stated in the introduction that one of the main motivations of our study in this work is the unique identifiability in inverse scattering problems. As we will see in the next section, we are able to develop a powerful mathematical strategy so that this condition is always fulfilled by making use of a linear combination of two eigenfunctions.

Proposition 7.1. *Let u be a Laplacian eigenfunction to (1.1), with its Fourier series given by (2.14). Suppose that there are two generalized singular lines Γ_h^+ and Γ_h^- from $\mathcal{M}_\Omega^\lambda$ such that (3.2) and (3.3) hold. Assume that $\eta_1 \equiv C_1$ and $\eta_2 \equiv C_2$ for two constants C_1 and C_2. Then if $\alpha = 1$ and $C_1 \neq C_2$, the Laplacian eigenfunction u fulfills $u(\mathbf{0}) = 0$. If $\alpha \neq 1$, two coefficients a_1 and b_1 in (2.14) can be expressed explicitly by*

$$a_1 = \frac{1}{\sqrt{\lambda}\sin\theta_0}(C_1 e^{-\mathrm{i}\theta_0} + C_2)u(\mathbf{0}), \quad b_1 = \frac{1}{\sqrt{\lambda}\sin\theta_0}(C_1 e^{\mathrm{i}\theta_0} + C_2)u(\mathbf{0}). \tag{7.1}$$

Proof. Recall the Laplacian eigenfunction u has the spherical wave expansion (2.14) in polar coordinates around the origin. Then we can obtain from Lemma 6.3, and noting that $\alpha = 1$ implies $\theta_0 = \pi$, the following equations

$$\begin{cases} 2C_2(a_0 + b_0) - \mathrm{i}\sqrt{\lambda}(a_1 - b_1) = 0, \\ 2C_1(a_0 + b_0) - \mathrm{i}\sqrt{\lambda}(a_1 - b_1) = 0. \end{cases}$$

Noting that $a_0 + b_0 = u(\mathbf{0})$ and using the assumption $C_1 \neq C_2$, we can derive from the above equations that $u(\mathbf{0}) = 0$. And (7.1) follows readily from (6.5) and (6.6) if $\alpha \neq 1$. □

Proposition 7.2. *Let u be a Laplacian eigenfunction to (1.1). Suppose that there are two generalized singular lines Γ_h^+ and Γ_h^- from $\mathcal{M}_\Omega^\lambda$ such that (3.2) and (3.3) hold. Assume that $\eta_1 \equiv C_1$ and $\eta_2 \equiv C_2$ for two constants C_1 and C_2. If $\alpha \neq \frac{1}{2}$, then a_2 and b_2 in (2.14) can be expressed explicitly as*

$$\begin{aligned} a_2 &= \frac{2u(\mathbf{0})}{\lambda \sin 2\theta_0 \sin\theta_0}(C_1C_2 + C_1C_2 e^{-\mathrm{i}2\theta_0} + C_2^2\cos\theta_0 + C_1^2\cos\theta_0 e^{-\mathrm{i}2\theta_0}), \\ b_2 &= \frac{2u(\mathbf{0})}{\lambda \sin 2\theta_0 \sin\theta_0}(C_1C_2 + C_1C_2 e^{\mathrm{i}2\theta_0} + C_2^2\cos\theta_0 + C_1^2\cos\theta_0 e^{\mathrm{i}2\theta_0}). \end{aligned} \tag{7.2}$$

Proof. Substituting (7.1) into (6.5) and (6.6) and taking $n = 2$, we can obtain that

$$\begin{cases} 2C_2(a_1 e^{\mathrm{i}\theta_0} + b_1 e^{-\mathrm{i}\theta_0}) + \mathrm{i}\sqrt{\lambda}(a_2 e^{\mathrm{i}2\theta_0} - b_2 e^{-\mathrm{i}2\theta_0}) = 0, \\ 2C_1(a_1 + b_1) - \mathrm{i}\sqrt{\lambda}(a_2 - b_2) = 0. \end{cases} \tag{7.3}$$

After rearranging the terms, (7.3) can be further simplified as

$$\begin{cases} a_2 e^{\mathrm{i}2\theta_0} - b_2 e^{-\mathrm{i}2\theta_0} = \frac{2C_2\mathrm{i}}{\sqrt{\lambda}}(a_1 e^{\mathrm{i}\theta_0} + b_1 e^{-\mathrm{i}\theta_0}), \\ a_2 - b_2 = -\frac{2C_1\mathrm{i}}{\sqrt{\lambda}}(a_1 + b_1). \end{cases} \tag{7.4}$$

Substituting (7.1) into (7.4), then we can derive by direct calculations (7.2) for the explicit expressions of a_2 and b_2. □

Proposition 7.3. *Let u be a Laplacian eigenfunction to (1.1). Suppose that there are two generalized singular lines Γ_h^+ and Γ_h^- from $\mathcal{M}_\Omega^\lambda$ such that (3.2) and (3.3) hold. Assume that $\eta_1 \equiv C_1$ and $\eta_2 \equiv C_2$ for two constants C_1 and C_2. Then if $\alpha = \frac{1}{3}$, $C_1 \neq C_2$ and*

$$1 + \frac{4}{3\lambda}\left(C_1^2 + C_1C_2 + C_2^2\right) \neq 0, \tag{7.5}$$

the Laplacian eigenfunction u fulfills $u(\mathbf{0}) = 0$. Furthermore, if $\alpha \neq \frac{1}{3}$, then a_3 and b_3 in (2.14) can be expressed explicitly as

$$a_3 = \frac{1}{\sqrt{\lambda}\sin 3\theta_0}(B_1 - B_2 e^{-\mathrm{i}3\theta_0}), \quad b_3 = \frac{1}{\sqrt{\lambda}\sin 3\theta_0}(B_1 - B_2 e^{\mathrm{i}3\theta_0}), \tag{7.6}$$

where B_1 and B_2 are given by

$$B_1 = \mathrm{i}\sqrt{\lambda}(a_1 e^{\mathrm{i}\theta_0} - b_1 e^{-\mathrm{i}\theta_0}) + C_2(a_2 e^{\mathrm{i}2\theta_0} + b_2 e^{-\mathrm{i}2\theta_0}) + C_2 u(\mathbf{0}), \tag{7.7}$$

$$B_2 = \mathrm{i}\sqrt{\lambda}(a_1 - b_1) - C_1(a_2 + b_2) - C_1 u(\mathbf{0}), \tag{7.8}$$

with (a_1, b_1) *and* (a_2, b_2) *defined in* (7.1) *and* (7.2), *respectively.*

Proof. Recall (6.7) and (6.8). Comparing the coefficients of r^2 on the both sides of these two equations, we can derive that

$$\mathrm{i}\sqrt{\lambda}(a_1 e^{\mathrm{i}\theta_0} - b_1 e^{-\mathrm{i}\theta_0}) + \frac{1}{2}\mathrm{i}\sqrt{\lambda}(a_3 e^{\mathrm{i}3\theta_0} - b_3 e^{-\mathrm{i}3\theta_0}) + C_2(a_2 e^{\mathrm{i}2\theta_0} + b_2 e^{-\mathrm{i}2\theta_0}) + C_2(a_0 + b_0) = 0 \quad \text{on } \Gamma_h^+, \tag{7.9}$$

and

$$\mathrm{i}\sqrt{\lambda}(a_1 - b_1) + \frac{1}{2}\mathrm{i}\sqrt{\lambda}(a_3 - b_3) - C_1(a_2 + b_2) - C_1(a_0 + b_0) = 0 \quad \text{on } \Gamma_h^-. \tag{7.10}$$

