UNIQUENESS AND A PRIORI ESTIMATES FOR SOME NONLINEAR ELLIPTIC NEUMANN EQUATIONS IN R^3

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Abstract. Under some conditions on $f(u)$, we show that for λ small, the only solutions to the following elliptic equation

$$\begin{cases} \Delta u - \lambda u + f(u) = 0 \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \end{cases}$$

are constants, provided that $\Omega \subset R^3$ is convex. The proof only uses integration by parts.

1. Introduction

In this paper, we consider the following nonlinear elliptic equation

$$\begin{cases} \Delta u - \lambda u + f(u) = 0 \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth and bounded domain in R^3, $f(u) \in C^{1+\theta}$ for some $0 < \theta < 1$. Moreover we suppose that $f(u)$ can be written as $f(u) = u^5(1+\rho(u)), \rho'(u) \leq 0, 0 \leq u^5\rho(u) \leq Cu^p$ for some $1 < p \leq \frac{13}{3}$ and some constant $C > 0$. A typical $f(u)$ can be $f(u) = u^5 + \sum_{i=1}^{K} a_i u^{p_i}, 1 < p_i \leq \frac{13}{3}, a_i \geq 0, i = 1, ..., K$.

Our main result is the following.

Theorem 1.1. *Suppose that $\Omega \subset R^3$ is convex. Then there exist $\lambda_0 > 0, C > 0$ such that for $\lambda < \lambda_0$, we have*

$$u \leq C\lambda^{1/4}$$

for all solutions u of (1.1), where C is independent of λ.

As a consequence, $u \equiv C_\lambda$ for some constant C_λ.

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Remark: The restriction that $u^5\rho(u) \le Cu^p$ for $p \le \frac{13}{3}$ just reflects a technical difficulty we have. We believe that the right condition should be that $0 \le u^5\rho(u) \le Cu^p$ for $1 < p < 5$.

Our proof only uses integration parts. It gives a new and simple proof of the following theorem

Theorem 1.2. *Suppose that $\Omega \subset R^3$ is convex and $f(u) = u^5$. Then for λ small, $u = \lambda^{1/4}$ for all solutions u of (1.1).*

Theorem (1.2) is a special case of Lin-Ni's conjecture [8]. In [8], Lin and Ni conjectured that for λ small and $f(u) = u^p, p > 1$, problem (1.1) admits only constant solutions. (Problem (1.1) with $f(u) = u^p$ arises in the study of steady-state solutions of Keller-Segel model in chemotaxis [7] and Gierer-Meinhardt system in pattern formation [5]. For more backgrounds on (1.1), see [10].) Lin and Ni proved the case when p is subcritical. In a recent paper [11], Zhu proved Lin-Ni's conjecture in three dimensional convex domains (i.e. Theorem (1.2)) by using very complicated blowing-up analysis. Zhu's proof is by contradiction and thus is indirect. It is unclear if Zhu's proof can be carried out to general $f(u)$. Here our proof is much simpler and elementary. In fact since our proof is direct, it can yield an explicit number for λ_0 in Theorem (1.1). We remark that when $\Omega = B_R(0)$ and u is radial, similar result was proved by Adimurthi and Yadava [1]. In higher dimensions, Lin-Ni's conjecture may not be true. For the radial case, see [2], [3]. For the nonradial case, see [6].

We make use an idea by Chang, Gursky and Yang developed in the study of three dimensional prescribed scalar curvature problem, [4]. Our starting point is to write equation (1.1) in the following way:

$$\Delta u - \lambda u + f(u) = \Delta u + R(u)u^5$$

where $R(u) = 1 + \rho(u) - \lambda u^{-4}$. Thus if we introduce the conformal transformation $g = u^4 g_0$, where g_0 is the usual Euclid metric, then R becomes the scalar curvature in the new metric.

We first make some preliminary notes. (Throughout this paper, the constant C will denote various generic constants independent of λ.)

From equation (1.1), we see that

$$\int_\Omega u^5 \le \int_\Omega f(u) = \lambda \int_\Omega u \le C\lambda (\int_\Omega u^5)^{1/5}$$

which implies that

$$\int_\Omega u^5 \le C\lambda^{5/4}, \int_\Omega u \le \lambda^{1/4} \tag{1.2}$$

and thus

$$u_{min} := \min_{x\in\bar\Omega} u(x) \le C\lambda^{1/4}. \tag{1.3}$$

From (1.2), we also obtain that

$$\int_\Omega u^q \le C\lambda^{q/4}, \text{ for all } 0 < q \le 5. \tag{1.4}$$

Next, we need the following well-known fact (see Lemma 5 of Appendix 1 of [9]): if Ω is convex and $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, then

$$\frac{\partial}{\partial\nu}|\nabla u|^2 \le 0 \text{ on } \partial\Omega. \tag{1.5}$$

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2. Proof of Theorem (1.1)

Let g_0 be the usual Euclidean metric and $g = u^4 g_0$. In the new metric, we consider the trace-free Ricci tensor B. In local coordinate system, we may write

$$B_{ij} = -u^{-2}[(u^2)_{g,ij} - \frac{1}{3}(\Delta_g u^2)g_{ij}]. \tag{2.1}$$

(Here the covariant derivatives are with respect to g (not g_0).)

