UNIQUENESS FOR TIME-DEPENDENT INVERSE PROBLEMS
WITH SINGLE DYNAMICAL DATA

IBTISSEM BEN AÏCHA[1], GUANG-HUI HU[2], MANMOHAN VASHISTH[3] AND JUN ZOU[4]

ABSTRACT. In this work, we investigate the shape identification and coefficient determination associated with two time-dependent partial differential equations in two dimensions. We consider the inverse problems of determining a convex polygonal obstacle and the coefficient appearing in the wave and Schrödinger equations from a single dynamical data. With the near-field data, we first prove that the sound speed of the wave equation together with its contrast support of convex-polygon type can be uniquely determined, then establish a uniqueness result for recovering an electric potential as well as its support appearing in the Schrödinger equation. As a consequence of these results, we demonstrate a uniqueness result for recovering the refractive index of a medium from a single far field pattern at a fixed frequency in the time-harmonic regime.

Keywords: Wave equation, Schrödinger equation, inverse problem, uniqueness, single measurement, time-domain, multi-frequency analysis, shape identification, coefficient determination.

Mathematics Subject Classifications (2010): 35R30, 35L15.

1. INTRODUCTION

This work aims at the mathematical understanding of the unique identifiability for three coefficient/obstacle inverse problems concerning the wave, the Schrödinger and the Helmholtz equations. We first introduce some notations which will be used throughout the work. For $R > 0$, we shall write $B_R := \{x \in \mathbb{R}^2 : |x| < R\}$, $\Gamma_R := \{x \in \mathbb{R}^2 : |x| = R\}$, and $\mathbb{S} := \{x \in \mathbb{R}^2 : |x| = 1\}$.

1.1. **Formulation of the inverse wave problem.** Consider the propagation of acoustic waves in an unbounded inhomogeneous background medium due to a compactly supported source term in $\mathbb{R}^2$. This can be modeled by the inhomogenous wave equation

$$(1.1) \qquad \frac{1}{c^2(x)} \frac{\partial^2 u}{\partial t^2} = \Delta u + f(x)g(t) \quad \text{in} \quad \mathbb{R}^2 \times \mathbb{R}_+,$$

together with the initial conditions

$$(1.2) \qquad u(x,0) = \partial_t u(x,0) = 0 \quad \text{on} \quad \mathbb{R}^2.$$

Throughout this manuscript, we will assume that $c \in L^\infty(\mathbb{R}^2)$, $c(x) \geq c_0 > 0$ for all $x \in \mathbb{R}^2$ and that $c \equiv 1$ in $\mathbb{R}^2 \backslash \overline{D}$, where $D \subset \mathbb{R}^2$ is an inhomogeneous medium such that $\mathbb{R}^2 \backslash \overline{D}$ is connected. Now from [23], we have that for $f \in L^2(\mathbb{R}^2)$ and $g \in L^1(\mathbb{R}_+)$, the initial value problem (1.1)-(1.2) admits a unique solution $u \in C\left(\mathbb{R}_+; H^1(\mathbb{R}^2)\right) \cap C^1\left(\mathbb{R}_+; L^2(\mathbb{R}^2)\right)$ (see also Theorem 8.1 in [27]). In the first part of the paper, we study the inverse problem of determining the sound speed $c|_D$ in Equation (1.1) and the shape of the unknown inhomogeneous medium, namely, D=supp$(1 - c(x))$, from the knowledge of a single dynamical data. More precisely, we prove that the sound speed and its contrast support are unique, under the knowledge of the solution u measured on $\Gamma_R \times \mathbb{R}_+$, provided they satisfy some a priori conditions (see Section 2 below).

1.2. **Formulation of the inverse Schrödinger problem.** The second part of the work deals with two inverse problems arising from the Schrödinger equation defined in $\mathbb{R}^2 \times \mathbb{R}_+$. More precisely, we intend to uniquely determine a compactly supported electric potential as well as its convex polygonal support, from the knowledge of the boundary observation. We shall consider the time-dependent Schrödinger equation

$$(1.3) \qquad (i\partial_t u + \Delta + q(x))u(x,t) = 0 \qquad \text{in } \mathbb{R}^2 \times \mathbb{R}_+,$$

together with the initial condition

$$(1.4) \qquad u(x,0) = u_0 \qquad \text{in } \mathbb{R}^2,$$

where $u_0 \in H^2(\mathbb{R}^2)$ and $q \in C(\mathbb{R}^2)$ is the compactly supported electric potential that is assumed to be a real-valued function. According to [33], the initial value problem (1.3)-(1.4) is well posed, with a unique solution $u \in C(\mathbb{R}_+; H^2(\mathbb{R}^2))$, also satisfying the energy identity

$$(1.5) \qquad \|u(\cdot,t)\|_{L^2(\mathbb{R}^2)} = \|u_0\|_{L^2(\mathbb{R}^2)} \quad \text{for all} \quad t > 0.$$

Our goal in the second part of the paper is to deal with the inverse problem of determining the electric potential q as well as its polygonal support $D := \operatorname{supp}(q) \subset B_R$ from the knowledge of the boundary measurement $u_{|\Gamma_R \times \mathbb{R}_+}$.

1.3. **Formulation of the inverse Helmholtz problem.** Consider the time-harmonic medium scattering problem for the total field $v = v^{in} + v^{sc}$:

$$(1.6) \qquad \Delta v + k^2 n(x)v = 0 \quad \text{in} \quad \mathbb{R}^2,$$

where $k > 0$ is the wave number of the homogeneous isotropic background medium, and the refractive index function n is supposed to satisfy $n \equiv 1$ in $|x| > R$ for some $R > 0$. The incident wave v^{in} is allowed to be either a plane wave of the form

$$v^{in}(x) = e^{ikx \cdot d}, \quad d = (\cos\theta, \sin\theta)^T \in \mathbb{S}$$

where d is a fixed incident direction d, and $\theta \in [0, 2\pi)$ is the incident angle, or a point source wave emitting from the fixed source position z, taking the form

$$v^{in}(x) = \frac{i}{4} H_0^{(1)}(k|x - z|), \quad x \neq z,$$

where $H_0^{(1)}(\cdot)$ is the Hankel function of the first kind of order zero. The scattered field v^{sc} is required to fulfill the Sommerfeld radiation condition

$$\sqrt{r} \left(\partial_r v^{sc} - ikv^{sc} \right) \to 0 \quad \text{as} \quad r = |x| \to \infty$$

uniformly in all directions $\widehat{x} := x/r$, leading to the far-field pattern v^∞ in the asymptotic behavior

$$v^{sc}(x) = \frac{e^{ikr}}{\sqrt{r}} \left(v^\infty(\widehat{x}) + O\left(\frac{1}{r}\right) \right) \quad \text{as} \quad r \to \infty.$$

Our aim in this part is to consider the inverse problem of determining the refractive index n and the shape of $\operatorname{supp}(1-n)$ from the knowledge of $v^\infty(\widehat{x})$ for $\widehat{x} \in \Gamma_1$. We prove that an admissible set of n and $\operatorname{supp}(1-n)$ can be determined uniquely from the far field pattern v^∞ of a single incident plane wave or a point source.

1.4. **Literature review.** Inverse problems of partial differential equations (PDEs) is a very broad field of various research directions, among which the inverse coefficients and/or obstacles problems have recently attracted a tremendous attention, particularly from the mathematical point of view. Inverse coefficient problems for time-dependent PDEs were widely studied, but still not much progress has been made for inverse transmission problems of recovering interfaces.

This paper will mainly be concerned with the simultaneous identification of both obstacles and coefficients appearing in some PDEs. More precisely, our main focus is on the uniqueness issue for time-dependent problems with a single dynamical data. There is a wide mathematical literature on this topic, but it is mostly concerned with the knowledge of sufficiently large measurement data; for example, the entire Dirichlet-to-Neumann map. We shall consider the cases that are very important in applications, namely, only one single dynamical data is available, and focus on the determination of both the geometrical shape of convex penetrable scatterers and some coefficients appearing in the wave, the dynamical Schrödinger and the Helmholtz equations.

The wave model (1.1)-(1.2) may be used in many applications, such as the thermoacoustic (TAT) and photoacoustic (PAT) tomography; see, e.g., [31, 32, 46, 47, 49] and references therein. There have been some results about the determination of the sound speed or the source term in the wave equation from a single measurement data. A uniqueness result was studied in [17] for determining the constant sound speed and in [45] for the recovery of the source term or the sound speed provided that one of them is known. The unique recovery of both the sound speed and the source term was considered in [14] for the case when the sound speed is radial. Recently, the result of [34] was improved in [30] to more general coefficients, indicating that the sound speed can be recovered from a single measurement provided it is a harmonic function. We consider in this work the uniqueness result for simultaneously recovering both the sound speed and its convex polygonal support from a single dynamical data. For more related works, we refer to [1, 18, 19, 37, 38, 39, 40, 41, 42, 48].

In regard with the determination of potentials appearing in the dynamical Schrödinger equation, there are many studies in the literature, but they are mostly concerned with the determination of the potentials from infinitely many boundary measurements. We refer to, e.g., [2, 3, 4, 6, 7, 9, 10, 11, 50, 51] and references therein. To the best of our knowledge, we are not aware of any existing studies of the determination of potentials from a single dynamical data, and this is one of the main motivations of this work. In the frequency domain, the determination of potentials appearing in the Schrödinger equation was studied in [44], where a uniqueness result was established for recovering the small potentials from the knowledge of the scattering amplitude.