Since $a_0 + b_0 = u(\mathbf{0})$, substituting (7.7) and (7.8) into (7.9) and (7.10), we can further obtain that

$$a_3 e^{\mathrm{i}3\theta_0} - b_3 e^{-\mathrm{i}3\theta_0} = \frac{2\mathrm{i}B_1}{\sqrt{\lambda}}, \quad a_3 - b_3 = \frac{2\mathrm{i}B_2}{\sqrt{\lambda}}. \tag{7.11}$$

Since $\alpha = \frac{1}{3}$, taking $\theta_0 = \frac{\pi}{3}$ in (7.11), we have

$$a_3 - b_3 = -\frac{2\mathrm{i}B_1}{\sqrt{\lambda}} = \frac{2\mathrm{i}B_2}{\sqrt{\lambda}},$$

which indicates that

$$B_1 + B_2 = 0, \tag{7.12}$$

where

$$B_1 = \mathrm{i}\sqrt{\lambda}(a_1 e^{\mathrm{i}\frac{\pi}{3}} - b_1 e^{-\mathrm{i}\frac{\pi}{3}}) + C_2(a_2 e^{\mathrm{i}\frac{2\pi}{3}} + b_2 e^{-\mathrm{i}\frac{2\pi}{3}}) + C_2 u(\mathbf{0}), \tag{7.13}$$

and B_2 is defined in (7.8). Substituting (7.1) and (7.2) into (7.13) and (7.8), after straightforward calculations, (7.12) can be reduced to

$$(C_1 - C_2)\left[1 + \frac{4}{3\lambda}\left(C_1^2 + C_1 C_2 + C_2^2\right)\right] u(\mathbf{0}) = 0.$$

This implies $u(\mathbf{0}) = 0$ by noting that $C_1 \neq C_2$ and using (7.5). Similarly we can deduce (7.6) from (7.11) directly for $\alpha \neq 1/3$. □

We end this section with two important remarks.

Remark 7.4. By tracing the proofs of Propositions 7.1–7.3 and repeating similar arguments, we can find that under some mild assumptions on C_1, C_2, the intersecting angle $\alpha \cdot \pi$ and the eigenvalue λ, the property $u(\mathbf{0}) = 0$ still holds for the rational intersecting angle $\alpha \cdot \pi$ generically except for $\alpha = \pi/2m$, where $m = 1, 2, \cdots$. The detailed arguments are rather tedious and technical, but straightforward.

Remark 7.5. In Propositions 7.1-7.3, we studied the property $u(\mathbf{0}) = 0$ for two intersected generalized singular lines only for some conditions on C_1, C_2, the intersecting angle $\alpha \cdot \pi$ and λ. Other situations may be analysed similarly, e.g., either $C_1 = 0$ or $C_2 = 0$. But as shown in Theorem 2.4, we can not guarantee $u(\mathbf{0}) = 0$ by imposing some conditions on the intersecting angle between two intersecting singular lines.

8. Unique identifiability for inverse scattering problems

In this section, we apply the spectral results we have established in the previous sections to study a fundamental mathematical topic, i.e., the unique identifiability, in a class of physically important inverse problems. These include the inverse obstacle problem and the inverse diffraction grating problem, which are concerned with imaging the shapes of some unknown or inaccessible objects from certain wave probing data in different physical settings. These inverse scattering problems may arise from a variety of important applications such as radar, sonar and medical imaging, as well as geophysical exploration and nondestructive testing.

8.1. **Unique recovery for the inverse obstacle problem.** We first consider the inverse obstacle problem. Let $k = \omega/c \in \mathbb{R}_+$ be the wavenumber of a time harmonic wave with $\omega \in \mathbb{R}_+$ and $c \in \mathbb{R}_+$, respectively, signifying the frequency and sound speed. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz-boundary $\partial\Omega$ and a connected complement $\mathbb{R}^2\backslash\overline{\Omega}$. Furthermore, let the incident field u^i be a plane wave of the form

$$u^i := \ u^i(\mathbf{x}; k, \mathbf{d}) = e^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}}, \quad \mathbf{x} \in \mathbb{R}^2\,, \tag{8.1}$$

where $\mathbf{d} \in \mathbb{S}^1$ denotes the incident direction of the impinging wave and $\mathbb{S}^1 := \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| = 1\}$ is the unit circle in $\mathbb{R}^2$. Physically, Ω is an impenetrable obstacle that is unknown or inaccessible, and u^i signifies the detecting wave field that is used for probing the obstacle. The presence of the obstacle interrupts the propagation of the incident wave, and generates the so-called scattered wave field u^s. Let $u := u^i + u^s$ be the resulting total wave field, then the forward scattering problem can be described by the following Helmholtz system:

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in} \quad \mathbb{R}^2\backslash\overline{\Omega}, \\ u = u^i + u^s & \text{in} \quad \mathbb{R}^2, \\ \mathcal{B}(u) = 0 & \text{on} \quad \partial\Omega, \\ \lim\limits_{r\to\infty} r^{\frac{1}{2}}\left(\dfrac{\partial u^s}{\partial r} - \mathrm{i}ku^s\right) = 0. \end{cases} \tag{8.2}$$

