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UNIFORM A PRIORI ESTIMATES FOR ELLIPTIC AND STATIC MAXWELL INTERFACE PROBLEMS

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Abstract. We present some new a priori estimates of the solutions to three-dimensional elliptic interface problems and static Maxwell interface system with variable coefficients. Different from the classical a priori estimates, the physical coefficients of the interface problems appear in these new estimates explicitly.

1. **Introduction.** Interface problems arise in many application areas, such as material sciences, fluid dynamics and electromagnetics. It is the case when different materials, fluids and media with different physical properties are involved. In this paper we are interested in the interface problems which may be modeled by the second-order elliptic equation

$$-\nabla\cdot(\beta(x)\nabla u(x)) = f(x) \quad \text{in} \quad \Omega \tag{1.1}$$

or by the following static Maxwell interface system (cf. [9, 10])

$$\nabla\times\mathbf{E} = 0 \quad \text{in} \quad \Omega, \tag{1.2}$$

$$\nabla\times\mathbf{H} = \mathbf{J} \quad \text{in} \quad \Omega, \tag{1.3}$$

$$\nabla\cdot(\varepsilon(x)\mathbf{E}) = \rho \quad \text{in} \quad \Omega, \tag{1.4}$$

$$\nabla\cdot(\mu(x)\mathbf{H}) = 0 \quad \text{in} \quad \Omega \tag{1.5}$$

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where Ω is an open simply-connected bounded domain in R^3, and is assumed to be either a convex polyhedron or a domain with C^2-smooth boundary. We shall consider that Ω is occupied by two different physical media (or materials) Ω_1 and Ω_2, see Fig. 1 for a two-dimensional sample. The coefficient $\beta(x)$ in (1.1) may represent physical parameters (e.g., diffusion, heat conductivity) of different materials in different applications, hence is only piecewise smooth in Ω. The coefficients $\varepsilon(x)$ and $\mu(x)$ in (1.4) and (1.5) represent the electric permittivity and magnetic permeability of two different media occupying the physical domain Ω, thus may have large jumps across the interface Γ between two different media Ω_1 and Ω_2 (cf. Fig. 1). Due to the importance of interface problems in applications, extensive studies have been devoted to the mathematical behaviors and numerical solutions of the systems (1.1) and (1.2)-(1.5) in the past few decades. The regularities of the solutions to the problem (1.1) and various a priori estimates of the solutions have been widely investigated (cf. [4, 17, 20, 23]), while regularities and edge/corner singularities of solutions were also studied, for instance, in [8, 28] for time-dependent and time-harmonic Maxwell equations, and in [2, 12, 14] for the static Maxwell system.

The primary interest of this paper is to study the mathematical behaviors of solutions to the interface systems (1.1) and (1.2)-(1.5) in terms of the discontinuous physical coefficients and how the solutions depend on the coefficients and their jumps across the interfaces, especially when the jumps are large. We shall achieve the goal through establishing some new a priori estimates of the interface solutions, in which the physical coefficients of the PDEs and their jumps appear explicitly, called *uniform a priori estimates* (with respect to the coefficients). Such uniform a priori estimates are essentially different from the existing estimates where physical coefficients are hidden and no any effects of coefficients on the solutions can be seen from the estimates. The new uniform a priori estimates are not only interesting mathematically, but may also provide more insights into physical behaviors of interface solutions. Moreover, a priori estimates are needed in convergence analysis of every numerical method [4, 6, 7, 23], so the new a priori estimates may help establish more accurate error estimates where one can see clearly how the convergence of numerical methods depends on and is affected by the physical coefficients and their jumps. It may further help construct more effective numerical methods for interface problems. Very little has been done in the literature about this topic. To our knowledge, the first such uniform a priori estimates were established in [15] for the elliptic interface problem (1.1) with piecewise constant coefficients.

In this paper, we shall establish the uniform a priori estimates for the system (1.1) with general variable coefficients, and this is much more difficult than the piecewise constant case treated in our early work [15]. The uniform a priori estimates are achieved by using some novel techniques, or an elegant combination of the theory of single and double layer potentials, Sobolev theory, maximum principle for elliptic equations, integral representation of piecewise harmonic functions and "formal" asymptotic expansions. The basic idea is to first reduce the a priori estimates of solutions to interface problems with variable coefficients into the estimates of piecewise harmonic solutions which are incorporated with the interface conditions of the original interface system; then the piecewise harmonic solutions will be represented by the single and double layer potentials through some integral interface equations. It is this new and elegant representation that enables us to trace closely

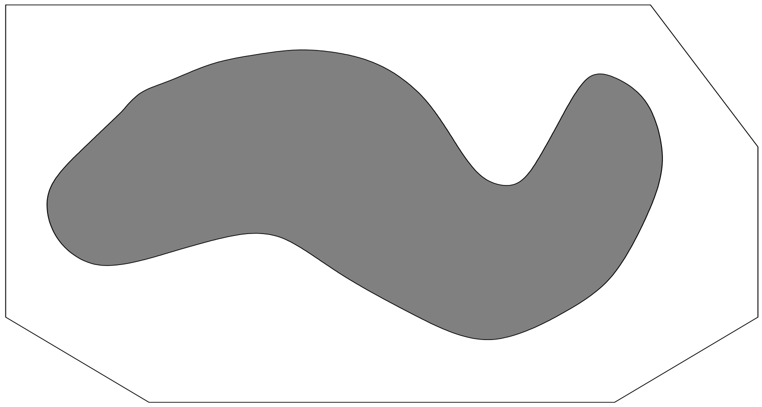

FIG. 1. domain Ω, its subregions Ω_1 (shaded region), Ω_2 (white region) and interface Γ (boundary of the shaded region)

the changes of coefficients from PDEs in every step of our estimates. Due to variable coefficients, the justification of representation of piecewise harmonic solutions by single and double layer potentials is done through two newly established uniqueness theorems for piecewise harmonic functions satisfying interface conditions; The well-posedness of the integral interface equation is demonstrated through Fredholm index theory, potential and Sobolev embedding theory as well as Hopf's maximum principles. The uniform a priori estimates obtained up to this stage are not yet *optimal* in terms of the jumps of coefficients in PDEs. These estimates are further improved by means of a new and powerful "formal" asymptotic expansions in terms of the jumps of coefficients. We emphasize that the final a priori estimates derived here are not only valid for variable coefficients, but also have greatly improved the results we obtained earlier in [15] for piecewise constant cases.

The new uniform a priori estimates for elliptic interface problems will be then applied to establish similar uniform estimates for the solutions to the static Maxwell interface system (1.2)-(1.5).

It is very interesting for us to notice a recent related work. Some uniform a priori estimates were obtained in [23] for the H^2-smooth part of the solution to elliptic interface problems with most general interfaces allowed but piecewise constant coefficients. The methodology there is completely different from ours, and can not be extended to deal with piecewise variable coefficients as the cases treated in the current work.

The usual notations on Sobolev spaces (cf. [13, 22]) will be adopted in the sequel. For any $m \geq 0$, $H^m(\Omega)$ denotes the standard Sobolev spaces of m-th order while $H^{-m}(\Omega)$ stands for the dual space of $H_0^m(\Omega)$. The norms and semi-norms of $H^m(\Omega)$ are denoted by $\|\cdot\|_{m,\Omega}$ and $|\cdot|_{m,\Omega}$ respectively. We shall write $<\cdot,\cdot>_\Gamma$ for the dual product between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$; similarly for $<\cdot,\cdot>_{\partial\Omega}$.

For the ease of exposition, we will frequently use the notation "$\lesssim \cdots$" to denote "$\leq C\cdots$" for some generic constant $C > 0$ which depends only on the geometric properties of Ω_1 and Ω_2.

2. **Interface problems.**

In this section we shall introduce a three-dimensional elliptic interface problem and static Maxwell interface system, which will be studied.

2.1. **Elliptic interface problems.** The elliptic interface problem to be considered is of the form

$$-\nabla \cdot (\beta(x)\nabla u(x)) = f(x) \quad \text{in} \quad \Omega \tag{2.1}$$

where Ω is an open simply-connected bounded domain in R^3, and is assumed to be either a convex polyhedron or a domain with C^2-smooth boundary. We will consider that Ω is occupied by two different materials Ω_1 and Ω_2, with Ω_1 strictly lying inside Ω, see Fig. 1. The major case of our interest is that the coefficient $\beta(x)$ represents physical parameters of two different materials in Ω_1 and Ω_2, so will be only piecewise smooth and possibly may have large jumps across the interface Γ between Ω_1 and Ω_2. With such background, we assume that $\beta_i \in C^1(\bar{\Omega}_i)$ for $i = 1, 2$, and satisfies the conditions

$$c_0\bar{\beta}_i \le \beta_i(x) \le c_1\bar{\beta}_i \quad \forall\, x \in \Omega_i\,, \tag{2.2}$$

where $\bar{\beta}_1$ and $\bar{\beta}_2$ are two positive constants measuring the magnitude of $\beta_1(x)$ and $\beta_2(x)$ in Ω_1 and Ω_2 respectively. For the interface problem, one is often more interested in the case that the magnitudes of $\beta_1(x)$ and $\beta_2(x)$ are of different scales, so it is reasonable to assume that $\beta_1(x) \ne \beta_2(x)$ for all $x \in \Gamma$.

The interface $\Gamma = \partial\Omega_1$ can be of arbitrary shape but is assumed to be C^2-smooth. For any vector-valued function $\mathbf{v}$ in Ω, we shall use $\mathbf{v}_1$ and $\mathbf{v}_2$ to denote its restrictions to Ω_1 and Ω_2 respectively. The same convention is adopted for a scalar function v. And for definiteness, we shall define $[\mathbf{v}](x) = \mathbf{v}_2(x) - \mathbf{v}_1(x)$ for $x \in \Gamma$.

Physically, the solution u needs to satisfy certain interface conditions. The frequently encountered interface conditions are of the form:

$$[u] = 0, \quad [\beta\partial_{\mathbf{n}}u] = g \quad \text{on} \quad \Gamma \tag{2.3}$$

where $\mathbf{n}$ is the unit outward normal to $\partial\Omega_1$. On the exterior boundary $\partial\Omega$, we shall consider both the Dirichlet boundary condition

$$u(x) = 0 \quad \text{on} \quad \partial\Omega, \tag{2.4}$$

and the Neumann boundary condition (with $\boldsymbol{\nu}$ being the unit outward normal to $\partial\Omega$)

$$\partial_{\boldsymbol{\nu}}u(x) = 0 \quad \text{on} \quad \partial\Omega\,. \tag{2.5}$$

To ensure the solvability of the problem (2.1)-(2.3) with Neumann boundary condition (2.5), the prescribed functions f and g must satisfy the consistency condition

$$\int_\Omega f dx = \int_\Gamma g d\sigma.$$

2.2. **Static Maxwell interface system.** Another system to be studied in this work is the following static Maxwell interface problem:

$$\nabla \times \mathbf{E} = 0 \quad \text{in} \quad \Omega, \tag{2.6}$$

$$\nabla \times \mathbf{H} = \mathbf{J} \quad \text{in} \quad \Omega, \tag{2.7}$$

$$\nabla \cdot (\varepsilon(x)\mathbf{E}) = \rho \quad \text{in} \quad \Omega, \tag{2.8}$$

$$\nabla \cdot (\mu(x)\mathbf{H}) = 0 \quad \text{in} \quad \Omega \tag{2.9}$$

where Ω is an open simply-connected bounded domain in R^3, and is assumed to be either a convex polyhedron or a domain with C^2-smooth boundary. We will consider that Ω is occupied by two dielectric materials Ω_1 and Ω_2, with Ω_1 strictly lying inside Ω, see Fig. 1. $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic fields, and $\mathbf{J}$ and ρ the current and charge density. The major case of our interest is that the magnetic permeability $\mu(x)$ and the electric permittivity $\varepsilon(x)$ of the medium are discontinuous across the interface Γ between medium Ω_1 and medium Ω_2. Hence, we may write

$$\varepsilon(x) = \begin{cases} \varepsilon_1(x) & \text{in } \Omega_1, \\ \varepsilon_2(x) & \text{in } \Omega_2, \end{cases} \qquad \mu(x) = \begin{cases} \mu_1(x) & \text{in } \Omega_1, \\ \mu_2(x) & \text{in } \Omega_2, \end{cases} \tag{2.10}$$

where $\varepsilon_i(x)$ and $\mu_i(x)$ are C^1-smooth in the individual subregion $\bar{\Omega}_i$ $(i = 1, 2)$. As illustrated for the elliptic interface problem in Subsection 2.1, we may assume the following conditions for the parameters $\varepsilon_i(x)$ and $\mu_i(x)$ $(i = 1, 2)$:

$$c_0 \bar{\varepsilon}_i \le \varepsilon_i(x) \le c_1 \bar{\varepsilon}_i, \quad c_0 \bar{\mu}_i \le \mu_i(x) \le c_1 \bar{\mu}_i \quad \forall\, x \in \Omega_i, \tag{2.11}$$

where $\bar{\varepsilon}_i$ and $\bar{\mu}_i$ are positive constants, and $\varepsilon_1(x) \ne \varepsilon_2(x)$, $\mu_1(x) \ne \mu_2(x)$ for all $x \in \Gamma$.

It is well-known (cf. [9, 10]) that the electric and magnetic fields $\mathbf{E}$ and $\mathbf{H}$ should satisfy the jump conditions across the interface Γ :

$$[\mathbf{E} \times \mathbf{n}] = 0, \quad [\varepsilon \mathbf{E} \cdot \mathbf{n}] = \rho_\Gamma, \tag{2.12}$$

$$[\mathbf{H} \times \mathbf{n}] = 0, \quad [\mu \mathbf{H} \cdot \mathbf{n}] = 0, \tag{2.13}$$

where ρ_Γ is the surface charge density. We supplement the system (2.6)-(2.9) with the perfect conductor boundary conditions

$$\boldsymbol{\nu} \times \mathbf{E} = 0, \quad \boldsymbol{\nu} \cdot (\mu \mathbf{H}) = 0 \quad \text{on } \partial\Omega, \tag{2.14}$$

where $\boldsymbol{\nu}$ is the unit outward normal to $\partial\Omega$.

3. **Preliminaries.** In this section, we shall present some fundamental results from the theory of single and double layer potentials, uniqueness theorems on piecewise harmonic functions and integral representations of piecewise harmonic functions with different boundary conditions. These serve important tools in our subsequent efforts of establishing uniform a priori estimates for the solution to the elliptic interface problem (2.1)-(2.3).