All the aforementioned studies are concerned only with inverse coefficient problems. For determining the shape of a sound-soft obstacle, the recent development in the time domain can be found in [8], in which an inverse obstacle problem for an acoustic transient wave equation is considered. The authors in [8] proved a uniqueness result for the determination of a sound-soft obstacle from the lateral Cauchy data given on a subboundary of an open bounded domain. In [28] the uniqueness and stability issues for recovering penetrable or impenetrable obstacles from various boundary data were considered. We shall restrict our consideration in this work to determine the shape of a convex polygonal penetrable scatterer using a single dynamical data. By means of the data along all the time, we prove in the theory that the Laplace transform can be applied to the measurement data and the time-dependent inverse problem is thus transformed to an equivalent problem in the Fourier-Laplace domain with many parameters (frequencies). To apply the recently developed shape identification theory [13, 22] to inverse coefficient problems in a corner domain, we shall show that there exists at least one parameter (frequency) for which the Laplace-transformed solution cannot vanish at the corner points (see Lemma 4.1, also [24]). We will then prove via asymptotic analysis that this parameter in two dimensions can be taken sufficiently small for wave equations and sufficiently

large for the Schrödinger equation (see Lemmas 4.2 and 5.1). We refer to [5, 12, 13, 21, 20, 29, 16, 26, 25, 24, 22, 35] for related works on target identification with a single measurement data in the stationary case. Finally, let us mention that although our results are restricted to contrast support of convex-polygon type, they remain valid for piecewise analytic interfaces with arbitrarily weakly singular points under additional assumptions (see e.g.,[36]).

The rest of the paper is organized as follow. In Sections 2 and 3, we state our main results and present some preliminary results that will be used to prove the main theorems of the work. In Sections 4, 5 and 6 we prove our main results, Theorems 2.1, 2.2 and 2.3 respectively. Some concluding remarks and open problems are summarized in Section 7.

2. STATEMENT OF THE MAIN RESULTS

In order to state our main results, we first introduce some notations. For some $R > 0$, let $D_i \subset B_R$ for $i = 1, 2$ be two convex polygons. For any corner point $\mathcal{O}$ of ∂D_i for $i = 1, 2$, we denote by

$$B_\epsilon(\mathcal{O}) := \{x \in \mathbb{R}^2 : |x - \mathcal{O}| < \epsilon\},$$

for some $\epsilon > 0$. We denote by E a subset of B_R satisfying $E \subset B_R \backslash (\overline{D_1 \cup D_2})$. Moreover, it is assumed that $B_R \backslash (\overline{D_1 \cup D_2 \cup E})$ is connected.
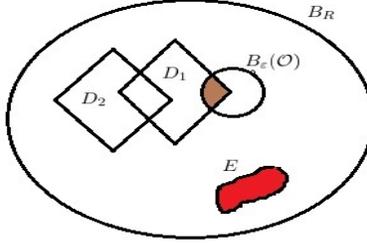
For $A(x) = (a_1(x), a_2(x)) \in (L^\infty(B_R))^2$ and $b \in L^\infty(B_R)$, we define a set $S(A, b)$ by

$$S(A, b) := \Big\{ v(x) : \Delta v(x) + A(x) \cdot \nabla v(x) + b(x) v(x) = 0 \text{ in } B_R \Big\}.$$

2.1. **Inverse wave problem.** Here we state the main result for the inverse problem related to wave Equations (1.1) and (1.2). For $\alpha \in (0, 1)$ we define the following admissible set of coefficient $c \in L^\infty(\mathbb{R}^2)$:

$$\mathcal{C}_D := \Big\{ c \in C^{0,\alpha}\left(\overline{B_\epsilon(\mathcal{O}) \cap D}\right) \text{ for each corner } \mathcal{O} \text{ of } \partial D \text{ and some } \epsilon > 0 :$$

$$supp(1 - c) = D \text{ and } c(\mathcal{O}) \neq 1 \text{ for each corner point } \mathcal{O} \text{ of } \partial D \Big\}.$$

Theorem 2.1. *Let (D_i, c_i) ($i = 1, 2$) be two pairs of convex polygonal scatterers D_i and sound speeds c_i such that $c_i \in \mathcal{C}_{D_i}$ for $i = 1, 2$. Let $f \in C_0^\infty(E)$ and $g \in C_0^\infty(0, T)$ be two non-vanishing functions, and $u_i(x, t)$ be the unique solution to Equations (1.1) and (1.2) with c replaced by c_i for $i = 1, 2$. If*

$$(2.7) \qquad\qquad u_1(x, t) = u_2(x, t), \text{ for all } \quad x \in \Gamma_R \text{ and } t > 0,$$

then $D_1 = D_2$. Moreover, Equation (2.7) also implies that $c_1 = c_2$ provided the following conditions hold:
  (i) *$1/c_i^2(x) = V_i|_{\overline{D_i}}$ for $x \in \overline{D_i}$ and some function $V_i \in S(A, b)$ where A and b are analytic functions near the corner points of D_i.*
  (ii) *$\widehat{g}(0) \neq 0$ and $\int_{\mathbb{R}^2} f(y) \, dy \neq 0$, where $\widehat{g}$ denotes the Laplace transform of g.*

2.2. Inverse Schrödinger problem.

In order to express the main statement of the second part of this work, we first introduce the set of the admissible unknown compactly supported coefficients q. For any $\alpha \in (0,1)$ and $M > 0$, we define an admissible set of q by

$$\mathcal{Q} := \Big\{ q : \ \text{supp}(q) = D \subset B_R \text{ is a convex polygon, } q(\mathcal{O}) \neq 0 \text{ at each corner } \mathcal{O} \text{ of } \partial D,$$

$$q \in C^{0,\alpha}(\overline{B_\varepsilon(\mathcal{O}) \cap D}) \text{ for some } \epsilon > 0 \Big\}.$$

Then we can state the uniqueness result for the inverse problem related to the Schrödinger equation below.

Theorem 2.2. *Let $u_0 \in C_0^\infty(E)$, (D_i, q_i) ($i = 1, 2$) be two pairs of convex polygonal obstacles D_i and potentials $q_i \in \mathcal{Q}$, and u_i ($i = 1, 2$) be the solutions to the initial value problems (1.3) and (1.4) with q replaced by q_i. If*

$$(2.8) \qquad\qquad\qquad u_1(x,t) = u_2(x,t) \qquad \text{on } \Gamma_R \times \mathbb{R}_+,$$

then $D_1 = D_2$. As a consequence, under the following additional assertions:

   i) *The coefficients $q_j \in S(A,b)$ for $j = 1, 2$ where A and b are analytic functions near each corner point of D_i.*

   ii) *There exists a $r_0 > 0$ such that $\displaystyle\int_0^{2\pi} u_0(r_0 \cos\theta, r_0 \sin\theta)\, d\theta \neq 0$,*

the relation (2.8) implies that $q_1 = q_2$ in D.

The above result claims the unique determination of the *obstacle D* from the boundary measurement data $u_{|\Gamma_R \times \mathbb{R}_+}$ and the recovery of the coefficient q under some additional conditions.

2.3. Inverse Helmholtz problem.

The main uniqueness result we will establish for the inverse Helmholtz problem can be stated in the following theorem.

Theorem 2.3. *Let $k > 0$ be fixed, and u_i for $i = 1, 2$ be the solution to (1.6) when n is replaced by n_i. Suppose that*

   (i) *$n_i \in L^\infty(\mathbb{R}^2)$, $D_i = supp(n_i - 1)$ is a convex polygon and $n_i \equiv 1$ in $\mathbb{R}^2 \backslash \overline{D_i}$.*

   (ii) *$n_i(x) = V_i(x)|_{\overline{D_i}}$ for all $x \in \overline{D_i}$, where $V_i \in S(A,b)$ for some functions A and b which are analytic near each corner point of D_i, and $n_i(\mathcal{O}) \neq 1$ for each corner point of ∂D_i.*

   (iii) *$|u_i(\mathcal{O})| > 0$ for each corner point $\mathcal{O}$ of ∂D_i.*

Then the equality $u_1^\infty(\hat{x}) = u_2^\infty(\hat{x})$ for all $\hat{x} \in \mathbb{S}$ implies that $D_1 = D_2$ and $n_1 = n_2$.

Remark 2.4. In the time-harmonic regime, the condition (iii) has already been used in [24] to prove uniqueness in recovering the support of the contrast function. For linear inverse source problems, the condition (ii) guarantees the unique identification of a source term having a convex-polygonal support (see [22]). The above Theorem 2.2 verifies that both the support and the refractive index can be uniquely identified under the conditions (i)-(iii).

3. PRELIMINARIES

An important idea in establishing our main results in this work is to transform the time-dependent problems into the equivalent frequency dependent problems with the help of the Laplace transform. The Laplace transform of a function u of time variable t is given by

$$(3.9) \qquad\qquad \hat{u}(x,s) := \int_0^\infty e^{-st} u(x,t)\, dt, \quad s > 0,\ x \in \mathbb{R}^2.$$

Our goal in this section is to study the *long time behavior* of the solutions to the wave and Schrödinger equations, in order to justify the use of the Laplace transform. It is well known that such kind of behaviors can be derived from classical energy estimates, which will be presented below for the shake of completeness of our arguments. Our emphasis will be placed upon the interpretation of the Laplace transform of u in (3.9).