The limiting equation above is known as the Sommerfeld radiation condition which holds uniformly in $\hat{\mathbf{x}} := \mathbf{x}/|\mathbf{x}| \in \mathbb{S}^1$ and characterizes the outgoing nature of the scattered wave field u^s. The boundary operator $\mathcal{B}$ could be Dirichlet type, $\mathcal{B}(u) = u$; or Neumann type, $\mathcal{B}(u) = \partial_\nu u$; or Robin type, $\mathcal{B}(u) = \partial_\nu u + \eta u$, corresponding to that Ω is a sound-soft, sound-hard or impedance obstacle, respectively. Here ν denotes the exterior unit normal vector to $\partial\Omega$ and $\eta \in L^\infty(\partial\Omega)$ signifies a boundary impedance parameter. It is required that $\Re\eta \geq 0$ and $\Im\eta \geq 0$. In what follows, we formally take $u = 0$ on $\partial\Omega$ as $\partial_\nu u + \eta u = 0$ on $\partial\Omega$ with $\eta = +\infty$. In doing so, we can unify all three boundary conditions as the generalized impedance boundary condition:

$$\mathcal{B}(u) = \partial_\nu u + \eta u = 0 \quad \text{on} \quad \partial\Omega, \tag{8.3}$$

where η could be ∞, corresponding to a sound-soft obstacle. The forward scattering problem (8.2) is well understood [16, 53] and there exists a unique solution $u \in H^1_{loc}(\mathbb{R}^2\backslash\overline{\Omega})$ that admits the following asymptotic expansion:

$$u^s(\mathbf{x}, \mathbf{d}, k) = \frac{e^{\mathrm{i}kr}}{r^{1/2}} u_\infty(\hat{\mathbf{x}}; k, \mathbf{d}) + \mathcal{O}\left(\frac{1}{r^{3/2}}\right) \quad \text{as } r \to \infty \tag{8.4}$$

which holds uniformly with respect to all directions $\hat{\mathbf{x}} := \mathbf{x}/|\mathbf{x}| \in \mathbb{S}^1$. The complex valued function u_∞ in (8.4) defined over the unit sphere $\mathbb{S}^1$ is known as the far-field pattern with $\hat{\mathbf{x}} \in \mathbb{S}^1$ signifying the observation direction. *The inverse obstacle scattering problem* is to recover Ω by using the knowledge of the far-field pattern $u_\infty(\hat{\mathbf{x}}, \mathbf{d}, k)$. By introducing an operator $\mathcal{F}$ which sends the obstacle to the corresponding far-field pattern through the Helmholtz system (8.2), the inverse obstacle problem can be formulated as the following abstract operator equation:

$$\mathcal{F}(\Omega, \eta) = u_\infty(\hat{\mathbf{x}}; k, \mathbf{d}), \tag{8.5}$$

where $\mathcal{F}$ is defined by the forward obstacle scattering system, and is nonlinear. That is, one intends to determine (Ω, η) from the knowledge of $u_\infty(\hat{\mathbf{x}}; k, \mathbf{d})$.

A primary issue for the inverse obstacle problem (8.5) is the unique identifiability, which is concerned with the sufficient conditions such that the correspondence between Ω and u_∞ is one-to-one. There is a widespread belief that one can establish uniqueness for (8.5) by a single or at most finitely many far-field patterns. We remark that by a single far-field pattern we mean that $u_\infty(\hat{\mathbf{x}}, k, \mathbf{d})$ is collected for all $\hat{\mathbf{x}} \in \mathbb{S}^1$, but is associated with a fixed incident $e^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}}$. Phrased in the geometric term, it states that the analytic function u_∞ on the unit sphere associated with at most finitely many k and d can supply a global parameterization of a generic domain Ω. This problem is known as the *Schiffer problem* in the inverse scattering community. It is named after M. Schiffer for his pioneering contribution around 1960 which is actually appeared as a private communication in the monograph by Lax and Phillips [45]. There is a long and colourful history on the study of the Schiffer problem, and we refer to a recent survey paper by Colton and Kress [17] which contains an excellent account of the historical development of this problem.

Recent progress on the Schiffer problem is made on general polyhedral obstacles in $\mathbb{R}^n$, $n \geq 2$. Uniqueness and stability results by using a finite number of far-field patterns can be found in [4, 15, 48–51]. The major idea is to make use of the reflection principle for the Laplacian eigenfunction to propagate the so-called Dirichlet or Neumann hyperplanes. In the two-dimensional case, the Dirichlet and Neumann hyperplanes are actually the nodal and singular lines introduced in the present paper. In [51], two of the authors of the present paper made an effort to answer the unique determination issue for impedance-type obstacles but gave only a partial solution to this fundamental problem. In this section, we develop a completely new approach that is able to provide a solution to this inverse obstacle problem in two dimensions, and the approach is uniform to sound-soft, sound-hard and impedance type obstacles.The new approach is completely local, and enables us to show in a rather general scenario that one can determine an impedance obstacle as well as its surface impedance by at most two far-field patterns.

Consider an obstacle Ω associated with the generalized impedance boundary condition (8.33). It is called an admissible polygonal obstacle if $\Omega \subset \mathbb{R}^2$ is an open polygon, and on each edge of $\partial\Omega$, η is either a constant (possibly zero) or ∞. That is, each edge $\mathcal{K}$ of an admissible polygonal obstacle is either sound-soft ($\eta \equiv \infty$ on $\mathcal{K}$), or sound-hard ($\eta \equiv 0$ on $\mathcal{K}$), or impedance-type (η is a constant on $\mathcal{K}$). It is emphasized that η may take different values on different edges of $\partial\Omega$. We write (Ω, η) to signify an admissible polygonal obstacle.

Definition 8.1. Let (Ω, η) be an admissible polygonal obstacle. If all the angles of its corners are irrational, then it is said to be an *irrational obstacle*. If there is a corner angle of Ω is rational, then it is called a *rational obstacle*. The smallest degree of the rational corner angles of Ω (cf. Definition 1.2) is referred to as the *rational degree* of Ω.

It is easy to see that for a rational polygonal obstacle Ω in Definition 8.1, the rational degree of Ω is at least 2.

Definition 8.2. Ω is said to be an admissible complex polygonal obstacle if it consists of finitely many admissible polygonal obstacles. That is,

$$(\Omega, \eta) = \bigcup_{j=1}^{l} (\Omega_j, \eta_j), \tag{8.6}$$

where $l \in \mathbb{N}$ and each (Ω_j, η_j) is an admissible polygonal obstacle. Here, we define

$$\eta = \sum_{j=1}^{l} \eta_j \chi_{\partial\Omega_j}. \tag{8.7}$$

Moreover, Ω is said to be irrational if all of its component polygonal obstacles are irrational, otherwise it is said to be rational. For the latter case, the smallest degree among all the degrees of its rational components is defined to be the degree of the complex obstacle Ω.

Next, we first consider the determination of an admissible complex irrational polygonal obstacle by at most two far-field patterns. We have the following local uniqueness result.

Theorem 8.3. *Let (Ω, η) and $(\widetilde{\Omega}, \widetilde{\eta})$ be two admissible complex irrational obstacles. Let $k \in \mathbb{R}_+$ be fixed and $\mathbf{d}_\ell$, $\ell = 1, 2$ be two distinct incident directions from $\mathbb{S}^1$. Let $\mathbf{G}$ denote the unbounded connected component of $\mathbb{R}^2 \backslash \overline{(\Omega \cup \widetilde{\Omega})}$. Let u_∞ and $\widetilde{u}_\infty$ be, respectively, the far-field patterns associated with (Ω, η) and $(\widetilde{\Omega}, \widetilde{\eta})$. If*

$$u_\infty(\hat{\mathbf{x}}, \mathbf{d}_\ell) = \widetilde{u}_\infty(\hat{\mathbf{x}}, \mathbf{d}_\ell), \quad \hat{\mathbf{x}} \in \mathbb{S}^1, \ell = 1, 2, \tag{8.8}$$

then one has that

$$\left(\partial\Omega \backslash \partial\overline{\widetilde{\Omega}}\right) \cup \left(\partial\widetilde{\Omega} \backslash \partial\overline{\Omega}\right)$$

cannot have a corner on $\partial\mathbf{G}$.