3.1. **Some fundamental results about single and double layer potentials.** We begin with some basic results on single and double layer potentials. Given a simply connected domain D with Lipschitz continuous boundary ∂D, let $\mathbf{n}_x$ be the unit outward normal to ∂D at x. Then the single and double layer potentials of any density function q are respectively defined by

$$\mathcal{S}_D q(x) = \int_{\partial D} E(x-y) q(y) d\sigma_y, \quad x \in R^3,$$

$$\mathcal{D}_D q(x) = \int_{\partial D} \partial_{\mathbf{n}_y} E(x-y) q(y) d\sigma_y, \quad x \in R^3,$$

where $E(x)$ is the fundamental solution associated with the Laplacian:

$$E(x-y) = -\frac{1}{4\pi} \frac{1}{|x-y|},$$

and $d\sigma_y$ the surface measure. Note that $\mathcal{S}_D q$ (resp. $\mathcal{D}_D q$) is defined in the entire space R^3, but we will also frequently use $\mathcal{S}_D q$ (resp. $\mathcal{D}_D q$), restricted on ∂D, as a boundary integral operator on ∂D when there is no confusion caused. For a function v defined in R^3 and any $x \in \partial D$, we shall adopt

$$v^+(x) = \lim_{y \to x, y \in R^3 \setminus \bar{D}} v(y), \quad v^-(x) = \lim_{y \to x, y \in D} v(y), \quad \partial_{\mathbf{n}_x^\pm} v(x) = \lim_{t \to 0^+} \mathbf{n}_x^T \nabla v(x \pm t \mathbf{n}_x)$$

whenever the limits exist. We have the following classical trace formulas (cf. [5, 18, 26]):

$$(\mathcal{S}_D q)^{\pm}(x) = \mathcal{S}_D q(x), \tag{3.1}$$

$$\partial_{\mathbf{n}^{\pm}} \mathcal{S}_D q(x) = (\pm\frac{1}{2}I + \mathcal{K}_D^*)q(x), \tag{3.2}$$

$$(\mathcal{D}_D q)^{\pm}(x) = (\mp\frac{1}{2}I + \mathcal{K}_D)q(x), \tag{3.3}$$

$$\partial_{\mathbf{n}^+} \mathcal{D}_D q(x) = \partial_{\mathbf{n}^-} \mathcal{D}_D q(x), \tag{3.4}$$

where $\mathcal{K}_D$ is the integral operator given by

$$\mathcal{K}_D q(x) = \frac{1}{4\pi}\text{p.v.}\int_{\partial D} \frac{\langle \mathbf{n}_y, y-x\rangle}{|x-y|^3} q(y) d\sigma_y,$$

and $\mathcal{K}_D^*$ is the L^2-adjoint of $\mathcal{K}_D$,

$$\mathcal{K}_D^* q(x) = \frac{1}{4\pi}\text{p.v.}\int_{\partial D} \frac{\langle \mathbf{n}_x, x-y\rangle}{|x-y|^3} q(y) d\sigma_y\,.$$

The following two lemmas collect some properties about the operators $\mathcal{S}_D$, $\mathcal{D}_D$ and $\mathcal{K}_D^*$:

Lemma 3.1. *If D is a bounded domain with Lipschitz boundary, we have*

1. *$\mathcal{S}_D$ maps $L^2(\partial D)$ into $H^1(\partial D)$, and has a bounded inverse (cf.* [18, p. 56]*).*
2. *For any $q \in L^2(\partial D)$, there holds (cf.* [5, p. 259 and p. 280]*)*

$$\lim_{|x|\to+\infty} \mathcal{S}_D q(x) = O\big(\frac{1}{|x|}\big), \qquad \lim_{|x|\to+\infty} \mathcal{D}_D q(x) = O\big(\frac{1}{|x|^2}\big).$$

Lemma 3.2. *If ∂D is of class C^{1+r} with some $r \in (0,1)$, we have (cf.* [25, pp. 165-169]*)*

1. *$\mathcal{S}_D$ is an isomorphism from $H^t(\partial D)$ onto $H^{1+t}(\partial D)$ for $-r < t < r$.*
2. *Both $\mathcal{K}_D$ and $\mathcal{K}_D^*$ are two linear bounded operators from $H^s(\partial D)$ into $H^{t+s}(\partial D)$ for $0 \le t < r$, $-r < s \le 0$.*
3. *$\frac{1}{2}I + \mathcal{K}_D$ is an isomorphism on $W^{s,p}(\partial D)$ for $-r < s < r$, $1 < p < \infty$.*

We shall need the boundedness of the single and double layer potentials as stated in the following two lemmas.

Lemma 3.3. *If D is a bounded (not necessarily convex) polyhedron or a bounded domain with a boundary of class C^2, then $\mathcal{S}_D q$ is a bounded function in R^3 for any function $q \in H^{1/2}(\partial D)$.*

Proof. We first consider the case where D is a bounded domain with a boundary of class C^2. For any $q \in H^{1/2}(\partial D)$, it follows from Lemma 3.2 that $\mathcal{S}_D q \in H^{3/2}(\partial D)$ and hence continuous on ∂D by the Sobolev embedding theorem (cf. [1, 3]). Since $\mathcal{S}_D q$ is harmonic in D, this implies the boundedness of $\mathcal{S}_D q$ in D by the maximum principle on harmonic functions (cf. [11], [24, p. 64]). To see the boundedness of $\mathcal{S}_D q$ in $R^3\backslash\bar{D}$, it suffices to show its boundedness in $\tilde{D}\backslash\bar{D}$ with $\tilde{D}$ being a suitably large domain (containing D), due to the fact that $\lim_{|x|\to+\infty} \mathcal{S}_D q(x) = O\big(\frac{1}{|x|}\big)$. But the conclusion follows directly from the infinite differentiability of $\mathcal{S}_D q$ on $\partial\tilde{D}$ and the harmonicity of $\mathcal{S}_D q$ in $\tilde{D}\backslash\bar{D}$.

We now consider the case where D is a bounded polyhedron. We start to show the single layer $\mathcal{S}_D q$ is well-defined for each $x \in \partial D$. For any ε such that $0 < \varepsilon \ll 1$, let

$$\Gamma_\varepsilon = \{z \in \partial D; |z - x| \le \varepsilon\}.$$

Then for all ε_1 and ε_2 satisfying $0 < \varepsilon_1 < \varepsilon_2 \ll 1$, we easily see

$$\left| \int_{\partial D \backslash \Gamma_{\varepsilon_1}} E(x-z)q(z)d\sigma_z - \int_{\partial D \backslash \Gamma_{\varepsilon_2}} E(x-z)q(z)d\sigma_z \right| \le \int_{\Gamma_{\varepsilon_2}} |E(x-z)q(z)|d\sigma_z. \tag{3.5}$$

Noting $q \in H^{1/2}(\partial D)$, we know $q \in L^4(\partial D)$ by the Sobolev embedding theorem. Thus we derive by the Hölder inequality that

$$\int_{\Gamma_{\varepsilon_2}} |E(x-z)q(z)|d\sigma_z \le \frac{1}{4\pi} \| \frac{1}{|x-z|} \|_{L^{4/3}(\Gamma_{\varepsilon_2})} \|q\|_{L^4(\partial D)}. \tag{3.6}$$

As D is a polyhedron, there are only three possible locations for the point x: in the interior of a face, or on an edge or at a vertex of D. In all three cases, we can easily show by direct computations that

$$\int_{\Gamma_{\varepsilon_2}} \frac{1}{|x-z|^{4/3}} d\sigma_z \lesssim \int_0^{\varepsilon_2} \frac{1}{r^{1/3}} dr = \frac{3}{2} (\varepsilon_2)^{2/3}.$$

Applying this to the inequality (3.6), we obtain

$$\int_{\Gamma_{\varepsilon_2}} |E(x-z)q(z)|d\sigma_z \lesssim \varepsilon_2^{1/2} \|q\|_{H^{1/2}(\partial D)}\,. \tag{3.7}$$

Now this, along with (3.5), ensures the existence of the limit

$$\lim_{\varepsilon \to 0+} \int_{\partial D \backslash \Gamma_\varepsilon} E(x-z)q(z)d\sigma_z\,,$$

i.e., $\mathcal{S}_D q$ is well-defined for every $x \in \partial D$. Finally, applying the technique in [5, p. 226] and the estimate (3.7), we immediately get

$$\lim_{y \to x} \mathcal{S}_D q(y) = \mathcal{S}_D q(x).$$

This shows $\mathcal{S}_D q$ is continuous in R^3, which with the decay property of $\mathcal{S}_D q$ (cf. Lemma 3.1) leads to the boundedness of $\mathcal{S}_D q$ in R^3. □

Lemma 3.4. *If D is a bounded (not necessarily convex) polyhedron or a bounded domain with a boundary of class C^2, then $\mathcal{D}_D q$ is a bounded function in R^3 for any q, which is the restriction of some function $v \in H^2(D)$ on ∂D.*

Proof. We first consider the case that D is a bounded domain with a C^2-smooth boundary. By the trace theorem, it is clear that $q \in H^{3/2}(\partial D)$ and thus embedded in $W^{1,4}(\partial D)$. Noting that $W^{1,4}(\partial D) \hookrightarrow W^{3/4,4}(\partial D)$ and the last statement of Lemma 3.2, we know $\mathcal{K}_D q$ is in $W^{3/4,4}(\partial D)$, thus continuous on ∂D by Sobolev embedding theorem. This further implies the continuity of $(\mathcal{D}_D q)^-(x)$ on ∂D using the evaluation formula (3.3). Then using the harmonicity of $\mathcal{D}_D q(x)$ in D, $\mathcal{D}_D q(x)$ must be bounded in $\bar{D}$ by the maximum principle on harmonic functions. Applying the same argument in the domain $R^3 \backslash \bar{D}$, and noting the evaluation formula (3.3) and the last statement of Lemma 3.1, we can show the boundedness of $\mathcal{D}_D q$ in $R^3 \backslash D$.

Now consider the domain D to be a bounded polyhedron. As $v \in H^2(D)$, we know $v \in C^{0,1/2}(\bar{D})$ by the Sobolev embedding theorem (cf. [1]). Here $C^{0,1/2}(\bar{D})$ (cf. [11]) consists of all continuous functions such that for all $x, y \in \bar{D}$,

$$|v(x) - v(y)| \lesssim |x-y|^{1/2} \|v\|_{C^{0,1/2}(\bar{D})},$$

with $\|\cdot\|_{C^{0,1/2}(\bar{D})}$ being the Hölder norm. Therefore, $q(x) \in C(\partial D)$. Let $\{F_i\}_{i=1}^M$ be M disjoint faces of D, then

$$\mathcal{K}_D q(x) = \frac{1}{4\pi} \sum_{i=1}^{M} \text{p.v.} \int_{F_i} \frac{\langle \mathbf{n}_y, y-x \rangle}{|x-y|^3} q(y) d\sigma_y. \tag{3.8}$$

Using a similar argument as for proving Theorem 6.5.2 in [5, pp. 231-236], we find that each function on the right-hand side of (3.8) is continuous in R^3, so is the function $\mathcal{K}_D q$. Now the desired boundedness follows from the same argument as used in the first part. □

3.2. **Uniqueness about piecewise harmonic functions.** In this subsection, we present two uniqueness results, which will play an important role in the justification of an integral representation of piecewise harmonic functions.

Theorem 3.1. *Let v be a function in R^3 with $\tilde{v}_1$ and $\tilde{v}_2$ being its restrictions to Ω_1 and $R^3 \backslash \bar{\Omega}_1$ respectively. Assume $\tilde{v}_1 \in H^2(\Omega_1)$, and $\tilde{v}_1$, $\tilde{v}_2$ solve the problem:*

$$\begin{aligned} \Delta \tilde{v}_1 &= 0 \quad \text{in } \Omega_1, \\ \Delta \tilde{v}_2 &= 0 \quad \text{in } R^3 \backslash \bar{\Omega}_1, \\ \tilde{v}_1(x) &= \tilde{v}_2(x),\ \beta_2(x) \partial_{\mathbf{n}} \tilde{v}_2 = \beta_1(x) \partial_{\mathbf{n}} \tilde{v}_1 \quad \text{on } \Gamma, \end{aligned} \tag{3.9}$$

$$\lim_{|x| \to +\infty} |v(x)| = O\big(\frac{1}{|x|}\big). \tag{3.10}$$

Then v is identically zero in R^3.

Proof. Let $p(x) = \tilde{v}_1(x) = \tilde{v}_2(x)$ on Γ. As $\tilde{v}_1 \in H^2(\Omega_1)$, so $p(x) \in H^{3/2}(\Gamma)$. By virtue of Lemma 3.2, there exists a unique density function $q(x)$ in $H^{1/2}(\Gamma)$ such that

$$\mathcal{S}_{\Omega_1} q(x) = p(x) \quad \text{on } \Gamma.$$

Then we get

$$v(x) \equiv \mathcal{S}_{\Omega_1} q(x) \quad \text{in } R^3,$$

since a harmonic function is uniquely determined by its boundary values. Now using the evaluation formula (3.2), we can write the second interface condition in (3.9) as

$$\beta_2(x)\{\frac{1}{2} q(x) + \mathcal{K}^*_{\Omega_1} q(x)\} - \beta_1(x)\{-\frac{1}{2} q(x) + \mathcal{K}^*_{\Omega_1} q(x)\} = 0,$$

which implies

$$q(x) = \frac{2(\beta_1(x) - \beta_2(x))}{\beta_1(x) + \beta_2(x)} \mathcal{K}^*_{\Omega_1} q(x) \quad \text{on } \Gamma. \tag{3.11}$$

Noting $q(x) \in H^{1/2}(\Gamma) \hookrightarrow L^2(\Gamma)$, we know from Lemma 3.2 that

$$\mathcal{K}^*_{\Omega_1} q(x) \in H^t(\Gamma), \quad \forall\, t \in [0,1),$$

which, together with (3.11) and the regularity assumptions on $\beta_1(x)$ and $\beta_2(x)$, yields

$$q(x) \in H^t(\Gamma), \quad \forall\, t \in [0,1).$$

Using this and Lemma 3.2, we know on the interface Γ,

$$p(x) = \mathcal{S}_{\Omega_1} q(x) \in H^{1+t}(\Gamma), \quad \forall\, t \in [0,1).$$

Taking $t = t^* = \frac{7}{8}$, then by the Sobolev embedding theorem we have

$$\mathcal{S}_{\Omega_1} q \in H^{1+t^*}(\Gamma) = W^{\frac{3}{2}+(t^*-\frac{1}{2}),2}(\Gamma) \hookrightarrow W^{\frac{3}{2},s^*}(\Gamma)$$

with $s^* = \frac{16}{5} > 3$.