3.1. **Long time behavior of solutions to the wave equation.**

Lemma 3.1. *Let $c \in L^\infty(\mathbb{R}^2)$ with $c(x) > c_0 \geq 0$ for all $x \in \mathbb{R}^2$ and let $F \in C_0^\infty\left(\mathbb{R}^2 \times \mathbb{R}_+\right)$ satisfy* $\sup_{t \in [0,\infty)} \|\partial_t^k F(t, \cdot)\|_{L^2(\mathbb{R}^+)} \leq C_k$ *for $k = 0, 1, 2$ and some constant $C_k > 0$ independent of t. Let u be a solution to the initial value problem*

(3.10)
$$\begin{cases} \frac{1}{c^2(x)} \partial_t^2 u(x,t) - \Delta_x u(x,t) = F(x,t), & (x,t) \in \mathbb{R}^2 \times \mathbb{R}^+ \\ u(x,0) = \partial_t u(x,0) = 0, & x \in \mathbb{R}^2. \end{cases}$$

Then the solution u has the asymptotic estimate

(3.11)
$$\|u(\cdot, t)\|_{H^2(\mathbb{R}^2)} = O(t^2) \quad as \quad t \to \infty,$$

leading to the well-definedness of the Laplace transform of u for all $x \in \mathbb{R}^2$. Moreover, the following estimate holds:

(3.12)
$$s^3 \|\widehat{u}(\cdot, s)\|_{H^2(\mathbb{R}^2)} \leq C, \quad s > 0,$$

for some constant $C > 0$ independent of s.

Proof. Since $c \in L^\infty(\mathbb{R}^2)$ and $F \in C_0^\infty(\mathbb{R}^2 \times \mathbb{R}_+)$, we have from [23] that the solution u to (3.10) satisfies $u \in \cap_{j=0}^2 C^j\left((0, \infty); H^{2-j}(\mathbb{R}^2)\right)$. Now multiplying the first equation in (3.10) by $2\partial_t u(x,t)$ and integrating over $(0, t) \times \mathbb{R}^2$, we get

$$\int_{\mathbb{R}^2} \left(\frac{1}{c^2(x)} |\partial_t u(x,t)|^2 + |\nabla_x u(x,t)|^2 \right) dx = 2 \int_0^t \int_{\mathbb{R}^2} F(x,s) \partial_t u(x,s) \mathrm{d}x \mathrm{d}s.$$

By using the Cauchy-Schwartz inequality, the hypothesis on $F(x,t)$, along with the fact that $0 < c(x) \leq c$ for some constant $c > 0$, we obtain

$$\mathcal{E}(t) := \int_{\mathbb{R}^2} \left(|\partial_t u(x,t)|^2 + |\nabla_x u(x,t)|^2 \right) \mathrm{d}x \leq C \int_0^t \|F(.,s)\|_{L^2(\mathbb{R}^2)} \|\partial_t u(.,s)\|_{L^2(\mathbb{R}^2)} \mathrm{d}s$$

$$\leq C \|F\|_{L^\infty(0,\infty; L^2(\mathbb{R}^2))} \int_0^t \mathcal{E}(s)^{1/2} \mathrm{d}s \leq C \int_0^t \mathcal{E}(s)^{1/2} \mathrm{d}s$$

for some constant $C > 0$, depending only on $c(x)$ and $F(x,t)$. Now let us define

$$\phi(T) := \max_{0 \leq t \leq T} \mathcal{E}(t).$$

Using this, we derive from the above estimate that

$$\phi(T) \leq C \max_{0 \leq t \leq T} \int_0^t \mathcal{E}(s)^{1/2} \mathrm{d}s \leq C \int_0^T \mathcal{E}(s)^{1/2} \mathrm{d}s \leq C \int_0^T \max_{0 \leq s \leq T} \mathcal{E}(s)^{1/2} \mathrm{d}s = C\phi(T)^{1/2} T.$$

This gives

$$\mathcal{E}(T) \leq \phi(T) \leq CT^2, \text{ for any } 0 \leq T < \infty.$$

Also the Minkowskii inequality for inetgrals ([15], page 194), we have

$$\|u(\cdot,t)\|_{L^2(\mathbb{R}^2)} = \left\|\int_0^t \partial_t u(\cdot,s)\mathrm{d}s\right\|_{L^2(\mathbb{R}^2)} \leq \int_0^t \|\partial_t u(\cdot,s)\|_{L^2(\mathbb{R}^2)}\mathrm{d}s \leq Ct^2.$$

This proves (3.11). Combining the estimate of $\mathcal{E}(t)$ together with Equation (3.11), we get that

(3.13) $\|u(.,t)\|_{H^1(\mathbb{R}^2)} \leq Ct^2,$ for t large enough and some constant $C > 0$ independent of t.

Now setting $\widetilde{u}(x,t) := \partial_t u$ and using the fact that $c(x)$ is independent of t, we have $\widetilde{u}$ solves (3.10) with right hand side equal to $\partial_t F(x,t)$ and hence $\|\widetilde{u}(.,t)\|_{H^1(\mathbb{R}^2)} \leq Ct^2$ for some constant $C > 0$ independent of t. Now repeating the previous arguments, we can have that $\partial_t^2 u(.,t) \in L^2(\mathbb{R}^2)$ and then solving the Poisson's equation for u at each fixed t, we can conclude that $u(.,t) \in H^2(\mathbb{R}^2)$ and

(3.14) $\|u(.,t)\|_{H^2(\mathbb{R}^2)} \leq Ct^2,$

holds for t large enough and $C > 0$ independent of t. Next we will show that the Laplace transform defined in Equation (3.9) makes sense for all $x \in \mathbb{R}^2$. To show this, it suffices to prove that

$$\|\widehat{u}(\cdot;s)\|_{L^2(\mathbb{R}^2)} = \left\|\int_0^\infty e^{-st}u(\cdot,t)dt\right\|_{L^2(\mathbb{R}^2)}$$

is finite. Again using the Minkowskii inequality for integrals ([15], page 194), we have

$$\left\|\int_0^\infty e^{-st}u(x,t)dt\right\|_{L^2(\mathbb{R}^2)} \leq \int_0^\infty e^{-st}\|u(\cdot,t)\|_{L^2(\mathbb{R}^2)}dt.$$

Now using Equation (3.11), we further derive

$$\|\widehat{u}(\cdot;s)\|_{L^2(\mathbb{R}^2)} = \left\|\int_0^\infty e^{-st}u(x,t)\mathrm{d}t\right\|_{L^2(\mathbb{R}^2)} \leq \int_0^\infty e^{-st}\|\widehat{u}(\cdot,t)\|_{L^2(\mathbb{R}^2)} \leq C\int_0^\infty e^{-st}t^2\mathrm{d}t \leq \frac{C}{s^3}.$$

Thus

$$\int_0^\infty e^{-st}u(x,t)\mathrm{d}t \in L^2(\mathbb{R}^2),$$

which implies that

$$\widehat{u}(x;s) = \int_0^\infty e^{-st}u(x,t)\mathrm{d}t$$

exists for almost every $x \in \mathbb{R}^2$. Since the solution u to (3.10) satisfies $u \in C\left((0,\infty); H^2(\mathbb{R}^2)\right)$ and in two dimensions the Sobolev space $H^2(B_R)$ is imbedded into $C^{0,\alpha}(B_R)$, the Laplace transform $\widehat{u}(x;s)$ of $u(x,t)$ can be defined pointwise. This completes the proof of Lemma 3.1. $\square$

Lemma 3.2. *Let $u(x,t)$ be the solution to (1.1)-(1.2) when $c \in \mathcal{C}_D$, f and g are two functions as given in Theorem 2.1. Then the relation that $u(x,t) = 0$ for $(x,t) \in \Gamma_R \times \mathbb{R}_+$ implies $\partial_\nu u(x,t) = 0$ for $(x,t) \in \Gamma_R \times \mathbb{R}_+$.*

Proof. It is easy to see that Equations (1.1)-(1.2) reduces to the following equations in $\left(\mathbb{R}^2\backslash B_R\right)\times(0,T)$:

$$(3.15) \quad \begin{cases} \partial_t^2 u(x,t) - \Delta_x u(x,t) = 0, \ (x,t) \in \left(\mathbb{R}^2\backslash B_R\right)\times\mathbb{R}^+ \\ u(x,0) = \partial_t u(x,0) = 0, \quad x \in \mathbb{R}^2\backslash B_R. \end{cases}$$

Now after multiplying (3.15) by $2\partial_t u(x,t)$ and integrating over $\left(\mathbb{R}^2\backslash B_R\right)\times(0,t)$, we get

$$\int\limits_{\mathbb{R}^2\backslash B_R} \left(|\partial_t u(x,t)|^2 + |\nabla_x u(x,t)|^2\right) \mathrm{d}x = 0, \text{ for any } t \in \mathbb{R}^+.$$

This gives

$$u(x,t) = 0, \text{ for } (x,t) \in \left(\mathbb{R}^2\backslash B_R\right)\times\mathbb{R}^+.$$

Now using the fact that solution $u \in C^\infty\left(\left(\mathbb{R}^2\backslash B_R\right)\times\mathbb{R}_+\right)$, we conclude that $\partial_\nu u(x,t) = 0$ for any for $(x,t) \in \Gamma_R \times \mathbb{R}_+$. This complete the proof of Lemma 3.2. $\qquad\square$

3.2. **Long time behavior of solutions to the Schrödinger equation.** The following lemma indicates that $\widehat{u}$ is well defined and that $\widehat{u}(\cdot,s) \in H^2(\mathbb{R}^2)$.

Lemma 3.3. *Suppose that $u_0 \in L^2(\mathbb{R}^2)$ and $q \in L^\infty(\mathbb{R}^2)$, and there exists $M > 0$ such that*

$$(3.16) \quad \|u_0\|_{L^2(\mathbb{R}^2)} + \|q\|_{L^\infty(\mathbb{R}^2)} \le M.$$

Then the following estimate holds

$$\|u(\cdot,t)\|_{L^2(\mathbb{R}^2)} \le C(1+t),$$

where C is a constant depending only on M. Moreover, we have $\widehat{u}(\cdot,s) \in H^2(\mathbb{R}^2)$.