Proof. We prove the theorem by contradiction. Assume (8.8) holds but $\left(\partial\Omega \backslash \partial\overline{\widetilde{\Omega}}\right) \cup \left(\partial\widetilde{\Omega} \backslash \partial\overline{\Omega}\right)$ has a corner $\mathbf{x}_c$ on $\partial\mathbf{G}$. Clearly, $\mathbf{x}_c$ is either a vertex of Ω or a vertex of $\widetilde{\Omega}$. Without loss of generality, we assume that $\mathbf{x}_c$ is a vertex of $\widetilde{\Omega}$. Moreover, we see that $\mathbf{x}_c$ lies outside Ω. Let $h \in \mathbb{R}_+$ be sufficiently small such that $B_h(\mathbf{x}_c) \Subset \mathbb{R}^2 \backslash \overline{\Omega}$. Moreover, since $\mathbf{x}_c$ is a vertex of $\widetilde{\Omega}$, we can assume that

$$B_h(\mathbf{x}_c) \cap \partial\widetilde{\Omega} = \Gamma_h^\pm, \tag{8.9}$$

where $\Gamma_h^\pm$ are the two line segments lying on the two edges of $\widetilde{\Omega}$ that intersect at $\mathbf{x}_c$.

Recall that $\mathbf{G}$ denotes the unbounded connected component of $\mathbb{R}^2 \backslash \overline{(\Omega \cup \widetilde{\Omega})}$. By (8.8) and the Rellich theorem (cf. [16]), we know that

$$u(\mathbf{x}; k, \mathbf{d}_\ell) = \widetilde{u}(\mathbf{x}; k, \mathbf{d}_\ell), \quad \mathbf{x} \in \mathbf{G}, \ \ell = 1, 2. \tag{8.10}$$

It is clear that $\Gamma_h^\pm \subset \partial\mathbf{G}$. Hence, by using (8.10) as well as the generalized boundary condition (8.33) on $\partial\widetilde{\Omega}$, we readily have

$$\partial_\nu u + \widetilde{\eta} u = \partial_\nu \widetilde{u} + \widetilde{\eta}\widetilde{u} = 0 \quad \text{on} \quad \Gamma_h^\pm. \tag{8.11}$$

It is also noted that in $B_h(\mathbf{x}_c)$, $-\Delta u = k^2 u$. Next, we consider two separate cases.

Case 1. Suppose that either $u(\mathbf{x}_c; k, \mathbf{d}_1)$ or $u(\mathbf{x}_c; k, \mathbf{d}_2)$ is zero. Without loss of generality, we assume that $u(\mathbf{x}_c; k, \mathbf{d}_1) = 0$. By the assumption of the theorem that $\widetilde{\Omega}$ is an irrational obstacle, we see that Γ_h^+ and Γ_h^- intersect with an irrational angle. Hence, by our results in Sections 2 and 4, one immediately has that

$$u(\mathbf{x}; k, \mathbf{d}_1) = 0 \quad \text{in} \quad B_h(\mathbf{x}_c), \tag{8.12}$$

which in turn yields by the analytic continuation that

$$u(\mathbf{x}; k, \mathbf{d}_1) = 0 \quad \text{in} \quad \mathbb{R}^2\backslash\overline{\Omega}. \tag{8.13}$$

In particular, one has from (8.13) that

$$\lim_{|\mathbf{x}|\to\infty} |u(\mathbf{x}; k, \mathbf{d}_1)| = 0. \tag{8.14}$$

But this contradicts to the fact that follows from (8.4):

$$\lim_{|\mathbf{x}|\to\infty} |u(\mathbf{x}; k, \mathbf{d}_1)| = \lim_{|\mathbf{x}|\to\infty} \left| e^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}_1} + u^s(\mathbf{x}; k, \mathbf{d}_1)\right| = 1. \tag{8.15}$$

Case 2. Suppose that both $u(\mathbf{x}_c; k, \mathbf{d}_1) \neq 0$ and $u(\mathbf{x}_c; k, \mathbf{d}_2) \neq 0$. Set

$$\alpha_1 = u(\mathbf{x}_c; k, \mathbf{d}_2) \quad \text{and} \quad \alpha_2 = -u(\mathbf{x}_c; k, \mathbf{d}_1), \tag{8.16}$$

and

$$v(\mathbf{x}) = \alpha_1 u(\mathbf{x}; k, \mathbf{d}_1) + \alpha_2 u(\mathbf{x}; k, \mathbf{d}_2), \quad \mathbf{x} \in B_h(\mathbf{x}_c). \tag{8.17}$$

Clearly, there hold

$$-\Delta v = k^2 v \quad \text{in} \quad B_h(\mathbf{x}_c); \quad \partial_\nu v + \widetilde{\eta} v = 0 \quad \text{on} \quad \Gamma_h^{\pm}. \tag{8.18}$$

Moreover, by the choice of α_1, α_2 in (8.16), one obviously has that $v(\mathbf{x}_c) = 0$. Hence, by our results in Sections 2 and 4, one immediately has that

$$v = 0 \quad \text{in} \quad B_h(\mathbf{x}_c), \tag{8.19}$$

which in turn yields by the analytic continuation that

$$\alpha_1 u(\mathbf{x}; k, \mathbf{d}_1) + \alpha_2 u(\mathbf{x}; k, \mathbf{d}_2) = 0 \quad \text{in} \quad \mathbb{R}^2\backslash\overline{\Omega}. \tag{8.20}$$

However, since $\mathbf{d}_1$ and $\mathbf{d}_2$ are distinct, we know from [16, Chapter 5] that $u(x; k, \mathbf{d}_1)$ and $u(\mathbf{x}; k, \mathbf{d}_2)$ are linearly independent in $\mathbb{R}^2\backslash\overline{\Omega}$. Therefore, one has from (8.20) that $\alpha_1 = \alpha_2 = 0$, which contracts to the assumption at the beginning that both α_1 and α_2 are nonzero. □

It is recalled that the convex hull of Ω, denoted by $\mathcal{CH}(\Omega)$, is the smallest convex set that contains Ω. As a direct consequence of Theorem 8.3, we next show that the convex hull of a complex irrational obstacle can be uniquely determined by at most two far-field measurements. Furthermore, the boundary impedance parameter η can be partially identified as well. In fact we have