Now applying the regularity theory for elliptic problems (cf. [11, p. 241]), we obtain

$$\mathcal{S}_{\Omega_1} q \in W^{2,s^*}(\Omega_1).$$

Then by the Sobolev embedding theorem again we know

$$\tilde{v}_1 = \mathcal{S}_{\Omega_1} q \in C^{1,r^*}(\bar{\Omega}_1)$$

with $r^* = 1 - 3/s^*$. In the same manner, we can show that $\tilde{v}_2 = \mathcal{S}_{\Omega_1} q \in C^{1,r^*}(\tilde{D} \backslash \bar{\Omega}_1)$ for any domain $\tilde{D}$ such that $\Omega_1 \subset\subset \tilde{D}$ by noting the infinite differentiability of $\mathcal{S}_{\Omega_1} q$ in $\tilde{D} \backslash \bar{\Omega}_1$.

We are now ready to show the desired result, using the classical maximum principle. We first assume that both functions $\tilde{v}_1$ and $\tilde{v}_2$ are not constant functions. By the maximum principle on harmonic functions, both $\tilde{v}_1$ and $\tilde{v}_2$ must take their maxima at a common point x_0 on Γ. But by the Hopf's maximum principle on harmonic functions (cf. [11], [24, p. 65]), we further have

$$\partial_{\mathbf{n}} \tilde{v}_1(x_0) > 0, \; \partial_{\mathbf{n}} \tilde{v}_2(x_0) < 0,$$

which contradicts with the second interface condition in (3.9). Hence it is possible that either $\tilde{v}_1$ or $\tilde{v}_2$ is a constant function.

If $\tilde{v}_1$ is constant in Ω_1, then the second interface condition in (3.9) gives $\partial_{\mathbf{n}} \tilde{v}_2 = 0$ on Γ. This with the decay condition (3.10) implies $\tilde{v}_2 \equiv 0$ in $R^3 \backslash \bar{\Omega}_1$, since $\tilde{v}_2$ is harmonic in $R^3 \backslash \bar{\Omega}_1$. Then the first interface condition also tells $\tilde{v}_1 = 0$ on Γ. But $\tilde{v}_1$ is harmonic in Ω_1, that proves $\tilde{v}_1 \equiv 0$ in Ω_1.

On the other hand, if $\tilde{v}_2$ is constant, then $\tilde{v}_2 \equiv 0$ in $R^3 \backslash \bar{\Omega}_1$ by the decay condition (3.10). Clearly we also have $\tilde{v}_1 = 0$ on Γ from the first interface condition, then $\tilde{v}_1$ must be identically zero as it is harmonic in Ω_1. □

Borrowing the proof of Theorem 3.1, we can now show the unique solvability of the following integral equation, which will be essential to our subsequent analysis:

Lemma 3.5. *The integral equation*

$$\left(\frac{\beta_1 + \beta_2}{2(\beta_1 - \beta_2)} I - \mathcal{K}^*_{\Omega_1} \right) \phi = h$$

is uniquely solvable on $L^2(\Gamma)$ *and* $H^{1/2}(\Gamma)$.

Proof. By Lemma 3.2 and noting that $H^t(\Gamma)$ ($\frac{1}{2} < t < 1$) is compactly embedded in $H^{1/2}(\Gamma)$ and $L^2(\Gamma)$, $\mathcal{K}^*_{\Omega_1}$ is a compact operator on $L^2(\Gamma)$ and $H^{1/2}(\Gamma)$ respectively. Hence, by the Fredholm theory for linear operators (cf. [5, p. 111]), $\frac{\beta_1+\beta_2}{2(\beta_1-\beta_2)} I - \mathcal{K}^*_{\Omega_1}$ is a Fredholm operator with zero index. Therefore, for Lemma 3.5 it suffices to prove

$$\mathrm{Ker} \left(\frac{\beta_1 + \beta_2}{2(\beta_1 - \beta_2)} I - \mathcal{K}^*_{\Omega_1} \right) = \{0\}.$$

Assume that q is an element in the above kernel and let $v = S_{\Omega_1} q$, then one can derive $v = 0$ following the same argument as used in the proof of Theorem 3.1 (see the derivations there starting from (3.11)). Now, $q = 0$ is a consequence of the isomorphism of S_{Ω_1} (cf. Lemma 3.2). □

We end up this subsection with another uniqueness result about piecewise harmonic functions, which is a direct consequence of Theorem 3.1.

Theorem 3.2. *Let $v \in H^1(R^3)$ be a bounded function in R^3, with $\tilde{v}_1$, $\tilde{v}_2$ and $\tilde{v}_3$ being its restrictions respectively to Ω_1, Ω_2 and $R^3\backslash\bar{\Omega}$. Assume that $\tilde{v}_1 \in H^2(\Omega_1)$, $\tilde{v}_1$, $\tilde{v}_2$ and $\tilde{v}_3$ solve the following interface problem:*

$$\Delta v = 0 \quad \text{in } \Omega_1 \cup \Omega_2 \cup (R^3\backslash\bar{\Omega}), \tag{3.12}$$

$$\tilde{v}_1(x) = \tilde{v}_2(x),\ \beta_2(x)\partial_{\mathbf{n}}\tilde{v}_2 = \beta_1(x)\partial_{\mathbf{n}}\tilde{v}_1 \quad \text{on } \Gamma, \tag{3.13}$$

$$\tilde{v}_2(x) = \tilde{v}_3(x),\ \partial_{\boldsymbol{\nu}}\tilde{v}_3 = \partial_{\boldsymbol{\nu}}\tilde{v}_2 \quad \text{on } \partial\Omega, \tag{3.14}$$

$$\lim_{|x|\to+\infty} |v(x)| = O\big(\frac{1}{|x|}\big). \tag{3.15}$$

Then v is identically zero in R^3.

Proof. By the usual arguments for removable singularity on harmonic functions (cf. [24, pp. 101-102]) and noting the interface conditions (3.14), we know that v is actually harmonic in the domain $R^3\backslash\bar{\Omega}_1$. This fact, along with (3.12), (3.13) and (3.15) shows $v \equiv 0$ in R^3 by virtue of Theorem 3.1. □

3.3. **Integral representation.** Using the uniqueness results in Subsection 3.2, we are now able to give an integral representation of the solution to the elliptic interface problem:

Given a function $h(x) \in H^{1/2}(\Gamma)$, find $v \in H_0^1(\Omega)$ such that

$$\Delta v = 0 \quad \text{in } \ \Omega_1 \cup \Omega_2\ , \tag{3.16}$$

$$[v] = 0,\ [\beta(x)\partial_{\mathbf{n}}v] = h \quad \text{on } \ \Gamma\,. \tag{3.17}$$

Theorem 3.3. *The solution v of the problem (3.16)-(3.17) can be characterized as*

$$v(x) = (\mathcal{S}_{\Omega_1}\phi)(x) - (\mathcal{S}_{\Omega}\psi)(x), \quad x \in \Omega, \tag{3.18}$$

where $\psi = \partial_{\boldsymbol{\nu}} v$ on $\partial\Omega$, and $\phi \in H^{1/2}(\Gamma)$ solves the integral equation:

$$\left(\frac{\beta_1+\beta_2}{2(\beta_1-\beta_2)}I - \mathcal{K}^*_{\Omega_1}\right)\phi = -\partial_{\mathbf{n}}(\mathcal{S}_{\Omega}\psi) + \frac{1}{\beta_1-\beta_2}h \quad \text{on } \ \Gamma\,. \tag{3.19}$$

Proof. It is easy to see that the right-hand side of (3.19) lies in $H^{1/2}(\Gamma)$. By Lemma 3.5, there exists a unique solution $\phi \in H^{1/2}(\Gamma)$ to the integral equation (3.19).

Applying Theorem 3.2, we know that the following interface problem has at most one bounded solution in $H^1(R^3)$ with $v_1 \in H^2(\Omega_1)$:

$$\Delta v = 0 \quad \text{in } \Omega_1 \cup \Omega_2 \cup (R^3 \setminus \bar{\Omega}), \tag{3.20}$$

$$v_1(x) = v_2(x),\ \beta_2(x)\partial_{\mathbf{n}}v_2 = \beta_1(x)\partial_{\mathbf{n}}v_1 + h \quad \text{on } \Gamma, \tag{3.21}$$

$$v_2 = v_3, \quad \partial_{\boldsymbol{\nu}}v_2 = \partial_{\boldsymbol{\nu}}v_3 + \psi \quad \text{on } \partial\Omega. \tag{3.22}$$

But by the evaluation formulas (3.1)-(3.2) and equation (3.19), one can check directly that

$$R_1(x) = \mathcal{S}_{\Omega_1}\phi(x) - \mathcal{S}_{\Omega}\psi(x)$$

is a solution to the system (3.20)-(3.22). Noting $\psi = \partial_{\boldsymbol{\nu}} v$, and $v_1 \in H^2(\Omega_1)$ and $v_2 \in H^2(\Omega_2)$ by the regularity theory for elliptic interface problems (cf. [15]), we know $\psi \in H^{1/2}(\partial\Omega)$. This with the fact that $\phi \in H^{1/2}(\Gamma)$ shows $R_1(x)$ is a bounded function in R^3 by Lemma 3.3, and its restriction to Ω_1 lies in $H^2(\Omega_1)$ by Lemma 3.2 and the trace theorem. On the other hand, one can easily verify that

$$R_2(x) = \begin{cases} v(x) & \text{in } \Omega, \\ 0 & \text{otherwise,} \end{cases}$$

is also a solution to the system (3.20)-(3.22). As $v_1 \in H^2(\Omega_1)$ and $v_2 \in H^2(\Omega_2)$, $R_2(x)$ is a bounded function in R^3. Thus $R_2(x)$ is another bounded solution to (3.20)-(3.22) with its restriction to Ω_1 lying in $H^2(\Omega_1)$. By the uniqueness of solutions to (3.20)-(3.22), we have $R_1 = R_2$ in R^3, and this confirms the relation (3.18). □

Following the same arguments as used in Theorem 3.3, and making use of Lemmas 3.1-3.4 and evaluation formulas (3.1)-(3.4), we can also establish an integral representation of the solution to the next elliptic interface problem:

Given a function $h(x) \in H^{1/2}(\Omega)$, find $v \in H^1(\Omega)$ such that $\partial_{\boldsymbol{\nu}} v = 0$ on $\partial\Omega$ and

$$\Delta v = 0 \quad \text{in } \Omega_1 \cup \Omega_2\,, \tag{3.23}$$

$$[v] = 0,\ [\beta(x)\partial_{\mathbf{n}} v] = h \quad \text{on } \Gamma\,. \tag{3.24}$$

Theorem 3.4. *The solution v of the problem (3.23)-(3.24) can be characterized as*

$$v(x) = (\mathcal{S}_{\Omega_1}\phi)(x) + (\mathcal{D}_\Omega\psi)(x), \quad x \in \Omega,$$

where $\psi = v$ on $\partial\Omega$, and $\phi \in H^{1/2}(\Gamma)$ solves the integral equation

$$\left(\frac{\beta_1+\beta_2}{2(\beta_1-\beta_2)} I - \mathcal{K}^*_{\Omega_1}\right)\phi = \partial_{\mathbf{n}}(\mathcal{D}_\Omega\psi) + \frac{1}{\beta_1-\beta_2} h \quad \text{on } \ \Gamma\,.$$

Remark 3.1. The integral representations in Theorems 3.3-3.4 were initiated by [16], where the representation in the piecewise constant coefficient case was discussed, and served as an essential tool in a different context, i.e., numerical identification of piecewise constant conductivity coefficients.

4. **Uniform a priori estimates for elliptic interface problems.** With preparations in Subsections 3.1-3.3, we are now ready to establish the main results of this paper, i.e., uniform a priori estimates for the solutions to the elliptic interface problem (2.1)-(2.3) with both Dirichlet and Neumann boundary conditions (2.4) and (2.5). We shall derive the a priori H^1-estimates in the next subsection and the H^2-estimates in Subsection 4.2.

4.1. **H^1-estimates.** We first present an auxiliary lemma which will be important to our subsequent analysis.

Lemma 4.1. *Let v be a function in $H^1(\Omega)$ such that $\int_{\partial\Omega} v d\sigma = 0$ and it satisfies*

$$-\nabla\cdot(\beta_i(x)\nabla v_i) = 0 \text{ in } \ \Omega_i \tag{4.1}$$

for $i = 1, 2$, then it holds that

$$\|\nabla v_1\|_{0,\Omega_1} \lesssim \|\nabla v_2\|_{0,\Omega_2}\,. \tag{4.2}$$

Proof. When the coefficient $\beta(x)$ is piecewise constant and $v \in H^1_0(\Omega)$, the estimate (4.2) follows immediately from the basic harmonic extension property (cf. [27]). For the general case, since the function v has zero integral average over $\partial\Omega$, we have by the trace theorem and the Poincaré inequality (cf. [1]) that

$$\|v_2\|_{1/2,\Gamma} \lesssim \|v_2\|_{1,\Omega_2} \lesssim \|\nabla v_2\|_{0,\Omega_2} + \left|\int_{\partial\Omega} v_2 d\sigma\right| = \|\nabla v_2\|_{0,\Omega_2}. \tag{4.3}$$

On the other hand, using the inverse trace inequality (cf. [1]), we can find a function $\hat{v}_1 \in H^1(\Omega_1)$ such that it equals v_1 on Γ and admits the estimate

$$\|\hat{v}_1\|_{1,\Omega_1} \lesssim \|v_1\|_{1/2,\Gamma}. \tag{4.4}$$

Noting (4.1) for v_1, we see $v_1 - \hat{v}_1 \in H_0^1(\Omega_1)$ and solves

$$-\nabla \cdot (\beta_1 \nabla (v_1 - \hat{v}_1)) = \nabla \cdot (\beta_1 \nabla \hat{v}_1) \quad \text{in } \Omega_1,$$

or equivalently,

$$\int_{\Omega_1} \beta_1 \nabla (v_1 - \hat{v}_1) \cdot \nabla e_1 dx = \int_{\Omega_1} \beta_1 \nabla \hat{v}_1 \cdot \nabla e_1 \, dx \quad \forall \, e_1 \in H_0^1(\Omega_1). \tag{4.5}$$

Taking $e_1 = v_1 - \hat{v}_1$ in (4.5), then using (2.2) and (4.4), we obtain

$$\bar{\beta}_1 |v_1 - \hat{v}_1|_{1,\Omega_1} \lesssim \bar{\beta}_1 \|\nabla \hat{v}_1\|_{0,\Omega_1} \lesssim \bar{\beta}_1 \|v_1\|_{1/2,\Gamma} \,,$$

hence

$$|v_1|_{1,\Omega_1} \le |\hat{v}_1|_{1,\Omega_1} + |v_1 - \hat{v}_1|_{1,\Omega_1} \lesssim \|v_1\|_{1/2,\Gamma} \,.$$

The combination of this, (4.3) and the fact that $v_1 = v_2$ on Γ implies (4.2) immediately. □

Remark 4.1. It is important to remark that inequality (4.2) does not hold when v_1 and v_2 are swapped. This can be verified from the following simple example: let v be a function in $H_0^1(\Omega)$ such that $v_1(x) \equiv 1$ in Ω_1 and v_2 is the solution of the following problem:

$$-\nabla \cdot (\beta_2(x) \nabla v_2) = 0 \quad \text{in} \quad \Omega_2; \quad v_2(x) = 1 \quad \text{on} \quad \Gamma.$$

We are now ready to establish the desired a priori estimates in H^1-norm. For this, we introduce a constant $\bar{k}(\beta) = \bar{\beta}_2/\bar{\beta}_1$. Clearly, $\bar{k}(\beta)$ measures the discrepancy between the coefficients $\beta_1(x)$ and $\beta_2(x)$. When no confusion is caused, we shall write $\bar{k}$ for $\bar{k}(\beta)$, and for any $w \in H^{-1}(\Omega)$ or $w \in (H^1(\Omega))'$, we may write the norm of w simply as $\|w\|_{-1,\Omega}$.

