ON THE EFFECT OF THE DOMAIN SHAPE ON THE EXISTENCE OF LARGE SOLUTIONS OF SOME SUPERLINEAR PROBLEMS

JUNCHENG WEI AND LIQUN ZHANG

ABSTRACT. We discuss the existence, uniqueness and multiplicity of *large* solutions of the equation

$$\Delta u + u^p = 0 \ \text{ in } \Omega_\varepsilon$$
$$u > 0 \text{ in } \Omega_\varepsilon, u = 0 \text{ on } \partial\Omega_\varepsilon,$$

where $p > 1$ and $\Omega_\varepsilon \subset R^2$ is a bounded smooth domain which is obtained by connecting two disconnected domains by a small tunnel. We show that the geometry of the corners dominates the properties of the *large* solutions (bump-like solutions). When two corners are both open, we prove that there are many solutions with arbitrary number of bumps, while when the two corners are both close, there exists at least one single-bump solution. When one corner is open and another corner is close, we prove that there exists no single-bump solutions.

1. INTRODUCTION

In this paper, we are concerned with the existence of large solutions of the following problem

$$\begin{cases} \Delta u + u^p = 0, u > 0 & \text{in } \Omega \\ u\big|_{\partial\Omega} = 0 \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a $C^{2,\alpha}$ smooth domain and is dumbbell-type, $p > 1$. (See Fig. 1.) Clearly, (1.1) is the simplest superlinear problems. However, it resembles most of the phenomenon occuring in the nonlinear problem. It is well known that there exist nontrivial solutions of problem (1.1). What is interesting is that there are some

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solutions blow up when domain is varying even its volume is uniformly bounded from below. These domains, in contrast to the convex domain where the solution of (1.1) has a priori estimates, have more complicated geometry. We are interested in how the domain shape affects the existence of *large* solutions of (1.1). We restrict our attention to the dumbbell-type domains. One of the motivations of this study is from the earlier works of Dancer. In [D1], Dancer studied the following problem

$$\begin{cases} \Delta u + u^p = 0, u > 0 \qquad \text{in } \Omega_\varepsilon \\ u\big|_{\partial\Omega_\varepsilon} = 0 \end{cases} \tag{1.2}$$

where $\Omega_\varepsilon \subset R^m$ is a smooth domain which is an approximation to a finite number of disjoint or touching balls, $1 < p < (m+2)/(m-2)$ for $m \geq 3$ and $1 < p < +\infty$ for $m = 2$. The domain Ω_ε is a singular perturbation of l disjoint open balls $\Omega_0 = \cup_{i=1}^l B_i$, where B_i are disjoint balls. (See Fig 2.) When $\Omega_\varepsilon = B_i$ is a ball, it is known that problem (1.2) admits a unique solution u_i. Ω_ε is obtained by connecting the l balls with small tunnels (we will make this more precise later). An interesting phenomenon is that the existence of these small tunnels also creates *large* solutions. (In domains with no small tunnels, there are no *large* solutions. See [D2] and [D3].) More precisely, Dancer studied the case $l = 2$, namely the dumbbell case. (See Fig. 3.) The assumption is the following: Ω_ϵ consists of two balls of radius a and centers $(\pm b, 0)$ (where $b > a > 0$) together with a joining strip $(-b, b) \times (-\frac{1}{2}\epsilon, \frac{1}{2}\epsilon)$. Under these conditions, Dancer [D1] proved the existence of *large* solutions, namely solutions of the form $u_\epsilon(x, y) \sim \epsilon^{-2/(p-1)}$. This result holds true in general dumbbell-like domains as long as it keeps Z_2 symmetry. Recently, J. Byeon [B] extended Dancer's result to higher dimensional case. However, he also needs the Z_2 symmetry assumption.

A natural question (see also [D1]) is asked as the following: is the Z_2 symmetry necessary in the existence of large solutions ?

The main purpose of this paper is to answer the above question. Here we only consider the two dimensional case. The reason is that in two dimensional case, the

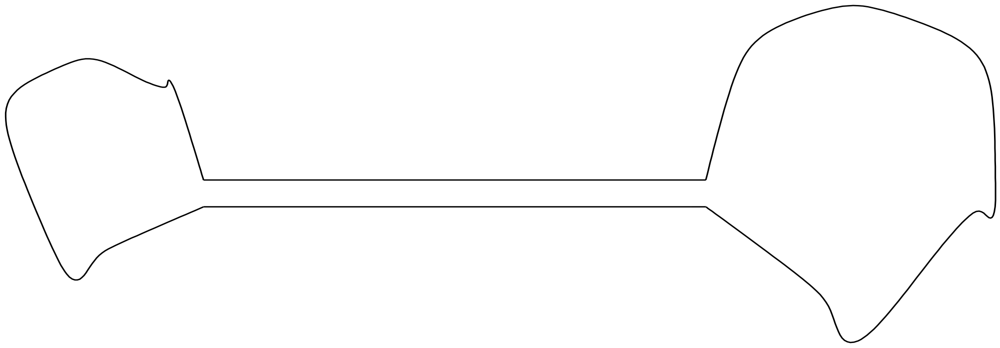

FIGURE 1. Fig 1: Dumbbell-domain without symmetry

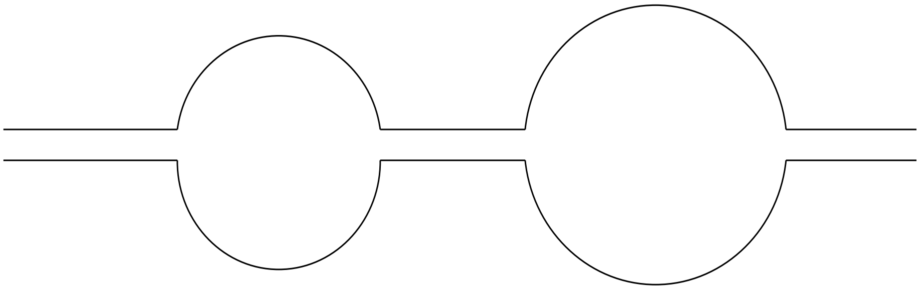

FIGURE 2. Fig 2:l-balls connected by small tunnels

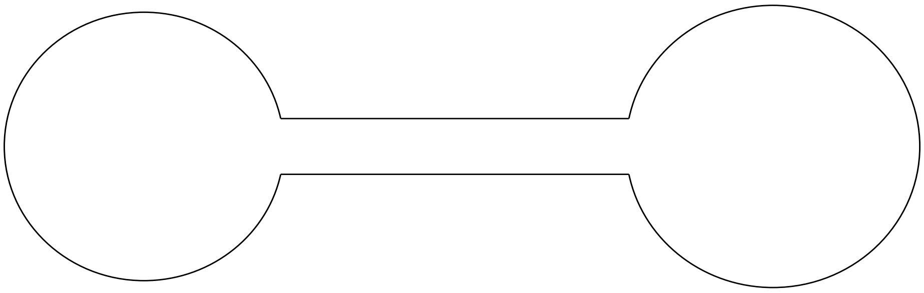

FIGURE 3. Fig 3: Dumbbell-domain with symmetry

kernel of the linearized operator of the solutions of problem (1.1) on an infinite strip is well understood. It turns out that the shape at the corners between the balls and the tunnel plays a very important role.

Our approach combines some geometric ideas and the Liapunov-Schmidt reduction method. It is known that the linearized operator of problem (1.3) has only one dimensional kernel. In order to kill the kernel, we need to solve an algebraic equation. We find out that the geometry at the corner points dominates the leading term of the algebraic equation. Therefore under some geometric conditions, we are able to give an criteria of the existence of large solutions. It is quite surprising that for some dumbbell-type domains there exist multiple large solutions. These solutions are asymptotically near to, in some sense, the positive solution of problem (1.3). In fact, the second author learned from Jianhua Zhou that he has some numerical results on the existence of multiple solutions, which indicates the existence of large solutions.

We first make some assumptions on the dumbbell type domain $\Omega = \Omega_\epsilon$. Without loss of generality, we assume that the length of the tunnel is 20 and the width is ϵ. Our basic assumptions are:

(H_1) $\Omega_\varepsilon \cap \{(x,y)| \quad |x| \leq 10\} = \{(x,y)| \quad |y| \leq \varepsilon, |x| \leq 10\}$, the volume of Ω_ε is uniformly bounded from below and above (independent of ϵ) and Ω_ε is connected.

(H_2) At points $(10,\varepsilon),(10,-\varepsilon),(-10,\varepsilon),(-10,-\varepsilon)$, the boundary can be written as a graph locally, say $(x, h_\varepsilon^\pm(x))$ where $h_\varepsilon^+, h_\varepsilon^- \in C^{2,\alpha}$,

$$
\begin{array}{ll}
\text{and } h_\varepsilon^\pm(x) \text{ satisfy at points } (10,\pm\varepsilon) \text{ as } x \to 0 \text{ uniformly in } \varepsilon, & x > 0,\\
h_\varepsilon^\pm(10+x) = \pm\varepsilon + A_1^\pm x^{2+\alpha_1^\pm} + O(x^3) & x > 0,\\
h_\varepsilon^{\pm'}(10+x) = (2+\alpha_1^\pm) A_1^\pm x^{1+\alpha_1^\pm} + O(x^2) & x > 0,\\
h_\varepsilon^{\pm''}(10+x) = (2+\alpha_1^\pm)(1+\alpha_1^\pm) A_1^\pm x^{\alpha_1^\pm} + O(x) & x > 0;
\end{array}
$$

similarly at points $(10,\pm\varepsilon)$ as $x \to 0$,

$$
\begin{array}{ll}
h_\varepsilon^\pm(-10-x) = \pm\varepsilon + A_2^\pm x^{2+\alpha_2^\pm} + O(x^3) & x > 0,\\
h_\varepsilon^{\pm'}(-10-x) = (2+\alpha_2^\pm) A_2^\pm x^{1+\alpha_2^\pm} + O(x^2) & x > 0,\\
h_\varepsilon^{\pm'}(-10-x) = (2+\alpha_2^\pm)(1+\alpha_2^\pm) A_2^\pm x^{\alpha_2^\pm} + O(x) & x > 0,
\end{array}
$$

where $h_\varepsilon^+ > 0$ $h_\varepsilon^- < 0, 0 < \alpha_i < 1$.

Now we introduce some notations. Let

$$D_1(\varepsilon) = \varepsilon^{\alpha_1^+}\pi^{-\alpha_1^+}\Gamma(\alpha_1^+ + 1)A_1^+(\alpha_1^+ + 2)(\alpha_1^+ + 1) - \varepsilon^{\alpha_1^-}\pi^{-\alpha_1^-}\Gamma(\alpha_1^- + 1)A_1^-(\alpha_1^- + 2)(\alpha_1^- + 1),$$

$$D_2(\varepsilon) = \varepsilon^{\alpha_2^+}\pi^{-\alpha_2^+}\Gamma(\alpha_2^+ + 1)A_2^+(\alpha_2^+ + 2)(\alpha_2^+ + 1) - \varepsilon^{\alpha_2^-}\pi^{-\alpha_2^-}\Gamma(\alpha_2^- + 1)A_2^-(\alpha_2^- + 2)(\alpha_2^- + 1),$$

where $\Gamma(\alpha)$ is the gamma function, i.e. $\Gamma(\alpha)=\int_0^\infty e^{-s}s^{\alpha-1}ds$.

Let $\alpha=\min(\alpha_1^+,\alpha_1^-,\alpha_2^+,\alpha_2^-)$.

Let $S=\{(x,y)\mid -\infty<x<+\infty, |y|\le 1\}$. Consider the problem

$$\begin{cases}\triangle u+u^p=0 \quad \text{in } S\\ u\mid_{\partial S}=0.\end{cases} \tag{1.3}$$

It was proved in [AT], that (1.3) has a positive solution which decays exponentially at infinity. After a translation, we may assume $u(x,y)$ is the positive solution of problem (1.3) which is even in x and y and exponentially decays as $|x|\to\infty$. We denote this solution as $u_0(x,y)$.

Put

$$u^*_{\varepsilon,\xi}(x,y)=u_0\big(\frac{x-\xi}{\varepsilon},\frac{y}{\varepsilon}\big)\varepsilon^{-\frac{2}{p-1}}.$$

Then $u^*_{\varepsilon,\xi}$ is a positive solution of problem (1.3) with S replaced by $S_\varepsilon=\{(x,y)|\quad |y|<\varepsilon,-\infty<x<+\infty\}$.

Theorem 1.1. *Let Ω_ε satisfy the assumptions (H_1) and (H_2). If $D_1D_2>0$ for $\varepsilon<<1$, then for $\varepsilon<\varepsilon_0$ problem (1.1) has a positive solution in the form $u^*_{\varepsilon,\xi_\varepsilon}+v_\varepsilon$ for $|x|<10$ and $|v_\varepsilon|<e^{-C/\varepsilon}$, $C>0$ with $|\xi_\varepsilon|<10$. Moreover ξ_ε satisfies as $\varepsilon\to 0$*

$$|\xi_\varepsilon|=O(\varepsilon\ln\varepsilon). \tag{1.4}$$

Remark 1.1: In [D1], the solution obtained in theorem 1.1 is called large solutions, since it blows up as $\varepsilon\to 0$. In [D2], Dancer proved the existence of large solutions of dumbbell type domain which is symmetric with respect to y-axis. We believe that the similar result is still true for higher dimensions.

Remark 1.2: For the dumbbell domain which has symmetry as in [D2], the condition $D_1D_2=D_1^2>0$ is always satisfied if it is non-degenerated namely, $D_1\neq 0$.

Remark 1.3: The assumption $\partial\Omega_\varepsilon\in C^{2,\alpha}$ is only a technique assumption and can be weakened. The non-degeneracy conditions at corner, that is, $A_1^\pm, A_2^\pm\neq 0$ in (H_2) can also be replaced by $A_1^\pm\varepsilon^M$ and $A_2^\pm\varepsilon^M$ with some constant M.

The condition $D_1 D_2 > 0$ is "almost" necessary. In fact we can prove

Theorem 1.2. *Suppose $D_1 D_2 < 0$. Then for ε sufficiently small, problem (1.1) doesn't admit a solution u in*

$$N_{\varepsilon^N} = \{u| \inf_{\xi \in (-10+\delta, 10-\delta)} \|u - u^*_{\varepsilon,\xi}\|_{L^2(\Omega_\varepsilon)} < \varepsilon^N e^{-\frac{\pi d_\xi}{2\varepsilon}}\}$$

for some constants N, $\delta > 0$, where $d_\xi = \min(10 - \xi, 10 + \xi)$.

Remark 1.4: The condition $D_1 D_2 > 0$ can be separated into two cases: (1) $D_1 > 0, D_2 > 0$; (2) $D_1 < 0, D_2 < 0$. It turns out that in case (1), we have multiplicity of large solutions.

Theorem 1.3. *Suppose $D_1 > 0, D_2 > 0$. Let K be any positive integer. Then for ε sufficiently small, problem (1.1) has a solution u with the form*

$$u = \sum_{i=1}^{K} u^*_{\varepsilon, P_{\varepsilon,i}} + v_\varepsilon$$

where

$$|P_{\varepsilon,i} - P_i^0| = o(1), \quad v_\varepsilon = O(e^{-\frac{\pi}{2}\frac{20-\delta}{K\varepsilon}}).$$

Here $P_1^0 = -10 + \frac{10}{K}, P_i^0 = -10 + \frac{10+20(i-1)}{K}, i = 2, \cdots, K-1, P_K^0 = 10 - \frac{10}{K}$.

Remark 1.5: Solutions in Theorem 1.3 can be called "multi-bump" solutions. For solutions of this kind arising from other problems, we refer to [AL], [BBF], [BF], [BDS], [BLO], [BP], [CR1], [CR2], [CW], [DF], [DFW], [F], [DY], [G], [GG], [GW], [K], [L1], [L2], [LN], [LW], [NT1], [NT2], [NT3], [NW], [Wa], [W1], [W2], [W2], [W3], [W4], [WW1], [WW2], [WW3], and the references therein. Our solutions are different from all the others since they do not produce point condensation or singularity.

In case (1): $D_1 < 0, D_2 < 0$, we believe that there exists only one large solution. Hence we have the following conjecture.

Conjecture: If $D_1 < 0, D_2 < 0$, then for ε sufficiently small, problem (1.1) has only one large solution which concentrates on the tunnel.

Remark 1.6: Geometrically speaking, $D_1(\varepsilon) > 0$ means that the tunnel "opens up" at the right hand. Hence Theorem 1.3 means if the domain "opens up" at both sides, there are many large solutions for (1.1) while the Conjecture means if the domain "closes" at both sides, there is only one large solution. Theorem 1.2 suggests if the domain "opens up" at one side and "closes" at the other side, then there are no large solutions.