Corollary 8.4. *Let (Ω, η) and $(\widetilde{\Omega}, \widetilde{\eta})$ be two admissible complex irrational obstacles. Let $k \in \mathbb{R}_+$ be fixed and $\mathbf{d}_\ell$, $\ell = 1, 2$ be two distinct incident directions from $\mathbb{S}^1$. Let $\mathbf{G}$ denote the unbounded connected component of $\mathbb{R}^2\backslash\overline{(\Omega \cup \widetilde{\Omega})}$. Let u_∞ and $\widetilde{u}_\infty$ be, respectively, the far-field patterns associated with (Ω, η) and $(\widetilde{\Omega}, \widetilde{\eta})$. If*

$$u_\infty(\hat{\mathbf{x}}, \mathbf{d}_\ell) = \widetilde{u}_\infty(\hat{\mathbf{x}}, \mathbf{d}_\ell), \quad \hat{\mathbf{x}} \in \mathbb{S}^1, \ell = 1, 2, \tag{8.21}$$

then one has that

$$\mathcal{CH}(\Omega) = \mathcal{CH}(\widetilde{\Omega}) := \Sigma, \tag{8.22}$$

and

$$\eta = \widetilde{\eta} \quad \text{on} \quad \partial\Omega \cap \partial\widetilde{\Omega} \cap \partial\Sigma. \tag{8.23}$$

Proof. From Theorem 8.3, we can immediately conclude (8.22). Next we prove (8.23). Let $\mathcal{E} \subset \partial\Omega \cap \partial\widetilde{\Omega} \cap \partial\Sigma$ be an open subset such that $\eta \neq \widetilde{\eta}$ on $\mathcal{E}$. By taking a smaller subset of $\mathcal{E}$ if necessary, we can assume that η (respectively, $\widetilde{\eta}$) is either a fixed constant or ∞ on $\mathcal{E}$. Clearly, one has $u = \widetilde{u}$ in $\mathbb{R}^2\backslash\overline{\Sigma}$. Hence, there hold that

$$\partial_\nu u + \eta u = 0, \quad \partial_\nu \widetilde{u} + \widetilde{\eta}\widetilde{u} = 0, \quad u = \widetilde{u}, \quad \partial_\nu u = \partial_\nu \widetilde{u} \quad \text{on} \quad \mathcal{E}. \tag{8.24}$$

By direct verification, one can show that

$$u = \partial_\nu u = 0 \quad \text{on} \quad \mathcal{E}, \tag{8.25}$$

which in turn yields by the Holmgren uniqueness result (cf. [50]) that $u = 0$ in $\mathbb{R}^2\backslash\overline{\Omega}$. Hence, we arrive at the same contradiction as that in (8.15), which implies (8.23). □

Remark 8.5. Let $\mathcal{V}(\Omega)$ and $\mathcal{V}(\mathcal{CH}(\Omega))$ denote, respectively, the sets of vertices of Ω and $\mathcal{CH}(\Omega)$. It is known that $\mathcal{V}(\mathcal{CH}(\Omega)) \subset \mathcal{V}(\Omega)$. Theorem 8.3 states that if the corner angle of the polygon Ω at any vertex in $\mathcal{V}(\Omega)$ is irrational, then $\mathcal{CH}(\Omega)$ can be uniquely determined by two far-field patterns. Indeed, from the proof of Theorem 8.3, we see that this requirement can be relaxed to that the corner angle of the polygon Ω at any vertex in $\mathcal{V}(\mathcal{CH}(\Omega))$ is irrational.

We proceed now to consider the unique determination of rational obstacles. Let Ω be a polygon in $\mathbb{R}^2$ and $\mathbf{x}_c$ be a vertex of Ω. In what follows, we define

$$\Omega_r(\mathbf{x}_c) = B_r(\mathbf{x}_c) \cap \mathbb{R}^2\backslash\overline{\Omega}, \quad r \in \mathbb{R}_+. \tag{8.26}$$

For a function $f \in L^2_{loc}(\mathbb{R}^2\backslash\overline{\Omega})$, we define

$$\mathcal{L}(f)(\mathbf{x}_c) := \lim_{r\to+0} \frac{1}{|\Omega_r(\mathbf{x}_c)|} \int_{\Omega_r(\mathbf{x}_c)} f(\mathbf{x})\, \mathrm{d}\mathbf{x} \tag{8.27}$$

if the limit exists. It is easy to see that if $f(\mathbf{x})$ is continuous in $\overline{\Omega_{\tau_0}(\mathbf{x}_c)}$ for a sufficiently small $\tau_0 \in \mathbb{R}_+$, then $\mathcal{L}(f)(\mathbf{x}_c) = f(\mathbf{x}_c)$.

Theorem 8.6. *Let (Ω, η) be an admissible complex rational obstacle of degree $p \geq 3$. Let $k \in \mathbb{R}_+$ be fixed and $\mathbf{d}_\ell$, $\ell = 1, 2$ be two distinct incident directions from $\mathbb{S}^1$. Set $u_\ell(\mathbf{x}) = u(\mathbf{x}; k, \mathbf{d}_\ell)$ to be the total wave fields associated with (Ω, η) and $e^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}_\ell}$, $\ell = 1, 2$, respectively. Recall that $\mathbf{G}$ denotes the unbounded connected component of $\mathbb{R}^2\backslash\overline{(\Omega \cup \widetilde{\Omega})}$. If the following condition is fulfilled,*

$$\mathcal{L}\left(u_2 \cdot \nabla u_1 - u_1 \cdot \nabla u_2\right)(\mathbf{x}_c) \neq 0, \tag{8.28}$$

where $\mathbf{x}_c$ is any vertex of Ω, then then one has that

$$\left(\partial\Omega\backslash\partial\overline{\widetilde{\Omega}}\right) \cup \left(\partial\widetilde{\Omega}\backslash\partial\overline{\Omega}\right)$$

cannot have a corner on $\partial\mathbf{G}$.

Proof. We prove the theorem by contradiction. Assume that there exists an admissible complex rational obstacle of degree $p \geq 3$, $(\widetilde{\Omega}, \widetilde{\eta})$, such that (8.8) holds but $\left(\partial\Omega\backslash\partial\overline{\widetilde{\Omega}}\right) \cup \left(\partial\widetilde{\Omega}\backslash\partial\overline{\Omega}\right)$ has a corner on $\partial\mathbf{G}$. In what follows, we adopt the same notation as those introduced in the proof of Theorem 8.3. Note that the total wave fields $\widetilde{u}_\ell$, $\ell = 1, 2$, associated with $(\widetilde{\Omega}, \widetilde{\eta})$, are also assumed to fulfill the condition (8.28).