Proof. From the Duhamel formula, we can express the solution u of the initial value problem (1.3)-(1.4) in the form

$$(3.17) \quad u(x,t) = e^{it\Delta}\, u_0(x) + i \int\limits_0^t e^{i(t-s)\Delta}\, u(x,s)\, q(x)\, ds.$$

Therefore, we readily get

$$\|u(\cdot,t)\|_{L^2(\mathbb{R}^2)} \le \|u_0\|_{L^2(\mathbb{R}^2)} + \int\limits_0^t \|u(\cdot,s)\, q\|_{L^2(\mathbb{R}^2)}\, ds.$$

In light of (1.5) and (3.16), we get

$$(3.18) \quad \|u(\cdot,t)\|_{L^2(\mathbb{R}^2)} \le \|u_0\|_{L^2(\mathbb{R}^2)} + Mt\|u_0\|_{L^2(\mathbb{R}^2)} \le C(1+t).$$

In view of (3.18), one can see that $\widehat{u}(\cdot,s) \in L^2(\mathbb{R}^2)$. Indeed, we have

$$\|\widehat{u}(x,s)\|_{L^2(\mathbb{R}^2)}^2 \le \int\limits_0^\infty e^{-2st}\, \|u(\cdot,t)\|_{L^2(\mathbb{R}^2)}^2\, dt\ < \infty.$$

Then, $\widehat{u}(\cdot,s)$ is well defined. Moreover, by applying the Laplace transform, one has

$$\Delta\, \widehat{u}(\cdot,s) = i\, u_0(\cdot) - is\, \widehat{u}(\cdot,s) - q(\cdot)\, \widehat{u}(\cdot,s) \in L^2(\mathbb{R}^2).$$

Therefore, $\widehat{u}(\cdot,s) \in H^2(\mathbb{R}^2)$, which completes the proof of Lemma 3.3. $\qquad\square$

Proceeding as we did in the proof of Lemma 3.2, we can come to the following claim.

Lemma 3.4. *Let $s > 0$ be fixed, and $U \in H^2(\mathbb{R}^2 \backslash \overline{B}_R)$ be a solution to the following elliptic equation*

$$(3.19) \qquad \Delta U(x) + is\, U(x) = 0, \qquad \forall\, x \in \mathbb{R}^2 \backslash \overline{B}_R.$$

Then, we have

$$U(x)\Big|_{\Gamma_R} = 0 \quad \text{implies that} \quad \partial_\nu U(x)\Big|_{\Gamma_R} = 0.$$

Proof. Multiplying the equation (3.19) by $\overline{U}(x)$ and integrating by parts lead to $U(x) \equiv 0$ in $\mathbb{R}^2 \backslash B_R$. This particularly implies the vanishing of the normal derivative of U on Γ_R. □

4. PROOF OF THEOREM 2.1

This section is devoted to the proof of our main results in Theorem 2.1, separated in two subsections. We first prove the unique identification of ∂D in subsection 4.1, and then show in subsection 4.2 two lemmas which are used to establish the uniqueness for identifying $c(x)$. The results of Theorem 2.1 are a consequence of the results from subsections 4.1 and 4.2.

4.1. **Shape identification.** Suppose that there are two convex polygonal obstacles (D_1, c_1) and (D_2, c_2) which generate the identical measurement data $u_1 = u_2$ on $\Gamma_R \times \mathbb{R}_+$. We will show that $D_1 = D_2$ in this subsection. Note that we have the following transmission conditions on ∂D_j (see (3.11)):

$$u_j^+ = u_j^-, \quad \partial_\nu u_j^+ = \partial_\nu u_j^- \quad \text{on} \quad \partial D_j \times \mathbb{R}^+, \quad j = 1, 2,$$

which are understood in the trace spaces $C(\mathbb{R}^+, H^{3/2}(\partial D_j))$ and $C(\mathbb{R}^+, H^{1/2}(\partial D_j))$, respectively. Here the symbols $(\cdot)^{\pm}$ denote the limits taking from outside (+) and inside (-) of D_j with respect to the space variable, respectively.

By Lemma 3.1, we can apply the Laplace transform to u_j to obtain

$$\widehat{u}_j(x, s) := \int_0^\infty u_j(x, t) e^{-st} dt \in H^2(\mathbb{R}^2), \quad \text{for every positive number} \quad s > 0.$$

Recalling the assumption that $u_1 = u_2$ on $\Gamma_R \times \mathbb{R}_+$ and Lemma 3.2, we obtain

$$\widehat{u}_1(x, s) = \widehat{u}_2(x; s), \quad \partial_\nu \widehat{u}_1(x, s) = \partial_\nu \widehat{u}_2(x; s), \quad \text{on} \quad \Gamma_R.$$

It is easy to deduce that u_j, for $j = 1, 2$, fulfills the inhomogeneous elliptic equation

$$\Delta \widehat{u}_j(x, s) - p_j(x, s)\, \widehat{u}_j(x; s) = -f(x)\widehat{g}(s), \quad \text{in} \quad B_R$$

for every fixed $s > 0$, where $p_j(x, s) := s^2 / c_j^2(x)$. Noting that $E \subset B_R \backslash (\overline{D_1 \cup D_2})$ is the support of the spatial source term f, we get $\widehat{u}_1 = \widehat{u}_2$ in $B_R \backslash \overline{D_1 \cup D_2 \cup E}$ by the unique continuation of elliptic equations.

If $D_1 \neq D_2$, without loss of generality, we may assume that there exists a corner point $\mathcal{O} \in \partial D_1 \backslash \partial D_2$ such that $B_\epsilon(\mathcal{O}) \cap (\overline{D}_2 \cup E) = \emptyset$ for some $\epsilon > 0$. Set $\Gamma := B_\epsilon(\mathcal{O}) \cap \partial D_1$. Then we obtain

$$(4.20) \qquad \begin{cases} \Delta \widehat{u}_1(x, s) - p_1(x, s)\widehat{u}_1(x, s) = 0, & \text{in} \quad B_\epsilon(\mathcal{O}), \\ \Delta \widehat{u}_2(x, s) - s^2 \widehat{u}_2(x, s) = 0, & \text{in} \quad B_\epsilon(\mathcal{O}), \\ \widehat{u}_1(x, s) = \widehat{u}_2(x, s), \quad \partial_\nu \widehat{u}_1(x, s) = \partial_\nu \widehat{u}_2(x, s) & \text{on} \quad \Gamma. \end{cases}$$

Note that $\widehat{u}_2$ is analytic in $B_\epsilon(\mathcal{O})$ and $\widehat{u}_1 \in H^2(B_\epsilon(\mathcal{O}))$. Since $c_1(\mathcal{O}) \neq 1$, it holds that $p_1(\mathcal{O}, s) \neq s^2$ for any $s > 0$. Applying [13, Lemma 1] we obtain $\widehat{u}_1 = \widehat{u}_2 \equiv 0$ in $B_\epsilon(\mathcal{O})$. By the unique continuation, we see $\widehat{u}_j(x, s) = 0$ in $B_R \backslash \overline{D_j \cup E}$ for $j = 1, 2$ and every $s > 0$. Let $E^* \supset E$ be a neighbouring area of E such that $E^* \subset B_R$ and $E^* \cap D_1 = \emptyset$. Then the function $\widehat{u}_1$ satisfies

$$(4.21) \qquad \Delta \widehat{u}_1(x, s) - s^2 \widehat{u}_1(x, s) = f(x)\widehat{g}(s) \quad \text{in} \quad E^*, \quad \widehat{u}_1 \equiv 0 \quad \text{in} \quad E^* \backslash \overline{E}$$

for all $s > 0$. Let v be any solution to the equation $\Delta v(x, s) - s^2 v(x, s) = 0$. Now multiplying (4.21) by $v(x, s)$ and integrating over $\mathbb{R}^2$, we have for all $s > 0$ that

$$\widehat{g}(s) \int\limits_{E^*} f(x) v(x, s) \mathrm{d}x = \int\limits_{E^*} \left(\Delta \widehat{u}_1(x, s) - s^2 \widehat{u}_1(x, s) \right) v(x, s) \mathrm{d}x$$

(4.22)

$$= \int\limits_{E^*} \left(\Delta \widehat{u}_1(x, s) \, v(x, s) - s^2 \widehat{u}_1(x, s) \, v(x, s) \right) \mathrm{d}x.$$

Now using the integration by parts and noting the vanishing of $\widehat{u}_1$ near ∂E^*, we get from (4.22) that

$$\widehat{g}(s) \int\limits_{\mathbb{R}^2} f(x) v(x, s) \mathrm{d}x = 0, \text{ for all } s > 0 \text{ and } v(x, s) \text{ as specified above.}$$

Since g does not vanish identically, there exists an open interval in which $\widehat{g} \neq 0$. Now for any $\omega \in \mathbb{S}$, taking the special solution $v(x, s) = e^{-sx \cdot \omega}$ to $\Delta v(x, s) - s^2 v(x, s) = 0$, we deduce

$$\widehat{f}(s\omega) = 0, \text{ for all } \omega \in \mathbb{S} \text{ and for } s > 0 \text{ belongs to some open interval.}$$

Since f is compactly supported, we have $f \equiv 0$ in $\mathbb{R}^2$, which is not true. This completes the proof that $D_1 = D_2$.

4.2. **Coefficient determination.** Having proved that $D_1 = D_2 := D$ in subsection 4.1, we can now verify that $c_1 = c_2$ on D, under the additional conditions (i) and (ii) as stated in Theorem 2.1. To this purpose, we first present two auxiliary results in Lemmas 4.1 and 4.2.