Remark 1.7: Another related work was done by Alikakos, Fusco and Kowalczyk ([AFK]). They studied the following problem in a dumbbell-type domain

$$\begin{cases} \varepsilon^2 \Delta u + f(u) = 0 \text{ in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \end{cases} \tag{1.5}$$

where $f(u)$ is a two-well potential (e.g., $f(u) = u - u^3$). They investigated the role of the corners on the existence of transitional layer solutions.

The paper is organized as follows. In section 2, the asymptotical behavior of solutions on the infinite strip is obtained, and then an approximated solution on Ω_ε is given which is in $W^{2,2}(\Omega_\varepsilon)$. Section 3 contains the result of the linearized operators and the key estimates including the geometry of the corners. Then we obtain a solution near the approximated solutions and prove Theorem 1.1 amd Theorem 1.2 in section 3. Finally, in section 5, we prove the existence of multiple-bump large solutions.

Throughout this paper, we use c, c_i, C to denote various generic constants (independent of ε). The number $\delta > 0$ is a very small number. $O(A)$ means $|O(A)| \leq C|A|$ and $o(a)$ means $|o(a)|/|a| \to 0$ as $\varepsilon \to 0$.

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2. Asymptoical Behavior and Approximated Solutions

We first recall some basic facts about problem (1.3). We are interested in the asymptotical behaviors of the solutions $u_0(x,y)$ of problem (1.3). By the result of [AT], we may assume $\frac{\partial u_0}{\partial x} < 0$ if $x > 0, y\frac{\partial u_0}{\partial y} < 0$ if $y \neq 0$ and there exist $c_1, c > 0$ such that $u_0(x,y) < c_1 \exp(-c|x|)$ on S.

Put

$$f(x) = \int_{-1}^{1} u_0(x,y)\cos(\frac{\pi}{2}y)dy, \tag{2.1}$$

$$F(x) = \int_{-1}^{1} u_0^p(x,y)\cos(\frac{\pi}{2}y)dy. \tag{2.2}$$

First we have the following lemma.

Lemma 2.1. *There exist $c_3, c_6 > 0$, such that as $x \to +\infty$*

$$f(x) = c_3 e^{-\frac{\pi}{2}x}\big(1 + O(e^{-c_6 x})\big). \tag{2.3}$$

Proof: Multiplying (1.3) by $\cos\frac{\pi}{2}y$ and integrating by parts we obtain

$$f''(x) - (\frac{\pi}{2})^2 f(x) + F(x) = 0. \tag{2.4}$$

Solving (2.4) with the boundary conditions which are exponentially decaying at infinity we have for $x \geq 0$

$$f(x) = c_4 e^{-\frac{\pi}{2}x} + \frac{1}{\pi}\int_0^x e^{-\frac{\pi}{2}(x-\tau)}F(\tau)d\tau + \frac{1}{\pi}\int_x^{+\infty} e^{\frac{\pi}{2}(x-\tau)}F(\tau)d\tau, \tag{2.5}$$

where $c_4 = f(0) - \frac{1}{\pi}\int_0^\infty e^{-\frac{\pi}{2}\tau}F(\tau)d\tau$.

Since $u_0(x,y)$ is even in x, so is $f(x)$, then $f'(0) = 0$. This implies

$$c_4 = \frac{1}{\pi}\int_0^\infty e^{-\frac{\pi}{2}\tau}F(\tau)d\tau > 0,$$

because $f'(0) = -\frac{\pi}{2}c_4 + \frac{1}{2}\int_0^\infty e^{-\frac{\pi}{2}\tau}F(\tau)d\tau$.

¿From (2.5), obviously for $x > R$

$$f(x) \geq (c_4 + \frac{1}{\pi}\int_0^R e^{-\frac{\pi}{2}\tau} F(\tau) d\tau) e^{-\frac{\pi}{2}x} = c_5 e^{-\frac{\pi}{2}x}. \tag{2.6}$$

Since

$$F(x) = \int_{-1}^{1} u_0^p(x,y) \cos\frac{\pi}{2} y dy \leq c_1^{p-1} e^{-(p-1)cx} f(x), \tag{2.7}$$

we may assume $F(x) \leq e^{-c_6 x} f(x)$ for $x \geq R$. Then

$$f(x) \leq c_5 e^{-\frac{\pi}{2}x} + \frac{1}{\pi}\int_R^x e^{-\frac{\pi}{2}(x-\tau)-c_6\tau} f(\tau) d\tau + \frac{1}{\pi}\int_x^\infty e^{\frac{\pi}{2}(x-\tau)-c_6\tau} d\tau f(x).$$

We may also assume $\displaystyle\int_x^\infty e^{+\frac{\pi}{2}(x-\tau)-c_6\tau} d\tau \leq \frac{\pi}{2}$ for $x \geq R$. We note that $f'(x) < 0$ for $x > 0$, then we have

$$f(x) \leq 2c_5 e^{-\frac{\pi}{2}x} + \frac{2}{\pi} e^{-\frac{\pi}{2}x} \int_R^x e^{\frac{\pi}{2}\tau - c_6\tau} f(\tau) d\tau = e^{-\frac{\pi}{2}x} H(x). \tag{2.8}$$

For $H(x)$, we have by (2.8)

$$H'(x) = \frac{2}{\pi} e^{\frac{\pi}{2}x - c_6 x} f(x) \leq \frac{2}{\pi} e^{-c_6 x} H(x). \tag{2.9}$$

Then (2.9) yields for $x \geq R$

$$H(x) \leq H(R) \exp\big(\frac{2}{\pi c_6} e^{-c_6 R}\big), \tag{2.10}$$

that is, for $x \geq R$

$$f(x) \leq c_7 e^{-\frac{\pi}{2}x}. \tag{2.11}$$

By (2.5), (2.7) and (2.11), we obtain as $x \to \infty$.

$$f(x) = e^{-\frac{\pi}{2}x} \frac{1}{\pi} \int_0^\infty (e^{\frac{\pi}{2}\tau} + e^{-\frac{\pi}{2}\tau}) F(\tau) d\tau (1 + O(e^{-c_6 x}))$$

which is (2.3), with $c_3 = \frac{1}{\pi}\int_0^\infty (e^{\frac{\pi}{2}\tau} + e^{-\frac{\pi}{2}\tau}) F(\tau) d\tau$. □

Corollary 2.2. *There exists a constant c_8, such that*

$$u_0(x,y) \leq c_8 e^{-\frac{\pi}{2}x}, \tag{2.12}$$

$$|\nabla u_0(x,y)| \leq c_8 e^{-\frac{\pi}{2}x}, \tag{2.13}$$

$$|u_{0,yy}(x,y)| \leq c_8 e^{-\frac{\pi}{2}x}. \tag{2.14}$$

Here $u_{0,yy}$ stands for $\frac{\partial^2 u_0}{\partial y^2}$.

Proof: By Harnack inequality, we have

$$u_0(x,0) \leq c_9 u_0(x, \frac{3}{4}),$$

where c_9 is independent of x. Then

$$u_0(x,y) \leq \ u_0(x,0) \leq c_9 u_0(x, \frac{3}{4}) \leq \frac{\pi c_9}{4 \sin \frac{3}{8}\pi} f(x). \tag{2.15}$$

We obtain (2.12) from (2.11) and (2.15). Then (2.13) and (2.14) are obtained from (2.12), by standard regularity arguments. □

Lemma 2.3. *There exists a constant $c_{10} > 0$ such that for $i > 1$ as $x \to +\infty$*

$$u_0(x,y) = c_3 e^{-\frac{\pi}{2}x} \cos \frac{\pi}{2} y + O(e^{-(\frac{\pi}{2}+c_{10})x}), \tag{2.16}$$

$$\frac{\partial u_0}{\partial y}(x,y) = -\frac{\pi}{2} c_3 e^{-\frac{\pi}{2}x} \sin \frac{\pi}{2} y + O(e^{-(\frac{\pi}{2}+c_{10})x}), \tag{2.17}$$

$$\frac{\partial^2 u_0}{\partial y^2}(x,y) = -(\frac{\pi}{2})^2 c_3 e^{-\frac{\pi}{2}x} \cos \frac{\pi}{2} y + O(e^{-(\frac{\pi}{2}+c_{10})x}). \tag{2.18}$$

Similar estimates hold for $\dfrac{\partial u_0}{\partial x}, \dfrac{\partial^2 u_0}{\partial x \partial y}$ and $\dfrac{\partial^2 u_0}{\partial x^2}$ by differentiating the leading term in (2.16).

Proof: Put

$$f_i(x) = \int_{-1}^{1} u_0(x,y) \cos(\frac{\pi}{2} y + i\pi y) dy, \tag{2.19}$$

$$F_i(x) = \int_{-1}^{1} u_0^p(x,y) \cos(\frac{\pi}{2} y + i\pi y) dy. \tag{2.20}$$

As in the proof of Lemma 2.1, we have for $x \geq 0$

$$f''_i - (\frac{\pi}{2} + i\pi)^2 f_i + F_i = 0. \tag{2.21}$$

We note that

$$f_i(x) = -\frac{1}{\frac{\pi}{2}+i\pi}\int_{-1}^{1} u_{0,y}(x,y)\sin(\frac{\pi}{2}y+i\pi y)dy$$
$$= -\frac{1}{(\frac{\pi}{2}+i\pi)^2}\int_{-1}^{1} u_{0,yy}(x,y)\cos(\frac{\pi}{2}y+i\pi y)dy.$$

By (2.14), we deduce

$$|f_i(x)| \le \frac{c_{12}}{(i\pi)^2}e^{-\frac{\pi}{2}x}. \tag{2.22}$$

Then we have the following Fourier series expansion

$$u_0(x,y) = f(x)\cos\frac{\pi}{2}y + \sum_{i=1}^{\infty} f_i(x)\cos\frac{1+2i}{2}\pi y. \tag{2.23}$$

By solving (2.21), we obtain

$$f_i(x) = \frac{1}{(1+2i)\pi}(\int_0^{\infty} e^{-(\frac{\pi}{2}+i\pi)\tau}F_i(\tau)d\tau + \int_0^{x} e^{-(\frac{\pi}{2}+i\pi)(x-\tau)}F_i(\tau)d\tau)$$
$$+ \frac{1}{(1+2i)\pi}\int_x^{\infty} e^{(\frac{\pi}{2}+i\pi)(x-\tau)}F_i(\tau)d\tau. \tag{2.24}$$

By (2.21) – (2.24), we have

$$|F_i(x)| = \left| \frac{1}{(\frac{\pi}{2}+i\pi)^2}\int_{-1}^{1}(u_0^p)_{yy}\cos(\frac{\pi}{2}y+i\pi y)dy \right| \le \frac{c_{13}}{(i\pi)^2}e^{-\frac{\pi}{2}px}. \tag{2.25}$$

¿From (2.24) and (2.25) we obtain for $i > 1$ and for x sufficiently large we have

$$|f_i(x)| \le \frac{1}{(i\pi)^3}e^{-(\frac{\pi}{2}+c_{14})x}, \tag{2.26}$$

where $c_{14} = \min\{i\pi, \frac{p-1}{2}\pi\} - 1$.

Then from (2.23), (2.26) and Lemma 2.1, we obtain (2.16). By differentiating (2.23) with respect to y, and by (2.26) we can deduce (2.17) similarly. In general, we can use the fact that $u_0(x,y)$ is exponentially decaying up to high order, for example fourth order derivatives, then the estimates (2.22) or (2.26) can be improved so that (2.23) is still true by differentiating it twice. Then we obtain the proof of Lemma 2.3.

□

Now we consider the construction of approximate solutions on Ω_ε. Put

$$h_\varepsilon(x,y)=\begin{cases}\varepsilon & |x|\le 10, y>0,\\ -\varepsilon & |x|\le 10, y<0,\\ \\ h_\varepsilon^+(x) & 11>|x|>10, y>0,\\ h_\varepsilon^-(x) & 11>|x|>10, y<0.\end{cases}$$

Let $\eta(x)$ be a positive C^∞ smooth function such that $\eta'(x)\ge 0$ and

$$\eta(x)=\begin{cases}1 & x>1,\\ 0 & x<\frac{1}{2}.\end{cases}$$

For convenience, we may assume $2>|h_\varepsilon(x,y)|>\varepsilon$ for $10<|x|\le 11$.

For $\delta>0$, we let $\beta^+(x,y)=\dfrac{y}{h_\varepsilon^+(x)}\eta(\dfrac{y}{\delta})+\dfrac{1}{2}\big(1-\eta(\dfrac{y}{\delta})\big)y$. We can check that $\beta^+(x,y)$ is a strictly monotone increasing in y for $y\ge 0$. Then the map $\Phi:[-11,11]\times[0,h_\varepsilon^+(x)]\longmapsto[-11,11]\times[0,1]$ given by

$$\Phi(x,y)=(x,\beta^+(x,y))$$

is a C^2 diffeomorphic map and onto.

If we let $\beta(x,y)=\dfrac{y}{h_\varepsilon(x,y)}\eta(\dfrac{|y|}{\delta})+\dfrac{1}{2}(1-\eta(\dfrac{|y|}{\delta}))y$. Then the map $\Phi:[-11,11]\times[h_\varepsilon^-(x),h_\varepsilon^+(x)]\longmapsto[-11,11]\times[-1,1]$ given by

$$\Phi(x,y)=(x,\beta(x,y)) \tag{2.27}$$

is an onto C^2 diffeomorphic map.

Then we can construct an approximate solution, for given $\xi\in(-10,10)$

$$u_{\varepsilon,\xi,\delta}(x-\xi,y)=\varepsilon^{-\frac{2}{p-1}}\eta(11.5+x)(1-\eta(x-10))u_0(\frac{x-\xi}{\varepsilon},\beta(x,y)). \tag{2.28}$$

Let

$$u_{\varepsilon,\xi}=\varepsilon^{-\frac{2}{p-1}}\eta(x+11.5)(1-\eta(x-10))u(\frac{x-\xi}{\varepsilon},\frac{y}{h_\varepsilon(x,y)}). \tag{2.29}$$

Note that $u_{\varepsilon,\xi}$ is the limit of $u_{\varepsilon,\xi,\delta}$ when $\delta \to 0$. Actually in the sequel we shall utilize $u_{\varepsilon,\xi}$ as the approximated solution.

Lemma 2.4. *For given $\xi \in (-10, 10)$, $u_{\varepsilon,\xi,\delta}$ and $u_{\varepsilon,\xi}$ satisfy*

(i) $u_{\varepsilon,\xi,\delta} \in C^2(\Omega_\varepsilon), u_{\varepsilon,\xi,\delta}\mid_{\partial\Omega_\varepsilon}= 0$ *when* $\delta < \varepsilon$;

(ii) $u_{\varepsilon,\xi,\delta} \to u_{\varepsilon,\xi}$ *in* $W^{2,2}(\Omega_\varepsilon)$ *as* $\delta \to 0$.

Therefore $u_{\varepsilon,\xi}$ is in $W^{2,2}(\Omega_\varepsilon)$.

Proof: (i) can be checked directly and easily.

Note that $h_\varepsilon(x, y)$ is C^2 smooth except at $y = 0$. By (2.29), we only need to show as $\delta \to 0$

$$\frac{\partial}{\partial y}u_{\varepsilon,\xi,\delta} \to \frac{\partial}{\partial y}u_{\varepsilon,\xi} \quad \text{ in } L^2(\Omega_\varepsilon), \tag{2.30}$$

$$\frac{\partial^2}{\partial y^2}u_{\varepsilon,\xi,\delta} \to \frac{\partial^2}{\partial y^2}u_{\varepsilon,\xi} \quad \text{ in } L^2(\Omega_\varepsilon). \tag{2.31}$$

We first show

$$\lim_{\delta\to 0}\int_{\substack{h_\varepsilon^+\geq y\geq 0\\ |x|\leq 11}} |u_{0,y}(\frac{x-\xi}{\varepsilon}, \beta^+(x,y))\beta_y^+ - u_{0,y}(\frac{x-\xi}{\varepsilon}, \frac{y}{h_\varepsilon^+(x)})\frac{1}{h_\varepsilon^+(x)}\ |^2 = 0. \tag{2.32}$$

In fact for $y \geq 0, \beta^+(x, y) \to \frac{y}{h_\varepsilon^+(x)}$, and

$$\beta_y^+ = \frac{1}{h_\varepsilon^+}\eta(\frac{y}{\delta}) + \frac{1}{2}\big(1-\eta(\frac{y}{\delta})\big) + \frac{y}{\delta}\big(\frac{1}{h_\varepsilon^+} - \frac{1}{2}\big)\eta'(\frac{y}{\delta}). \tag{2.33}$$

We can calculate by recalling $\eta(s) = 1$ when $s \geq 1$

$$\lim_{\delta\to 0}\int_0^{h_\varepsilon^+} \big(1-\eta(\frac{y}{\delta})\big)^2 dy = \lim_{\delta\to 0}\delta\int_0^{h_\varepsilon^+}(1-\eta(s))^2 ds = 0, \tag{2.34}$$

$$\lim_{\delta\to 0}\int_0^{h_\varepsilon^+} (\frac{y}{\delta})^2\big(\eta'(\frac{y}{\delta})\big)^2 dy = \lim_{\delta\to 0}\delta\int_0^{h_\varepsilon^+} s^2\eta'^2(s)ds = 0. \tag{2.35}$$

Therefore

$$\lim_{\delta\to 0}\int_0^{h_\varepsilon^+} |\beta_y^+(x,y) - \frac{1}{h_\varepsilon^+(x)}\ |^2\ dy = 0. \tag{2.36}$$

Then (2.32) follows easily from (2.36). Similarly we consider (2.32) on the parts of Ω_ε where $y \le 0$ and then we have proved (2.30).