Theorem 4.1. *Assume that u is a solution to the interface problem (2.1)-(2.3) with Dirichlet boundary condition (2.4) or Neumann boundary condition (2.5), $g \in H^{-1/2}(\Gamma)$, and $f \in H^{-1}(\Omega)$ when (2.4) holds and $f \in (H^1(\Omega))'$ when (2.5) holds. Then*

$$\bar{\beta}_2 \|\nabla u_2\|_{0,\Omega_2} \lesssim \|g\|_{-1/2,\Gamma} + \|f\|_{-1,\Omega}, \tag{4.6}$$

$$\bar{\beta}_1 \|\nabla u_1\|_{0,\Omega_1} \lesssim \|f\|_{-1,\Omega_1} + \bar{k}^{-1} \big(\|g\|_{-1/2,\Gamma} + \|f\|_{-1,\Omega}\big). \tag{4.7}$$

Proof. We first prove for Neumann boundary condition (2.5). Observing that the solution u is unique up to an additive constant, it suffices to derive the estimates (4.6) and (4.7) for the solution u with vanishing average on $\partial\Omega$:

$$\int_{\partial\Omega} u \, d\sigma = 0. \tag{4.8}$$

By the Poincaré inequality, we have for all $v_2 \in H^1(\Omega_2)$ satisfying (4.8) that

$$\|v_2\|_{1,\Omega_2} \lesssim \|\nabla v_2\|_{0,\Omega_2}. \tag{4.9}$$

Next, we assume $f \in L^2(\Omega)$ and introduce two auxiliary functions $\tilde{u}_i \in H_0^1(\Omega_i)$ satisfying

$$-\nabla \cdot (\beta_i(x) \nabla \tilde{u}_i) = f_i, \quad i = 1, 2. \tag{4.10}$$

It is clear that $u_i \in H^2(\Omega_i)$ (cf. [11, 13]) and there hold

$$\int_{\Omega_i} \beta_i(x)\nabla \tilde{u}_i \cdot \nabla v_i dx = \int_{\Omega_i} f_i\, v_i dx, \quad \forall v_i \in H_0^1(\Omega_i), \quad i = 1, 2.$$

Taking $v_i = \tilde{u}_i$ above and noting the assumption (2.2), we have

$$\bar{\beta}_i \|\tilde{u}_i\|_{1,\Omega_i} \lesssim \|f_i\|_{-1,\Omega_i}. \tag{4.11}$$

Let $\bar{u}_i = u_i - \tilde{u}_i$, $i = 1, 2$. It is easy to show from (4.10) that

$$-\nabla \cdot (\beta_i(x)\nabla \bar{u}_i(x)) = 0 \quad \text{in } \Omega_i, \quad i = 1, 2, \tag{4.12}$$

$$[\bar{u}] = 0, \quad [\beta \partial_{\mathbf{n}} \bar{u}] = g + g_1 \quad \text{on } \Gamma, \tag{4.13}$$

$$\beta_2 \partial_{\boldsymbol{\nu}} \bar{u}(x) = -\beta_2 \partial_{\boldsymbol{\nu}} \tilde{u}_2 \quad \text{on } \partial\Omega, \tag{4.14}$$

where $g_1 = \beta_1 \partial_{\mathbf{n}} \tilde{u}_1 - \beta_2 \partial_{\mathbf{n}} \tilde{u}_2$. The variational form of (4.12)-(4.14) is

$$\sum_{i=1}^{2} \int_{\Omega_i} \beta_i \nabla\, \bar{u}_i \cdot \nabla\, v_i dx = - < g + g_1, v >_\Gamma - < \beta_2 \partial_{\boldsymbol{\nu}} \tilde{u}_2, v >_{\partial\Omega} \quad \forall\, v \in H^1(\Omega)\,. \tag{4.15}$$

Taking $v = \bar{u}$ in the above equation and noting the fact that $\int_{\partial\Omega} \bar{u}_2 d\sigma = 0$, we have by (4.9) and the definition of norms of linear functionals that

$$\bar{\beta}_2 \|\nabla \bar{u}_2\|_{0,\Omega_2} \lesssim \|g\|_{-1/2,\Gamma} + \|g_1\|_{-1/2,\Gamma} + \|\beta_2 \partial_{\boldsymbol{\nu}} \tilde{u}_2\|_{-1/2,\Gamma}. \tag{4.16}$$

To estimate the last two terms in (4.16), for any $\eta \in H^{1/2}(\Gamma)$ we introduce v_η to be a function in $H_0^1(\Omega)$ satisfying

$$\Delta v_\eta = 0 \quad \text{in } \Omega_1 \cup \Omega_2\,; \quad v_\eta = \eta \quad \text{on } \Gamma.$$

Owing to the Green's formula (2.17) in [12, p. 28] and (4.10) we have

$$\begin{aligned}
\langle g_1, \eta \rangle_\Gamma =& \langle \beta_1 \partial_{\mathbf{n}} \tilde{u}_1, v_\eta \rangle_\Gamma - \langle \beta_2 \partial_{\mathbf{n}} \tilde{u}_2, v_\eta \rangle_\Gamma \\
=& \int_{\Omega_1} \{\nabla \cdot (\beta_1 \nabla\, \tilde{u}_1) v_\eta + \beta_1 \nabla\, \tilde{u}_1 \cdot \nabla v_\eta\} dx \\
&+ \int_{\Omega_2} \{\nabla \cdot (\beta_2 \nabla \tilde{u}_2) v_\eta + \beta_2 \nabla\, \tilde{u}_2 \cdot \nabla v_\eta\} dx \\
=& - \int_\Omega f v_\eta dx + \sum_{i=1}^{2} \int_{\Omega_i} \beta_i \nabla\, \tilde{u}_i \cdot \nabla v_\eta dx.
\end{aligned}$$

Using (4.11) and the basic estimate $\|v_\eta\|_{1,\Omega} \lesssim \|\eta\|_{1/2,\Gamma}$ (cf. [27]), we are further led to

$$\begin{aligned}
|\langle g_1, \eta \rangle_\Gamma| &\lesssim \|f\|_{-1,\Omega} \|v_\eta\|_{1,\Omega} + \|\nabla v_\eta\|_{0,\Omega} \sum_{i=1}^{2} \|\beta_i \nabla \tilde{u}_i\|_{0,\Omega_i} \\
&\lesssim \{\|f\|_{-1,\Omega} + \sum_{i=1}^{2} \bar{\beta}_i \|\nabla \tilde{u}_i\|_{0,\Omega_i}\} \|\eta\|_{1/2,\Gamma} \\
&\lesssim \{\|f\|_{-1,\Omega} + \sum_{i=1}^{2} \|f\|_{-1,\Omega_i}\} \|\eta\|_{1/2,\Gamma} \lesssim \|f\|_{-1,\Omega} \|\eta\|_{1/2,\Gamma},
\end{aligned}$$

which implies

$$\|g_1\|_{-1/2,\Gamma} \lesssim \|f\|_{-1,\Omega}. \tag{4.17}$$

Similarly, we can deduce

$$\|\beta_2 \partial_{\boldsymbol{\nu}} \tilde{u}_2\|_{-1/2,\Gamma} \lesssim \|f\|_{-1,\Omega}. \tag{4.18}$$

Combining the last two inequalities with (4.11) and (4.16), we find

$$\bar{\beta}_2 \|\nabla u_2\|_{0,\Omega_2} \lesssim \|g\|_{-1/2,\Gamma} + \|f\|_{-1,\Omega},$$

which proves (4.6).

By Lemma 4.1 and (4.16)-(4.18) we derive

$$\bar{\beta}_1 \|\nabla \bar{u}_1\|_{0,\Omega_1} \lesssim \bar{k}^{-1} \bar{\beta}_2 \|\nabla \bar{u}_2\|_{0,\Omega_2} \lesssim \bar{k}^{-1} \left(\|g\|_{-1/2,\Gamma} + \|f\|_{-1,\Omega}\right),$$

which, together with (4.11), proves (4.7).

Now, the desired estimates for the general $f \in (H^1(\Omega))'$ can be obtained by the established results (4.6) and (4.7) for $f \in L^2(\Omega)$ and the usual density argument (cf. [1]).

The situation with Dirichlet boundary condition (2.4) can be handled in a same way as for the Neumann case. We first assume $f \in L^2(\Omega)$ and introduce two auxiliary functions $\tilde{u}_i \in H_0^1(\Omega_i)$ $(i = 1, 2)$ satisfying (4.10) and then define $\bar{u}_i = u_i - \tilde{u}_i$ $(i = 1, 2)$. Then following the same derivations as for getting (4.15), we can show that $\bar{u} \in H_0^1(\Omega)$ satisfies the variational equation

$$\sum_{i=1}^{2} \int_{\Omega_i} \beta_i \nabla \bar{u}_i \cdot \nabla v_i dx = - < g + g_1, v >_\Gamma \quad \forall v \in H_0^1(\Omega).$$

Letting $v = \bar{u}$ and using (4.9) we know

$$\bar{\beta}_2 \|\nabla \bar{u}_2\|_{0,\Omega_2} \lesssim \|g\|_{-1/2,\Gamma} + \|g_1\|_{-1/2,\Gamma}, \tag{4.19}$$

which, in combination with (4.11) and (4.17), leads to (4.6). (4.7) follows from Lemma 4.1, (4.11), (4.17) and (4.19). The results for the general case $f \in H^{-1}(\Omega)$ can be obtained by the density argument. □

4.2. **H^2-estimates.** This subsection is devoted to the uniform a priori H^2-estimates for the solution u to the interface problem (2.1)-(2.3), with either Dirichlet boundary condition (2.4) or Neumann boundary condition (2.5). To begin with, we introduce two parameters

$$d_1(\beta) = \frac{|\beta_1|_{1,\infty,\Omega_1}}{\bar{\beta}_1}, \quad d_2(\beta) = \frac{|\beta_2|_{1,\infty,\Omega_2}}{\bar{\beta}_2}$$

to measure the relative oscillation of the coefficient $\beta(x)$ in each individual subregion, Ω_1 and Ω_2. For the ease of exposition, we shall assume that

$$d_1(\beta) \le \tilde{c}_1, \quad d_2(\beta) \le \tilde{c}_2, \tag{4.20}$$

where $\tilde{c}_1$ and $\tilde{c}_2$ are two positive constants independent of $\beta_1(x)$ and $\beta_2(x)$. By the mean value theorem, the assumption (4.20) implies

$$\frac{|\beta_i(x) - \beta_i(y)|}{\bar{\beta}_i \, |x - y|} \le \tilde{c}_i \qquad \forall\, x, y \in \Omega_i \text{ with } x \ne y$$

for $i = 1, 2$. So the relative oscillation of $\beta(x)$ in each subregion Ω_1 and Ω_2 is bounded independent of $\beta(x)$. Assumption (4.20) only helps avoid some unnecessary technical complications in the subsequent estimates. In fact, by slightly more careful derivations, one can achieve explicit dependence on $d_1(\beta)$ and $d_2(\beta)$ in the a priori estimates which follow.

The next estimates will be frequently used and can be checked directly according to the definition of fractional Sobolev norms:

Lemma 4.2. *For any $F \in W^{1,\infty}(\Gamma)$, $G_1 \in H^{1/2}(\Gamma)$ and $G_2 \in H^{-1/2}(\Gamma)$, one has $FG_1 \in H^{1/2}(\Gamma)$ and $FG_2 \in H^{-1/2}(\Gamma)$, and there hold the estimates:*

$$\|FG_1\|_{1/2,\Gamma} \lesssim \|F\|_{1,\infty,\Gamma}\|G_1\|_{1/2,\Gamma}, \quad \|FG_2\|_{-1/2,\Gamma} \lesssim \|F\|_{1,\infty,\Gamma}\|G_2\|_{-1/2,\Gamma}.$$

Now we start with the H^2-estimates for the interface system (2.1)-(2.3) with Neumann boundary condition (2.5). For the purpose, we introduce two auxiliary functions w_1 and w_2 such that $w_1 \in H_0^1(\Omega_1)$ and satisfies

$$-\Delta w_1 = \frac{1}{\beta_1}\{f_1 + \nabla\beta_1 \cdot \nabla u_1\} \quad \text{in} \quad \Omega_1, \tag{4.21}$$

while $w_2 \in H^1(\Omega_2)$ satisfies $w_2 = 0$ on Γ, $\partial_{\boldsymbol{\nu}} w_2 = 0$ on $\partial\Omega$ and

$$-\Delta w_2 = \frac{1}{\beta_2}\{f_2 + \nabla\beta_2 \cdot \nabla u_2\} \quad \text{in} \quad \Omega_2, \tag{4.22}$$

where u_i and f_i are restrictions of solution u and function f to Ω_i $(i = 1, 2)$ respectively. By the standard a priori estimates for elliptic problems, we have

$$\|w_i\|_{2,\Omega_i} \lesssim \|\frac{1}{\beta_i}\{f + \nabla\beta_i \cdot \nabla u_i\}\|_{0,\Omega_i} \lesssim \frac{1}{\bar{\beta}_i}(\|f\|_{0,\Omega_i} + |\beta_i|_{1,\infty,\Omega_i}\|\nabla u_i\|_{0,\Omega_i}). \tag{4.23}$$

Let $\bar{w}_i = u_i - w_i$ in Ω_i and $\tilde{g} = \beta_1\partial_{\mathbf{n}} w_1 - \beta_2\partial_{\mathbf{n}} w_2$ on Γ, it is easy to see that $\partial_{\boldsymbol{\nu}}\bar{w}_2 = 0$ on $\partial\Omega$ and

$$\Delta\bar{w}_i = 0 \quad \text{in} \quad \Omega_i, \quad i = 1, 2, \tag{4.24}$$

$$[\bar{w}] = 0, \quad [\beta\partial_{\mathbf{n}}\bar{w}] = g + \tilde{g} \quad \text{on } \Gamma\,. \tag{4.25}$$