We first obtain an integral estimate for $\int_\Omega |B|^{3/2} dV_g$. (Here we use dV_g to denote the integration with respect to the new metric g. It is easy to see that $dV_g = u^6 dx$.) To this end, we consider

$$\int_\Omega |B|^2 u^2 dV_g = \int_\Omega g^{ik} g^{jl} B_{ij} B_{kl} u^2 dV_g$$

$$= -\int_\Omega (u^2)_{g,ij} B_{kl} g^{ik} g^{jl} dV_g + \frac{1}{3}\int_\Omega (\Delta_g u^2) g_{ij} g^{ik} g^{jl} B_{kl} dV_g$$

where the second term vanishes because B is trace-free.

Now integrating by parts (using the Neumann boundary condition) we have

$$\int_\Omega |B|^2 u^2 dV_g = \int_\Omega (u^2)_{g,i} B_{kl;j} g^{ik} g^{jl} dV_g + \int_{\partial\Omega} \frac{\partial}{\partial \nu} |\nabla u|^2 d\sigma_g$$

where $d\sigma_g$ is the surface element in the new metric g.

Using the contracted second Bianchi identity and (1.5) we obtain

$$\int_\Omega |B|^2 u^2 dV_g \leq \frac{1}{6} \int_\Omega (u^2)_{g,i} R_{,k} g^{ijk} dV_g$$

$$= \frac{1}{6} \int_\Omega < \nabla_g (u^2), \nabla_g R > dV_g = \frac{1}{12} \int_\Omega < \nabla(u^4), \nabla R > dx$$

$$= \frac{1}{12} \int_\Omega \nabla u^4 \nabla (1 + \rho(u) - \lambda u^{-4})$$

$$\leq \frac{4}{3} \lambda \int_\Omega u^{-2} |\nabla u|^2 dx \quad (\text{since } \rho'(u) \leq 0 \ )$$

$$= \frac{4}{3} \lambda \int_\Omega u^{-1} \Delta u dx = \frac{4}{3} \lambda \int_\Omega u^{-1} (\lambda u - f(u)) \leq C \lambda^2.$$

Thus we obtain the following key estimate (this is the place where we have to consider R^3 only)

$$\int_\Omega |B|^{3/2} dV_g \leq (\int_\Omega |B|^2 u^2 dV_g)^{3/4} (\int_\Omega u^{-6} dV_g)^{1/4} \leq C \lambda^{3/2}. \tag{2.2}$$

Next we estimate $\int_\Omega |\nabla u^{-2}|^6 dx$.

Let $v = 1/u$. Recall the Bochner identity, which states that

$$\Delta_g |\nabla_g v|^2 = 2 |\nabla_g^2 v|^2 + 2 < \nabla_g v, \nabla_g (\Delta_g v) > + 2 Ric_g \ (\nabla_g v, \nabla_g v). \tag{2.3}$$

If Ω is convex, then by (1.5)

$$\int_\Omega \Delta_g |\nabla_g v|^2 dV_g \leq 0. \tag{2.4}$$

So integrating (2.3) by parts, we obtain

$$\int_\Omega |\nabla_g^2 v|^2 dV_g \leq \int_\Omega (\Delta_g v)^2 dV_g + \int_\Omega |Ric| |\nabla_g v|^2 dV_g \tag{2.5}$$

Recall the Sobolev inequality in R^3

$$(\int_\Omega \psi^6 dV_g)^{1/3} \leq C \int_\Omega |\nabla_g \psi|^2 dV_g + C \int_\Omega \psi^2 dV_g \tag{2.6}$$

Take $\psi = |\nabla_g v|$. We obtain

$$(\int_\Omega |\nabla_g v|^6 dV_g)^{1/3} \le C\int_\Omega |\nabla_g |\nabla_g v||^2 + C\int_\Omega |\nabla_g v|^2$$

$$\le C\int_\Omega |\nabla_g^2 v|^2 dV_g + C\int_\Omega |\nabla_g v|^2 dV_g \quad \text{(by Kato's inequality)}$$

$$\le C\int_\Omega |\Delta_g v|^2 dV_g + C\int_\Omega |Ric||\nabla_g v|^2 dV_g + C\int_\Omega |\nabla_g v|^2 dV_g$$

$$\le C\int_\Omega |\Delta_g v|^2 dV_g + C\int_\Omega |B||\nabla_g v|^2 dV_g + C\int_\Omega |R||\nabla_g v|^2 dV_g + C\int_\Omega |\nabla_g v|^2 dV_g, \tag{2.7}$$

where we have used the fact that $|Ric| \le |B| + \frac{1}{3}|R|$.

Let us estimate each term in (2.7).