Recall the assumption $h_\varepsilon^+(x) < 2$ for $|x| \le 11$, then

$$\int_0^{h_\varepsilon^+} \eta^4(\frac{y}{\delta})dy \le 2.$$

And similarly as in the proof of (2.34) and (2.35), we have

$$\lim_{\delta\to 0}\int_0^{h_\varepsilon^+} (1-\eta(\frac{y}{\delta}))^4 dy = 0,$$
$$\lim_{\delta\to 0}\int_0^{h_\varepsilon^+} (\frac{y}{\delta})^4(\eta'(\frac{y}{\delta}))^4 dy = 0.$$

Therefore

$$\lim_{\delta\to 0}\int_0^{h_\varepsilon^+} \mid \beta_y^{+2}(x,y) - \frac{1}{h_\varepsilon^{+2}(x)} \mid^2 dy = 0. \tag{2.37}$$

¿From (2.37), we deduce

$$\lim_{\delta\to 0}\int_{\substack{h_\varepsilon^+\ge y\ge 0\\ |x|\le 11}} \left|u_{0,yy}(\frac{x-\xi}{\varepsilon}, \beta^+(x,y))\beta^{+2}_y - u_{0,yy}(\frac{x-\xi}{\varepsilon}, \frac{y}{h_\varepsilon^+(x)})\frac{1}{h_\varepsilon^{+2}(x)}\right|^2 = 0. \tag{2.38}$$

Since $u(x,y)$ is even in y, then there exists a constant c_{15} such that for $(x,y) \in S$

$$|u_{0,y}(x,y)| \le c_{15}|y|.$$

Again as in (2.34) and (2.35) we have

$$\lim_{\delta\to 0}\int_0^{h_\varepsilon^+} (y\beta_{yy}^+)^2 dy = 0. \tag{2.39}$$

Then we deduce

$$\lim_{\delta\to 0}\int_{\substack{h_\varepsilon^+\ge y\ge 0\\ |x|\le 11}} \mid u_{0,y}(\frac{x-\xi}{\varepsilon}, \beta^+(x,y))\beta_{yy}^+ \mid^2 dxdy = 0. \tag{2.40}$$

Similarly we can consider (2.38) and (2.40) for the parts of Ω_ε where $y \le 0$ and then we obtain the proof of (2.31). Then (ii) is followed easily. $\square$

3. Linearized Operators and Some Estimates

In [D2], Dancer proved the invertibility of the linearized operators of problem (1.3) in even function class. Here is the only place that we need the dimension is two. The following lemma is proved in [D2].

Lemma 3.1 (Dancer). *There is no non-trivial function v even in x, $v \in L^\infty(S) \cap W^{1,2}_{loc}(S)$ such that*

$$\begin{cases} \triangle v + pu_0^{p-1} v = 0 \quad \text{in} \quad S \\ v \mid_{\partial S} = 0. \end{cases} \tag{3.1}$$

Here $W^{1,2}_{loc}$ means the functions which are in $W^{1,2}$ on compact subsets of $\bar{S}$ and which vanish on ∂S.

As pointed out in [D2] by Dancer that problem (3.1) has exactly one dimensional kernel. This is actually a direct corollary of Lemma 3.1, we have

Corollary 3.2. *Problem (3.1) has exactly one dimensional kernel $u_{0,x}$ in $L^\infty \cap W^{1,2}_{loc}(S)$.*

Proof: Suppose that $v(x,y)$ is a nontrivial solution of problem (3.1). Then by Lemma 3.1, we have

$$v(-x,y) = -v(x,y)$$

in particular, $v(0,y) = 0$. In fact, $v(-x,y) + v(x,y)$ is even in x and satisfies (3.1), then it must be identically zero.

Note that $u_{0,xx}(0,y) \not\equiv 0$. Otherwise $u_{0,yy}(0,y) \le 0$ and then $u_{0,y}(0,y)$ is decreasing which contradicts $u_0(0,\pm 1) = 0$. We may assume $u_{0,xx}(0,\frac{1}{2}) \neq 0$. Put

$$w(x,y) = v(x,y) - cu_{0,x}(x,y),$$

where c is chosen so that $w_x(0,\frac{1}{2}) = 0$. Note that $w(0,y) = 0$. If $w \not\equiv 0$, then w has a nodal line in $S^+ = \{(x,y) \in S \mid x > 0\}$ through point $(0,\frac{1}{2})$. In particular, w changes sign in S^+. Let

$$\Omega_+ = \{(x,y) \mid (x,y) \in S^+ \text{ and } w(x,y) > 0\}.$$

Then as in [D2], by considering $u_{0,x}$ and w, we have

$$\int_{\partial\Omega_+} u_{0,x}\frac{\partial w}{\partial \nu} = 0$$

which is impossible since $u_{0,x} \leq 0, u_{0,x} \not\equiv 0$ and $\frac{\partial w}{\partial \nu} < 0$ on $\partial\Omega_+$. □

Remark 3.3: As mentioned in [D2], Dancer's proof of Lemma 3.1 can also be carried over to show Corollary 3.2 directly. As remarked in [D2], the positive solution of problem (2.1) which tends to zero as $|x| \to \infty$ is unique up to a translation.

Lemma 3.4. *Let $u_{\varepsilon,\xi}$ and Ω_ε be given as In Section 2. Then there exist $c_{15}, \varepsilon_0 > 0$ such that for any $\varphi \in W_0^{1,2}(\Omega_\varepsilon)$ and*

$$\int_{\Omega_\varepsilon} \varphi \frac{\partial u_{\varepsilon,\xi}}{\partial \xi} = 0, \tag{3.2}$$

$\xi \in (-10+\delta, 10-\delta)$, and $\varepsilon \leq \varepsilon_0$, we have

$$|\int_{\Omega_\varepsilon} (|\bigtriangledown \varphi|^2 - pu_{\varepsilon,\xi}^{p-1}\varphi^2)| \geq c_{15}\int_{\Omega_\varepsilon} \varphi^2. \tag{3.3}$$

Proof: Suppose not. Then there exist $\varepsilon_n \to 0, \beta_n \to 0, \xi_n \to \xi \in [10-\delta, 10-\delta]$ as $n \to \infty$, $\varphi_n \in W_0^{1,2}(\Omega_{\varepsilon_n})$, $|\varphi_n|_{L^2(\Omega_{\varepsilon_n})} = 1$ and $\int_{\Omega_{\varepsilon_n}} \varphi_n \frac{\partial u_{\varepsilon_n}}{\partial \xi} = 0$,

$$\begin{cases} \triangle \varphi_n + pu_{\varepsilon_n,\xi_n}^{p-1}\varphi_n = \beta_n \varphi_n & \text{in } \Omega_{\varepsilon_n} \\ \varphi_n \mid_{\partial\Omega_{\varepsilon_n}} = 0. \end{cases} \tag{3.4}$$

Note that $|u_{\varepsilon_n}^{p-1}\varphi_n|_{L^2(\Omega_{\varepsilon_n})}$ is uniformly bounded and the volume of Ω_{ε_n} is uniformly upper and lower bounded, then as in [D1] there exists constant c_{16}

$$|\varphi_n|_{L^\infty(\Omega_{\varepsilon_n})} < c_{16}. \tag{3.5}$$

Now we scale the variables by letting

$$\psi_n(x,y) = \varphi_n(\varepsilon_n x + \xi_n, \varepsilon_n y),$$

$$\tilde{\Omega}_{\varepsilon_n} = \{(\frac{x-\xi_n}{\varepsilon_n}, \frac{y}{\varepsilon_n})| \quad (x,y) \in \Omega_{\varepsilon_n}\}.$$

Then we have

$$\begin{cases} \triangle \psi_n + pu_0^{p-1}(\frac{x-\xi_n}{\varepsilon_n}, \frac{y}{\varepsilon})\psi_n = \varepsilon_n^2 \beta_n \psi_n & |\varepsilon_n x + \xi_n| \leq 10, \quad (x,y) \in \tilde{\Omega}_{\varepsilon_n}, \\ \psi_n \mid_{\partial\tilde{\Omega}_{\varepsilon_n}} = 0 \end{cases} \tag{3.6}$$

and $|\psi_n|_{L^2(\tilde\Omega_{\varepsilon_n})} = 1$.

Without loss of generality, we may assume $\xi_n = 0$ in the following proof of this lemma.

By a standard regularity argument, we know that the $C^{2,\alpha}$ norm of ψ_n is uniformly bounded on any compact subsets of $\tilde\Omega_{\varepsilon_n}$. We may assume that $\{\psi_n\}$ converges uniformly on any compact subsets of the infinite strip S to a solution w of

$$\begin{cases} \triangle w + pu_0^{p-1}w = 0 \quad \text{in } S \\ w\mid_{\partial S} = 0 \end{cases} \tag{3.7}$$

Since $\int_{\Omega_{\varepsilon_n}} \varphi_n \frac{\partial u_{\varepsilon_n,\xi_n}}{\partial \xi_n} = 0$ and $u_{0,x}$ is exponentially decaying as $|x| \to \infty$, we deduce by (3.5)

$$\int_S w \frac{\partial u_0}{\partial x} dxdy = 0. \tag{3.8}$$

By Corollary 3.2, we deduce $w \equiv 0$ in S. Therefore similar reasoning

$$\lim_{n\to\infty} \int_{\substack{\tilde\Omega_{\varepsilon_n} \\ |\frac{\xi_n}{\varepsilon_n}+x|<\frac{10}{\varepsilon_n}}} u_{\varepsilon,\xi_n}^{p-1}(\frac{x-\xi_n}{\varepsilon_n}, \frac{y}{\varepsilon_n})\psi_n^2 dxdy = 0. \tag{3.9}$$

When $\frac{9}{\varepsilon_n} - 2 < |x| < \frac{9}{\varepsilon_n} + 2$, $u_0(x,y)$ is exponentially decaying as $\varepsilon_n \to 0$, by the regularity arguments in (3.6) as before we obtain

$$|\psi_n|_{C^1(\tilde\Omega_{\varepsilon_n} \cap \{\frac{9}{\varepsilon_n}-1<|x|<\frac{9}{\varepsilon_n}+1\})} < c_{17}\varepsilon_n^2|\beta_n|. \tag{3.10}$$

Let η_n be the smooth function such that

$$\eta_n(x) = \begin{cases} 1 & |x| < \frac{9}{\varepsilon_n} \\ 0 & |x| > \frac{9}{\varepsilon_n} + 1. \end{cases}$$

¿From (3.6), (3.9) and (3.10), we deduce

$$\lim_{n\to\infty} \int_{\tilde\Omega_{\varepsilon_n}} |\bigtriangledown (\eta_n\psi_n)|^2 = 0.$$

Note that on the domains $\tilde\Omega_{\varepsilon_n} \cap \{\frac{9}{\varepsilon_n} > |x|\}$, the first Dirichlet eigenvalue is uniformly lower bounded. Then

$$\lim_{n\to\infty} \int_{\tilde\Omega_{\varepsilon_n}} \eta_n^2\psi_n^2 = 0.$$

Therefore

$$\lim_{n\to\infty}\int_{\tilde{\Omega}_{\varepsilon_n}}(1-\eta_n)^2\psi_n^2 = 1. \tag{3.11}$$

Again from (3.6), (3.10) and the exponential decay of $u(x,y)$ as $\varepsilon_n \to \infty$ when $|x| > \dfrac{9}{\varepsilon_n}$, we deduce

$$\int_{\tilde{\Omega}_{\varepsilon_n}} |\bigtriangledown\big((1-\eta_n)\psi_n\big)|^2 \le c_{18}\varepsilon_n^2|\beta_n|. \tag{3.12}$$

Now we consider that first eigenvalue of Dirichlet problem on $\tilde{\Omega}_{\varepsilon_n}\cap\{|x| > \frac{9}{\varepsilon_n}\}$. By scaling back into the original variables and note that the volume of Ω_{ε_n} is uniformly lower and upper bounded, then there exists C_{19} such that

$$\lambda_1(\tilde{\Omega}_{\varepsilon_n}\cap\{|x| > \frac{9}{\varepsilon_n}\}) > c_{19}\varepsilon_n^2. \tag{3.13}$$

Thus (3.11), (3.12) and (3.13) give a contradiction.

□

Remark 3.5 The proof of Lemma 3.4 is essentially due to Dancer [D2]. We make some simplifications in the decay estimates.

Corollary 3.6. *Under the assumptions of Lemma 3.4, there exists a constant c_{20} independent of ε, such that*

$$|\int_{\Omega_\varepsilon}(|\bigtriangledown\varphi|^2 - pu_{\varepsilon,\xi}^{p-1}\varphi^2)| \ge c_{20}|\varphi|^2_{W^{1,2},(\Omega_\varepsilon)}.$$

By the same proof as in Lemma 3.4, we can obtain the following lemma which will be used in Section 5.

Lemma 3.7. *Let $\xi_1,...,\xi_K$ be K points in $[-10+\delta, 10-\delta]$. Then there exist $c_{21},\varepsilon_0 > 0$ such that for any $\varphi\in W_0^{1,2}(\Omega_\varepsilon)$ and*

$$\int_{\Omega_\varepsilon}\varphi\frac{\partial u_{\varepsilon,\xi_i}}{\partial\xi_i} = 0, i = 1,...,K, \tag{3.14}$$

$\xi_i \in (-10-\delta, 10+\delta), |\xi_k - \xi_l| \geq \frac{10}{K} - 2\delta$ *for* $i, k, l = 1, ..., K, k \neq l,$ *and* $\varepsilon \leq \varepsilon_0$, *we have*

$$|\int_{\Omega_\varepsilon} (|\nabla \varphi|^2 - p(\sum_{i=1}^{K} u_{\varepsilon,\xi_i})^{p-1}\varphi^2)| \geq c_{21}\|\varphi\|^2_{W_0^{1,2}(\Omega_\varepsilon)}. \tag{3.15}$$

Now let $d_\xi = \min\{10-\xi, 10+\xi\} > 0$.

The following lemma gives the key estimates.