For uniqueness of the solution to (4.24)-(4.25), we consider the solution satisfying

$$\int_\Gamma \bar{w}_2 d\sigma = 0. \tag{4.26}$$

We shall need the next estimate for $\tilde{g}$, which follows from Lemma 4.2 and the trace theorem:

$$\|\tilde{g}\|_{1/2,\Gamma} \lesssim \sum_{i=1}^{2}(\bar{\beta}_i + |\beta_i|_{1,\infty,\Omega_i})\|w_i\|_{2,\Omega_i}. \tag{4.27}$$

Theorem 4.2. *Assume that $f \in L^2(\Omega)$, $g \in H^{1/2}(\Gamma)$, and u is the solution to the interface problem (2.1)-(2.3) with Neumann boundary condition (2.5), then the following a priori estimates hold:*

$$\bar{\beta}_1\|u_1\|_{2,\Omega_1} \lesssim \big(1 + \bar{k}^{-1}(\beta)\big)(\|f\|_{0,\Omega} + \|g\|_{1/2,\Gamma}), \tag{4.28}$$

$$\bar{\beta}_2\|u_2\|_{2,\Omega_2} \lesssim \big(1 + \bar{k}^{-1}(\beta)\big)(\|f\|_{0,\Omega} + \|g\|_{1/2,\Gamma})\,. \tag{4.29}$$

Proof. By Theorem 3.4, we can represent $\bar{w}_1$ and $\bar{w}_2$ in (4.24)-(4.25) as

$$\bar{w}(x) = (\mathcal{S}_{\Omega_1}\phi)(x) + (\mathcal{D}_\Omega\psi)(x), \quad \forall\, x \in \Omega, \tag{4.30}$$

where $\psi = \bar{w}$ on $\partial\Omega$, and $\phi \in H^{1/2}(\Gamma)$ solves the integral equation

$$\left(\frac{\beta_1 + \beta_2}{2(\beta_1 - \beta_2)}I - \mathcal{K}^*_{\Omega_1}\right)\phi = \partial_{\mathbf{n}}(\mathcal{D}_\Omega\psi) + \frac{g + \tilde{g}}{\beta_1 - \beta_2} \quad \text{on} \quad \Gamma\,. \tag{4.31}$$

Noting that $\bar{w}$ is harmonic in both Ω_1 and Ω_2, we drive from (4.30) that

$$\|\bar{w}_1\|_{2,\Omega_1} + \|\bar{w}_2\|_{2,\Omega_2} \lesssim \|\bar{w}\|_{3/2,\Gamma} \lesssim \|\mathcal{S}_{\Omega_1}\phi\|_{3/2,\Gamma} + \|\mathcal{D}_\Omega\psi\|_{3/2,\Gamma}. \tag{4.32}$$

But for $\mathcal{S}_{\Omega_1}\phi$ we get by Lemma 3.2 that

$$\|\mathcal{S}_{\Omega_1}\phi\|_{3/2,\Gamma} \lesssim \|\phi\|_{1/2,\Gamma}. \tag{4.33}$$

To estimate $\mathcal{D}_\Omega\psi$, we note for any C^2-smooth surface $\Gamma' \subset\subset \Omega$ and $x \in \Gamma'$, the kernel function $\partial_{\mathbf{n}_y}E(x-y)$ of the operator $\mathcal{D}_\Omega$ is C^∞-smooth. So it is easy to see from the definition of $\mathcal{D}_\Omega$ that

$$\|\mathcal{D}_\Omega\psi\|_{3/2,\Gamma'} \lesssim \|\mathcal{D}_\Omega\psi\|_{2,\Gamma'} \lesssim \|\psi\|_{\alpha,\partial\Omega}, \quad \forall \alpha \in R. \tag{4.34}$$

Noting the fact that $\psi = \bar{w}$ on $\partial\Omega$, we have by (4.32)-(4.34) that

$$\|\bar{w}_1\|_{2,\Omega_1} + \|\bar{w}_2\|_{2,\Omega_2} \lesssim \|\phi\|_{1/2,\Gamma} + \|\bar{w}\|_{1/2,\partial\Omega}\,. \tag{4.35}$$

It remains to bound $\|\phi\|_{1/2,\Gamma}$. We rewrite (4.31) as

$$\phi = \frac{2(\beta_1-\beta_2)}{\beta_1+\beta_2}\left\{\mathcal{K}^*_{\Omega_1}\phi + \partial_{\mathbf{n}}(\mathcal{D}_\Omega\psi)\right\} + \frac{2}{\beta_1+\beta_2}(g+\tilde{g}). \tag{4.36}$$

By direct computations and (4.20) we derive

$$\left\|\frac{2(\beta_1-\beta_2)}{\beta_1+\beta_2}\right\|_{1,\infty,\Gamma} \lesssim 1 + \frac{|\beta_1|_{1,\infty,\Omega_1} + |\beta_2|_{1,\infty,\Omega_2}}{\bar{\beta}_1+\bar{\beta}_2} \lesssim 1 \tag{4.37}$$

and

$$\left\|\frac{2}{\beta_1+\beta_2}\right\|_{1,\infty,\Gamma} \lesssim \frac{1}{\bar{\beta}_1+\bar{\beta}_2} + \frac{|\beta_1|_{1,\infty,\Omega_1} + |\beta_2|_{1,\infty,\Omega_2}}{(\bar{\beta}_1+\bar{\beta}_2)^2} \lesssim \frac{1}{\bar{\beta}_1+\bar{\beta}_2}. \tag{4.38}$$

Now it follows from Lemma 4.2, (4.36)-(4.38) that

$$\|\phi\|_{1/2,\Gamma} \lesssim \|\mathcal{K}^*_{\Omega_1}\phi + \partial_{\mathbf{n}}(\mathcal{D}_\Omega\psi)\|_{1/2,\Gamma} + (\bar{\beta}_1+\bar{\beta}_2)^{-1}\|g+\tilde{g}\|_{1/2,\Gamma}. \tag{4.39}$$

On the other hand, since $\mathcal{D}_\Omega\psi$ is harmonic and H^2-smooth in any bounded domain $\tilde{\Omega}_2$, with its interior boundary being Γ and an exterior boundary being a C^∞-smooth surface Γ_1 such that $\tilde{\Omega}_2$ strictly lies in Ω_2. Then by the regularity estimates for harmonic functions, and the same argument as used for deriving (4.34) we obtain

$$\|\partial_{\mathbf{n}}(\mathcal{D}_\Omega\psi)\|_{1/2,\Gamma} \lesssim \|\mathcal{D}_\Omega\psi\|_{2,\tilde{\Omega}_2} \lesssim \|\mathcal{D}_\Omega\psi\|_{3/2,\Gamma} + \|\mathcal{D}_\Omega\psi\|_{3/2,\Gamma_1} \lesssim \|\psi\|_{1/2,\partial\Omega}. \tag{4.40}$$

We have by Lemma 3.2 and the Sobolev intermediate inequality (cf. [1, 3]) that

$$\|\mathcal{K}^*_{\Omega_1}\phi\|_{1/2,\Gamma} \lesssim \|\phi\|_{0,\Gamma} \lesssim \|\phi\|_{1/2,\Gamma} + \|\phi\|_{-1/2,\Gamma}\,. \tag{4.41}$$

But noting that $\psi = \bar{w}$ on $\partial\Omega$, by Lemma 3.2 and the relation (4.30) we derive

$$\begin{aligned}\|\phi\|_{-1/2,\Gamma} &\lesssim \|\mathcal{S}_{\Omega_1}\phi\|_{1/2,\Gamma} \lesssim \|\bar{w}\|_{1/2,\Gamma} + \|\mathcal{D}_\Omega\psi\|_{1/2,\Gamma} \\ &\lesssim \|\bar{w}\|_{1/2,\Gamma} + \|\psi\|_{1/2,\partial\Omega} \lesssim \|\bar{w}\|_{1/2,\Gamma} + \|\bar{w}\|_{1/2,\partial\Omega} \lesssim \|\bar{w}_2\|_{1,\Omega_2}\,,\end{aligned}$$

which with (4.39)-(4.41) gives

$$\|\phi\|_{1/2,\Gamma} \lesssim \|\bar{w}_2\|_{1,\Omega_2} + (\bar{\beta}_1+\bar{\beta}_2)^{-1}\|g+\tilde{g}\|_{1/2,\Gamma}. \tag{4.42}$$

Combining this and (4.35) leads to

$$\|\bar{w}_1\|_{2,\Omega_1} + \|\bar{w}_2\|_{2,\Omega_2} \lesssim \|\bar{w}_2\|_{1,\Omega_2} + (\bar{\beta}_1+\bar{\beta}_2)^{-1}(\|g\|_{1/2,\Gamma} + \|\tilde{g}\|_{1/2,\Gamma}). \tag{4.43}$$

To estimate $\|\bar{w}_2\|_{1,\Omega_2}$, we know by the definition of w_2 that $\bar{w}_2 = u_2 - w_2 \in H^1(\Omega_2)$ satisfies

$$-\Delta\bar{w}_2 = 0 \quad \text{in } \Omega_2; \quad \bar{w}_2 = u_2 \quad \text{on } \Gamma; \quad \partial_{\boldsymbol{\nu}}\bar{w}_2 = 0 \quad \text{on } \partial\Omega.$$

Multiplying $-\Delta\bar{w}_2 = 0$ by $\bar{w}_2$ and u_2 respectively and integrating over Ω_2 yield

$$\|\nabla\bar{w}_2\|^2_{0,\Omega_2} = -\langle\partial_{\boldsymbol{\nu}}\bar{w}_2, \bar{w}_2\rangle_\Gamma = -\langle\partial_{\boldsymbol{\nu}}\bar{w}_2, u_2\rangle_\Gamma = \int_{\Omega_2}\nabla\bar{w}_2\cdot\nabla u_2 dx,$$

thus we have $\|\nabla\bar{w}_2\|_{0,\Omega_2} \lesssim \|\nabla u_2\|_{0,\Omega_2}$. This with (4.43) leads to

$$\|\bar{w}_1\|_{2,\Omega_1} + \|\bar{w}_2\|_{2,\Omega_2} \lesssim \|\nabla u_2\|_{0,\Omega_2} + (\bar{\beta}_1+\bar{\beta}_2)^{-1}(\|g\|_{1/2,\Gamma} + \|\tilde{g}\|_{1/2,\Gamma}). \tag{4.44}$$

We are now ready to prove (4.28)-(4.29). Using (4.23) and Theorem 4.1, we know

$$\begin{aligned}\bar\beta_1\|w_1\|_{2,\Omega_1} &\lesssim \|f_1\|_{0,\Omega_1}+d_1(\beta)\big\{\|f\|_{-1,\Omega_1}+\bar k^{-1}(\|g\|_{-1/2,\Gamma}+\|f\|_{-1,\Omega})\big\},\\ \bar\beta_2\|w_2\|_{2,\Omega_2} &\lesssim \|f_2\|_{0,\Omega_2}+d_2(\beta)\big\{\|g\|_{-1/2,\Gamma}+\|f\|_{-1,\Omega}\big\}.\end{aligned} \tag{4.45}$$

Combining these estimates with (4.27) yields

$$\|\tilde g\|_{1/2,\Gamma}\lesssim\sum_{i=1}^{2}(1+d_i(\beta))\bar\beta_i\|w_i\|_{2,\Omega_i}\lesssim\|f\|_{0,\Omega}+(1+\bar k)^{-1}(\|f\|_{-1,\Omega}+\|g\|_{-1/2,\Gamma}). \tag{4.46}$$

Now, with direct computations, it follows from Theorem 4.1, (4.44) and (4.46) that

$$\bar\beta_1\|\bar w_1\|_{2,\Omega_1}\lesssim(1+\bar k^{-1})(\|g\|_{1/2,\Gamma}+\|f\|_{0,\Omega}),\quad \bar\beta_2\|\bar w_2\|_{2,\Omega_2}\lesssim(1+\bar k^{-1})(\|g\|_{1/2,\Gamma}+\|f\|_{0,\Omega}),$$

which, along with (4.45) and the relation $u_i=\bar w_i+w_i$ lead directly to (4.28)-(4.29). □

Following the proof of Theorem 4.2 but with $f=0$, $w_1=w_2=0$ and $\tilde g=0$, we come immediately to the following simple results:

Theorem 4.3. *Assume that $f=0$, $g\in H^{1/2}(\Gamma)$ and the coefficient $\beta(x)$ in (2.1) is equal to constant $\bar\beta_1$ in Ω_1 and constant $\bar\beta_2$ in Ω_2, then the solution u to the interface problem (2.1)-(2.3) with Neumann boundary condition (2.5) admits the a priori estimates:*

$$\bar\beta_1\,\|u_1\|_{2,\Omega_1}\lesssim\bar k^{-1}(\beta)\|g\|_{1/2,\Gamma},\quad \bar\beta_2\,\|u_2\|_{2,\Omega_2}\lesssim\|g\|_{1/2,\Gamma}.$$

For the Dirichlet boundary condition (2.4), we have the following similar results.