Lemma 4.1. *Define*

$$\Sigma := \left\{ (r, \theta) : |\theta| < \theta_0, \ r < 1 \text{ and } \theta_0 \in \left(0, \frac{\pi}{2} \right) \right\},$$

$$\Gamma := \left\{ (r, \theta) : \theta = \pm \theta_0, \ r < 1 \right\}.$$

Suppose that $p \in C^{0,\alpha}(\overline{\Sigma})$, $f \in C^{0,\alpha}(\overline{\Sigma})$ with $f(\mathcal{O}) \neq 0$ is not identically zero and that $u \in H^2(\Sigma)$ is a solution to the boundary value problem

(4.23)
$$\begin{cases} \Delta u(x) + p(x) u(x) = h(x) f(x), \ x \in \Sigma, \\ u(x) = \partial_\nu u(x) = 0, \ x \in \Gamma. \end{cases}$$

If the source component h above belongs to $S(A, b)$ in a neighborhood of $\overline{\Sigma}$ and both A and b are analytic on $\overline{\Sigma}$, then $u = h \equiv 0$ in Σ.

Proof. Obviously, the right hand side hf belongs to $C^{0,\alpha}(\overline{\Sigma})$. Since $h \in S(A, b)$ is analytic , its lowest order Taylor expansion at O must be harmonic (see e.g., [22, Corollary 2.2]), that is, the asymptotic expansion

(4.24) $h(x) = r^N \left(A \cos(N\theta) + B \sin(N\theta) \right) + \mathcal{O}(r^{N+1}), \quad |x| \to 0, \ x \in \Sigma$

holds uniformly in all $\theta \in (-\theta_0, \theta_0)$ for some $N \in \mathbb{N}_0$ and $A, B \in \mathbb{C}$. Using the fact that $f(O) \neq 0$ and $f \in C^{0,\alpha}(\overline{\Sigma})$, it follows that the lowest order Taylor expansion of hf at O is also harmonic. Applying [22, Lemma 2.3] or [13, Lemma 2] we get $h(x) f(x) \equiv 0$ for $x \in \overline{\Sigma}$, which implies $h \equiv 0$ and thus $u \equiv 0$. This completes the proof of Lemma 4.1. $\qquad \square$

Lemma 4.2. *Let u be the unique solution to the initial value problem (1.1)-(1.2), with f and g as in Theorem 2.1 fulfilling the conditions $\widehat{g}(0) \neq 0$ and $\int_{\mathbb{R}^2} f(y) \, dy \neq 0$. Let $\widehat{u}$ be the Laplace transform of $u(x, t)$ with respect to the time variable. Then for any point $\mathcal{O}$ of ∂D, there exists a small number $s_0 > 0$ depending on $\mathcal{O}$ such that $\widehat{u}(\mathcal{O}, s_0) \neq 0$.*

Proof. It is easy to check that

$$\Delta \widehat{u}(x,s) - s^2 \widehat{u}(x,s) = -f(x)\widehat{g}(s) + s^2 \left(\frac{1}{c^2(x)} - 1 \right) \widehat{u}(x,s) \quad \text{in} \quad \mathbb{R}^2. \tag{4.25}$$

Recall that

$$\left(\Delta - s^2 \right) \frac{i}{4} H_0^{(1)} \left(is|x-y| \right) = \delta(x-y), \text{ for fixed } y \in \mathbb{R}^2 \text{ and } x \neq y$$

where $H_0^{(1)}$ is the Hankel function of first kind of order zero, which has the following asymptotic expansion as $s \to 0$ (cf. [43]):

$$\frac{i}{4} H_0^{(1)}(is|x-y|) = -\frac{1}{2\pi} \ln|x-y| + \frac{i}{4} - \ln \frac{is}{2} - \frac{C}{2\pi} + O(s^2|x-y|^2 \ln s|x-y|). \tag{4.26}$$

Then the solution to Equation (4.25) can be given by

$$\widehat{u}(x,s) = -\frac{i}{4}\widehat{g}(s) \int_{\mathbb{R}^2} f(y)H_0^{(1)}(is|x-y|)\, \mathrm{d}y + \frac{i}{4}s^2 \int_{\mathbb{R}^2} \left(\frac{1}{c^2(y)} - 1 \right) \widehat{u}(y,s)H_0^{(1)}(is|x-y|)\mathrm{d}y.$$

Taking $x, y \in B_R$, sufficiently small $s > 0$ and using the asymptotic expansion (4.26), we get

$$\widehat{u}(x,s) = -\frac{i}{4}\widehat{g}(s) \int_{\mathbb{R}^2} f(y) \left(-\frac{1}{2\pi}\ln|x-y| + \frac{i}{4} - \ln \frac{is}{2} - \frac{C}{2\pi} \right) \mathrm{d}y$$

$$+ \frac{i}{4}s^2 \int_{\mathbb{R}^2} \left(\frac{1}{c^2(y)} - 1 \right) \widehat{u}(y,s) \left(-\frac{1}{2\pi}\ln|x-y| + \frac{i}{4} - \ln \frac{is}{2} - \frac{C}{2\pi} \right) \mathrm{d}y$$

$$- \frac{i}{4}\widehat{g}(s) \int_{\mathbb{R}^2} f(y)O(s^2|x-y|^2 \ln s|x-y|)\mathrm{d}y \tag{4.27}$$

$$+ \frac{i}{4}s^2 \int_{\mathbb{R}^2} \left(\frac{1}{c^2(y)} - 1 \right) \widehat{u}(y,s)O(s^2|x-y|^2 \ln s|x-y|)\mathrm{d}y.$$

Multiplying Equation (4.27) by s^2 and using Equation (3.12) together with the fact that f, g and $\left(\frac{1}{c^2(x)} - 1 \right)$ are compactly supported, we derive

$$\lim_{s \to 0} s^2 \widehat{u}(x,s) = 0, \text{ for } x \in \overline{B_R}. \tag{4.28}$$

Then noting that $s^2\widehat{u}(x;s)$ is a continuous function for $x \in \mathbb{R}^2$ and $s > 0$, we know there exist a constant $M > 0$ such that

$$|s^2 \widehat{u}(x,s)| \leq M, \text{ for } s \text{ close to } 0 \text{ and } x \in \overline{B_R}. \tag{4.29}$$

Now multiplying Equation (4.27) by s and using Equation (4.29), we get

$$\lim_{s \to 0} s\widehat{u}(x,s) = 0, \text{ for } x \in B_R. \tag{4.30}$$

Using this and repeating the same argument as above, we know the existence of a constant $M_1 > 0$ such that

$$|s\widehat{u}(x,s)| \leq M_1, \text{ for } s \text{ close to } 0 \text{ and } x \in \overline{B_R}. \tag{4.31}$$

Finally using Equation (4.31) in (4.27) and the fact that $\widehat{g}(0) \neq 0$, $\int_{\mathbb{R}^2} f(y)\mathrm{d}y \neq 0$, we further deduce

$$\lim_{s \to 0} \widehat{u}(x,s) = \infty, \text{ for } x \in \overline{B_R}.$$

This implies that for $\mathcal{O} \in \partial D$, there exists a sufficiently small s_0 such that $\widehat{u}(\mathcal{O}, s_0) \neq 0$, hence completes the proof of Lemma 4.2. $\qquad\square$

Proof of Theorem 2.1. The uniqueness for ∂D was already established in subsection 4.1. Next we prove the uniqueness of identifying $c(x)$, namely, $c_1(x) = c_2(x)$ for all $x \in \overline{D}$. Taking the Laplace transform to u_j, it follows from the arguments as in subsection 4.1 that $\widehat{u}_1 = \widehat{u}_2$ in $B_R \backslash \overline{D \cup E}$. Let $\mathcal{O} \in \partial D$ be a corner point and set $\Sigma := B_\epsilon(\mathcal{O}) \cap D$ for some small $\epsilon > 0$. Then it is easy to see

$$\begin{cases} \Delta \widehat{u}_1(x, s) - p_1(x, s)\widehat{u}_1(x, s) = 0 & \text{in} \quad \Sigma, \\ \Delta \widehat{u}_2(x, s) - p_2(x, s)\widehat{u}_2(x, s) = 0 & \text{in} \quad \Sigma, \\ \widehat{u}_1(x, s) = \widehat{u}_2(x, s), \ \partial_\nu \widehat{u}_1(x, s) = \partial_\nu \widehat{u}_2(x, s) & \text{on} \quad \Gamma := B_\epsilon(\mathcal{O}) \cap \partial D. \end{cases}$$

Setting $w := \widehat{u}_1 - \widehat{u}_2$, we get

(4.32)
$$\begin{cases} \Delta w - p_1(x, s)w = s^2 h(x)\widehat{u}_2(x, s) & \text{in} \quad \Sigma, \\ w = \partial_\nu w = 0 & \text{on} \quad \Gamma, \end{cases}$$

where h is defined by

$$h(x) := 1/c_1^2(x) - 1/c_2^2(x) = V_1(x) - V_2(x), \quad x \in \overline{\Sigma}.$$

By the assumption on c_j, we know $h \in S(A, b)$. Now from Lemma 4.2 there exists $s_0 > 0$ such that $s_0^2\widehat{u}_2(\mathcal{O}, s_0) \neq 0$. Hence using Lemma 4.1, we have that $h \equiv 0$. This completes the proof of Theorem 2.1. $\qquad\square$

5. PROOF OF THEOREM 2.2

5.1. **Shape identification.** Our goal in this subsection is to deal with the obstacle identification problem for the time dependent Schrödinger equation (1.3). More precisely, we aim to prove that the measurement data $u_{|\Gamma_R \times \mathbb{R}_+}$ can uniquely determine the object D defined as the support of the coefficient q. We will make some appropriate changes of the proof of Theorem 2.1 for the wave equation to be applicable to the Schrödinger equation.