Lemma 3.8. *For* $\varepsilon \to 0$ *we have*

$$\varepsilon^{\frac{4}{p-1}} \int_{\substack{10<|x|<10.5\\ h_\varepsilon^- <y<h_\varepsilon^+}} \triangle u_{\varepsilon,\xi} \frac{\partial u_{\varepsilon,\xi}}{\partial \xi} dxdy = \frac{1}{8}c_3^2 D_1(\varepsilon)e^{-\frac{\pi}{\varepsilon}(10-\xi)} - \frac{1}{8}c_3^2 D_2(\varepsilon)e^{-\frac{\pi}{\varepsilon}(10+\xi)} + O(\varepsilon e^{-\frac{\pi d_\xi}{\varepsilon}}). \tag{3.16}$$

Proof: By Lemma 2.3, we can calculate for $10.5 > |x| > 10, y > 0$. Let us denote $h_\varepsilon^\pm$ by h_ε.

$$\varepsilon^{\frac{2}{p-1}} \frac{\partial^2 u_{\varepsilon,\xi}}{\partial x^2} = (\frac{\pi}{2\varepsilon})^2 c_3 e^{-\frac{\pi}{2\varepsilon}(x-\xi)} \cos\frac{\pi y}{2h_\varepsilon} - \frac{\pi^2}{2\varepsilon} c_3 \frac{y h_\varepsilon'}{h_\varepsilon^2} e^{-\frac{\pi}{2\varepsilon}(x-\xi)} \sin\frac{\pi y}{2h_\varepsilon}$$
$$+\frac{\pi}{2}c_3(\frac{h_\varepsilon'}{h_\varepsilon^2})' y e^{-\frac{\pi}{2\varepsilon}(x-\xi)} \sin\frac{\pi y}{2h_\varepsilon} + (\frac{\pi}{2})^2 c_3 (\frac{h_\varepsilon'}{h_\varepsilon^2})^2 y^2 e^{-\frac{\pi}{2\varepsilon}(x-\xi)} \cos\frac{\pi}{2h_\varepsilon} y + O(e^{-\frac{\pi d_\xi + 2c_{10}}{2\varepsilon}}), \tag{3.17}$$

$$\varepsilon^{\frac{2}{p-1}} \frac{\partial^2 u_{\varepsilon,\xi}}{\partial y^2} = -c_3\Big(\frac{\pi}{2h_\varepsilon}\Big)^2 e^{-\frac{\pi}{2\varepsilon}(x-\xi)} \cos\frac{\pi y}{2h_\varepsilon} + O(e^{-\frac{\pi d_\xi + 2c_{10}}{2\varepsilon}}), \tag{3.18}$$

$$\varepsilon^{\frac{2}{p-1}} \frac{\partial u_{\varepsilon,\xi}}{\partial \xi} = c_3 \frac{\pi}{2\varepsilon} e^{-\frac{\pi}{2\varepsilon}(x-\xi)} \cos\frac{\pi y}{2h_\varepsilon} + O(e^{-\frac{\pi d_\xi + 2c_{10}}{2\varepsilon}}). \tag{3.19}$$

(3.17)-(3.18) imply that for $|x| > 10$,

$$\Delta u_{\varepsilon,\xi} = O(\varepsilon^{-\frac{2}{p-1}} \varepsilon^{-1+\alpha} e^{-\frac{\pi d_\xi}{2\varepsilon}}). \tag{3.20}$$

Since

$$\int_0^{h_\varepsilon} \cos^2 \frac{\pi y}{2h_\varepsilon} dy = \frac{1}{2}\int_0^{h_\varepsilon} (1+\cos\frac{\pi y}{h_\varepsilon}) dy = \frac{1}{2} h_\varepsilon, \tag{3.21}$$

$$\int_0^{h_\varepsilon} y \sin\frac{\pi y}{2h_\varepsilon}\cos\frac{\pi y}{2h_\varepsilon} dy = \frac{1}{2}\int_0^{h_\varepsilon} y\sin\frac{\pi y}{h_\varepsilon} dy = \frac{h_\varepsilon^2}{2\pi}, \tag{3.22}$$

$$\int_0^{h_\varepsilon} y^2 \cos^2\frac{\pi y}{2h_\varepsilon} dy = \frac{1}{2}\int_0^{h_\varepsilon} y^2 (1+\cos\frac{\pi y}{h_\varepsilon}) dy = (\frac{1}{6}-\frac{1}{\pi^2}) h_\varepsilon^3, \tag{3.23}$$

then from (3.17) – (3.23), we obtain

$$\int_{\substack{10<x<10.5\\ 0<y<h_\varepsilon}} \varepsilon^{\frac{4}{p-1}} \triangle u_\varepsilon \frac{\partial u_{\varepsilon,\xi}}{\partial \xi} = c_3^2 \frac{\pi}{2\varepsilon}\int_{10}^{10.5} e^{-\frac{\pi}{\varepsilon}(x-\xi)} \big[(\frac{\pi}{2\varepsilon})^2 (1-\frac{\varepsilon^2}{h_\varepsilon^2}) \frac{h_\varepsilon}{2} - \frac{\pi}{4\varepsilon} h'_\varepsilon$$
$$+\frac{h_\varepsilon^2}{4}(\frac{h'_\varepsilon}{h_\varepsilon^2})' - (\frac{\pi}{2})^2 (\frac{1}{6}-\frac{1}{\pi^2}) \frac{{h'_\varepsilon}^2}{h_\varepsilon}\big] + O(e^{-\frac{\pi d_\xi + c_{10}}{\varepsilon}}).$$

That is

$$\int_{\substack{10<x<10.5\\ 0<y<h_\varepsilon}} \varepsilon^{\frac{4}{p-1}} \triangle u_{\varepsilon,\xi} \frac{\partial u_{\varepsilon,\xi}}{\partial \xi} = c_3^2 \frac{\pi}{2\varepsilon}\int_{10}^{10.5} e^{-\frac{\pi}{\varepsilon}(x-\xi)} \big[\frac{\pi^2}{8\varepsilon^2}(h_\varepsilon - \frac{\varepsilon^2}{h_\varepsilon}) - \frac{\pi {h'}_\varepsilon}{4\varepsilon}$$
$$+\underbrace{\frac{1}{4}({h''}_\varepsilon - \frac{2{h'}_\varepsilon^2}{h_\varepsilon}) - \frac{1}{4}(\frac{\pi^2}{6}-1)\frac{{h'}_\varepsilon^2}{h_\varepsilon}\big]}_{\frac{h''_\varepsilon}{4} - \frac{\pi^2}{24}\frac{{h'}_\varepsilon^2}{h_\varepsilon} - \frac{1}{4}\frac{{h'}_\varepsilon^2}{h_\varepsilon}}$$
$$+O(e^{-\frac{\pi d_\xi + c_{10}}{\varepsilon}}). \tag{3.24}$$

By assumption (H_2), putting $x = 10+\varepsilon t$, we have

$$\int_0^{\frac{1}{2\varepsilon}} e^{-\pi t}\big[\frac{\pi^2}{8\varepsilon^2}(h_\varepsilon - \frac{\varepsilon^2}{h_\varepsilon}) - \frac{\pi h'_\varepsilon}{4\varepsilon}\big] dt = \int_0^{\frac{1}{2\varepsilon}} e^{-\pi t}\big[\frac{\pi^2}{4}\varepsilon^{\alpha_1^+} t^{2+\alpha_1^+} A_1^+ - \frac{\pi}{4}(\alpha_1^+ + 2)\varepsilon^{\alpha_1^+} t^{1+\alpha_1^+} A_1^+ + O(\varepsilon)\big]$$
$$= O(\varepsilon), \tag{3.25}$$

$$\int_0^{\frac{1}{2\varepsilon}} e^{-\pi t} \frac{{h''}_\varepsilon}{4} dt = \frac{A_1^+}{4\pi}\varepsilon^{\alpha_1^+}\pi^{-\alpha_1^+}(\alpha_1^+ + 2)(\alpha_1^+ + 1)\Gamma(\alpha_1^+ + 1) + O(\varepsilon), \tag{3.26}$$

$$\int_0^{\frac{1}{2\varepsilon}} e^{-\pi t} \frac{{h'}_\varepsilon^2}{h_\varepsilon} dt = O(\varepsilon). \tag{3.27}$$

From (3.24) – (3.27), we obtain

$$\int_{\substack{10<x<10.5\\0<y<h_\varepsilon}} \triangle u_{\varepsilon,\xi}\frac{\partial u_{\varepsilon,\xi}}{\partial \xi} = c_3^2\frac{\varepsilon^{-\frac{4}{p-1}}}{8}e^{-\frac{10-\xi}{\varepsilon}}\left[\varepsilon^{\alpha_1^+}\pi^{-\alpha_1^+}(\alpha_1^++2)(\alpha_1^++1)\Gamma(\alpha_1^++1)A_1^+ + O(\varepsilon)\right]. \tag{3.28}$$

Similarly we can calculate the remaining integrals. Then we obtain (3.16) from (3.28).

□

The following lemma is also a consequence of the proof of Lemma 3.8.

Lemma 3.9. *For $\varepsilon < \varepsilon_0$, we have*

$$\int_{\substack{|x|>10\\\Omega_\varepsilon}} |\triangle u_{\varepsilon,\xi}|^2 \le c_{22}\varepsilon^{-\frac{4}{p-1}}\varepsilon^{2\alpha}e^{-\frac{\pi}{\varepsilon}d_\xi}. \tag{3.29}$$

Proof: By (3.17) and (3.18), we have for $|x| > 10, |y| \le h_\varepsilon(x)$,

$$\varepsilon^{\frac{2}{p-1}}\Delta u_{\varepsilon,\xi} = O(e^{-1+\alpha}e^{-\frac{\pi}{2\varepsilon}(x-\xi)}) \tag{3.30}$$

Hence

$$\begin{aligned}\int_{\substack{|x|>10\\\Omega_\varepsilon}} |\triangle u_{\varepsilon,\xi}|^2 &= \varepsilon^2 O(\varepsilon^{-2+2\alpha}e^{-\frac{\pi d_\xi}{\varepsilon}})\\ &= O(\varepsilon^{2\alpha}e^{-\frac{\pi d_\xi}{\varepsilon}}).\end{aligned}$$

(3.29) follows easily.

□

4. Proof of Theorem 1.1 and Theorem 1.2

In this section, we shall find a solution of problem (1.1) near the approximate solution $u_{\varepsilon,\xi}$. We first introduce a metric space. Let

$$X = \{v \mid W_0^{1,2}(\Omega_\varepsilon), \int_{\Omega_\varepsilon} v\frac{\partial u_{\varepsilon,\xi}}{\partial \xi} = 0\},$$

$$X_\varepsilon = \{v \mid v \in X, |v|_{W^{1,2}(\Omega_\varepsilon)} + \varepsilon^3 |v|_{L^\infty(\Omega_\varepsilon)} < \varepsilon^{-N} e^{-\frac{\pi}{2\varepsilon} d_\xi}\}$$

where N is a constant to be given later.

We want to solve,

$$\begin{cases} \triangle v_\varepsilon + p u_{\varepsilon,\xi}^{p-1} v_\varepsilon = -(\triangle u_{\varepsilon,\xi} + u_{\varepsilon,\xi}^p) - [(u_{\varepsilon,\xi} + v_\varepsilon)_+^p - u_\varepsilon^p - p u_{\varepsilon,\xi}^{p-1} v_\varepsilon] \\ v_\varepsilon \mid_{\partial\Omega_\varepsilon} = 0, \end{cases} \tag{4.1}$$

where $(u_{\varepsilon,\xi} + v_\varepsilon)_+ = \max\{(u_{\varepsilon,\xi} + v_\varepsilon), 0\}$.

Put $F(v_\varepsilon) = (u_{\varepsilon,\xi} + v_\varepsilon)_+^p - u_{\varepsilon,\xi}^p - p u_{\varepsilon,\xi}^{p-1} v_\varepsilon$ and

$$c(\varepsilon, \xi) = \int_{\Omega_\varepsilon} \left[\triangle u_{\varepsilon,\xi} + u_{\varepsilon,\xi}^p + F(v_\varepsilon)\right] \frac{\partial u_{\varepsilon,\xi}}{\partial \xi} / |\frac{\partial u_{\varepsilon,\xi}}{\partial \xi}|_{L^2(\Omega_\varepsilon)},$$

where $u_{\varepsilon,\xi}$ depends on ξ and $\xi \in (-10 + \delta, 10 - \delta)$. Consider the problem

$$\begin{cases} \triangle w_\varepsilon + p u_{\varepsilon,\xi}^{p-1} w_\varepsilon = -(\triangle u_{\varepsilon,\xi} + u_{\varepsilon,\xi}^p) - F(v_\varepsilon) + c(\varepsilon, \xi) \frac{\frac{\partial u_{\varepsilon,\xi}}{\partial \xi}}{|\frac{\partial u_{\varepsilon,\xi}}{\partial \xi}|_{L^2(\Omega_\varepsilon)}} \\ w_\varepsilon \mid_{\partial\Omega_\varepsilon} = 0. \end{cases} \tag{4.2}$$

By Lemma 3.4, we know that for $v_\varepsilon \in W^{1,2}(\Omega_\varepsilon), \varepsilon < \varepsilon_0$, (4.2) has a unique solution $w_\varepsilon \in X$. Let

$$w_\varepsilon = T v_\varepsilon.$$

Lemma 4.1. *For $\varepsilon < \varepsilon_0, v_\varepsilon \in X_\varepsilon$, we have*

$$|T v_\varepsilon|_{W_0^{1,2}(\Omega_\varepsilon)} + \varepsilon^3 |T v_\varepsilon|_{L^\infty(\Omega_\varepsilon)} < \varepsilon^{-N} e^{-\frac{d_\xi}{2\varepsilon}\pi}. \tag{4.3}$$

where $N = \frac{2}{p-1} + 2 - \alpha$.

Proof: By Corollary 3.6 and (4.2), we have

$$C_0 |T v_\varepsilon|_{W^{1,2}(\Omega_\varepsilon)}^2 \leq \int_{\Omega_\varepsilon} \left(|\triangle u_{\varepsilon,\xi} + u_{\varepsilon,\xi}^p| + |F(v_\varepsilon)|\right) |T v_\varepsilon|. \tag{4.4}$$

For $\varepsilon < \varepsilon_0$

$$\begin{aligned} \int_{\Omega_\varepsilon} |\triangle u_{\varepsilon,\xi} + u_{\varepsilon,\xi}^p| |T v_\varepsilon| &= \int_{\substack{\Omega_\varepsilon \\ |x| > 10}} (\triangle u_{\varepsilon,\xi} + u_{\varepsilon,\xi}^p) |T v_\varepsilon| \\ &\leq c_{41} |T v_\varepsilon|_{L^2(\Omega_\varepsilon)} [\int_{\substack{\Omega_\varepsilon \\ |x| > 10}} (|\triangle u_{\varepsilon,\xi} + u_{\varepsilon,\xi}^p|)^2]^{\frac{1}{2}} \end{aligned}$$

$$\leq c_{42}\varepsilon^{-\frac{2}{p-1}}\varepsilon^{\alpha}e^{-\frac{d_\xi}{2\varepsilon}\pi}|Tv_\varepsilon|_{W_0^{1,2}(\Omega_\varepsilon)}, \tag{4.5}$$

where we used (3.29) and (2.16).

Now we estimate $F(v_\varepsilon)$. If $p \geq 2$, we have for $\varepsilon < \varepsilon_0$.

$$|(u_{\varepsilon,\xi}+v_\varepsilon)_+^p - u_\varepsilon^p - pu_{\varepsilon,\xi}^{p-1}v_\varepsilon| \leq p(p-1)(|u_{\varepsilon,\xi}|+|v|_\varepsilon)^{p-2}v_\varepsilon^2 \leq e^{-\frac{d_\xi}{4\varepsilon}\pi}|v_\varepsilon|. \tag{4.6}$$

If $1 < p < 2$ and $v_\varepsilon \geq 0$, we have

$$0 \leq (u_{\varepsilon,\xi}+v_\varepsilon)^p - u_{\varepsilon,\xi}^p - pu_{\varepsilon,\xi}^{p-1}v_\varepsilon \leq c_{43}v_\varepsilon^p. \tag{4.7}$$

If $1 < p < 2$ and $0 \geq v_\varepsilon \geq -u_{\varepsilon,\xi}$, we have

$$0 \leq (u_{\varepsilon,\xi}+v_\varepsilon)^p - u_{\varepsilon,\xi}^p - pu_{\varepsilon,\xi}^{p-1}v_\varepsilon \leq c_{44}(-v_\varepsilon)^p. \tag{4.8}$$

If $1 < p < 2$ and $v_\varepsilon < -u_{\varepsilon,\xi}$, we also have (4.8). Therefore for $\varepsilon < \varepsilon_0$, $1 < p < 2$, we have

$$|F(v_\varepsilon)| \leq c_{45}e^{-\frac{(p-1)\pi}{4\varepsilon}d_\xi}|v_\varepsilon|. \tag{4.9}$$

Then for $\varepsilon < \varepsilon_0$, we obtain from (4.4) – (4.9)

$$|Tv_\varepsilon|_{W^{1,2}(\Omega_\varepsilon)} \leq \frac{1}{2}\varepsilon^{-N}e^{-\frac{d_\xi}{2\varepsilon}\pi}. \tag{4.10}$$

Next we note that

$$|u_{\varepsilon,\xi}^{p-1}Tv_\varepsilon|_{L^2(\Omega_\varepsilon)} \leq \frac{c_{46}}{\varepsilon^2}|Tv_\varepsilon|_{L^2(\Omega_\varepsilon)}.$$

Then by apply the regularity result to (4.2), we obtain for $\varepsilon < \varepsilon_0$

$$|Tv_\varepsilon|_{L^\infty(\Omega_\varepsilon)} < \frac{1}{2}\varepsilon^{-N-3}e^{-\frac{d_\xi}{2\varepsilon}\pi}. \tag{4.11}$$

We have finished the proof of Lemma 4.1.