Theorem 4.4. *Assume that that $f\in L^2(\Omega)$, $g\in H^{1/2}(\Gamma)$, and u is the solution to the interface problem (2.1)-(2.3) with Dirichlet boundary condition (2.4), then the following a priori estimates hold:*

$$\bar\beta_1\|u_1\|_{2,\Omega_1}\lesssim\big(1+\bar k^{-1}(\beta)\big)(\|f\|_{0,\Omega}+\|g\|_{1/2,\Gamma}), \tag{4.47}$$

$$\bar\beta_2\|u_2\|_{2,\Omega_2}\lesssim\big(1+\bar k^{-1}(\beta)\big)(\|f\|_{0,\Omega}+\|g\|_{1/2,\Gamma})\,. \tag{4.48}$$

Proof. The proof is basically the same as the one for Neumann boundary condition in Theorem 4.2, with some natural modifications. First, let $w_i\in H_0^1(\Omega_i)$ $(i=1,2)$ be the two functions uniquely determined by (4.21) and (4.22), respectively. And define $\bar w_i=u_i-w_i$ in Ω_i and $\tilde g=\beta_1\partial_{\mathbf n}w_1-\beta_2\partial_{\mathbf n}w_2$ on Γ. Thus by means of a priori estimates for elliptic problems and Theorem 4.1, the estimates (4.45) and (4.46) still hold in the present case. Moreover, it is easy to see that (4.24) and (4.25) are also valid. Therefore, by Theorem 3.3 we can write $\bar w$ as

$$\bar w(x)=(\mathcal S_{\Omega_1}\phi)(x)-(\mathcal S_\Omega\psi)(x),\quad\forall\,x\in\Omega, \tag{4.49}$$

where $\psi=\partial_{\boldsymbol\nu}\bar w$ on $\partial\Omega$, and $\phi\in H^{1/2}(\Gamma)$ solves the integral equation

$$\left(\frac{\beta_1+\beta_2}{2(\beta_1-\beta_2)}I-\mathcal K^*_{\Omega_1}\right)\phi=\partial_{\mathbf n}(\mathcal S_\Omega\psi)+\frac{g+\tilde g}{\beta_1-\beta_2}\quad\text{on }\ \Gamma\,. \tag{4.50}$$

Noting that $\bar w_2=0$ on $\partial\Omega$, it follows from (4.49) that

$$(\mathcal S_{\Omega_1}\phi)(x)=(\mathcal S_\Omega\psi)(x)\quad\forall\,x\in\partial\Omega.$$

Using this identity, (4.49), Lemmas 3.1-3.2 and the derivations as for getting (4.34), we find

$$\begin{aligned}\|\bar{w}_1\|_{2,\Omega_1} + \|\bar{w}_2\|_{2,\Omega_2} &\lesssim \|\bar{w}\|_{3/2,\Gamma} \lesssim \|\mathcal{S}_{\Omega_1}\phi\|_{3/2,\Gamma} + \|\mathcal{S}_{\Omega}\psi\|_{3/2,\Gamma}\\ &\lesssim \|\phi\|_{1/2,\Gamma} + \|\psi\|_{0,\partial\Omega}\\ &\lesssim \|\phi\|_{1/2,\Gamma} + \|\mathcal{S}_{\Omega}\psi\|_{1,\partial\Omega}\\ &\lesssim \|\phi\|_{1/2,\Gamma} + \|\mathcal{S}_{\Omega_1}\phi\|_{1,\partial\Omega} \lesssim \|\phi\|_{1/2,\Gamma}. \end{aligned} \tag{4.51}$$

To further our estimates, we use the fact that $\bar{w}_2 \in H^1(\Omega_2)$ satisfies

$$-\Delta\bar{w}_2 = 0 \quad \text{in } \Omega_2; \quad \bar{w}_2 = u_2 \quad \text{on } \Gamma \quad \text{and} \quad \partial\Omega,$$

so we know by the standard a priori estimate for elliptic problems, the Sobolev trace theorem and the Poincaré inequality that

$$\|\bar{w}_2\|_{1,\Omega_2} \lesssim \|u_2\|_{1/2,\Gamma} + \|u_2\|_{1/2,\partial\Omega} \lesssim \|\nabla u_2\|_{0,\Omega_2}.$$

Using this and (4.51), and following the same arguments as for deriving (4.42) from (4.36), we obtain

$$\begin{aligned}\|\phi\|_{1/2,\Gamma} &\lesssim \|\bar{w}_2\|_{1,\Omega_2} + (\bar{\beta}_1 + \bar{\beta}_2)^{-1}\|g + \tilde{g}\|_{1/2,\Gamma}\\ &\lesssim \|\nabla u_2\|_{0,\Omega_2} + (\bar{\beta}_1 + \bar{\beta}_2)^{-1}\|g + \tilde{g}\|_{1/2,\Gamma}, \end{aligned} \tag{4.52}$$

Now, the estimates (4.47)-(4.48) follow from (4.45)-(4.46), (4.51)-(4.52), and Theorem 4.1. □

4.3. **Improved a priori estimates.** In this section, we shall study whether we can improve the uniform a priori estimates established in Subsections 4.1-4.2 for the elliptic interface problem (2.1)-(2.3) with either Dirichlet boundary condition (2.4) or Nerumann condition (2.5). A natural question is whether the factor $\bar{k}^{-1}(\beta)$ appearing in those estimates of Theorems 4.1-4.4 is necessary. By considering a special example of the interface problem (2.1)-(2.3) in spherical coordinates (cf. [15, p. 581]), we find that the factor $\bar{k}^{-1}(\beta)$ appears necessary as long as the norms used contain the L^2-norm part, and unnecessary when the H^1 and H^2 semi-norms are considered. The removal of this factor is of essential importance as it can be very large if $\beta_1(x)$ is much larger than $\beta_2(x)$ in magnitude. This may happen often in applications and is in fact more interesting from the physical point of view.

Indeed, as we shall demonstrate, the factor $\bar{k}^{-1}$ appearing in the a priori estimates of Theorems 4.1-4.4 can be removed. The improvements will be achieved based on the established a priori estimates in Theorems 4.1-4.4 and by means of a novel technique, which mimics the standard asymptotic analysis (cf. [21]) but is in fact not an actual asymptotic expansion.

Dirichlet boundary condition. We start with the analysis on Dirichlet boundary condition case. For simplicity, we shall write $\bar{k}$ for $\bar{k}(\beta)$, and $\bar{d}_i$ for $\bar{d}_i(\beta)$ below. Clearly for our purpose, we need only to consider the case where $\bar{k}(\beta) < 1$.

Dividing both sides of equations (2.1) and (2.3) by $\bar{\beta}_1$, we can rewrite (2.1)-(2.3) as follows: Find $u \in H_0^1(\Omega)$ such that

$$-\nabla\cdot(\tilde{\beta}_1\nabla u_1) = \tilde{f}_1 \quad \text{in } \Omega_1; \quad -\bar{k}\,\nabla\cdot(\tilde{\beta}_2\nabla u_2) = \tilde{f}_2 \quad \text{in } \Omega_2; \tag{4.53}$$

$$u_2 = u_1, \quad \bar{k}\tilde{\beta}_2\partial_{\mathbf{n}}u_2 - \tilde{\beta}_1\partial_{\mathbf{n}}u_1 = \tilde{g} \quad \text{on } \Gamma, \tag{4.54}$$

where $\tilde{g} = g/\bar{\beta}_1,\ \tilde{f} = f/\bar{\beta}_1,\ \tilde{\beta}_i = \beta_i/\bar{\beta}_i,\ i = 1, 2.$

Next, we expand the solution u formally in powers of $\bar{k}$ in the form

$$u \sim \bar{k}^{-1}u^{-1} + u^0 + O(\bar{k}).$$

Substituting it into the equations (4.53)-(4.54), then comparing the terms of same power of $\bar{k}$, we obtain

$$-\nabla\cdot(\tilde{\beta}_1\nabla u_1^{-1})=0 \quad \text{in} \quad \Omega_1\,; \quad \partial_{\mathbf{n}}u_1^{-1}=0 \quad \text{on} \quad \Gamma, \tag{4.55}$$

$$-\nabla\cdot(\tilde{\beta}_2\nabla u_2^{-1})=\tilde{f}_2 \quad \text{in} \quad \Omega_2\,; \quad u_2^{-1}=u_1^{-1} \quad \text{on} \quad \Gamma; \quad u_2^{-1}=0 \quad \text{on} \quad \partial\Omega, \tag{4.56}$$

$$-\nabla\cdot(\tilde{\beta}_1\nabla u_1^{0})=\tilde{f}_1 \quad \text{in} \quad \Omega_1\,; \quad \tilde{\beta}_1\partial_{\mathbf{n}}u_1^0=\tilde{\beta}_2\partial_{\mathbf{n}}u_2^{-1}-\tilde{g} \quad \text{on} \quad \Gamma\,; \quad \int_\Gamma u_1^0 d\sigma=0, \tag{4.57}$$

$$-\nabla\cdot(\tilde{\beta}_2\nabla u_2^{0})=0 \quad \text{in} \quad \Omega_2\,; \quad u_2^0=u_1^0 \quad \text{on} \quad \Gamma; \quad u_2^0=0 \quad \text{on} \quad \partial\Omega\,, \tag{4.58}$$

where the condition $\int_\Gamma u_1^0 d\sigma=0$ in (4.57) is imposed to ensure the uniqueness of u_1^0 which is governed by a Neumann boundary value problem.

By a straightforward computation, we see from (4.53)-(4.58) that the error function

$$u^r=u-(\bar{k}^{-1}u^{-1}+u^0) \quad \text{in} \quad \Omega \tag{4.59}$$

satisfies $u^r(x)=0$ on $\partial\Omega$ and

$$-\nabla\cdot(\tilde{\beta}_1\nabla u_1^r)=0 \quad \text{in} \quad \Omega_1\,; \quad -\bar{k}\,\nabla\cdot(\tilde{\beta}_2\nabla u_2^r)=0 \quad \text{in} \quad \Omega_2\,; \tag{4.60}$$

$$u_2^r=u_1^r, \quad \bar{k}\,\tilde{\beta}_2\partial_{\mathbf{n}}u_2^r-\tilde{\beta}_1\partial_{\mathbf{n}}u_1^r=-\bar{k}\tilde{\beta}_2\partial_{\mathbf{n}}u_2^0 \quad \text{on} \quad \Gamma\,. \tag{4.61}$$

Now we try to further simplify the functions u^{-1} and u^0. It is easy to see from (4.55) that $u_1^{-1}=\alpha$, a constant, which with the interface condition in (4.56) indicates $u_2^{-1}=\alpha$ on Γ. Using this, we can express u_2^{-1} in the form $u_2^{-1}=w_2^{-1}+\alpha v_2^{-1}$, where w_2^{-1} and v_2^{-1} solve respectively the systems

$$-\nabla\cdot(\tilde{\beta}_2\nabla w_2^{-1})=\tilde{f}_2 \quad \text{in} \quad \Omega_2\,; \quad w_2^{-1}=0 \quad \text{on}\,\Gamma\,; \quad w_2^{-1}=0 \quad \text{on} \quad \partial\Omega, \tag{4.62}$$

$$-\nabla\cdot(\tilde{\beta}_2\nabla v_2^{-1})=0 \quad \text{in} \quad \Omega_2\,; \quad v_2^{-1}=1 \quad \text{on} \quad \Gamma\,;\; v_2^{-1}=0 \quad \text{on} \quad \partial\Omega. \tag{4.63}$$

Next, we try to determine the above constant $\alpha=u_1^{-1}$. Noting that problem (4.57) is of Neumann type, we must have the consistency condition:

$$\int_\Gamma \tilde{\beta}_1\partial_{\mathbf{n}}u_1^0 d\sigma=-\int_{\Omega_1}\tilde{f}_1 dx,$$

which with the interface condition in (4.57) implies

$$\int_\Gamma \tilde{\beta}_2\partial_{\mathbf{n}}u_2^{-1} d\sigma=\int_\Gamma \tilde{g}d\sigma-\int_{\Omega_1}\tilde{f}_1 dx.$$

Using this relation and the expression $u_2^{-1}=w_2^{-1}+\alpha\,v_2^{-1}$, we have

$$\int_\Gamma \tilde{\beta}_2\partial_{\mathbf{n}}w_2^{-1} d\sigma+\alpha\int_\Gamma \tilde{\beta}_2\partial_{\mathbf{n}}v_2^{-1} d\sigma=\int_\Gamma \tilde{g}d\sigma-\int_{\Omega_1}\tilde{f}_1 dx,$$

which gives the formula to evaluate the constant α:

$$\alpha=\left\{\int_\Gamma \tilde{g}d\sigma-\int_{\Omega_1}\tilde{f}_1 dx-\int_\Gamma \tilde{\beta}_2\partial_{\mathbf{n}}w_2^{-1}d\sigma\right\}\Big/\int_\Gamma \tilde{\beta}_2\partial_{\mathbf{n}}v_2^{-1}d\sigma. \tag{4.64}$$

Below we shall mention a few lemmas to present some a priori estimates for the auxiliary functions u^{-1}, u^0 and u^r.

Lemma 4.3. *For the function w_2^{-1} defined by (4.62), we have*

$$\|w_2^{-1}\|_{1,\Omega_2}\lesssim\|\tilde{f}_2\|_{-1,\Omega_2}, \quad \|w_2^{-1}\|_{2,\Omega_2}\lesssim\|\tilde{f}_2\|_{0,\Omega_2}. \tag{4.65}$$

Proof. The first estimate in (4.65) follows readily from the weak formulation of (4.62). For the second estimate, by a direct computation, we know from (4.62) that

$$-\Delta w_2^{-1} = \tilde{\beta}_2^{-1}\{\tilde{f}_2 + \nabla\tilde{\beta}_2 \cdot \nabla w_2^{-1}\}.$$

Now by the standard estimates for elliptic problems (cf. [13, 22]) and noting the assumption (2.2) for β, we can derive

$$\|w_2^{-1}\|_{2,\Omega_2} \lesssim \|\tilde{\beta}_2^{-1}\{\tilde{f}_2 + \nabla\tilde{\beta}_2 \cdot \nabla w_2^{-1}\}\|_{0,\Omega_2} \lesssim \|\tilde{f}_2\|_{0,\Omega_2}.$$

□

Lemma 4.4. *For the function v_2^{-1} defined by (4.63), we have*

$$1 \lesssim \left|\int_\Gamma \tilde{\beta}_2 \partial_{\mathbf{n}} v_2^{-1} d\sigma\right|, \quad \|v_2^{-1}\|_{1,\Omega_2} \lesssim 1, \quad \|v_2^{-1}\|_{2,\Omega_2} \lesssim 1. \tag{4.66}$$

Proof. The proof of the last two estimates in (4.66) are rather standard (cf. [11]). To see the first estimate in (4.66), using (4.63) and applying the Green's formula to $\tilde{\beta}_2\nabla v_2^{-1}$ we obtain

$$0 = -(\nabla \cdot (\tilde{\beta}_2 \nabla v_2^{-1}),\, v_2^{-1})_{\Omega_2} = \int_{\Omega_2} \tilde{\beta}_2 |\nabla v_2^{-1}|^2 dx + \int_\Gamma \tilde{\beta}_2 \partial_{\mathbf{n}} v_2^{-1} d\sigma,$$

which implies

$$\left|\int_\Gamma \tilde{\beta}_2 \partial_{\mathbf{n}} v_2^{-1} d\sigma\right| = \int_{\Omega_2} \tilde{\beta}_2 |\nabla v_2^{-1}|^2 dx. \tag{4.67}$$

Then the desired estimate follows directly from (4.67), the fact that

$$J(v_2^{-1}) = \min_{z\in Z} J(z) = \frac{1}{2}\int_{\Omega_2} \tilde{\beta}_2 |\nabla z|^2 dx$$

and $\tilde{\beta}_2 = \beta_2/\bar{\beta}_2 \geq c_0$ by (2.2), where Z is a set of functions given by

$$Z = \Big\{z \in H^1(\Omega_2);\ z = 1 \text{ on } \Gamma,\ z = 0 \text{ on } \partial\Omega\Big\}.$$

□

Lemma 4.5. *For the constant α in (4.64) and function u_2^{-1} defined by (4.56), we have*

$$|\alpha| \lesssim \|\tilde{g}\|_{-1/2,\Gamma} + \|\tilde{f}\|_{0,\Omega}, \tag{4.68}$$

$$\|u_2^{-1}\|_{1,\Omega_2} \lesssim \|\tilde{g}\|_{-1/2,\Gamma} + \|\tilde{f}\|_{0,\Omega}, \tag{4.69}$$

$$\|u_2^{-1}\|_{2,\Omega_2} \lesssim \|\tilde{g}\|_{-1/2,\Gamma} + \|\tilde{f}\|_{0,\Omega}. \tag{4.70}$$