By following a similar argument to the proof of Theorem 8.3, one can show that there exist two line segments $\Gamma_h^\pm$ in $\mathbb{R}^2\backslash\overline{\Omega}$ such that $\partial_\nu u + \widetilde{\eta} u = 0$ on $\Gamma_h^\pm$, and Γ_h^+ and Γ_h^- intersect at a point $\mathbf{x}_c$ which is a vertex of $\widetilde{\Omega}$. Using the fact that $u = \widetilde{u}$ near $\mathbf{x}_c$ and the condition (8.28) on $(\widetilde{\Omega}, \widetilde{\eta})$, we actually have

$$u(\mathbf{x}_c; \mathbf{d}_2) \cdot \nabla u(\mathbf{x}_c; \mathbf{d}_1) - u(\mathbf{x}_c; \mathbf{d}_1) \cdot \nabla u(\mathbf{x}_c; \mathbf{d}_2) \neq \mathbf{0}. \tag{8.29}$$

Clearly, (8.29) implies that $\alpha_1 := u(\mathbf{x}_c; \mathbf{d}_2)$ and $\alpha_2 := -u(\mathbf{x}_c; \mathbf{d}_1)$ cannot be identically zero. Set v to be the one introduced in (8.17), then it clearly satisfies

$$v(\mathbf{x}_c) = 0 \quad \text{and} \quad \nabla v(\mathbf{x}_c) \neq \mathbf{0}. \tag{8.30}$$

Since $\widetilde{\Omega}$ is rational of degree $p \geq 3$, we know that Γ_h^+ and Γ_h^- intersect either at an irrational angle or at a rational angle of degree $p \geq 3$. In either case, by our results in Sections 2, 3 and 4, we can see that v is vanishing at least to the second order at $\mathbf{x}_c$. Hence, there holds $\nabla v(\mathbf{x}_c) = 0$, which is a contradiction to (8.30). ☐

Similar to Corollary 8.4, as a direct consequence of Theorem 8.6, under the condition (8.28), we next show that the convex hull of a complex rational obstacle of degree $p \geq 3$ can be uniquely determined by at most two far-field measurements. Indeed we have

Corollary 8.7. *Let (Ω, η) be an admissible complex rational obstacle of degree $p \geq 3$. Let $k \in \mathbb{R}_+$ be fixed and $\mathbf{d}_\ell$, $\ell = 1, 2$ be two distinct incident directions from $\mathbb{S}^1$. Set $u_\ell(\mathbf{x}) = u(\mathbf{x}; k, \mathbf{d}_\ell)$ to be the total wave fields associated with (Ω, η) and $e^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}_\ell}$, $\ell = 1, 2$, respectively. If (8.28) is fulfilled, then $\mathcal{CH}(\Omega)$ is uniquely determined by $u_\infty(\hat{\mathbf{x}}, \mathbf{d}_\ell)$, $\ell = 1, 2$. Similar to Corollary 8.4, the boundary impedance parameter η can be partially identified as well.*

Remark 8.8. As mentioned earlier that a general rational obstacle is at least of order 2. By Remark 3.8, we can easily extend the proof of Theorem 8.6 to cover the general case that $p = 2$. However, as discussed in Remark 3.8, we need to exclude the case that $\eta \equiv \infty$ and η is a finite number (possibly being zero), respectively on the two intersecting line segments $\Gamma_h^\pm$ (as appeared in the proof of Theorem 8.6).

Remark 8.9. Similar to Remark 8.5, the condition (8.28) can be relaxed to hold only at any vertex in $\mathcal{V}(\mathcal{CH}(\Omega))$ in Theorem 8.6 and Corollary 8.7. Furthermore, since in the proof of Theorem 8.6, we only make use of the vanishing up to the second order. By our results in Section 5, we know that Theorem 8.6 actually holds for a more general case where the surface impedance η can be a C^1 function.

It would be interesting to investigate the sufficient conditions for (8.28) to hold. From a practical point of view, the condition (8.28) depends on the a-priori knowledge of the underlying obstacle as well as the choice of the incident waves. As an illustrating scenario, suppose that the obstacle Ω is sufficiently small compared with the wavelength, namely $k \cdot \mathrm{diam}(\Omega) \ll 1$. Then from a physical viewpoint, the scattered wave field due to the obstacle is of a much smaller magnitude than that of the incident field, and the incident plane wave dominates in the total wave field $u = u^i + u^s$. In such a case, one can verify that the condition (8.28) is fulfilled in the setup described in Theorem 8.6 (thanks to the fact that the condition is actually satisfied by two incident plane waves). However, we shall not explore more about this point. Finally, we also would like to point out that our arguments for the uniqueness results in Theorems 8.3 and 8.6 are "localized" around the corner point $\mathbf{x}_c$. Therefore one may consider other different types of wave incidences from the incident plane wave (8.1), e.g., the point source of the form,

$$u^i(\mathbf{x}; \mathbf{z}_0) = H_0^1(k|\mathbf{x} - \mathbf{z}_0|), \ \mathbf{x}, \mathbf{z}_0 \in \mathbb{R}^2,$$

where H_0^1 is the zeroth-order Hankel function of the first kind, and $\mathbf{z}_0$ signifies the location of the source $u^i(\mathbf{x}, \mathbf{z}_0)$. $u^i(\mathbf{x}; \mathbf{z}_0)$ blows up at the point $\mathbf{z}_0$. By direct verifications, we can show that both the uniqueness results in Theorem 8.3 and 8.6 still hold for this point source incidence.

8.2. **Unique recovery for the inverse diffraction grating problem.** In this subsection, we consider the unique recovery for the inverse diffraction grating problem. First we give a brief review of the basic mathematical model for this inverse problem. Let the profile of a diffraction grating be described by the curve

$$\Lambda_f = \{(x_1, x_2) \in \mathbb{R}^2;\ x_2 = f(x_1)\}, \tag{8.31}$$

where f is a periodic Lipschitz function with period 2π. Let

$$\Omega_f = \{\mathbf{x} \in \mathbb{R}^2; x_2 > f(x_1), x_1 \in \mathbb{R}\}$$

be filled with a material whose index of refraction (or wave number) k is a positive constant. Suppose further that the incident wave given by

$$u^i(\mathbf{x}; k, \mathbf{d}) = e^{\mathrm{i}k\mathbf{d}\cdot\mathbf{x}}, \quad \mathbf{d} = (\sin\theta, -\cos\theta)^\top, \quad \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

propagates to Λ_f from the top. Then the total wave satisfies the following Helmholtz system:

$$\Delta u + k^2 u = 0 \quad \text{in } \Omega_f; \quad \mathcal{B}(u)\big|_{\Lambda_f} = 0 \quad \text{on } \Lambda_f, \tag{8.32}$$

with the generalized impedance boundary condition

$$\mathcal{B}(u) = \partial_\nu u + \eta u = 0 \quad \text{on } \partial\Omega, \tag{8.33}$$

where η can be ∞ or 0, corresponding to a sound-soft or sound-hard grating, respectively.