□

Lemma 4.2. *For $\varepsilon < \varepsilon_0$, $v_\varepsilon^1, v_\varepsilon^2 \in X_\varepsilon$ we have*

$$|T(v_\varepsilon^1 - v_\varepsilon^2)|_{W^{1,2}(\Omega_\varepsilon)} + \varepsilon^3|T(v_\varepsilon^1 - v_\varepsilon^2)|_{L^\infty(\Omega_\varepsilon)} < \lambda|v_\varepsilon^1 - v_\varepsilon^2|_{W^{1,2}(\Omega_\varepsilon)} \tag{4.12}$$

with some constant $0 < \lambda < 1$.

Proof: Let $w_\varepsilon = v_\varepsilon^1 - v_\varepsilon^2$, then it satisfies

$$\triangle w_\varepsilon + p u_{\varepsilon,\xi}^{p-1} w_\varepsilon = F(v_\varepsilon^2) - F(v_\varepsilon^1) + \Big(\int_{\Omega_\varepsilon} (F(v_\varepsilon^1) - F(v_\varepsilon^2)) \frac{\frac{\partial u_{\varepsilon,\xi}}{\partial \xi}}{|\frac{\partial u_{\varepsilon,\xi}}{\partial \xi}|_{L^2(\Omega_\varepsilon)}} \Big) \frac{\frac{\partial u_{\varepsilon,\xi}}{\partial \xi}}{|\frac{\partial u_{\varepsilon,\xi}}{\partial \xi}|_{L^2(\Omega_\varepsilon)}}. \tag{4.13}$$

As in (4.4), we have

$$\begin{aligned} C_0 |w_\varepsilon|^2_{W^{1,2}(\Omega_\varepsilon)} \leq & \Big| \int_{\Omega_\varepsilon} (F(v_\varepsilon^2) - F(v_\varepsilon^1)) w_\varepsilon \Big| \\ & + \Big| \int_{\Omega_\varepsilon} (F(v_\varepsilon^1) - F(v_\varepsilon^2)) \frac{\partial u_{\varepsilon,\xi}}{\partial \xi} / |\frac{\partial u_{\varepsilon,\xi}}{\partial \xi}|_{L^2(\Omega_\varepsilon)} \Big| |w_\varepsilon|_{L^2(\Omega_\varepsilon)}. \end{aligned} \tag{4.14}$$

If $p \geq 2, \varepsilon < \varepsilon_0$, as in (4.6), we have

$$|F(v_\varepsilon^1) - F(v_\varepsilon^2)| \leq e^{-\frac{(p-1)d_\xi}{4\varepsilon}\pi} |v_\varepsilon^1 - v_\varepsilon^2|. \tag{4.15}$$

If $1 < p < 2$, and $(u_{\varepsilon,\xi} + v_\varepsilon^1)_+ = (u_{\varepsilon,\xi} + v_\varepsilon^2)_+ = 0$, then

$$|F(v_\varepsilon^1) - F(v_\varepsilon^2)| \leq e^{-\frac{(p-1)}{4\varepsilon} d_\xi \pi} |v_\varepsilon^1 - v_\varepsilon^2|. \tag{4.16}$$

If $1 < p < 2$ and $(u_{\varepsilon,\xi} + v_\varepsilon^1)_+ > 0$ or $(u_{\varepsilon,\xi} + v_\varepsilon^2)_+ > 0$ as in (4.7) and (4.8) we have

$$|F(v_\varepsilon^1) - F(v_\varepsilon^2)| \leq |v_\varepsilon^1 - v_\varepsilon^2|^p + |p(u_{\varepsilon,\xi} + v_\varepsilon^1)_+^{p-1} - p u_\varepsilon^{p-1}| |v_\varepsilon^1 - v_\varepsilon^2|. \tag{4.17}$$

Then for $\varepsilon < \varepsilon_0, p > 1$, we have

$$|F(v_\varepsilon^1) - F(v_\varepsilon^2)| < \frac{\lambda}{2} |v_\varepsilon^1 - v_\varepsilon^2|. \tag{4.18}$$

¿From (4.14) and (4.17) we obtain

$$|T v_\varepsilon^1 - T v_\varepsilon^2|_{W^{1,2}(\Omega_\varepsilon)} < \frac{\lambda}{2} |v_\varepsilon^1 - v_\varepsilon^2|_{L^2(\Omega_\varepsilon)}. \tag{4.19}$$

Again note that

$$|u_\varepsilon^{p-1} w_\varepsilon|_{L^2(\Omega_\varepsilon)} \leq \frac{c_{20}}{\varepsilon^2} |w_\varepsilon|_{L^2(\Omega_\varepsilon)}.$$

Then we obtain for $\varepsilon < \varepsilon_0$

$$\varepsilon^3 |T v_\varepsilon^1 - T v_\varepsilon^2|_{L^\infty(\Omega_\varepsilon)} < \frac{\lambda}{2} |v_\varepsilon^1 - v_\varepsilon^2|_{L^2(\Omega_\varepsilon)}. \tag{4.20}$$

Then we have proved Lemma 4.2. □

By Lemma 4.1, Lemma 4.2 and a contraction mapping theorem, we obtain the following

Lemma 4.3. *For $\xi \in (-10+\delta, 10-\delta)$ and ε sufficiently small, problem (4.2) has a unique solution $v_\varepsilon \in X_\varepsilon$. Moreover we have*

$$|v_\varepsilon|_{W_0^{1,2}(\Omega_\varepsilon)} + \varepsilon^3 |v_\varepsilon|_{L^\infty(\Omega_\varepsilon)} < \varepsilon^{-\frac{2}{p-1}-2+\alpha} e^{-\frac{d_\xi}{2\varepsilon}\pi}. \tag{4.21}$$

The estimates in Lemma 4.3 are not good enough for the proof of our main theorem. We need the following refined estimates of v_ε.

Lemma 4.4. *Let v_ε be the solution constructed in Lemma 4.3. Then there exists some constants $c_{47}, c_{48} > 0$ such that*

$$|v_\varepsilon| \le c_{47} e^{-\frac{\pi}{2\varepsilon} d_\xi} e^{-\frac{c}{\varepsilon}} \tag{4.22}$$

for $|x| \le 10 - \frac{\delta}{2}$ and

$$|v_\varepsilon| \le c_{48} \varepsilon^{-\frac{2}{p-1}} e^{-\frac{\pi}{2\varepsilon} d_\xi} \varepsilon^{1+\alpha} \tag{4.23}$$

for $x \in \Omega_\varepsilon \cap \{|x| \ge 10 - \frac{\delta}{2}\}$.

Proof: Let $G(x, x_0)$ be the Green's function on the strip S. Namely $G(x, x_0)$ satisfies

$$-\Delta G = \delta(x - x_0), \ x \in S, G(x, x_0) = 0 \text{ on } \partial S. \tag{4.24}$$

It is known that

$$|\nabla G(x, x_0)| + |G(x, x_0)| \le c_{49} e^{-\frac{\pi}{2}|x_1 - x_{1,0}|}. \tag{4.25}$$

We claim that

$$|v_\varepsilon(x_0)| \le c_{50} e^{-\frac{\pi d_\xi}{2\varepsilon}} e^{-\frac{c}{\varepsilon}} \tag{4.26}$$

for $10 - \frac{3\delta}{4} \le |x_{0,1}| \le 10 - \frac{\delta}{4}$, where $c > 0$ is a number which may depend on $\delta > 0$ and $x_0 = (x_{0,1}, y_{0,1})$.

In fact, we let $\Omega_{\varepsilon,\delta} = \{(x, y) \in \Omega_\varepsilon | \quad 10 - \frac{13\delta}{16} \le |x| \le 10 - \frac{\delta}{16}\}$. Recall that v_ε satisfies (4.2) with $w_\varepsilon = v_\varepsilon$ in $\Omega_{\varepsilon,\delta}$

$$\Delta v_\varepsilon = f_\varepsilon := -p u_{\varepsilon,\xi}^{p-1} v_\varepsilon + F(v_\varepsilon) + c(\varepsilon, \xi) \frac{\frac{\partial u_{\varepsilon,\xi}}{\partial \xi}}{|\frac{\partial u_{\varepsilon,\xi}}{\partial \xi}|_{L^2(\Omega_\varepsilon)}} \tag{4.27}$$

where f_ε is defined at above.

Note that for $(x,y)\in\Omega_{\varepsilon,\delta}$, by Lemma 4.3 and Lemma 3.7, we have

$$|u_{\varepsilon,\xi}^{p-1}v_\varepsilon|, |F_\varepsilon| = O(e^{-\frac{\min(p,2)-\delta}{2\varepsilon}d_\xi}).$$

On the other hand, by Lemma 3.7 and Lemma 4.3,

$$c(\varepsilon,\xi)=\int_{\Omega_\varepsilon}(\Delta u_{\varepsilon,\xi}+u_{\varepsilon,\xi}^p+F(v_\varepsilon))\frac{\frac{\partial u_{\varepsilon,\xi}}{\partial\xi}}{|\frac{\partial u_{\varepsilon,\xi}}{\partial\xi}|_{L^2(\Omega_\varepsilon)}}=O(e^{-\frac{\pi-\delta}{\varepsilon}d_\xi})$$

and hence

$$c(\varepsilon,\xi)\frac{\frac{\partial u_{\varepsilon,\xi}}{\partial\xi}}{|\frac{\partial u_{\varepsilon,\xi}}{\partial\xi}|_{L^2(\Omega_\varepsilon)}}=O(e^{-\frac{3\pi-\delta}{2\varepsilon}d_\xi}),\ \text{for } (x,y)\in\Omega_{\varepsilon,\delta}.$$

Thus,

$$|f_\varepsilon|\le c_{51}e^{-\frac{\min(p,2)\pi-\delta}{2\varepsilon}d_\xi},\quad \text{for } (x,y)\in\Omega_{\varepsilon,\delta}. \tag{4.28}$$

Multiplying (4.27) by $G(x,x_0)$ and integrating by parts over $\Omega_{\varepsilon,\delta}$ we obtain

$$\begin{aligned}v_\varepsilon(x_0)&=\int_{\Omega_{\varepsilon,\delta}}f_\varepsilon(x)G(x,x_0)dx+\int_{\partial\Omega_{\varepsilon,\delta}}(-G\frac{\partial v_\varepsilon}{\partial\nu}+\frac{\partial G}{\partial\nu}v_\varepsilon)d\sigma\\&=E_1+E_2\end{aligned}$$

where E_1,E_2 are defined at the last equality.

By (4.28) and (4.25),

$$|E_1|\le c_{53}e^{-\frac{\min(p,2)\pi-\delta}{2\varepsilon}d_\xi}.$$

On $\partial\Omega_{\varepsilon,\delta}$, $|x-x_{0,1}|\ge\frac{\delta}{16}$ and hence by (4.25) and Lemma 4.3,

$$|E_2|\le c_{52}e^{-\frac{\pi d_\xi}{2\varepsilon}}e^{-\frac{\delta}{16\varepsilon}}.$$

Combining the estimates of E_1 and E_2, we obtain (4.26).

Let $\Omega_\varepsilon^1=\{(x,y)\in\Omega_\varepsilon||x|\le 10-\frac{\delta}{8}\}$. We now multiply (4.2) by v_ε and integrate over Ω_ε^1, we obtain (noting that on $\partial\Omega_\varepsilon^1, v_\varepsilon=O(e^{-\frac{\pi d_\xi}{2\varepsilon}}e^{-\frac{c}{\varepsilon}})$)

$$\begin{aligned}\int_{\Omega_\varepsilon^1}[|\nabla v_\varepsilon|^2-pu_{\varepsilon,\xi}^{p-1}v_\varepsilon^2]&=\int_{\Omega_\varepsilon^1}F(v_\varepsilon)v_\varepsilon+O(e^{-\frac{\pi d_\xi}{\varepsilon}}e^{-\frac{c}{\varepsilon}})\\&=O(e^{-\frac{\pi d_\xi}{\varepsilon}}e^{-\frac{c}{\varepsilon}}),\end{aligned}$$

by Lemma 4.3.

Note that

$$\int_{\Omega_\varepsilon^1} v_\varepsilon \frac{\partial u_{\varepsilon,\xi}}{\partial \xi} = O(e^{-\frac{\pi d_\xi}{2\varepsilon}} e^{-\frac{c}{\varepsilon}})$$

By the same proof as in Lemma 3.4, we can obtain

$$\int_{\Omega_\varepsilon^1} [|\nabla v_\varepsilon|^2 - p u_{\varepsilon,\xi}^{p-1} v_\varepsilon^2] \geq c_{54} \|v_\varepsilon\|^2_{W^{1,2}(\Omega_\varepsilon^1)} + e^{-\frac{\pi d_\xi}{\varepsilon}} e^{-\frac{c}{\varepsilon}}$$

Therefore

$$|v_\varepsilon|_{L^\infty(\Omega_\varepsilon^1)} \leq \varepsilon^{-2} c_{55} \|v_\varepsilon\|_{W^{1,2}(\Omega_\varepsilon^1)} \leq c_{56} e^{-\frac{\pi d_\xi}{2\varepsilon}} e^{-\frac{c}{\varepsilon}},$$

which proves (4.22).

Next, we prove (4.23). First we note that similar arguments as in the proof of (4.22) shows that

$$|v_\varepsilon|_{L^\infty(\Omega_\varepsilon)\cap\{10+\frac{\delta}{4}<|x|<10+\delta\}} \leq c_{57} e^{-\frac{\pi d_\xi}{2\varepsilon}} e^{-\frac{c}{\varepsilon}}. \tag{4.29}$$

Set $\Omega_\varepsilon^2 := \{(x,y)|(x,y) \in \Omega_\varepsilon, 10 - \frac{\delta}{2} \leq |x| \leq 10 + \frac{\delta}{2}, x > 0\}$ and $\tilde{v}_\varepsilon(\tilde{x},\tilde{y}) = v_\varepsilon(x,y)$ where $\tilde{x} = \frac{x-10}{\varepsilon}, \tilde{y} = \frac{y}{\varepsilon}$. Let $\tilde{\Omega}_\varepsilon^2 = \{(\varepsilon\tilde{x}+10, \varepsilon\tilde{y}) \in \Omega_\varepsilon^2\}$. Then in $\tilde{\Omega}_\varepsilon^2$, $\tilde{v}_\varepsilon$ satisfies, by (3.20) in Lemma 3.7,

$$\Delta \tilde{v}_\varepsilon = \varepsilon^2 f_\varepsilon = \varepsilon^2(\Delta u_{\varepsilon,\xi} + u_{\varepsilon,\xi}^p + F(v_\varepsilon)) + O(\varepsilon^{-\frac{2}{p-1}} e^{-\frac{\pi d_\xi}{\varepsilon}})$$

$$= O(\varepsilon^{1+\alpha} \varepsilon^{-\frac{2}{p-1}} e^{-\frac{\pi d_\xi}{2\varepsilon}})$$

and $|\tilde{v}_\varepsilon| = O(e^{-\frac{\pi d_\xi}{2\varepsilon}} e^{-\frac{c}{\varepsilon}})$ on $\tilde{\Omega}_\varepsilon^2$ by (4.26) and (4.29). By apriori bounds estimates (see Theorem 3.7 of [GT]), we have

$$|\tilde{v}_\varepsilon| \leq O(\varepsilon^{-\frac{2}{p-1}} \varepsilon^{1+\alpha} e^{-\frac{\pi d_\xi}{2\varepsilon}}) \text{ in } \tilde{\Omega}_\varepsilon^2,$$

which proves (4.23) for $x > 0$. Similarly we can prove the case for $x < 0$.

□

By Lemma 4.4, we have the following asymptotic expansion of $c(\varepsilon,\xi)$.

Lemma 4.5. *For $\xi \in (-10+\delta, 10-\delta)$, we have*

$$c(\varepsilon,\xi) = \varepsilon^{-\frac{4}{p-1}} \frac{1}{8} c_3^2 D_1(\varepsilon) e^{-\frac{\pi}{\varepsilon}(10-\xi)} - \varepsilon^{-\frac{4}{p-1}} \frac{1}{8} c_3^2 D_2(\varepsilon) e^{-\frac{\pi}{\varepsilon}(10+\xi)} + O(\varepsilon^{-\frac{4}{p-1}} \varepsilon e^{-\frac{\pi d_\xi}{\varepsilon}}). \tag{4.30}$$

Proof: Note that

$$\int_{\Omega_\varepsilon} F(v_\varepsilon)\frac{\partial u_{\varepsilon,\xi}}{\partial \xi} = \int_{\Omega_\varepsilon^1} F(v_\varepsilon)\frac{\partial u_{\varepsilon,\xi}}{\partial \xi} + O(e^{-\frac{(p+1)\pi}{2\varepsilon}d_\xi})$$

By (4.22) of Lemma 4.4 and the definition of $c(\varepsilon,\xi)$, this finishes the proof of Lemma 4.5. $\square$

Finally we can prove Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1.