Proof. For the estimate (4.68) of α, we easily see from (4.64), Lemmas 4.3-4.4 and the Sobolev trace theorem (cf. [1, 13]) that

$$|\alpha| \lesssim \|\tilde{g}\|_{-1/2,\Gamma} + \|\tilde{f}_1\|_{0,\Omega_1} + \|w_2^{-1}\|_{2,\Omega_2} \lesssim \|\tilde{g}\|_{-1/2,\Gamma} + \|\tilde{f}\|_{0,\Omega}.$$

The estimates (4.69) and (4.70) follow directly from the expression $u_2^{-1} = w_2^{-1} + \alpha v_2^{-1}$, Lemmas 4.3-4.4 and the estimate of α. □

Lemma 4.6. *For the function u_1^0 defined by (4.57), we have*

$$\|u_1^0\|_{1,\Omega_1} \lesssim \|\tilde{g}\|_{-1/2,\Gamma} + \|\tilde{f}\|_{0,\Omega}, \tag{4.71}$$

$$\|u_1^0\|_{2,\Omega_1} \lesssim \|\tilde{g}\|_{1/2,\Gamma} + \|\tilde{f}\|_{0,\Omega}. \tag{4.72}$$

Proof. By (4.57), we know $u_1^0 \in H^1(\Omega_1)$ solves the variational problem:

$$\int_{\Omega_1} \tilde{\beta}_1 \nabla u_1^0 \cdot \nabla \omega_1 dx = \int_{\Omega_1} \tilde{f}_1 \omega_1 dx + < \tilde{\beta}_2 \partial_{\mathbf{n}} u_2^{-1} - \tilde{g}, \omega_1 >_\Gamma \quad \forall\, \omega_1 \in H^1(\Omega_1).$$

Taking $\omega_1 = u_1^0$ above and using the Poincaré inequality with condition (4.20) and the trace theorem imply

$$\begin{aligned} \|u_1^0\|_{1,\Omega_1} &\lesssim \|\tilde{f}_1\|_{0,\Omega_1} + \|\tilde{\beta}_2 \partial_{\mathbf{n}} u_2^{-1} - \tilde{g}\|_{-1/2,\Gamma} \\ &\lesssim \|\tilde{f}_1\|_{0,\Omega_1} + \|\tilde{g}\|_{-1/2,\Gamma} + \|\tilde{\beta}_2 \partial_{\mathbf{n}} u_2^{-1}\|_{-1/2,\Gamma}. \end{aligned} \tag{4.73}$$

Next we estimate $\|\tilde{\beta}_2 \partial_{\mathbf{n}} u_2^{-1}\|_{-1/2,\Gamma}$ by the duality argument. For all $\bar{\omega} \in H^{1/2}(\Gamma)$, let ω_2 be the harmonic extension of $\bar{\omega}$ into Ω_2 such that $\omega_2 = \bar{\omega}$ on Γ, and $\omega_2 = 0$ on $\partial\Omega$. Clearly, we know $\|\omega_2\|_{1,\Omega_2} \lesssim \|\bar{\omega}\|_{1/2,\Gamma}$. Thus, it follows from (4.56) and the Green's formula that

$$\int_{\Omega_2} \tilde{f}_2 \omega_2 dx = \int_{\Omega_2} \tilde{\beta}_2 \nabla u_2^{-1} \cdot \nabla \omega_2 dx + \int_\Gamma \tilde{\beta}_2\, \omega_2\, \partial_{\mathbf{n}} u_2^{-1}\, d\sigma,$$

so we can further derive

$$\begin{aligned} \left| \int_\Gamma \tilde{\beta}_2 \partial_{\mathbf{n}} u_2^{-1} \omega_2 d\sigma \right| &\le \left| \int_{\Omega_2} \tilde{\beta}_2 \nabla u_2^{-1} \cdot \nabla \omega_2 dx \right| + \left| \int_{\Omega_2} \tilde{f}_2 \omega_2 dx \right| \\ &\lesssim (\|\tilde{f}_2\|_{0,\Omega_2} + \|u_2^{-1}\|_{1,\Omega_2}) \|\bar{\omega}\|_{1/2,\Gamma}. \end{aligned}$$

This indicates with Lemma 4.5 that

$$\|\tilde{\beta}_2 \partial_{\mathbf{n}} u_2^{-1}\|_{-1/2,\Gamma} \lesssim \|\tilde{f}_2\|_{0,\Omega_2} + \|u_2^{-1}\|_{1,\Omega_2} \lesssim \|\tilde{g}\|_{-1/2,\Gamma} + \|\tilde{f}\|_{0,\Omega}. \tag{4.74}$$

Now (4.71) follows from this estimate and (4.73).

To estimate the term $\|u_1^0\|_{2,\Omega_1}$, we first rewrite equations (4.57) as

$$-\Delta u_1^0 = \tilde{\beta}_1^{-1} \{\tilde{f}_1 + \nabla \tilde{\beta}_1 \cdot \nabla u_1^0\} \quad \text{in} \quad \Omega_1\,; \quad \partial_{\mathbf{n}} u_1^0 = \tilde{\beta}_1^{-1} \{\tilde{\beta}_2 \partial_{\mathbf{n}} u_2^{-1} - \tilde{g}\} \quad \text{on} \quad \Gamma.$$

Then we deduce by virtue of Lemmas 4.2 and 4.5, the estimate of $\|u_1^0\|_{1,\Omega_2}$ that

$$\begin{aligned} \|\Delta u_1^0\|_{0,\Omega_1} &= \|\tilde{\beta}_1^{-1} \{\tilde{f}_1 + \nabla \tilde{\beta}_1 \cdot \nabla u_1^0\}\|_{0,\Omega_1} \\ &\lesssim \|\tilde{f}_1\|_{0,\Omega_1} + d_1 \|\tilde{g}\|_{-1/2,\Gamma} + (1 + d_2)^2 d_1 \|\tilde{f}\|_{0,\Omega}, \\ \|\partial_{\mathbf{n}} u_1^0\|_{1/2,\Gamma} &= \|\tilde{\beta}_1^{-1} \{\tilde{\beta}_2 \partial_{\mathbf{n}} u_2^{-1} - \tilde{g}\}\|_{1/2,\Gamma} \lesssim \|\tilde{\beta}_1^{-1}\|_{1,\infty,\Omega_1} \|\tilde{\beta}_2 \partial_{\mathbf{n}} u_2^{-1} - \tilde{g}\|_{1/2,\Gamma} \\ &\lesssim \|\tilde{g}\|_{1/2,\Gamma} + \|u_2^{-1}\|_{2,\Omega_2} \lesssim \|\tilde{g}\|_{1/2,\Gamma} + \|\tilde{f}\|_{0,\Omega}. \end{aligned}$$

Finally, the Poincaré inequality and standard a priori estimates for elliptic problems lead to

$$\|u_1^0\|_{2,\Omega_1} \lesssim \|\Delta u_1^0\|_{0,\Omega_1} + \|\partial_{\mathbf{n}} u_1^0\|_{1/2,\Gamma} \lesssim \|\tilde{g}\|_{1/2,\Gamma} + \|\tilde{f}\|_{0,\Omega}.$$

This completes the proof of (4.72) □

The estimates in the following lemma about u_2^0 follow from the standard a priori estimates of elliptic problems and Lemma 4.6:

Lemma 4.7. *For the function u_2^0 defined by (4.58), we have*

$$\|u_2^0\|_{1,\Omega_2} \lesssim \|\tilde{g}\|_{-1/2,\Gamma} + \|\tilde{f}\|_{0,\Omega}\,, \quad \|u_2^0\|_{2,\Omega_2} \lesssim \|\tilde{g}\|_{1/2,\Gamma} + \|\tilde{f}\|_{0,\Omega}.$$

Lemma 4.8. *For the error function u^r defined in (4.59), we have*

$$\begin{aligned} \|\nabla u_1^r\|_{0,\Omega_1} &\lesssim \|\tilde{g}\|_{-1/2,\Gamma} + \|\tilde{f}\|_{0,\Omega}, \\ \|u_1^r\|_{2,\Omega_1} + \bar{k}\, \|u_2^r\|_{2,\Omega_2} &\lesssim \|\tilde{g}\|_{1/2,\Gamma} + \|\tilde{f}\|_{0,\Omega}. \end{aligned}$$

Proof. By the same argument as for bounding $(\tilde{\beta}_2\partial_{\mathbf{n}}u_2^{-1})$ in (4.74) and using Lemma 4.7, we obtain

$$\|\tilde{\beta}_2\partial_{\mathbf{n}}u_2^0\|_{-1/2,\Gamma} \lesssim \|\tilde{g}\|_{-1/2,\Gamma} + \|\tilde{f}\|_{0,\Omega}. \tag{4.75}$$

Now applying Theorem 4.1 and Theorem 4.4 to the elliptic interface problem (4.60)-(4.61) and noting $\bar{k} < 1$, and (2.2), we derive

$$\|\nabla u_1^r\|_{0,\Omega_1} \lesssim \bar{k}^{-1}\|\bar{k}\tilde{\beta}_2\partial_{\mathbf{n}}u_2^0\|_{-1/2,\Gamma} = \|\tilde{\beta}_2\partial_{\mathbf{n}}u_2^0\|_{-1/2,\Gamma},$$
$$\|u_1^r\|_{2,\Omega_1} \lesssim (1+\bar{k}^{-1})\|\bar{k}\tilde{\beta}_2\partial_{\mathbf{n}}u_2^0\|_{1/2,\Gamma} \lesssim \|\tilde{\beta}_2\partial_{\mathbf{n}}u_2^0\|_{1/2,\Gamma},$$
$$\bar{k}\,\|u_2^r\|_{2,\Omega_2} \lesssim (1+\bar{k}^{-1})\|\bar{k}\tilde{\beta}_2\partial_{\mathbf{n}}u_2^0\|_{1/2,\Gamma} \lesssim \|\tilde{\beta}_2\partial_{\mathbf{n}}u_2^0\|_{1/2,\Gamma}.$$

Then the desired estimates follow directly from (4.75), the trace theorem and Lemma 4.7. □

With the above preparations, we are now ready to present our improved a priori estimates.

Theorem 4.5. *For the solution to the interface problem (2.1)-(2.3) with Dirichlet boundary condition (2.4), there hold*

$$\bar{\beta}_1\|\nabla u_1\|_{0,\Omega_1} + \bar{\beta}_2\|\nabla u_2\|_{0,\Omega_2} \lesssim \|g\|_{-1/2,\Gamma} + \|f\|_{-1,\Omega} \tag{4.76}$$

for $f \in H^{-1}(\Omega)$ *and* $g \in H^{-1/2}(\Gamma)$, *and*

$$\bar{\beta}_1|u_1|_{2,\Omega_1} + \bar{\beta}_2|u_2|_{2,\Omega_2} \lesssim \|g\|_{1/2,\Gamma} + \|f\|_{0,\Omega} \tag{4.77}$$

for $f \in L^2(\Omega)$ *and* $g \in H^{1/2}(\Gamma)$.

Proof. Observing that u_1^{-1} in (4.55) is a constant, and noting u^r in (4.59) and the definitions of $\tilde{g}$ and $\tilde{f}$, we derive from Lemmas 4.5-4.8 that

$$\bar{\beta}_1|u_1|_{2,\Omega_1} \lesssim \bar{\beta}_1|u_1^r|_{2,\Omega_1} + \bar{\beta}_1\|u_1^0\|_{2,\Omega_1} \lesssim \|g\|_{1/2,\Gamma} + \|f\|_{0,\Omega},$$
$$\bar{\beta}_2|u_2|_{2,\Omega_2} = \bar{\beta}_1\bar{k}\,|u_2|_{2,\Omega_2} \lesssim \bar{\beta}_1\{\bar{k}\,|u_1^r|_{2,\Omega_1} + \|u_2^{-1}\|_{2,\Omega_2} + \|u_2^0\|_{2,\Omega_2}\}$$
$$\lesssim \|g\|_{1/2,\Gamma} + \|f\|_{0,\Omega},$$

which prove (4.77). The same reasoning as above gives

$$\bar{\beta}_1\|\nabla u_1\|_{0,\Omega_1} \le \bar{\beta}_1\|\nabla u_1^r\|_{0,\Omega_1} + \bar{\beta}_1\|u_1^0\|_{1,\Omega_1} \lesssim \|g\|_{-1/2,\Gamma} + \|f\|_{0,\Omega}. \tag{4.78}$$

This is not the optimal estimate as required in (4.76), where only the H^{-1}-norm is needed. To improve this estimate, we introduce two functions $\omega_i \in H_0^1(\Omega_i)$ $(i = 1, 2)$ such that

$$-\nabla\cdot(\beta_i\nabla\omega_i) = f_i \quad \text{in } \Omega_i. \tag{4.79}$$

By the standard duality argument as used for bounding $(\tilde{\beta}_2\partial_{\mathbf{n}}u_2^{-1})$ in (4.74), we deduce

$$\|\beta_i\partial_{\mathbf{n}}\omega_i\|_{-1/2,\Gamma} \lesssim \|\beta_i\nabla\omega_i\|_{0,\Omega_i} + \|f_i\|_{-1,\Omega_i} \lesssim \|f\|_{-1,\Omega_i}. \tag{4.80}$$

Now letting $v = u - \omega$, we know from (2.1)-(2.4), (4.79) that $v = 0$ on $\partial\Omega$ and satisfies

$$-\nabla\cdot(\beta_1\nabla v_1) = 0 \quad \text{in } \Omega_1; \quad -\nabla\cdot(\beta_2\nabla v_2) = 0 \quad \text{in } \Omega_2;$$
$$[v] = 0, \quad [\beta\partial_{\mathbf{n}}v] = g + \beta_1\partial_{\mathbf{n}}\omega_1 - \beta_2\partial_{\mathbf{n}}\omega_2 \quad \text{on } \Gamma.$$

Applying the estimate (4.78) to the above interface problem and using (4.80), we obtain

$$\bar{\beta}_1\|\nabla v_1\|_{0,\Omega_1} \lesssim \|g + \beta_1\partial_{\mathbf{n}}\omega_1 - \beta_2\partial_{\mathbf{n}}\omega_2\|_{-1/2,\Gamma} \lesssim \|g\|_{-1/2,\Gamma} + \|f\|_{-1,\Omega},$$

which, with (4.80) and the relation $u = v + \omega$, gives

$$\bar{\beta}_1 \|\nabla u_1\|_{0,\Omega_1} \lesssim \|g\|_{-1/2,\Gamma} + \|f\|_{-1,\Omega}.$$

The combination of this and (4.6) implies (4.76). □

Remark 4.2. The estimates in Theorem 4.5 are sharp in terms of $\bar{\beta}_1$ and $\bar{\beta}_2$ and for general f and g. These have even greatly improved our previous results obtained in [15] for the case with piecewise constant coefficients.