Let us consider two convex polygonal obstacles D_1 and D_2 corresponding to the two electric potentials q_1 and q_2 respectively. Let u_1 and u_2 be two respective solutions to the initial value problem(1.3) and (1.4) for the Schrödinger equations corresponding to the coefficients q_1 (with support D_1) and q_2 (with support D_2). After applying the Laplace transform, one can see that for any fixed $s > 0$, the solutions $\widehat{u}_j$ for $j = 1, 2$ satisfy

$$\Delta \widehat{u}_j(x, s) + (q_j(x) + is)\, \widehat{u}_j(x, s) = i\, u_0(x), \quad \text{for all } x \in B_R$$

and $\widehat{u}_1 = \widehat{u}_2$ on Γ_R for any fixed $s > 0$. Let us recall that the function u_0 satisfies supp$(u_0) \subset E$. In view of the proof of Lemma 3.4 we obtain $\widehat{u}_1 = \widehat{u}_2$ in $(\mathbb{R}^2 \backslash B_R) \times \mathbb{R}_+$. Thus, by the unique continuation principle for elliptic equations, one can see that for any fixed $s > 0$, we have the following identity

(5.33)
$$\widehat{u}_1(x, s) = \widehat{u}_2(x, s) \quad \text{in } B_R \backslash (\overline{D_1 \cup D_2 \cup E}).$$

On the other hand, by assuming that $D_1 \neq D_2$, one can see that there exists (without loss of generality) a corner point $\mathcal{O} \in \partial D_1 \backslash \partial D_2$. For $\varepsilon > 0$, we recall that $B_\varepsilon(\mathcal{O})$ is the ball centred in $\mathcal{O}$ satisfying

$$B_\varepsilon(\mathcal{O}) \cap (E \cup \overline{D_2}) = \emptyset,$$

and $\Gamma := \partial D_1 \cap B_\varepsilon(\mathcal{O})$. Since $D_2 \cap B_\varepsilon(\mathcal{O}) = \emptyset$, then for any fixed $s > 0$, u_1 and u_2 satisfy

(5.34) $\Delta \widehat{u}_1(x, s) + (q_1(x) + is)\, \widehat{u}_1(x, s) = 0, \quad \text{and} \quad \Delta \widehat{u}_2(x, s) + is\, \widehat{u}_2(x, s) = 0, \quad \text{for all } x \in B_\varepsilon(\mathcal{O}).$

Moreover, taking into account the fact that $\widehat{u}_j(\cdot, s) \in H^2(\mathbb{R}^2)$ for $j = 1, 2$, one can see in light of (5.33) and Lemma 3.4 that we have for any fixed $s > 0$,

$$\text{(5.35)} \qquad \widehat{u}_1(x, s) = \widehat{u}_2(x, s), \quad \text{and} \quad \partial_\nu \widehat{u}_1(x, s) = \partial_\nu \widehat{u}_2(x, s), \quad \text{for all } x \in \Gamma.$$

By our assumption, $q_1(O) \neq 0$. Now, applying [13, Lemma 1] (see also Lemma 4.1) to the Cauchy problem (5.34) and (5.35), we obtain the following identity (cf. (4.20) in the wave equation case):

$$\text{(5.36)} \qquad \widehat{u}_1(x, s) = \widehat{u}_2(x, s) = 0, \quad \forall x \in B_\varepsilon(O)$$

for any fixed $s > 0$. In view of the unique continuation principle, we have

$$\text{(5.37)} \qquad \widehat{u}_1(x, s) = 0, \quad \forall x \in B_R \backslash (\overline{D_1 \cup E}).$$

To derive the desired contradiction, we still denoted by E^* a neighborhood of E that satisfies the conditions

$$E \subset E^* \subset B_R, \quad \text{and} \quad E^* \cap D_1 = \emptyset.$$

Let v be an arbitrary solution to the homogeneous equation

$$\Delta v(x, s) + is\, v(x, s) = 0.$$

Multiplying v to the equation of $\widehat{u}_1$:

$$\text{(5.38)} \qquad \Delta \widehat{u}_1(x, s) + is\, \widehat{u}_1(x, s) = i\, u_0(x), \quad \text{for all } x \in B_R$$

and integrating over E^*, one gets the following identity

$$i \int_{E^*} u_0(x)\, v(x, s)\, dx = \int_{E^*} (\Delta \widehat{u}_1(x, s) + is\, \widehat{u}_1(x, s)) v(x, s) dx$$

$$= \int_{\partial E^*} (\partial_\nu \widehat{u}_1(x, s) v(x, s) - \partial_\nu v(x, s) \widehat{u}_1(x, s))\, ds.$$

Now, taking $v(x, s) = i\, e^{-i\sqrt{s}\, x \cdot \theta}$ with $\theta \in \mathbb{S}$ above and using (5.37), we get

$$\mathcal{F}(u_0)(\sqrt{s}\, \theta) = \int_{\mathbb{R}^2} u_0(x) e^{-i\sqrt{s}\, \theta\, x}\, dx = 0, \quad \text{for all } \theta \in \mathbb{S}, \ s > 0,$$

where $\mathcal{F}(\cdot)$ denotes the Fourier transform of u_0. This implies that $u_0 \equiv 0$ in $\mathbb{R}^2$, which is a contradiction. Thus we have proved $D_1 = D_2$.

5.2. **Coefficient identification.** Our goal in this subsection is to pursue the proof of Theorem 2.2. For $q_1, q_2 \in \mathcal{Q}$, having already proved that $D_1 = D_2 =: D$, we now move forward to show that $q_1 = q_2$ in D. We start first with one of the main key ingredients in our proof.

Lemma 5.1. *Let $O \in \partial D$ be a corner point, $\widehat{u}$ be the solution to the equation*

$$\text{(5.39)} \qquad \Delta\, \widehat{u}(x, s) + (q(x) + is)\, \widehat{u}(x, s) = iu_0(x), \quad \text{for all } x \in \mathbb{R}^2, \ s > 0.$$

Then there exists a sufficiently large $s_0 > 0$ such that $\widehat{u}(O, s_0) \neq 0$.

Proof. Let us decompose the solution $\widehat{u}$ into the sum $\widehat{u} = \widehat{v} + \widehat{w}$, where $\widehat{v}$ solves the equation

$$\text{(5.40)} \qquad \Delta\, \widehat{v}(x, s) + is\, \widehat{v}(x, s) = i\, u_0(x), \quad \text{for all } x \in \mathbb{R}^2, \ s > 0,$$

and $\widehat{w}$ satisfies

$$\text{(5.41)} \qquad \Delta\, \widehat{w}(x, s) + is\, \widehat{w}(x, s) + q(x)\, \widehat{w}(x, s) = 0, \quad \text{for all } x \in \mathbb{R}^2, \ s > 0.$$

Note that $v(x, t)$ satisfies the initial value problem (1.3)-(1.4) for the Schrödinger equation with $q \equiv 0$ in $\mathbb{R}^2$ and $w := u - v$ denotes the scattered field incited by the potentials $v(x, t)$ and $q(x)$. The proof will be

divided into two steps.

Step 1: We prove that there exists a large $s_0 > 0$ such that $\widehat{v}(\mathcal{O}, s_0) \neq 0$. Indeed, one can easily see that the solution $\widehat{v}$ to (5.40) solves the following integral equation

$$\widehat{v}(x, s) = \int_E i\,\phi(x, y; s)\, u_0(y)\, dy,$$

where the function ϕ is given by

$$(5.42) \qquad\qquad \phi(x, y; s) = \frac{i}{4} H_0^{(1)}(\sqrt{is}\,|x - y|) = \frac{i}{4} H_0^{(1)}(\sqrt{s}\,e^{i\pi/4}|x - y|).$$

Here $H_0^{(1)}$ denotes the Hankel function of the first kind of order zero. This yields the following identity

$$\widehat{v}(x, s) = \int_E \frac{-1}{4} H_0^{(1)}(\sqrt{s}\,e^{i\pi/4}|x - y|)\, u_0(y)\, dy.$$

Then, by taking s to the infinity, we will get in view of the asymptotic behavior of $H_0^{(1)}$ at infinity, together with the identity (47) in [43], that only the principal part of $\widehat{v}$ will dominate and it will be equivalent to

$$(5.43) \qquad \widehat{v}(x, s) \simeq -\frac{\sqrt{2}\,e^{-i\pi/4}}{4\sqrt{\pi}\,e^{i\pi/4}} \int_{\mathbb{R}^2} \frac{u_0(y)}{\sqrt{\sqrt{s}\,|x - y|}}\, e^{i\sqrt{s}\,|x-y|\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)}\, dy, \qquad \text{as } s \to \infty.$$

Thus, at the corner point $\mathcal{O}$ which is assumed to be, without loss of generality, the origin of $\mathbb{R}^2$, we have

$$(5.44) \qquad\qquad \widehat{v}(\mathcal{O}, s) \;\simeq\; -\frac{\sqrt{2}\,e^{-i\pi/4}}{4\sqrt{\pi}\,e^{i\pi/4}} \int_{\mathbb{R}^2} \frac{u_0(y)}{\sqrt{\sqrt{s}|y|}}\, e^{i\sqrt{s}\,|y|\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)}\, dy,$$

$$\;=\; -C_0 \int_{\mathbb{R}^2} \frac{u_0(y)}{\sqrt{\sqrt{s}|y|}}\, e^{i\sqrt{s}\,|y|\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)}\, dy$$

as $s \to \infty$, where $C_0 \in \mathbb{C}$ is a constant. Let us denote by

$$I(s) := \int_{\mathbb{R}^2} \frac{u_0(y)}{\sqrt{\sqrt{s}|y|}}\, e^{i\sqrt{s}\,|y|\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)}\, dy.$$