To achieve uniqueness of (8.32), the total wave field u should be α-quasiperiodic in the x_1-direction, with $\alpha = k\sin\theta$, which means that

$$u(x_1 + 2\pi, x_2) = e^{2\mathrm{i}\alpha\pi} \cdot u(x_1, x_2),$$

and the scattered field u^s satisfies the Rayleigh expansion (cf. [55,56]):

$$u^s(\mathbf{x}; k, \mathbf{d}) = \sum_{n=-\infty}^{+\infty} u_n e^{\mathrm{i}\xi_n(\theta)\cdot\mathbf{x}} \quad \text{for} \quad x_2 > \max_{x_1\in[0,2\pi]} f(x_1), \tag{8.34}$$

where $u_n \in \mathbb{C}(n \in \mathbb{Z})$ are called the Rayleigh coefficient of u^s, and

$$\begin{aligned} \xi_n(\theta) &= (\alpha_n(\theta), \beta_n(\theta))^\top, \quad \alpha_n(\theta) = n + k\sin\theta, \\ \beta_n(\theta) &= \begin{cases} \sqrt{k^2 - \alpha_n^2(\theta)}, & \text{if } |\alpha_n(\theta)| \le k \\ \mathrm{i}\sqrt{\alpha_n^2(\theta) - k^2}, & \text{if } |\alpha_n(\theta)| > k \end{cases}. \end{aligned} \tag{8.35}$$

The existence and uniqueness of the α-quasiperiodic solution to (8.32) for the sound-soft or impedance boundary condition with $\eta \in \mathbb{C}$ being a constant satisfying $\Im(\eta) > 0$ can be found in [2,14,37,38]. It should be pointed out that the uniqueness of the direct scattering problem associated with the sound-hard condition is not always true (see [35]). In our subsequent study, we assume the well-posedness of the forward scattering problem and focus on the study of the inverse grating problem.

Introduce a measurement boundary as

$$\Gamma_b := \{(x_1, b) \in \mathbb{R}^2;\ 0 \le x_1 \le 2\pi,\, b > \max_{x_1\in[0,2\pi]} |f(x_1)|\}.$$

The inverse diffraction grating problem is to determine (Λ_f, η) from the knowledge of $u(\mathbf{x}|_{\Gamma_b}; k, \mathbf{d})$, and can be formulated as the operator equation:

$$\mathcal{F}(\Lambda_f, \eta) = u(\mathbf{x}; k, \mathbf{d}), \quad \mathbf{x} \in \Gamma_b,$$

where $\mathcal{F}$ is defined by the forward diffraction scattering system, and is nonlinear.

The unique recovery result on the inverse diffraction grating problem with the sound-soft boundary condition by a finite number of incident plane waves can be found in [38,39]. But the unique identifiability still open for the impedance or generalized impedance cases, and will be the focus of the remaining task in this work. To do so, we propose the following admissible polygonal gratings associated with the inverse diffraction grating problem.

Definition 8.10. Let (Λ_f, η) be a periodic grating as described in (8.31). Suppose there is a partition, $[0, 2\pi] = \cup_{i=1}^{\ell}[a_i, a_{i+1}]$ with $a_i < a_{i+1}$, $a_1 = 0$ and $a_{\ell+1} = 2\pi$. If on each piece $[a_i, a_{i+1}]$, $1 \leq i \leq \ell$, f is a linear polynomial and η is either a constant (possibly zero) or ∞, then (Λ_f, η) is said to be an admissible polygonal grating.

Definition 8.11. Let (Λ_f, η) be an admissible polygonal grating. Let Γ^+ and Γ^- be two adjacent pieces of Λ_f. The intersecting point of Γ^+ and Γ^- is called a corner point of Λ_f, and $\angle(\Gamma^+, \Gamma^-)$ is called a corner angle. If all the corner angles of Λ_f are irrational, then it is said to be an *irrational polygonal grating*. If a corner angle of Λ_f is rational, it is called a *rational polygonal grating*. The smallest degree of the rational corner angles of Λ_f is referred to as the *rational degree* of Λ_f.

Clearly for a rational polygonal grating Λ_f in Definition 8.11, the rational degree of Λ_f is at least 2. Next, we establish our uniqueness result in determining an admissible irrational polygonal grating by at most two incident waves. We first present a useful lemma, whose proof follows from a completely similar argument to that of [16, Theorem 5.1].

Lemma 8.12. *Let $\xi_\ell \in \mathbb{R}^2$, $\ell = 1, \ldots, n$, be n vectors which are distinct from each other, D be an open set in $\mathbb{R}^2$. Then all the functions in the following set are linearly independent:*

$$\{e^{\mathrm{i}\xi_\ell\cdot\mathbf{x}};\ \mathbf{x} \in D,\quad \ell = 1, 2, \ldots, n\}$$

Theorem 8.13. *Let (Λ_f, η) and $(\Lambda_{\widetilde{f}}, \widetilde{\eta})$ be two admissible irrational polygonal gratings, and $\mathbf{G}$ be the unbounded connected component of $\Omega_f \cap \Omega_{\widetilde{f}}$. Let $k \in \mathbb{R}_+$ be fixed and $\mathbf{d}_\ell$, $\ell = 1, 2$ be two distinct incident directions from $\mathbb{S}^1$, with*

$$\mathbf{d}_\ell = (\sin\theta_\ell, -\cos\theta_\ell)^\top, \quad \theta_\ell \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Let $u(\mathbf{x}; k, \mathbf{d}_\ell)$ and $\widetilde{u}(\mathbf{x}; k, \mathbf{d}_\ell)$ denote the total fields associated with (Λ_f, η) and $(\Lambda_{\widetilde{f}}, \widetilde{\eta})$ respectively and let Γ_b be a measurement boundary given by

$$\Gamma_b := \left\{(x_1, b) \in \mathbb{R}^2;\ 0 \leq x_1 \leq 2\pi,\ b > \max\left\{\max_{x_1\in[0,2\pi]}|f(x_1)|,\ \max_{x_1\in[0,2\pi]}|\widetilde{f}(x_1)|\right\}\right\},$$

If it holds that

$$u(\mathbf{x}; k, \mathbf{d}_\ell) = \widetilde{u}(\mathbf{x}; k, \mathbf{d}_\ell), \quad \ell = 1, 2, \quad \mathbf{x} = (x_1, b) \in \Gamma_b, \tag{8.36}$$

then it cannot be true that there exists a corner point of Λ_f lying on $\partial\mathbf{G}\backslash\partial\Lambda_{\widetilde{f}}$, or a corner point of $\Lambda_{\widetilde{f}}$ lying on $\partial\mathbf{G}\backslash\partial\Lambda_f$.

Proof. The proof follows from a similar argument to that for Theorem 8.3, and we only sketch the necessary modifications in this new setup. By contradiction and without loss of generality, we assume that there exists a corner point $\mathbf{x}_c$ of Λ_f which lies on $\partial\mathbf{G}\backslash\Lambda_{\widetilde{f}}$.