By Lemma 4.3, we know that for any $\xi \in (-10+\delta, 10-\delta)$, problem (4.2) has a unique solution $v_\varepsilon \in X_\varepsilon$, such that $Tv_\varepsilon = v_\varepsilon$.

By Lemma 4.5, if $D_1(\varepsilon)\cdot D_2(\varepsilon) > 0$, then there exists $\xi_\varepsilon \in (-10+\delta, 10-\delta)$, such that

$$c(\varepsilon, \xi_\varepsilon) = 0. \tag{4.31}$$

Then we can choose ξ_ε so that (4.31) is satisfied. Thus we obtain a solution of (1.1). By (3.16) and(4.31) we also deduce

$$\xi_\varepsilon = O(\varepsilon \ln \varepsilon).$$

Thus we have proved our Theorem 1.1.

$\square$

Proof of Theorem 1.2. Suppose that u is a solution in N_{ε^N}. Let ξ be the number such that $u_{\varepsilon,\xi}$ achieves the minimum of $|u-u_{\varepsilon,\xi}|_{L^2(\Omega_\varepsilon)}$ for $\xi \in (-10-\delta, 10+\delta)$. For this ξ we put $v_\varepsilon = u - u_{\varepsilon,\xi}$, for $\varepsilon < \varepsilon_0$, then we know that v_ε satisfies (4.1). Multiplying $\frac{\partial u_{\varepsilon,\xi}}{\partial \xi}/|\frac{\partial u_{\varepsilon,\xi}}{\partial \xi}|_{L^2(\Omega_\varepsilon)}$ to (4.1) and integrating by parts, we have

$$c(\varepsilon,\xi) = \int_{\substack{|x|>10 \\ \Omega_\varepsilon}} [(\triangle \frac{\partial u_{\varepsilon,\xi}}{\partial \xi} + p u_{\varepsilon,\xi}^{p-1}\frac{\partial u_{\varepsilon,\xi}}{\partial \xi})v_\varepsilon + F(v_\varepsilon)v_\varepsilon]/|\frac{\partial u_{\varepsilon,\xi}}{\partial \xi}|_{L^2(\Omega_\varepsilon)}. \tag{4.32}$$

Since $u_{\varepsilon,\xi}$ has exponential decay, we have for $\varepsilon < \varepsilon_0$

$$c(\varepsilon,\xi) \leq \varepsilon^N e^{-\frac{d_\xi \pi}{\varepsilon}} / |\frac{\partial u_{\varepsilon,\xi}}{\partial \xi}|_{L^2(\Omega_\varepsilon)}.$$

Similar to Lemma 4.4 and 4.5, we can prove that v_ε has better decay inside and we obtain

$$c(\varepsilon,\xi) = \frac{\varepsilon^{-\frac{4}{p-1}}}{|\frac{\partial u_{\varepsilon,\xi}}{\partial \xi}|_{L^2(\Omega_\varepsilon)}} \frac{1}{8}[D_1(\varepsilon)e^{-\frac{\pi}{\varepsilon}(10-\xi)} - D_2(\varepsilon)e^{-\frac{\pi}{\varepsilon}(10+\xi)} + O(\varepsilon e^{-\frac{\pi d_\xi}{\varepsilon}})]. \tag{4.33}$$

If $D_1 D_2 < 0$ then we see from (4.33) that (4.31) can not be satisfied for $N > 1$, we obtain a contradiction. Then we have obtained the proof of Theorem 1.2. $\square$

5. Multiple-bump Case: Proof of Theorem 1.3

In this section, we consider multiple-bump case and prove Theorem 1.3. Our idea is similar to that of [GW]. Namely, we first use the results of Section 3 and Section 4 to reduce the problem into a finite dimensional problem and then we use variational method to obtain a critical point.

Fix an integer $K > 0$. Let P_i^0 be K points in $[-10, 10]$ such that $-10 < P_1^0 < P_2^0 < \cdots < P_K^0 < 10$, $|P_i^0 - P_j^0| = \frac{20}{K}$, $i \neq j$, $P_1^0 + 10 = \frac{10}{K}, 10 - P_K^0 = \frac{10}{K}$. We now construct K-bump solutions with K-bump near P_i^0.

As a first step, we introduce some notations. Let u_{ε,P_i} be defined in Section 2. Set

$$\mathbf{P} = (P_1, \cdots, P_K),\ u_{\varepsilon,\mathbf{P}} = \sum_{i=1}^{K} u_{\varepsilon,P_i},$$

$$d_{P_i} = \min(10 - P_i, 10 + P_i),\quad \varphi_K(\mathbf{P}) = \min_{i,k,l;k\neq\ell}(2d_{P_i}, |P_k - P_\ell|),$$

$$\Lambda_\delta = \{(P_1, \cdots, P_K) \in [-10, 10]^K|\quad \varphi_K(P_1, \cdots, P_K) \geq \frac{20}{K} - \delta\}.$$

Let

$$J_\varepsilon(u) = \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla u|^2 - \frac{1}{p+1}\int_{\Omega_\varepsilon} u_+^{p+1}$$

be the energy of $u \in W_0^{1,2}(\Omega_\varepsilon)$.

We first need some estimates.

Lemma 5.1. *For* $\mathbf{P} = (P_1, ..., P_K) \in \Lambda_\delta$, *we have*

$$\int_{\Omega_\varepsilon} | \nabla u_{\varepsilon,P_i}|^2 = \varepsilon^{-\frac{4}{p-1}} \int_S u_0^{p+1} dxdy + \varepsilon^{-\frac{4}{p-1}} \frac{1}{4\pi} c_3^2 \{(-\varepsilon D_1(\varepsilon) + O(\varepsilon^2))e^{-\pi\frac{10-P}{\varepsilon}} + (-\varepsilon D_2(\varepsilon) + O(\varepsilon^2))e^{-\pi\frac{10+P}{\varepsilon}}\}, \tag{5.1}$$

$$\int_\Omega u_{\varepsilon,P_i}^{p+1} = \varepsilon^{-\frac{4}{p-1}} \int_S u_0^{p+1} dxdy + O(\varepsilon^{-\frac{4}{p-1}} e^{-\frac{(\pi+c_{10})d_{P_i}}{\varepsilon}}). \tag{5.2}$$

Proof: For notational convenience, we let $P = P_i$. We first have by Lemma 2.3, for $\tilde{x} >> 1$,

$$u_0(\tilde{x}, \tilde{y}) = c_3 e^{-\frac{\pi}{2}\tilde{x}} \cos\frac{\pi}{2}\tilde{y} + O(e^{-(\frac{\pi}{2}+c_{10})\tilde{x}}).$$

Hence

$$u_{\varepsilon,P} = c_3 e^{-\frac{\pi}{2}\frac{x-P}{\varepsilon}} \cos\frac{\pi}{2}\frac{y}{h_\varepsilon} + O(e^{-(\frac{\pi}{2}+c_{10})d_P/\varepsilon}),$$

$$\frac{\partial u_{\varepsilon,P}}{\partial x} = -c_3 \frac{\pi}{2\varepsilon} e^{-\frac{\pi}{2}\frac{x-P}{\varepsilon}} \cos(\frac{\pi}{2}\frac{y}{h_\varepsilon}) + c_3 \frac{\pi}{2}\frac{y}{h_\varepsilon^2} h_\varepsilon' e^{-\frac{\pi}{2}\frac{x-P}{\varepsilon}} \sin(\frac{\pi}{2}\frac{y}{h_\varepsilon}) + O(e^{-(\frac{\pi}{2}+c_{10})d_P/\varepsilon}),$$

$$\frac{\partial u_{\varepsilon,P}}{\partial y} = -c_3 e^{\frac{\pi}{2}\frac{x-P}{\varepsilon}} \frac{\pi}{2}\frac{1}{h_\varepsilon} \sin(\frac{y}{h_\varepsilon}\frac{\pi}{2}) + O(e^{-(\frac{\pi}{2}+c_{10})d_P/\varepsilon}).$$

We now prove (5.1). Set

$$\int_{\Omega_\varepsilon} | \nabla u_{\varepsilon,P}|^2 = \int_{\Omega_\varepsilon\cap\{(x)<10\}} | \nabla u_{\varepsilon,P}|^2 + \int_{\Omega_\varepsilon\cap\{|x|>10\}} | \nabla u_{\epsilon,P}|^2$$

$$= I_1 + I_2$$

where I_1 and I_2 are defined at the last equality.

We now compute each term:

$$
\begin{aligned}
I_1 &= \varepsilon^{-\frac{4}{p-1}} \int_{\{|x|<10\}} | \bigtriangledown u_0(\frac{x-P}{\varepsilon}, \frac{y}{\varepsilon})|^2 dxdy \\
&= \varepsilon^{-\frac{4}{p-1}} \int_{\{-\frac{10+P}{\varepsilon}<\tilde{x}<\frac{10-P}{\varepsilon}, |\tilde{y}|<1\}} | \bigtriangledown u_0(\tilde{x},\tilde{y})|^2 d\tilde{x}d\tilde{y} \quad (x = P+\varepsilon\tilde{P}, y = \varepsilon\tilde{y}) \\
&= \varepsilon^{-\frac{4}{p-1}} \int_{\{\frac{-10-P}{\varepsilon}<\tilde{x}<\frac{10-P}{\varepsilon}, |\tilde{y}|<1\}} [\bigtriangledown(u_0 \bigtriangledown u_0) + u_0^{p+1}] \\
&= \varepsilon^{-\frac{4}{p-1}} \int_{\{|\tilde{y}|<1, \tilde{x}=\frac{-10-P}{\varepsilon} \text{ or } \frac{10-P}{\varepsilon}\}} u_0 \frac{\partial u_0}{\partial \nu} + \varepsilon^{-\frac{4}{p-1}} \Big[\int_S u_0^{p+1} - O(e^{-\frac{\pi(p+1)}{2\varepsilon} d_P}) \Big] \\
&= \varepsilon^{-\frac{4}{p-1}} \int_S u_0^{p+1} + \varepsilon^{-\frac{4}{p-1}} \int_{\{|\tilde{y}|<1, \tilde{x}=\frac{10-P}{\varepsilon}\}} u_0 \frac{\partial u_0}{\partial \nu} \\
&+ \varepsilon^{-\frac{4}{p-1}} \int_{\{|\tilde{y}|<1, \tilde{x}=\frac{-10-P}{\varepsilon}\}} u_0 \frac{\partial u_0}{\partial \nu} + O(e^{-\frac{\pi(p+1)}{2\varepsilon} d_P}) \\
&= \varepsilon^{-\frac{4}{p-1}} \int_S u_0^{p+1} - \varepsilon^{\frac{4}{p-1}} \int_{-1}^{1} c_3^2 (\frac{\pi}{2}) e^{-\frac{\pi(10-P)}{\varepsilon}} \cos^2 \frac{\pi}{2}\tilde{y} d\tilde{y} \\
&- \varepsilon^{-\frac{4}{p-1}} \int_{-1}^{1} c_3^2 (\frac{\pi}{2}) e^{-\frac{\pi(10+P)}{\varepsilon}} (\cos \frac{\pi}{2}\tilde{y})^2 d\tilde{y} + O(e^{-\frac{\pi+c_{10}}{\varepsilon} d_P}) \\
&= \varepsilon^{-\frac{4}{p-1}} \int_S u_0^{p+1} dxdy - \varepsilon^{-\frac{4}{p-1}} c_3^2 \big[\frac{\pi}{2} e^{-\frac{\pi(10-P)}{\varepsilon}} + \frac{\pi}{2} e^{-\frac{\pi(10+P)}{\varepsilon}} \big] + O(e^{-\frac{\pi+c_{10}}{\varepsilon} d_P}).
\end{aligned}
$$

We next compute I_2:

$$
I_2 = \int_{10 \le x < 10.5} | \bigtriangledown u_{\varepsilon,P}|^2 + \int_{-10.5<x\le -10} | \bigtriangledown u_{\varepsilon,P}|^2 = I_3 + I_4
$$

where I_3, I_4 are defined at the last equality.

We only need to compute I_3. The computation of I_4 is similar to that of I_3.

To compute I_3, we obtain by (3.18), (3.19), (3.27)

$$
\begin{aligned}
&\varepsilon^{\frac{4}{p-1}}\int_{10<x<10.5,0<y<h_\varepsilon^+}|\frac{\partial u_{\varepsilon,P}}{\partial x}|^2dxdy\\
&=\big(\frac{\pi}{2\varepsilon}c_3\big)^2\int_{10<x<10.5}e^{-\pi\frac{x-P}{\varepsilon}}\int_0^{h_\varepsilon^+}\cos^2(\frac{\pi}{2}\frac{y}{h_\varepsilon^+})dydx\\
&-2\frac{\pi^2}{2^2\varepsilon}c_0^2\int_{10<x<10.5}e^{-\pi\frac{x-P}{\varepsilon}}\int_0^{h_\varepsilon^+}y\cos(\frac{\pi}{2}\frac{y}{h_\varepsilon^+})\sin(\frac{\pi}{2}\frac{y}{h_\varepsilon^+})\frac{(h_\varepsilon^+)'}{(h_\varepsilon^+)^2}\\
&+\big(\frac{\pi}{2}c_3\big)^2\int_{10<x<10.5}e^{-\pi\frac{x-P}{\varepsilon}}\int_0^{h_\varepsilon^+}(y^2\sin^2(\frac{\pi}{2}\frac{y}{h_\varepsilon^+}))\big(\frac{(h_\varepsilon^+)'}{(h_\varepsilon^+)^2}\big)^2+O(e^{-\frac{\pi+c_{10}}{\varepsilon}d_P})\\
&=\big(\frac{\pi}{2\varepsilon}c_3\big)^2\int_{10<x<10.5}e^{-\pi\frac{x-P}{\varepsilon}}\big[\frac{1}{2}h_\varepsilon^+-2\varepsilon\cdot\frac{(h_\varepsilon^+)^2}{2\pi}\cdot\frac{(h_\varepsilon^+)'}{(h_\varepsilon^+)^2}+\varepsilon^2\big(\frac{1}{6}+\frac{1}{\pi^2}\big)(h_\varepsilon^+)^3\big(\frac{(h_\varepsilon^+)'}{(h_\varepsilon^+)^2}\big)\big]+e^{-\frac{\pi+c_{10}}{\varepsilon}d_P})\\
&=\big(\frac{\pi}{2\varepsilon}c_3\big)^2e^{-\pi\frac{10-P}{\varepsilon}}\varepsilon\int_0^{\frac{0.5}{\varepsilon}}e^{-\pi t}\big[\frac{1}{2}h_\varepsilon^+-\frac{\varepsilon}{\pi}(h_\varepsilon^+)'+\varepsilon^2\big(\frac{1}{6}+\frac{1}{\pi^2}\big)\frac{((h_\varepsilon^+)')^2}{h_\varepsilon^+}\big]dt+O(e^{-\frac{\pi+c_{10}}{\varepsilon}d_P})\\
&=\big(\frac{\pi}{2\varepsilon}c_3\big)^2e^{-\pi\frac{10-P}{\varepsilon}}\varepsilon\int_0^{\frac{0.5}{\varepsilon}}e^{\pi t}\big[\frac{1}{2}\varepsilon+\frac{1}{2}A_1^+\varepsilon^{2+\alpha_1^+}t^{2+\alpha_1^+}-\frac{\varepsilon}{\pi}A_1^+(2+\alpha_1^+)\varepsilon^{1+\alpha_1^+}t^{1+\alpha_1^+}+O(\varepsilon^3t^3)\big]\\
&=\big(\frac{\pi}{2}c_3\big)^2e^{-\pi\frac{10-P}{\varepsilon}}\big[\frac{1}{2\pi}-\frac{1}{2}A_1^+\varepsilon^{1+\alpha_1^+}\int_0^\infty e^{-\pi t}t^{2+\alpha_1^+}+O(\varepsilon^2)\big].
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
&\varepsilon^{\frac{4}{p-1}}\int_{10<x<10.5,0<y<h_\varepsilon^+}|\frac{\partial u_{\varepsilon,P}}{\partial y}|^2dxdy\\
&=\big(\frac{\pi}{2}c_3\big)^2\int_{10}^{10.5}e^{-\pi\frac{x-P}{\varepsilon}}\frac{1}{(h_\varepsilon^+)^2}\int_0^{h_\varepsilon^+}\sin^2(\frac{y}{h_\varepsilon^+}\frac{\pi}{2})+O(e^{-\frac{(\pi+c_{10})d_P}{\varepsilon}})\\
&=\big(\frac{\pi}{2}c_3\big)^2\int_{10}^{10.5}e^{-\pi\frac{x-P}{\varepsilon}}\frac{1}{(h_\varepsilon^+)^2}\cdot\frac{1}{2}h_\varepsilon^++O(e^{-\frac{(\pi+c_{20})d_P}{\varepsilon}})=\big(\frac{\pi}{2}c_3\big)^2\int_{10}^{10.5}e^{-\pi\frac{x-P}{\varepsilon}}\frac{1}{2h_\varepsilon^+}+O(e^{-\frac{(\pi+c_{10})d_P}{\varepsilon}})\\
&=\big(\frac{\pi}{2}c_3\big)^2e^{-\pi\frac{10-P}{\varepsilon}}\varepsilon\int_0^{\frac{0.5}{\varepsilon}}e^{-\pi t}\frac{1}{2h_\varepsilon^+}dt+O(e^{-\frac{(\pi+c_{10})d_P}{\varepsilon}})\\
&=\big(\frac{\pi}{2}c_3\big)^2e^{-\pi\frac{10-P}{\varepsilon}}\int_0^{\frac{0.5}{\varepsilon}}e^{-\pi t}\frac{1}{2}(1-A_1^+\varepsilon^{1+\alpha_1^+}t^{2+\alpha_1^+}+O(\varepsilon^2))dt\\
&=\big(\frac{\pi}{2}c_3\big)^2e^{-\pi\frac{10-P}{\varepsilon}}\big[\frac{1}{2\pi}-\frac{A_1^+}{2}(\int_0^\infty e^{-\pi t}t^{2+\alpha_1^+})\varepsilon^{1+\alpha_1^+}+O(\varepsilon^2)\big].
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\varepsilon^{\frac{4}{p-1}}\int_{10<x<10.5,0<y<h_\varepsilon^+}|\bigtriangledown u_{\varepsilon,P}|^2dxdy\\
&=\varepsilon^{\frac{4}{p-1}}\int_{10<x<10.5,0<y<h_\varepsilon^+}[(\frac{\partial u_{\varepsilon,P}}{\partial x})^2+(\frac{\partial u_{\varepsilon,P}}{\partial y})^2]dxdy\\
&=\big(\frac{\pi}{2}c_3\big)^2e^{-\pi\frac{10-P}{\varepsilon}}\frac{1}{\pi}-\big(\frac{\pi}{2}c_3\big)^2e^{-\pi\frac{10-P}{\varepsilon}}(A_1^+\int_0^\infty e^{-\pi t}t^{2+\alpha_1^+}\varepsilon^{1+\alpha_1^+}+O(\varepsilon^2)).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\varepsilon^{\frac{4}{p-1}}\int_{10<x<10.5,h_\varepsilon^-<y<0}|\bigtriangledown u_{\varepsilon,P}|^2dxdy\\
&=\big(\frac{\pi}{2}c_3\big)^2e^{-\pi\frac{10-P}{\varepsilon}}\frac{1}{\pi}+\big(\frac{\pi}{2}c_3\big)^2e^{-\pi\frac{10-P}{\varepsilon}}(A_1^-\int_0^\infty e^{-\pi t}t^{2+\alpha_1^-}\varepsilon^{1+\alpha_1^-}+O(\varepsilon^2)).
\end{aligned}
$$