Recall that $g_0 = (\frac{1}{u})^4 g = v^4 g$. Note that the scalar curvature R_0 with respect to g_0 is 0. So we have

$$\Delta_g v = (\Delta_g + \frac{1}{8}R)v - \frac{1}{8}Rv$$

$$= -\frac{1}{8}R_0 v^5 - \frac{1}{8}Rv$$

$$= -\frac{1}{8}Rv = -\frac{1}{8}(1+\rho(u) - \lambda u^{-4})u^{-1}$$

So we have

$$(\Delta_g v)^2 \le C(u^{-2} + u^{-2}\rho^2(u) + \lambda^2 u^{-10}) \le C(u^{-2} + u^{2p-12} + \lambda^2 u^{-10})$$

and

$$\int_\Omega (\Delta_g v)^2 dV_g \le C + C\int_\Omega u^{2p-6}dx + C\lambda^2\int_\Omega u^{-4}dx$$

$$\le C(1 + u_{min}^{-2\tau} + \lambda^2 u_{min}^{-4}),$$

where $\tau = \min(3-p, 0) < 2$ since $1 < p < 5$.

For the second term at (2.7), we have

$$\int_\Omega |B||\nabla_g v|^2 dV_g \le C(\int_\Omega |B|^{3/2} dV_g)^{2/3} (\int_\Omega |\nabla_g v|^6 dV_g)^{1/3}$$

$$\le C\lambda(\int_\Omega |\nabla_g v|^6 dV_g)^{1/3}$$

by (2.2).

The third and fourth terms at (2.7) can be estimated as follows:

$$\int_\Omega |\nabla_g v|^2 dV_g = \int_\Omega |\nabla_0 u|^2 u^{-2} \le C\lambda,$$

$$\int_\Omega |R||\nabla_g v|^2 dV_g \leq \int_\Omega C(1+\rho(u)+\lambda u^{-4})|\nabla_g v|^2 dV_g$$
$$\leq C\lambda + (\int_\Omega |\rho(u)+\lambda u^{-4}|^{3/2} dV_g)^{2/3} (\int_\Omega |\nabla_g v|^6 dV_g)^{1/3}$$
$$\leq C\lambda + C(\int_\Omega (u^{3/2(p-5)+6} + \lambda^{3/2}) dx)^{2/3} (\int_\Omega |\nabla_g v|^6 dV_g)^{1/3}$$
$$\leq C\lambda + C\lambda^{1/4(p-5)+1} (\int_\Omega |\nabla_g v|^6 dV_g)^{1/3}$$

by (1.4), since $0 < \frac{3}{2}(p-5)+6 \leq 5$ by our assumption that $p \leq \frac{13}{3}$. (This is the place where we need $p \leq \frac{13}{3}$.)

Combing all estimates together, we have that

$$(\int_\Omega |\nabla_g v|^6 dV_g)^{1/3} \leq C(1 + u_{min}^{-2\tau} + \lambda^2 u_{min}^{-4}). \tag{2.8}$$

Note that

$$\int_\Omega |\nabla_g v|^6 dV_g = \int_\Omega |\nabla_0 u|^6 u^{-18} dx.$$

So

$$(\int_\Omega |\nabla_g u^{-2}|^6 dx)^{1/3} \leq C(1 + u_{min}^{-2\tau} + \lambda^2 u_{min}^{-4}). \tag{2.9}$$

By Sobolev embedding theorem, for any $p, q \in \bar{\Omega}$, we have

$$|u^{-2}(p) - u^{-2}(q)| \leq C\|\nabla(u^{-2})\|_{L^6(\Omega)} |p-q|^{1/2}$$
$$\leq C\|\nabla(u^{-2})\|_{L^6(\Omega)} \leq C(1 + u_{min}^{-\tau} + \lambda u_{min}^{-2}).$$

Therefore we have that for any $p, q \in \bar{\Omega}$,

$$u^{-2}(p) \geq u^{-2}(q) - C(1 + u_{min}^{-\tau} + \lambda u_{min}^{-2}). \tag{2.10}$$

Let us now choose q so that $u(q) = u_{min} = \min_{\bar{\Omega}} u$. Since $u_{min} \leq C\lambda^{1/4}$ (by (1.3)) and $\tau < 2$, we see that

$$u^{-2}(q) - C(1 + u_{min}^{-\tau} + \lambda u_{min}^{-2}) \geq \frac{1}{2} u_{min}^{-2} \tag{2.11}$$

which implies that

$$u(p) \leq C u_{min} \leq C\lambda^{1/4}, \text{ for all } p \in \bar{\Omega}. \tag{2.12}$$

Now let $w = u - \bar{u}$, where $\bar{u} = \frac{1}{|\Omega|}\int_\Omega u$. Then w satisfies

$$\Delta w - \lambda w + f(w+\bar{u}) - f(\bar{u}) - \lambda \bar{u} + f(\bar{u}) = 0 \tag{2.13}$$

Multiplying (2.13) by w and integrating by parts, we have that

$$\int_\Omega (|\nabla w|^2 + \lambda w^2 + c(w) w^2) = 0 \tag{2.14}$$

where $c(w) = \frac{f(w+\bar{u})-f(\bar{u})}{w} \to 0$ as $\lambda \to 0$. Since $\int_\Omega w = 0$, (2.14) implies that $w \equiv 0$ and hence $u \equiv \bar{u}$. Theorem (1.1) is thus proved.

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