First, by the well-posedness of the diffraction grating problem (8.32)-(8.34) as well as the unique continuation, we show that $u(\mathbf{x}; k, \mathbf{d}_\ell) = \widetilde{u}(\mathbf{x}; k, \mathbf{d}_\ell)$ for $\mathbf{x} \in \mathbf{G}$. In fact, by introducing $w(\mathbf{x}; k, \mathbf{d}_\ell) := u(\mathbf{x}; k, \mathbf{d}_\ell) - \widetilde{u}(\mathbf{x}; k, \mathbf{d}_\ell)$, $\ell = 1, 2$, we see from (8.36) that w fulfils

$$\Delta w + k^2 w = 0 \text{ in } \mathbf{U};\ w = 0 \text{ on } \Gamma_b \text{ and } w \text{ satisfies the Rayleigh expansion (8.34)},$$

where $\mathbf{U} := \mathbf{G} \cap \{\mathbf{x} \in \mathbb{R}^2; x_2 > b, x_1 \in \mathbb{R}\}$ with $\partial\mathbf{U} = \Gamma_b$. Hence, by the uniqueness of the solution to the diffraction grating problem, we readily know $w = 0$ in $\mathbf{U}$. On the other hand, since $u(\mathbf{x};k,\mathbf{d}_\ell)$ and $\widetilde{u}(\mathbf{x};k,\mathbf{d}_\ell)$ are analytic in $\mathbf{G}$, we know $w(\mathbf{x};k,\mathbf{d}_\ell)=0$ in $\mathbf{G}$ by means of the analytic continuation, that is, $u(\mathbf{x};k,\mathbf{d}_\ell)=\widetilde{u}(\mathbf{x};k,\mathbf{d}_\ell)$ for $\mathbf{x}\in\mathbf{G}$.

Next, using a similar argument to the proof of Theorem 8.3, we can prove that

$$u(\mathbf{x};k,\mathbf{d}_\ell)=0 \quad \text{or} \quad v(\mathbf{x})=0 \quad \text{for} \quad x_2>\max_{x_1\in[0,2\pi]} f(x_1),$$

where v is similarly defined to (8.17) and (8.16). Next, when $x_2>\max_{x_1\in[0,2\pi]}|f(x_1)|$, $u(\mathbf{x};k,\mathbf{d}_\ell)$ has the Rayleigh expansion (cf. [55,56]):

$$u(\mathbf{x};k,\mathbf{d}_\ell)=e^{\mathrm{i}k\mathbf{d}_\ell\cdot\mathbf{x}}+\sum_{n=-\infty}^{+\infty} u_n e^{\mathrm{i}\xi_n(\theta_\ell)\cdot\mathbf{x}} \quad \text{for} \quad x_2>\max_{x_1\in[0,2\pi]} f(x_1), \tag{8.37}$$

where $\xi_n(\theta_\ell)$, $\alpha_n(\theta_\ell)$, $\beta_n(\theta_\ell)$ are defined in (8.35). Using the definition of $\alpha_0(\theta_\ell)$ and $\beta_0(\theta_\ell)$ in (8.35), we can easily show that

$$k\mathbf{d}_\ell=(\alpha_0(\theta_\ell),-\beta_0(\theta_\ell))^\top. \tag{8.38}$$

We proceed to consider two separate cases.

Case 1. Suppose that either $u(\mathbf{x}_c;k,\mathbf{d}_1)$ or $u(\mathbf{x}_c;k,\mathbf{d}_2)$ is zero. Without loss of generality, we assume the former case. Then

$$u(\mathbf{x};k,\mathbf{d}_1)=0 \quad \text{for} \quad x_2>\max_{x_1\in[0,2\pi]} f(x_1).$$

Clearly any two vectors of $\{\xi_n(\theta_1)\mid n\in\mathbb{Z}\}$ are distinct from each other. Moreover, in view of (8.38), $k\mathbf{d}_1\notin\{\xi_n(\theta_1)\mid n\in\mathbb{Z}\}$ since $|\theta_1|<\pi/2$. In view of (8.37), from Lemma 8.12 we can arrive at a contradiction.

Case 2. Suppose that both $u(\mathbf{x}_c;k,\mathbf{d}_1)\neq 0$ and $u(\mathbf{x}_c;k,\mathbf{d}_2)\neq 0$. Then it holds that

$$\alpha_1 u(\mathbf{x};k,\mathbf{d}_1)+\alpha_2 u(\mathbf{x};k,\mathbf{d}_2)=0 \quad \text{for} \quad x_2>\max_{x_1\in[0,2\pi]} f(x_1), \tag{8.39}$$

where $\alpha_\ell\neq 0$, $\ell=1,2$, are defined in (8.16). Substituting (8.37) into (8.39), we derive that

$$\sum_{\ell=1}^{2}\alpha_\ell e^{\mathrm{i}k\mathbf{d}_\ell\cdot\mathbf{x}}+\sum_{n=-\infty}^{+\infty}\sum_{\ell=1}^{2} u_n(\theta_\ell)\alpha_\ell e^{\mathrm{i}\xi_n(\theta_\ell)\cdot\mathbf{x}}=0 \quad \text{for} \quad x_2>\max_{x_1\in[0,2\pi]} f(x_1), \tag{8.40}$$

where $u_n(\theta_\ell)\in\mathbb{C}(n\in\mathbb{Z})$ are the Rayleigh coefficients of $u^s(\mathbf{x};k,\mathbf{d}_\ell)$ associated with the incident wave $e^{\mathrm{i}k\mathbf{d}_\ell\cdot\mathbf{x}}$. Clearly, any two vectors of the set

$$\{k\mathbf{d}_1\}\bigcup\{k\mathbf{d}_2\}\bigcup\{\xi_n(\theta_1)\mid n\in\mathbb{Z}\}\bigcup\{\xi_n(\theta_2)\mid n\in\mathbb{Z}\}$$

are distinct since $|\theta_\ell|<\pi/2$ and (8.38). Using Lemma 8.12 and (8.40), we can see $\alpha_\ell=0$ for $\ell=1,2$, which is a contradiction to $\alpha_\ell\neq 0$, $\ell=1,2$. □

For the polygonal gratings, one can introduce a certain notion of "convexity" in the sense that if two such gratings are different, then their difference must contain a corner point lying outside their union. Clearly, by Theorem 8.13, if a polygonal grating is "convex", then both the grating and its surface impedance can be uniquely determined by at most two measurements.

As the result in Theorem 8.6 for the inverse obstacle problem, we may consider the unique determination of an admissible rational polygonal grating by two measurement if a similar condition to (8.28) is introduced in this new setup. In such a case, one can establish the local unique recovery result, similar to Theorem 8.13.

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