Together we have

$$
\begin{aligned}
I_3&=\int_{10<x<10.5}|\bigtriangledown u_{\varepsilon,P}|^2\\
&=\varepsilon^{-\frac{4}{p-1}}\big(\frac{\pi}{2}c_3\big)^2e^{-\pi\frac{10-P}{\varepsilon}}\big[\frac{2}{\pi}-A_1^+\varepsilon^{1+\alpha_1^+}\int_0^\infty e^{-\pi t}t^{2+\alpha_1^+}+A_1^-\varepsilon^{1+\alpha_1^-}\int_0^\infty e^{-\pi t}t^{2+\alpha_1^-}+O(\varepsilon^2)\big]\\
&=\varepsilon^{-\frac{4}{p-1}}\big(\frac{\pi}{2}c_3\big)^2e^{-\pi\frac{10-P}{\varepsilon}}\big[\frac{2}{\pi}-\frac{\varepsilon}{\pi^3}D_1(\varepsilon)+O(\varepsilon^2)\big].
\end{aligned}
$$

In a similar way, we can also obtain

$$
\begin{aligned}
I_4&=\int_{-10.5<x<-10}|\bigtriangledown u_{\varepsilon,P}|^2\\
&=\varepsilon^{-\frac{4}{p-1}}\big(\frac{\pi}{2}c_3\big)^2e^{-\pi\frac{10+P}{\varepsilon}}\big[\frac{2}{\pi}-\frac{\varepsilon}{\pi^3}D_2(\varepsilon)+O(\varepsilon^2)\big].
\end{aligned}
$$

So

$$
I_2=\varepsilon^{-\frac{4}{p-1}}\big(\frac{\pi}{2}c_3\big)^2\big\{e^{-\pi\frac{10-P}{\varepsilon}}\big[\frac{2}{\pi}-\frac{\varepsilon}{\pi^3}D_1(\varepsilon)+O(\varepsilon^2)\big]\ +e^{-\pi\frac{10+P}{\varepsilon}}\big[\frac{2}{\pi}-\frac{\varepsilon}{\pi^3}D_2(\varepsilon)+O(\varepsilon^2)\big]\big\}.
$$

Thus

$$
I_1+I_2=\varepsilon^{-\frac{4}{p-1}}\int_S u_0^{p+1}+\varepsilon^{-\frac{4}{p-1}}\frac{1}{4\pi}c_3^2\big\{(-\varepsilon D_1(\varepsilon)+O(\varepsilon^2))e^{-\pi\frac{10-P}{\varepsilon}}+(-\varepsilon D_2(\varepsilon)+O(\varepsilon^2))e^{-\pi\frac{10+P}{\varepsilon}}\big\}
$$

which proves (5.1).

Finally we prove (5.2)

$$\begin{aligned}
\int_{\Omega_\varepsilon} u_{\varepsilon,P}^{p+1} &= \varepsilon^{-\frac{2(p+1)}{p-1}} \int_{\Omega_\varepsilon} u_0^{p+1}\big(\frac{x-P}{\varepsilon}, \beta(x,y))\big)dxdy \\
&= \varepsilon^{-\frac{2(p+1)}{p-1}} \int_{|x|<10} u_0^{p+1}\big(\frac{x-P}{\varepsilon}, \frac{y}{\varepsilon}\big)dxdy + O(\varepsilon^{-\frac{4}{p-1}} e^{-\frac{\pi+c_{10}}{\varepsilon}d_P}) \\
&= \varepsilon^{-\frac{4}{p-1}} \int_{|x|<10} u_0^{p+1}(\tilde{x},\tilde{y})d\tilde{x}d\tilde{y} + O(\varepsilon^{-\frac{4}{p-1}} e^{-\frac{\pi+c_{10}}{\varepsilon}d_P}) \\
&= \varepsilon^{-\frac{4}{p-1}} \int_S u_0^{p+1}(\tilde{x},\tilde{y})d\tilde{x}d\tilde{y} + O(\varepsilon^{-\frac{4}{p-1}} e^{-\frac{\pi+c_{10}}{\varepsilon}d_P}).
\end{aligned}$$

□

By Lemma 5.1, we have the following corollary.

Corollary 5.2. *For* $\mathbf{P} \in \Lambda_\delta$*, we have*

$$\frac{1}{2}\int_{\Omega_\varepsilon} |\bigtriangledown u_{\varepsilon,P_i}|^2 - \frac{1}{p+1}\int_{\Omega_\varepsilon} u_{\varepsilon,P_i}^{p+1} = \varepsilon^{-\frac{4}{p-1}}\big[\frac{1}{2} - \frac{1}{p+1}\big]\int_S u_0^{p+1}$$
$$+\varepsilon^{-\frac{4}{p-1}}\frac{1}{8\pi}c_3^2\big\{(-\varepsilon D_1(\varepsilon) + O(\varepsilon^2))e^{-\pi\frac{10-P}{\varepsilon}} + (-\varepsilon D_2(\varepsilon) + O(\varepsilon^2))e^{-\pi\frac{10+P}{\varepsilon}}\big\}$$

Next we compute the interaction of two bumps.

Lemma 5.3. *For* $\mathbf{P} \in \Lambda_\delta$*, we have*

$$\int_{\Omega_\varepsilon} \bigtriangledown u_{\varepsilon,P_1} \bigtriangledown u_{\varepsilon,P_2} = \varepsilon^{-\frac{4}{p-1}} e^{-\frac{\pi}{2}|P_1-P_2|}\gamma_1 + O\big(\varepsilon^{-\frac{4}{p-1}}(e^{-\frac{\pi}{2}\frac{20-P_1-P_2}{\varepsilon}} + e^{-\frac{\pi}{\varepsilon}\frac{20+P_1+P_2}{\varepsilon}})\big). \tag{5.3}$$

$$\int_{\Omega_\varepsilon} u_{\varepsilon,P_1}^p u_{\varepsilon,P_2} = \varepsilon^{-\frac{4}{p-1}} e^{-\frac{\pi}{2}|P_1-P_2|}\gamma_1 + O\big(\varepsilon^{-\frac{4}{p-1}}(e^{-\frac{\pi}{2}\frac{20-P_1-P_2}{\varepsilon}} + e^{-\frac{\pi}{2}\frac{20+P_1+P_2}{\varepsilon}})\big). \tag{5.4}$$

where

$$\gamma_1 := c_3^2 \int_S u_0^p(\tilde{x},\tilde{y})\cos\frac{\pi}{2}\tilde{y}e^{-\tilde{x}}d\tilde{x}d\tilde{y} > 0. \tag{5.5}$$

Proof:

$$\begin{aligned}
\int_{\Omega_\varepsilon} \bigtriangledown u_{\varepsilon,P_1} \bigtriangledown u_{\varepsilon,P_2} &= \int_{|x|<10} \bigtriangledown u_{\varepsilon,P_1} \bigtriangledown u_{\varepsilon,P_2} + \int_{|x|>10} \bigtriangledown u_{\varepsilon,P_1} \bigtriangledown u_{\varepsilon,P_2} \\
&= J_1 + J_2
\end{aligned}$$

where J_1 and J_2 are defined at the last equality. We have

$$J_2 = O\big(e^{-\frac{\pi}{2}\frac{20-P_1-P_2}{\varepsilon}}\varepsilon^{-\frac{4}{p-1}} + e^{-\frac{\pi}{2}\frac{20+P_1+P_2}{\varepsilon}}\varepsilon^{-\frac{4}{p-1}}\big),$$

$$\begin{aligned}
J_1 &= \int_{|x|<10} \nabla(u_{\varepsilon,P_2}\nabla u_{\varepsilon,P_1}) - u_{\varepsilon,P_2}\triangle u_{\varepsilon,P_1}\\
&= \int_{\partial\{|x|<10\}} u_{\varepsilon,P_2}\frac{\partial u_{\varepsilon,P_1}}{\partial\nu} + \int_{|x|<10} u_{\varepsilon,P_2}u^p_{\varepsilon,P_1}\\
&= O\big(\varepsilon^{-\frac{4}{p-1}}e^{-\frac{\pi}{2}\frac{20-P_1-P_2}{\varepsilon}} + \varepsilon^{-\frac{4}{p-1}}e^{-\frac{\pi}{2}\frac{20+P_1+P_2}{\varepsilon}}\big) + \int_{|x|<10} u^p_{\varepsilon,P_1}u_{\varepsilon,P_2}\\
&= O\big(\varepsilon^{-\frac{4}{p-1}}\big(e^{-\frac{\pi}{2}\frac{20-P_1-P_2}{\varepsilon}} + e^{-\frac{\pi}{2}\frac{20+P_1+P_2}{\varepsilon}}\big)\big)\\
&+ \varepsilon^{-\frac{4}{p-1}}\int_{\frac{-10-P}{\varepsilon}<\tilde{x}<\frac{10-P}{\varepsilon},-1<\tilde{y}<1} u_0^p(\tilde{x},\tilde{y})u_0(\tilde{x}+\frac{P_1-P_2}{\varepsilon},\tilde{y})dy\\
&= O\big(\varepsilon^{-\frac{4}{p-1}}\big(e^{-\frac{\pi}{2}\frac{20-P_1-P_2}{\varepsilon}} + e^{-\frac{\pi}{2}\frac{20+P_1+P_2}{\varepsilon}}\big)\big) + \varepsilon^{-\frac{4}{p-1}}\int_S u_0^p(\tilde{x},\tilde{y})u_0(\tilde{x}+\frac{P_1-P_2}{\varepsilon},\tilde{y})dy\\
&= O\big(\varepsilon^{-\frac{4}{p-1}}\big(e^{-\frac{\pi}{2}\frac{20-P_1-P_2}{\varepsilon}} + e^{-\frac{\pi}{2}\frac{20+P_1+P_2}{\varepsilon}}\big)\big) + \varepsilon^{-\frac{4}{p-1}}\int_S u_0^p(\tilde{x},\tilde{y})c_3e^{-\frac{\pi}{2}|\frac{P_1-P_2}{\varepsilon}+\tilde{x}|}\cos\frac{\pi}{2}\tilde{y}dy\\
&= O\big(\varepsilon^{-\frac{4}{p-1}}\big(e^{-\frac{\pi}{2}\frac{20-P_1-P_2}{\varepsilon}} + e^{-\frac{\pi}{2}\frac{20+P_1+P_2}{\varepsilon}}\big)\big) + \varepsilon^{-\frac{4}{p-1}}c_3e^{-\frac{\pi}{2}|\frac{P_1-P_2}{\varepsilon}|}\int_S u_0^p(\tilde{x},\tilde{y})\cos\frac{\pi}{2}\tilde{y}e^{-e_{12}\tilde{x}}dx,
\end{aligned}$$

where $e_{12} = \frac{P_1-P_2}{|P_1-P_2|} = 1$ or -1.

So

$$\begin{aligned}
\int_{\Omega_\varepsilon} \nabla u_{\varepsilon,P_1}\nabla u_{\varepsilon,P_2} &= \varepsilon^{-\frac{4}{p-1}}c_3e^{-\frac{\pi}{2}\frac{|P_1-P_2|}{2}}\int_S u_0^p(\tilde{x},\tilde{y})\cos\frac{\pi}{2}\tilde{y}e^{-e_{12}\tilde{x}}d\tilde{x}d\tilde{y}\\
&+ O\big(\varepsilon^{-\frac{4}{p-1}}\big(e^{-\frac{\pi}{2}\frac{20-P_1-P_2}{\varepsilon}} + e^{-\frac{\pi}{2}\frac{20+P_1+P_2}{\varepsilon}}\big)\big)\\
&= \varepsilon^{-\frac{4}{p-1}}c_3e^{-\frac{\pi}{2}\frac{|P_1-P_2|}{2}}\int_S u_0^p(\tilde{x},\tilde{y})\cos\frac{\pi}{2}\tilde{y}e^{-\tilde{x}}d\tilde{x}d\tilde{y}\\
&+ O\big(\varepsilon^{-\frac{4}{p-1}}\big(e^{-\frac{\pi}{2}\frac{20-P_1-P_2}{\varepsilon}} + e^{-\frac{\pi}{2}\frac{20+P_1+P_2}{\varepsilon}}\big)\big).
\end{aligned}$$

(5.4) can be proved similarly. □

Finally, we have the following asymptotic expansion of energy.