In polar coordinates with $y = (r\cos\theta, r\sin\theta)$ and $r = |y|$, the integral I will be given by

$$I(s) \;=\; s^{\frac{-1}{4}} \int_0^\infty e^{-\left(\frac{\sqrt{s}}{\sqrt{2}} - i\frac{\sqrt{s}}{\sqrt{2}}\right)r}\, \sqrt{r}\left(\int_0^{2\pi} u_0(r\cos\theta, r\sin\theta)\, d\theta\right)\, dr$$

$$\;=\; s^{\frac{-1}{4}} \int_0^\infty e^{-Z_s\,r}\, g(r)\, dr$$

$$:=\; s^{\frac{-1}{4}} \widehat{g}(Z_s),$$

where $Z_s := \sqrt{s}/\sqrt{2} - i\sqrt{s}/\sqrt{2}$ and $\widehat{g}$ denotes the Laplace transform of g with respect to r, defined by

$$g(r) := \sqrt{r} \int_0^{2\pi} u_0(r\cos\theta, r\sin\theta)\, d\theta.$$

Assume on the contrary that $\widehat{v}(O, s) \equiv 0$ for all $s \geq M_0$ for a large constant $M_0 > 0$. Then the principal part $I(s)$ must also vanish for $s \geq M_0$, implying that $\widehat{g}(Z_s) = 0$ for $s \geq M_0$. Since u_0 has a compact support, we have $g(r) \equiv 0$ for large r. Therefore, the analyticity of $\widehat{g}$ leads to $g(r) = 0$ for any $r \geq 0$. Consequently,

$$\int_0^{2\pi} u_0(r \cos \theta, r \sin \theta) \, d\theta = 0 \quad \text{for any } r \geq 0,$$

which contradicts the condition ii).

Step 2: We prove that the principal part of $\widehat{u}(O, s)$ is dominated by $\widehat{v}(O, s)$ as $s \to \infty$, namely, $\widehat{w}(O, s)$ decays faster than $\widehat{v}(O, s)$ when s goes to the infinity. In the frequency domain, it is well known the solution $\widehat{w}$ can be represented via the integral equation

$$\widehat{w}(x, s) = \int_{B_R} \phi(x, y; s) \, q(y) \, \widehat{u}(y, s) \, dy,$$

where ϕ is given by (5.42). Define the integral operator

$$K_s : L^\infty(B_R) \longrightarrow L^\infty(B_R)$$
$$\widehat{u} \longmapsto \int_{B_R} \phi(x, y; s) \, q(y) \, \widehat{u}(y, s) \, dy, \qquad .$$

Similar to (5.43), we have for sufficiently large s that

(5.45) $$\|\widehat{w}\|_{L^\infty(B_R)} = \|K_s \, \widehat{u}\|_{L^\infty(B_R)} \leq \frac{C}{s^{1/4}} \, \|\widehat{u}\|_{L^\infty(B_R)}$$

where C depends on M and B_R. Thus, for large s it holds that

$$\|K_s\| < \frac{C}{s^{1/4}} < 1.$$

This entails that $(I - K_s)$ is an invertible operator. Since $\widehat{u} = \widehat{v} + \widehat{w} = \widehat{v} + K_s \widehat{u}$, then one can get

(5.46) $$\widehat{u} = (I - K_s)^{-1} \widehat{v} = \sum_{n=0}^{\infty} K_s^n \widehat{u} = \widehat{v} + K_s \widehat{v} + \cdots + K_s^n \widehat{v} + \cdots$$

We recall that $B_\varepsilon(\mathcal{O})$ is a neighborhood of the corner point $\mathcal{O}$. Therefore, for any $n \geq 1$ we have

(5.47) $$\|K_s^n \widehat{v}\|_{L^\infty(B_\varepsilon(\mathcal{O}))} \leq \|K_s^n \widehat{v}\|_{L^\infty(B_R)} \leq \frac{C}{s^{n/4}} \|\widehat{u}\|_{L^\infty(B_R)} \leq \frac{C}{s^{n/4}} \|\widehat{v}\|_{L^\infty(B_R)}.$$

In view of (5.47), we can see that the second part $\widehat{w}$ of the solution $\widehat{u}$ decays faster to zero than $\widehat{v}$ as s intends to infinity. This together with the first step completes the proof of Lemma 5.1. $\qquad \square$

Proof of Theorem 2.2. Let $\widehat{u}_j$ ($j = 1, 2$) be the solution to the equation corresponding to q_j, namely

$$\Delta \widehat{u}_j(x, s) + (q_j(x) + is)\widehat{u}_j(x, s) = i \, u_0(x), \quad \forall \, x \in B_\varepsilon(\mathcal{O}) \cap D.$$

Arguing like in the previous section, we can get that

(5.48) $$\widehat{u}_1 = \widehat{u}_2, \quad \text{and} \quad \partial_\nu \widehat{u}_1 = \partial_\nu \widehat{u}_2, \qquad \forall \, x \in \Gamma = B_\varepsilon \cap \partial D.$$

Denote $u := u_1 - u_2$. Then u is a solution to

(5.49) $$\Delta u(x, s) - q_1(x) \, u(x, s) = (q_1 - q_2)(x) u_2(x, s), \quad \forall \, x \in B_\varepsilon(\mathcal{O}).$$

By Lemma 5.1, we may chose a large $s > 0$ such that $u_2(O, s) \neq 0$. On the other hand, since $q_1 - q_2$ lies in the admissible set $S(A, b)$, by arguing analogously to the proof of Theorem 2.1, we obtain $q_1 = q_2$ in $B_\varepsilon(\mathcal{O})$. Now applying the unique continuation argument concludes that $q_1 = q_2$ in D. $\square$

Remark 5.2. The analogue of Lemma 5.1 for wave equations was proved via asymptotic analysis with a small number $s > 0$ (see Lemma 4.2). For the Schrödinger equation, we shall carry out the proof by taking a large number $s > 0$, as the proof of Lemma 4.2 does not apply to the Schrödinger equation.

6. PROOF OF THEOREM 2.3

Proof of Theorem 2.3. We give a sketch of the proof. Suppose $D_1 \neq D_2$. Without loss of generality, we may assume that there exists a corner point $\mathcal{O} \in \partial D_1 \backslash \partial D_2$ and $B_\epsilon(\mathcal{O}) \cap \overline{D}_2 = \emptyset$ for some $\epsilon > 0$. We recall the notations that $\Sigma := B_\epsilon(\mathcal{O}) \cap D_1$ and $\Gamma := B_\epsilon(\mathcal{O}) \cap \partial D_1$. Then we get

$$\begin{cases} \Delta u_1 + k^2 n_1(x) u_1 = 0 \text{ in } \Sigma, \\ \Delta u_2 + k^2 u_2 = 0 \text{ in } \Sigma, \\ u_1 = u_2, \ \partial_\nu u_1 = \partial_\nu u_2 \text{ on } \Gamma, \end{cases}$$

and the difference $u := u_1 - u_2$ solves the Cauchy problem

$$(6.50) \qquad \begin{cases} \Delta u + k^2 n_1(x) u = k^2 \left(1 - n_1(x)\right) u_2 \text{ in } B_\epsilon(\mathcal{O}), \\ u = \partial_\nu u = 0 \text{ on } \Gamma. \end{cases}$$

Noting that u_2 solves $\Delta u_2 + k^2 u_2 = 0$ in $B_\epsilon(\mathcal{O})$, its lowest order expansion around O is harmonic. Then using Lemma 4.1, we get that $u_2 \equiv 0$ in Σ, and further by unique continuation, $u_2 \equiv 0$ in $\mathbb{R}^2$, which is impossible. Therefore we have $D_1 = D_2 := D$. Now setting $\Sigma := B_\varepsilon(\mathcal{O}) \cap D$ and $\Gamma := B_\varepsilon(\mathcal{O}) \cap \partial D$, then

$$(6.51) \qquad \begin{cases} \Delta u + k^2 n_1(x) u = k^2 \left(n_2 - n_1\right) u_2, \text{ in } \Sigma \\ u = \partial_\nu u = 0, \text{ on } \Gamma. \end{cases}$$

Since $n_i \in S(A, b)$ and $u_2(\mathcal{O}) \neq 0$, again applying Lemma 4.1 we get $n_1 = n_2$ in $\mathbb{R}^2$. $\square$

7. CONCLUDING REMARKS

This work has been mainly devoted to the target identification and coefficient recovery problems for the time-dependent wave and Schrödinger equations as well as the Helmholtz equation. We have considered the penetrable scatterers with transmission conditions on the interface which are not much studied in the literature. As we are interested in the important case when only a single dynamical data is available, our investigations have been restricted to convex polygonal scatterers and an admissible set of coefficients that include harmonic functions.

There are several interesting topics that deserve further investigation. The first topic is the uniqueness for the important cases when the data is available only on a finite period of time as well as for general penetrable scatterers. Note that our general idea of applying the Laplace transform relies heavily on the data available over the infinite time, which can not be carried out to the case of the dynamical data available only on a finite period of time. However, we believe that the uniqueness results in recovering the shape (the first part in Theorems 2.1 and 2.2) can be generalized to non-polygonal convex penetrable scatterers, while the Laplace transform provides the measurement data for each parameter (frequency). The second topic is how to design efficient inversion algorithms for recovering convex polygonal scatterers with only a single dynamical data, based on the theory that has been developed here and some existing numerical schemes for time-harmonic inverse problems.