Neumann boundary conditions. For the interface problem (2.1)-(2.3) with Neumann boundary condition (2.5), we can also improve the uniform a priori estimates established in Theorem 4.1 and Theorem 4.2. We shall state only the results below but omit the lengthy derivations. In fact, the derivations follow almost exactly the same arguments as used earlier for the Dirichlet boundary condition (2.4), with only some minor modifications, e.g., the boundary conditions $u_2^{-1} = 0$ and $u_2^0 = 0$ on $\partial\Omega$ in (4.56) and (4.58) should be changed into the Neumann boundary conditions $\partial_{\boldsymbol{\nu}} u_2^{-1} = 0$ and $\partial_{\boldsymbol{\nu}} u_2^0 = 0$ respectively.

Theorem 4.6. *For the solution u to the interface problem (2.1)-(2.3) with Neumann boundary condition (2.5), there hold*

$$\bar{\beta}_1 \|\nabla u_1\|_{0,\Omega_1} + \bar{\beta}_2 \|\nabla u_2\|_{0,\Omega_2} \lesssim \|g\|_{-1/2,\Gamma} + \|f\|_{-1,\Omega} \tag{4.81}$$

for $f \in (H^1(\Omega))'$ and $g \in H^{-1/2}(\Gamma)$, and

$$\bar{\beta}_1 |u_1|_{2,\Omega_1} + \bar{\beta}_2 |u_2|_{2,\Omega_2} \lesssim \|g\|_{1/2,\Gamma} + \|f\|_{0,\Omega} \tag{4.82}$$

for $f \in L^2(\Omega)$ and $g \in H^{1/2}(\Gamma)$.

5. **Uniform a priori estimates for static Maxwell interface problems.** In this section, we will present some uniform a priori estimates for the electric and magnetic field $\mathbf{E}$ and $\mathbf{H}$ to the static Maxwell system. In this case, the electric and magnetic fields $\mathbf{E}$ and $\mathbf{H}$ are uniquely determined by two independent systems, and thus their estimates can be obtained separately. As showed in Subsection 2.2, we assume that conditions (2.10) and (2.11) hold for the material parameters $\varepsilon(x)$ and $\mu(x)$.

5.1. **Electric field E.** It follows from the equations (2.6), (2.8), and the interface and boundary conditions (2.12)-(2.14) that the electric field $\mathbf{E}$ satisfies the condition $\mathbf{n} \times \mathbf{E} = \mathbf{0}$ on $\partial\Omega$ and is governed by the following system:

$$\nabla \cdot (\varepsilon(x)\mathbf{E}) = \rho\,, \quad \nabla \times \mathbf{E} = 0 \qquad \text{in } \Omega\,; \tag{5.1}$$

$$[\mathbf{E} \times \mathbf{n}] = 0, \quad [\varepsilon \mathbf{E} \cdot \mathbf{n}] = \rho_\Gamma \text{ on } \Gamma\,. \tag{5.2}$$

Using $\nabla \times \mathbf{E} = 0$, we know that there exists a scalar potential $u \in H_0^1(\Omega)$ such that (cf. [12, p. 31])

$$\mathbf{E} = -\nabla u. \tag{5.3}$$

Substituting this into the first equation in (5.1) yields

$$-\nabla \cdot (\varepsilon(x)\nabla u) = \rho \quad \text{in} \ \ \Omega.$$

On the other hand, by direct computations we find using (5.2) and (5.3) that

$$[u] = 0, \quad [\varepsilon(x)\partial_{\mathbf{n}} u] = -\rho_\Gamma \quad \text{on} \ \ \Gamma.$$

Now we can see that the potential function $u \in H_0^1(\Omega)$ satisfies the elliptic interface problem (2.1)-(2.3) with $f = \rho$, $\beta = \varepsilon$ and $g = -\rho_\Gamma$. Then an application of Theorem 4.5 leads directly to the following uniform a priori estimates on $\mathbf{E}$:

Theorem 5.1. *For the electric field* $\mathbf{E}$ *governed by the system (5.1)-(5.2), we have*

$$\bar{\varepsilon}_1\|\mathbf{E}_1\|_{0,\Omega_1}+\bar{\varepsilon}_2\|\mathbf{E}_2\|_{0,\Omega_2}\lesssim\|\rho_\Gamma\|_{-1/2,\Gamma}+\|\rho\|_{-1,\Omega} \tag{5.4}$$

for $\rho_\Gamma\in H^{-1/2}(\Gamma)$ *and* $\rho\in H^{-1}(\Omega)$, *and*

$$\bar{\varepsilon}_1|\mathbf{E}_1|_{1,\Omega_1}+\bar{\varepsilon}_2|\mathbf{E}_2|_{1,\Omega_2}\lesssim\|\rho_\Gamma\|_{1/2,\Gamma}+\|\rho\|_{0,\Omega} \tag{5.5}$$

for $\rho_\Gamma\in H^{1/2}(\Gamma)$ *and* $\rho\in L^2(\Omega)$.

5.2. **Magnetic field H.** From the equations (2.7), (2.9), and the interface and boundary conditions (2.12)-(2.14), we see that the magnetic field $\mathbf{H}$ satisfies the condition $\boldsymbol{\nu}\cdot(\mu\mathbf{H})=0$ on $\partial\Omega$ and is governed by the following system:

$$\nabla\times\mathbf{H}=\mathbf{J}\,,\quad \nabla\cdot(\mu\mathbf{H})=0\quad\text{in}\ \ \Omega\,; \tag{5.6}$$

$$[\mathbf{H}\times\mathbf{n}]=0,\quad [\mu\mathbf{H}\cdot\mathbf{n}]=0\ \text{on}\ \ \Gamma\,. \tag{5.7}$$

In general, the magnetic field $\mathbf{H}$ can be described by introducing a vector field which satisfying some gauge conditions (cf. [9, 10]). But we shall use a different way to represent $\mathbf{H}$ (cf. [19]). Noting the first equation in (5.6), we should have the following consistency condition:

$$\nabla\cdot\mathbf{J}=0.$$

Since the domain Ω is a simply-connected convex polyhedron or a domain with a smooth boundary, we know by Theorem 3.12 and Theorem 2.17 in [2] that there exists a vector potential $\mathbf{W}$ in $(H^1(\Omega))^3$ satisfying that $\mathbf{W}\cdot\boldsymbol{\nu}=0$ on $\partial\Omega$ and

$$\mathbf{J}=\nabla\times\mathbf{W}\quad\text{in}\ \ \Omega\,,\quad \nabla\cdot\mathbf{W}=0\quad\text{in}\ \ \Omega, \tag{5.8}$$

with the stability estimate

$$\|\mathbf{W}\|_{1,\Omega}\lesssim\|\mathbf{J}\|_{0,\Omega}. \tag{5.9}$$

Combining (5.8) with (5.6) yields

$$\nabla\times(\mathbf{H}-\mathbf{W})=0\,,$$

hence there exists (cf. Theorem 2.9, [12, p. 31]) a scalar potential $\omega\in H^1(\Omega)$ such that

$$\mathbf{H}=\mathbf{W}+\nabla\omega\quad\text{in}\ \ \Omega. \tag{5.10}$$

Substituting this into the second equation of (5.6) and noting $\nabla\cdot\mathbf{W}=0$, we obtain

$$-\nabla\cdot(\mu_1\nabla\omega_1)=\nabla\mu_1\cdot\mathbf{W}_1\quad\text{in}\ \ \Omega_1, \tag{5.11}$$

$$-\nabla\cdot(\mu_2\nabla\omega_2)=\nabla\mu_2\cdot\mathbf{W}_2\quad\text{in}\ \ \Omega_2. \tag{5.12}$$

On the other hand, using (5.10), the interface and boundary conditions on $\mathbf{H}$ and $\mathbf{W}$, we can see that the potential function ω satisfies the boundary condition $\partial_{\boldsymbol{\nu}}\omega=0$ on $\partial\Omega$ and the interface conditions

$$[\omega]=0,\quad [\mu\partial_{\mathbf{n}}\omega]=(\mu_1-\mu_2)\mathbf{W}\cdot\mathbf{n}\quad\text{on}\ \ \Gamma.$$

Therefore, we find that the scalar potential ω satisfies the elliptic interface problem (2.1)-(2.3) with $\beta(x)=\mu(x)$, $f(x)=\nabla\mu_1\cdot\mathbf{W}_1$ in Ω_1, $f(x)=\nabla\mu_2\cdot\mathbf{W}_2$ in Ω_2, and

$$g(x)=(\mu_1-\mu_2)\mathbf{W}\cdot\mathbf{n}\,.$$

Then applying Theorem 4.6 and (5.9)-(5.10), we come immediately to the following conclusion.

Theorem 5.2. *Assume that* $\mathbf{J} \in (L^2(\Omega))^3$*, and* $\mathbf{H}$ *is the solution to the system (5.6)-(5.7). Then the following a priori estimates hold*

$$\begin{aligned}
\|\mathbf{H}_1\|_{0,\Omega_1} &\le \Big\{1 + \frac{|\mu_1|_{1,\infty,\Omega_1} + |\mu_2|_{1,\infty,\Omega_2} + \|\mu_1 - \mu_2\|_{0,\infty,\Gamma}}{\bar{\mu}_1}\Big\}\|\mathbf{J}\|_{0,\Omega}\,,\\
\|\mathbf{H}_2\|_{0,\Omega_2} &\le \Big\{1 + \frac{|\mu_1|_{1,\infty,\Omega_1} + |\mu_2|_{1,\infty,\Omega_2} + \|\mu_1 - \mu_2\|_{0,\infty,\Gamma}}{\bar{\mu}_2}\Big\}\|\mathbf{J}\|_{0,\Omega}\,,\\
|\mathbf{H}_1|_{1,\Omega_1} &\le \Big\{1 + \frac{|\mu_1|_{1,\infty,\Omega_1} + |\mu_2|_{1,\infty,\Omega_2} + \|\mu_1 - \mu_2\|_{1,\infty,\Gamma}}{\bar{\mu}_1}\Big\}\|\mathbf{J}\|_{0,\Omega}\,,\\
|\mathbf{H}_2|_{1,\Omega_2} &\le \Big\{1 + \frac{|\mu_1|_{1,\infty,\Omega_1} + |\mu_2|_{1,\infty,\Omega_2} + \|\mu_1 - \mu_2\|_{1,\infty,\Gamma}}{\bar{\mu}_2}\Big\}\|\mathbf{J}\|_{0,\Omega}\,.
\end{aligned}$$

When the magnetic permeability $\mu(x)$ is piecewise constant, we have

Theorem 5.3. *Assume that* $\mathbf{J} \in (L^2(\Omega))^3$*, and that the permeability parameter* $\mu(x)$ *is piecewise constant, equal to* $\bar{\mu}_i$ *in* Ω_i $(i = 1, 2)$*, then the magnetic field* $\mathbf{H}$ *to the system (5.6)-(5.7) admits the following a priori estimates*

$$\begin{aligned}
&\|\mathbf{H}_1\|_{1,\Omega_1} \lesssim \|\mathbf{J}\|_{0,\Omega}\,, \quad \|\mathbf{H}_2\|_{1,\Omega_2} \lesssim \|\mathbf{J}\|_{0,\Omega} \quad \textit{for } \bar{\mu}_2 > \bar{\mu}_1,\\
&\|\mathbf{H}_1\|_{1,\Omega_1} \lesssim \|\mathbf{J}\|_{0,\Omega}, \quad \|\mathbf{H}_2\|_{1,\Omega_2} \lesssim \frac{\bar{\mu}_1}{\bar{\mu}_2}\|\mathbf{J}\|_{0,\Omega} \quad \textit{for } \bar{\mu}_1 > \bar{\mu}_2\,.
\end{aligned}$$

Proof. For $\bar{\mu}_2 > \bar{\mu}_1$, we apply Theorem 4.1 (with $f = 0$) to the system (5.11)-(5.12) to obtain that

$$\bar{\mu}_2\,(|\omega_1|_{1,\Omega_1} + |\omega_2|_{1,\Omega_2}) \lesssim \|(\mu_1 - \mu_2)\mathbf{W}\cdot\mathbf{n}\|_{-1/2,\Gamma} \lesssim |\mu_1 - \mu_2|\,\|\mathbf{J}\|_{0,\Omega}\,,$$

which gives

$$|\omega_1|_{1,\Omega_1} + |\omega_2|_{1,\Omega_2} \lesssim \frac{\bar{\mu}_2 - \bar{\mu}_1}{\bar{\mu}_2}\|\mathbf{J}\|_{0,\Omega} \lesssim \|\mathbf{J}\|_{0,\Omega}\,.$$

On the other hand, one can apply Theorem 4.3 to the system (5.11)-(5.12) and use (5.9) to derive that

$$\bar{\mu}_2\,(|\omega_1|_{2,\Omega_1} + |\omega_2|_{2,\Omega_2}) \lesssim \|(\mu_1 - \mu_2)\mathbf{W}\cdot\mathbf{n}\|_{1/2,\Gamma} \lesssim (\bar{\mu}_2 - \bar{\mu}_1)\|\mathbf{J}\|_{0,\Omega}\,,$$

thus we have

$$|\omega_1|_{2,\Omega_1} + |\omega_2|_{2,\Omega_2} \lesssim \frac{\bar{\mu}_2 - \bar{\mu}_1}{\bar{\mu}_2}\|\mathbf{J}\|_{0,\Omega} \lesssim \|\mathbf{J}\|_{0,\Omega}\,.$$

With these estimates, the results for the case that $\bar{\mu}_2 > \bar{\mu}_1$ comes readily from (5.10).

To treat the case with $\bar{\mu}_1 > \bar{\mu}_2$, we apply Theorem 4.6 to the system (5.11)-(5.12) to obtain that

$$\begin{aligned}
\bar{\mu}_1\,|\omega_1|_{1,\Omega_1} + \bar{\mu}_2\,|\omega_2|_{1,\Omega_2} &\lesssim \|(\mu_1 - \mu_2)\mathbf{W}\cdot\mathbf{n}\|_{-1/2,\Gamma} \lesssim (\bar{\mu}_1 - \bar{\mu}_2)\|\mathbf{J}\|_{0,\Omega}\,,\\
\bar{\mu}_1\,|\omega_1|_{2,\Omega_1} + \bar{\mu}_2\,|\omega_2|_{2,\Omega_2} &\lesssim \|(\mu_1 - \mu_2)\mathbf{W}\cdot\mathbf{n}\|_{1/2,\Gamma} \lesssim (\bar{\mu}_1 - \bar{\mu}_2)\|\mathbf{J}\|_{0,\Omega}\,,
\end{aligned}$$

now the desired estimate follows immediately from these and (5.10). □

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