ACKNOWLEDGEMENTS

REFERENCES

[1] M. Belishev and Y.V. Kurylev, Boundary control, wave field continuation and inverse problems for the wave equation, Multi-dimensional inverse problems, Comput. Math. Appl. 22 (1991): 27-52.

[2] M. Bellassoued, Stable determination of coefficients in the dynamical Schrödinger equation in a magnetic field, Inverse Problems 33 (2017): 055009.

[3] M. Bellassoued and M. Choulli, Stability estimate for an inverse problem for the magnetic Schödinger equation from the Dirichlet-to-Neumann map, Journal of Functional Analysis 258 (2010): 161-195.

[4] M. Bellassoued and D. Dos Santos Ferreira, Stable determination of coefficients in the dynamical anisotropic Schrödinger equation from the Dirichlet-to-Neumann map, Inverse Problems 26 (2010): 125010.

[5] M. Bellassoued, M. Choulli and A. Jbalia, Stability of the determination of the surface impedance of an obstacle from the scattering amplitude, Mathematical Methods in the Applied Sciences 36 (2013): 2429-2448.

[6] I. Ben Aïcha, Stability estimate for an inverse problem for the Schrödinger equation in a magnetic field with time-dependent coefficient, Journal of Mathematical Physics 58 (2017): 071508.

[7] I. Ben Aïcha and Y. Mejri, Simultaneous determination of the magnetic field and the electric potential in the Schrodinger equation by a finite number of boundary observations, Journal of Inverse and Ill-posed Problems 26 (2018): 928-219.

[8] L. Bourgeois and D. Ponomarev, An inverse obstacle for the wave equation in a finite time domain, Inverse Problems & Imaging, (2019): 377-400.

[9] M. Choulli, Y. Kian and E. Soccorsi, Stable determination of time-dependent scalar potential from boundary measurements in a periodic quantum waveguide, SIAM J. Math. Anal., 47 (2015): 4536-4558.

[10] M. Cristofol and E. Soccorsi, Stability estimate in an inverse problem for non autonomous magnetic Schrödinger equations, Applicable Analysis 90 (2011): 1499-1520.

[11] G. Eskin, Inverse problems for the Schödinger equations with time-dependent electromagnetic potentials and the Aharonov-Bohn effect, J. Math. Phys. 49 (2008): 022105.

[12] J. Elschner and G. Hu, Corners and edges always scatter, Inverse Problems, 31 (2015): 015003.

[13] J. Elschner and G. Hu, Acoustic scattering from corners, edges and circular cones, Archive for Rational Mechanics and Analysis, 228 (2018): 653-690.

[14] D. Finch and K.S. Hickmann, Transmission eigenvalues and thermoacoustic tomography, Inverse Problems 29 (2013): 104016.

[15] G.B. Folland, Real Analysis, Modern techniques and their applications, 2nd edition, A wiley interscience publication.

[16] A. Friedman and V. Isakov, On the uniqueness in the inverse conductivity problem with one measurement, Indiana Univ. Math. J. 38 (1989): 563-579.

[17] Y. Hristova, Time reversal in thermoacoustic tomography - an error estimate, Inverse Problems 25 (2009): 055008.

[18] T. Helin, M. Lassas and L. Oksanen, An inverse problem for the wave equation with one measurement and the pseudorandom source, Anal. PDE 5(2012): 887-912.

[19] G. Hu and Y. Kian, Uniqueness and stability for the recovery of a time-dependent source and initial conditions in elastodynamics, Inverse Problems and Imaging 14 (2020): 463-487.

[20] G. Hu, L. Li and J. Zou, Unique determination of a penetrable scatterer of rectangular type for inverse Maxwell equations by a single incoming wave, Inverse Problems 35 (2019): 035006.

[21] G. Hu, M. Salo and E. V. Vesalainen, Shape identification in inverse medium scattering problems with a single far-field pattern, SIAM J. Math. Anal. 48 (2016): 152-165.

[22] G. Hu and J. Li, Uniqueness to inverse source problems in an inhomogeneous medium with a single far-field pattern, to appear in: SIAM Math. Anal., 2020.

[23] M. Ikawa, Hyperbolic partial differential equations and wave phenomena. Translated from the 1997 Japanese original by Bohdan I. Kurpita. Translations of Mathematical Monographs, 189. Iwanami Series in Modern Mathematics. American Mathematical Society, Providence, RI, 2000. xxii+190 pp. ISBN: 0-8218-1021-9.

[24] M. Ikehata, Reconstruction of a source domain from the Cauchy data, Inverse Problems, 15 (1999): 637-645.

[25] M. Ikehata, On reconstruction in the inverse conductivity problem with one measurement, Inverse Problems, 16 (2000): 785-793.

[26] M. Ikehata, An inverse transmission scattering problem and the enclosure method, Computing, 75 (2005): 133-156.

[27] V. Isakov, Inverse problems for partial differential equations, Second edition. Applied Mathematical Sciences, 127. Springer, New York, 2006. xiv+344 pp.

[28] V. Isakov, Inverse obstacle problems, Inverse problems 25 (2009): 123002.

[29] V. Isakov, New stability result for soft obstacles in inverse scattering, Inverse Problems 9 (1993): 535-543.

[30] C. Knox and A. Moradifam, Determining both the source of a wave and its speed in a medium from boundary measurements, arXiv:1803.06750.

[31] P. Kuchment and L. Kunyansky, Mathematics of thermoacoustic tomography, European J. Appl. Math. 19 (2008): 191-224, 2008.

[32] P. Kuchment, The Radon Transform and Medical Imaging, SIAM, 2014.

[33] J.L. Lions and E. Magenes, Non-Homogeneous Boundary Value Problems and Applications vol 1, Berlin: Springer, 1972.

[34] H. Liu and G. Uhlmann, Determining both sound speed and internal source in thermo- and photo-acoustic tomography, Inverse Problems, 31 (2015): 105005.

[35] L. Li, G. Hu and J. Yang, Interface with weakly singular points always scatter, Inverse Problems 34 (2018): 075003.

[36] L. Li, G. Hu and J. Yang, Piecewise-analytic interfaces with weakly singular points of arbitrary order always scatter, arXiv:2010.00748.

[37] S. Liu and L. Oksanen, A Lipschitz stable reconstruction formula for the inverse problem for the wave equation, Trans. Amer. Math. Soc. 368 (2016): 319-335.

[38] L. Oksanen and G. Uhlmann, Photoacoustic and thermoacoustic tomography with an uncertain wave speed, Math. Res. Lett. 21 (2014): 1199-1214.

[39] L. Pestov, On reconstruction of the speed of sound from a part of boundary, Journal of inverse and ill-posed problems 7 (1999): 481-486.

[40] L. Pestov, On determining an absorption coefficient and a speed of sound in the wave equation by the BC method, Journal of Inverse and Ill-posed Problems 22 (2014): 245-250.

[41] Rakesh and M. Salo, Fixed angle inverse scattering for almost symmetric or controlled perturbations, arXiv:1905.03974.

[42] Rakesh and T. Yuan, Recovering initial values from light cone traces of solutions of the wave equation, Inverse Problems 34 (2018): 075003.

[43] N. Sekeljic, Assymptotic expansion of bessel functions, Applications to electromagnetics Dynamics at the Horsetooth, Volume 2 A, Focussed Issue: Assymptotics and Perturbations.

[44] P.D. Stefanov, Generic uniqueness for two inverse problems in potential scattering, Communications in partial differential equations 17 (1992): 55-68.

[45] P.D. Stefanov and G. Uhlmann, Recovery of a source term or a speed with one measurement and applications, Trans. Amer. Math. Soc. 365(2013): 5737-5758.

[46] P.D. Stefanov, Conditionally stable unique continuation and applications to thermoacoustic tomography, Mathematics in Engineering 1(2019): 789-799.

[47] P.D. Stefanov and Y. Tang, Thermo and Photoacoustic Tomography with variable speed and planar detectors, SIAM J. Math. Anal. 49 (2017): 297-310.

[48] P.D. Stefanov and G. Uhlmann, Instability of the linearized problem in multiwave tomography of recovery both the source and the speed, Inverse Problems and Imaging, 7(2013): 1367-1377.

[49] O. Scherzer, Handbook of Mathematical Methods in Imaging, 2nd edition, springer.

[50] Z. Sun, An inverse boundary value problem for the Schrödinger operator with vector potentials, Trans. Amer. Math. Soc, 338(1992): 953-969.

[51] L. Tzou, Stability estimates for coefficients of magnetic Shrödinger equation from full and partial boundary measurements, Comm. Partial Differential Equations, 11 (2008): 1911-1952.

[1] BEIJING COMPUTATIONAL SCIENCE RESEARCH CENTER, BEIJING 100193, CHINA.
E-MAIL: ibtissem@csrc.ac.cn

[2] SCHOOL OF MATHEMATICAL SCIENCES AND LPMC, TIANJIN 300071, CHINA.
E-MAIL: ghhu@nankai.edu.cn

[3] DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF TECHNOLOGY, JAMMU, 181221, INDIA.
E-MAIL: manmohan.vashisth@iitjammu.ac.in

[4] DEPARTMENT OF MATHEMATICS, THE CHINESE UNIVERSITY OF HONG KONG, HONG KONG, HONG KONG SPECIAL ADMINISTRATIVE REGION OF CHINA.
E-MAIL: zou@math.cuhk.edu.hk