Corollary 5.4. *For* $\mathbf{P} = (P_1, ..., P_K) \in \Lambda_\delta$, *we have*

$$
\begin{aligned}
J_\varepsilon(u_{\varepsilon,\mathbf{P}}) &= \frac{1}{2}\int_{\Omega_\varepsilon} |\bigtriangledown u_{\varepsilon,\mathbf{P}}|^2 - \frac{1}{p+1}\int_{\Omega_\varepsilon} u_{\varepsilon,\mathbf{P}}^{p+1} \\
&= \varepsilon^{-\frac{4}{p+1}} K[\frac{1}{2} - \frac{1}{p+1}]\int_S u_0^{p+1} + \varepsilon^{-\frac{4}{p-1}} \frac{1}{8\pi} c_3^2 \sum_{i=1}^K \{(-\varepsilon D_1(\varepsilon) + O(\varepsilon^2))e^{-\pi\frac{10-P_i}{\varepsilon}} \\
&+ (-\varepsilon D_2(\varepsilon) + O(\varepsilon^2))e^{-\pi\frac{10+P_i}{\varepsilon}}\} \\
&- \frac{1}{2}(\gamma_1 + o(1))\varepsilon^{-\frac{4}{p-1}} \sum_{i\neq j} e^{-\frac{\pi}{2}|P_i - P_j|}
\end{aligned}
$$

Proof: This follows from Corollary 5.2 and Lemma 5.3. In fact

$$
\begin{aligned}
\int_{\Omega_\varepsilon} |\bigtriangledown u_{\varepsilon,\mathbf{P}}|^2 &= \sum_{i=1}^K \int_{\Omega_\varepsilon} |\bigtriangledown u_{\varepsilon,P_i}|^2 + \sum_{i\neq j}\int_{\Omega_\varepsilon} \bigtriangledown u_{\varepsilon,P_i} \bigtriangledown u_{\varepsilon,P_j} \\
&= K\varepsilon^{-\frac{4}{p-1}} \int_S u_0^{p+1} + \varepsilon^{-\frac{4}{p-1}} \frac{1}{4\pi} c_3^2 \sum_{i=1}^K \{(-\varepsilon D_1(\varepsilon) + O(\varepsilon^2))e^{-\pi\frac{10-P_i}{\varepsilon}} \\
&+ (-\varepsilon D_2(\varepsilon) + O(\varepsilon^2))e^{-\pi\frac{10+P_i}{\varepsilon}}\} + \varepsilon^{-\frac{4}{p-1}}\gamma_1 \sum_{i\neq j} e^{-\frac{\pi}{2}|P_i-P_j|} \\
&+ O\big(e^{-\frac{4}{p-1}} \sum_{i\neq j}(e^{-\frac{\pi}{2}\frac{20-P_i-P_j}{\varepsilon}} + e^{\frac{20+P_i+P_j}{\varepsilon}})\big).
\end{aligned}
$$

By our choices of P_i, for $i \neq j$

$$
|20 \pm (P_i + P_j)| \geq \frac{30}{K} - 2\delta > \min_{i\neq j} |P_j - P_j| \frac{1+\sigma}{2}
$$

for some $\sigma > 0$. Hence

$$
O\big(e^{-\frac{\pi}{2}\frac{20\pm(P_i+P_j)}{\varepsilon}}\big) = o(\sum_{i\neq j} e^{-\frac{\pi}{2}|P_i-P_j|}).
$$

On the other hand,

$$\int_{\Omega_\varepsilon} u_{\varepsilon,P}^{p+1} = \int_{\Omega_\varepsilon} (u_{\varepsilon,P_1} + \cdots + u_{\varepsilon,P_K})^{p+1}$$

$$= \sum_{i=1}^{K} \int_{\Omega_\varepsilon} u_{\varepsilon,P_i}^{p+1} + (p+1) \sum_{i \neq j} \int_{\Omega_\varepsilon} u_{\varepsilon,P_i}^{p} u_{\varepsilon,P_j} + O(\varepsilon^{-\frac{4}{p-1}} e^{-\frac{(1+c_{23})\pi \min_{i\neq j} |P_i - P_j|}{2\varepsilon}})$$

$$= \sum_{i=1}^{K} \int_{\Omega_\varepsilon} (u_{\varepsilon,P_i})^{p+1} + (p+1)\varepsilon^{-\frac{4}{p-1}} (\gamma_1 + o(1)) \sum_{i\neq j} e^{-\frac{\pi}{2}|P_i - P_j|}$$

$$+ O(\varepsilon^{-\frac{4}{p-1}} e^{-\frac{(1+c_{21})\pi \min_{i\neq j} |P_i - P_j|}{2\varepsilon}}).$$

By Lemmas 5.1 and 5.3, we finish the proof of (5.4).

□

As in Section 4, we now set

$$Y = \{v | v \in w_0^{1,2}(\Omega_\varepsilon), \int_{\Omega_\varepsilon} v \frac{\partial u_{\varepsilon,\mathbf{P}}}{\partial P_i} = 0, \ i = 1, \cdots, K\},$$

$$Y_\varepsilon = \{v | v \in Y, |v|_{W^{1,2}(\Omega_\varepsilon)} + \varepsilon^3 |v|_{L^\infty(\Omega_\varepsilon)} < \varepsilon^{-\frac{2}{p-1}-2} e^{-\frac{\pi}{4\varepsilon}\varphi_K(\mathbf{P})}\}.$$

We now solve the following equation

$$\begin{cases} \triangle v_\varepsilon + p u_{\varepsilon,\mathbf{P}}^{p-1} v_\varepsilon = -(\triangle u_{\varepsilon,\mathbf{P}} + u_{\varepsilon,\mathbf{P}}^p) - [(u_{\varepsilon,\mathbf{P}} + v_\varepsilon)_+^p - u_{\varepsilon,\mathbf{P}}^p - p u_{\varepsilon,\mathbf{P}}^{p-1} v_\varepsilon] \\ v_\varepsilon \mid_{\partial\Omega_\varepsilon} = 0 \end{cases} \tag{5.6}$$

To solve (5.6), we need to have the following error estimates.

Lemma 5.5. *We have*

$$\int_{\Omega_\varepsilon} (\triangle u_{\varepsilon,P} + u_{\varepsilon,\mathbf{P}}^p) \frac{\partial u_{\varepsilon,\mathbf{P}}}{\partial P_i} = O\big(\varepsilon^{-\frac{4}{p-1}} e^{-\frac{\pi}{2}\varphi_K(\mathbf{P})}\big). \tag{5.7}$$

Proof: This follows from direct computations, which is similar to that of Lemma 3.7 and Lemma 5.3. We omit the details. □

By following the proof of Lemmas 4.1-4.5, we have the following

Proposition 5.6. *For $\mathbf{P} \in \wedge_\delta$, problem (5.6) has a unique solution $v_{\varepsilon,\mathbf{P}} \in Y_\varepsilon$, such that the following holds.*

$$\begin{cases} \triangle v_{\varepsilon,\mathbf{P}} + p u_{\varepsilon,\mathbf{P}}^{p-1} v_{\varepsilon,\mathbf{P}} = & -(\triangle u_{\varepsilon,\mathbf{P}} + u_{\varepsilon,\mathbf{P}}^p) - ((u_{\varepsilon,\mathbf{P}} + v_{\varepsilon,\mathbf{P}})_+^p - u_{\varepsilon,\mathbf{P}}^p - p u_{\varepsilon,\mathbf{P}}^{p-1} v_{\varepsilon,\mathbf{P}}) \\ & + \sum_{j=1}^K c_j(\varepsilon,\mathbf{P}) \frac{\frac{\partial u_{\varepsilon,\mathbf{P}}}{\partial P_j}}{\|\frac{\partial u_{\varepsilon,\mathbf{P}}}{\partial P_j}\|_{L^2(\Omega_\varepsilon)}} \\ v_{\varepsilon,\mathbf{P}} = 0 & on\ \partial\Omega_\varepsilon. \end{cases} \tag{5.8}$$

Moreover,

$$\|v_{\varepsilon,\mathbf{P}}\|_{W^{1,2}(\Omega_\varepsilon)} \leq C\varepsilon^{-\frac{2}{p-1}-2+\alpha} e^{-\frac{\pi}{4\varepsilon}\varphi_K(\mathbf{P})}, \tag{5.9}$$

$$|v_{\varepsilon,\mathbf{P}}|_{L^\infty(\Omega_\varepsilon^1)} = O(e^{-\frac{\pi}{4\varepsilon}\varphi_K(\mathbf{P})} e^{-\frac{c}{\varepsilon}}), \tag{5.10}$$

$$|v_{\varepsilon,\mathbf{P}}|_{L^\infty(\Omega_\varepsilon^2)} = O(\varepsilon^{1+\alpha}\varepsilon^{-\frac{2}{p-1}} e^{-\frac{\pi}{4\varepsilon}\varphi_K(\mathbf{P})}), \tag{5.11}$$

for some $c > 0$, where $\Omega_\varepsilon^1 = \{(x,y) \in \Omega_\varepsilon \| x| \leq 10 - \frac{\delta}{2}\}, \Omega_\varepsilon^2 = \Omega_\varepsilon \cap \{10 - \frac{\delta}{2} \leq |x| \leq 10 + \frac{\delta}{2}\}$.

Next we show that $v_{\varepsilon,\mathbf{P}} \in C^1$ in $\mathbf{P} \in \wedge_\delta$.

Lemma 5.7. *For ε sufficiently small, $v_{\varepsilon,\mathbf{P}} \in C^1$ in $\mathbf{P} \in \wedge_\delta$.*

Proof: The proof follows from same arguments as in Lemma 3.5 in [GW]. □

Set

$$K_\varepsilon(\mathbf{P}) = \frac{1}{2}\int_{\Omega_\varepsilon} |\bigtriangledown (u_{\varepsilon,\mathbf{P}} + v_{\varepsilon,\mathbf{P}})|^2 - \frac{1}{p+1}\int_{\Omega_\varepsilon} (u_{\varepsilon,\mathbf{P}} + v_{\varepsilon,\mathbf{P}})^{p+1} : \bar{\wedge}_\delta \to R.$$

Lemma 5.8. *Let $\mathbf{P} \in \Lambda_\delta$ be a critical point of $K_\varepsilon(\mathbf{P})$, then $u_{\varepsilon,\mathbf{P}} + v_{\varepsilon,\mathbf{P}}$ is a critical point of J_ε.*

Proof: Let $\mathbf{P} \in \wedge_\delta$ be a critical point of $K_\varepsilon(\mathbf{P})$. Then we have $\dfrac{\partial K_\varepsilon}{\partial \mathbf{P}} = 0$. Namely,

$$\int_{\Omega_\varepsilon} \bigtriangledown(u_{\varepsilon,\mathbf{P}} + v_{\varepsilon,\mathbf{P}}) \frac{\partial(u_{\varepsilon,\mathbf{P}} + v_{\varepsilon,\mathbf{P}})}{\partial \mathbf{P}_i} - \int_{\Omega_\varepsilon} (u_{\varepsilon,\mathbf{P}} + v_{\varepsilon,\mathbf{P}})^p \frac{\partial(u_{\varepsilon,\mathbf{P}} + v_{\varepsilon,\mathbf{P}})}{\partial P_i} = 0.$$

By (5.8)

$$\int_{\Omega_\varepsilon} \Big(\sum_{j=1}^K c_j(\varepsilon,\mathbf{P}) \frac{\frac{\partial u_{\varepsilon,\mathbf{P}}}{\partial \mathbf{P}_j}}{\|\frac{\partial u_{\varepsilon,\mathbf{P}}}{\partial \mathbf{P}_j}\|_{L^2}}\Big) \frac{\partial (u_{\varepsilon,\mathbf{P}} + v_{\varepsilon,\mathbf{P}})}{\partial \mathbf{P}_i} = 0. \tag{5.12}$$

Note that

$$\begin{aligned}
\int_{\Omega_\varepsilon} \frac{\partial u_{\varepsilon,\mathbf{P}}}{\partial \mathbf{P}_j} \frac{\partial v_{\varepsilon,\mathbf{P}}}{\partial \mathbf{P}_i} &= -\int_{\Omega_\varepsilon} v_{\varepsilon,\mathbf{P}} \frac{\partial^2 u_{\varepsilon,\mathbf{P}}}{\partial \mathbf{P}_j \partial \mathbf{P}_j}, \\
&= O\big(\varepsilon^{-2} e^{-\frac{\pi}{4\varepsilon}\varphi_K(\mathbf{P})}\big), \\
\int_{\Omega_\varepsilon} \frac{\partial u_{\varepsilon,\mathbf{P}}}{\partial \mathbf{P}_j} \frac{\partial u_{\varepsilon,\mathbf{P}}}{\partial \mathbf{P}_i} &= \varepsilon^{-\frac{4}{p-1}} \varepsilon^{-2} \Big(\int_S (\frac{\partial u_0}{\partial x})^2 dxdy \quad \delta_{ij} + o(1)\Big).
\end{aligned}$$

Hence (5.12) is a system of equation in $c_j(\varepsilon,\mathbf{P})$ where the matrix is diagonally dominant and hence is nonsingular. So $c_j(\varepsilon,\mathbf{P}) = 0, \ j = 1,\cdots,K$.

□

By Lemma 5.8, to find a critical point of J_ε, we just need to find a critical point of $K_\varepsilon(\mathbf{P})$.

We need an asymptotic expansion of $K_\varepsilon(\mathbf{P})$.

Lemma 5.9.

$$\begin{aligned}
K_{\varepsilon,\mathbf{P}} &= \varepsilon^{-\frac{4}{p-1}} K \big[\frac{1}{2} - \frac{1}{p+1}\big] \int_S u_0^{p+1} \\
&+ \varepsilon^{-\frac{4}{p-1}} \frac{1}{8\pi} c_3^2 \sum_{i=1}^K \{(-\varepsilon D_1(\varepsilon) + O(\varepsilon^2)) e^{-\pi\frac{10-P_i}{\varepsilon}} + (-\varepsilon D_2(\varepsilon) + O(\varepsilon^2)) e^{-\pi\frac{10+P_i}{\varepsilon}}\} \\
&- \frac{1}{2}(1+o(1))\gamma_1 \varepsilon^{-\frac{4}{p-1}} \sum_{i\neq j} e^{-\frac{\pi}{2}|P_i-P_j|}
\end{aligned}$$

Proof:

$$
\begin{aligned}
K_\varepsilon(\mathbf{P}) &= \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla(u_{\varepsilon,\mathbf{P}} + v_{\varepsilon,\mathbf{P}})|^2 - \frac{1}{p+1}\int_{\Omega_\varepsilon} (u_{\varepsilon,\mathbf{P}} + v_{\varepsilon,\mathbf{P}})^{p+1} \\
&= J_\varepsilon(u_{\varepsilon,\mathbf{P}}) + \int_{\Omega_\varepsilon} (\nabla u_{\varepsilon,\mathbf{P}} \nabla v_{\varepsilon,\mathbf{P}} - u_{\varepsilon,\mathbf{P}}^p v_{\varepsilon,\mathbf{P}}) \\
&+ \frac{1}{2}\int_{\Omega_\varepsilon} (|\nabla v_{\varepsilon,\mathbf{P}}|^2 - p u_{\varepsilon,\mathbf{P}}^{p-1} v_{\varepsilon,\mathbf{P}}^2) + O(e^{-\frac{\pi}{2\varepsilon}\varphi_K(\mathbf{P})} e^{-\frac{c}{\varepsilon}}) \\
&= J_\varepsilon(u_{\varepsilon,\mathbf{P}}) - \frac{1}{2}\int_{\Omega_\varepsilon} ((\Delta u_{\varepsilon,\mathbf{P}} + u_{\varepsilon,\mathbf{P}}^p) v_{\varepsilon,\mathbf{P}}) + O(e^{-\frac{\pi}{2\varepsilon}\varphi_K(\mathbf{P})} e^{-\frac{c}{\varepsilon}}) \\
&= J_\varepsilon(u_{\varepsilon,\mathbf{P}}) - \frac{1}{2}\int_{\Omega_\varepsilon \cap \{|x|>10\}} ((\Delta u_{\varepsilon,\mathbf{P}}) v_{\varepsilon,\mathbf{P}}) + O(e^{-\frac{\pi}{2\varepsilon}\varphi_K(\mathbf{P})} e^{-\frac{c}{\varepsilon}}) \\
&= J_\varepsilon(u_{\varepsilon,\mathbf{P}}) + O(\varepsilon^{2+2\alpha}\varepsilon^{-\frac{4}{p-1}} e^{-\frac{\pi\varphi_K(\mathbf{P})}{2\varepsilon}}) + O(e^{-\frac{\pi}{2\varepsilon}\varphi_K(\mathbf{P})} e^{-\frac{c}{\varepsilon}}).
\end{aligned}
$$

(By (5.10) and (5.11)).

By Corollary 5.4, this finishes the proof of Lemma 5.9.

□

Lemma 5.10. *If $D_1 > 0, D_2 > 0$, then the following problem*

$$\max_{\mathbf{P}\in\bar{\wedge}_\delta} K_\varepsilon(\mathbf{P}) \tag{5.13}$$

has a solution $\mathbf{P} = \mathbf{P}_\varepsilon = (P_{\varepsilon,1}, ..., P_{\varepsilon,K}) \in \wedge_{\frac{\delta}{2}}$. Moreover, $\varphi_K(\mathbf{P}_\varepsilon) \to \max_{\mathbf{P}\in\Lambda_\delta} \varphi_K(\mathbf{P})$ as $\varepsilon \to 0$.

Proof: By Lemma 5.7, $K_\varepsilon(\mathbf{P})$ is continuous. Therefore the maximizing problem (5.13) has a maximum point $\mathbf{P}_\varepsilon \in \bar{\wedge}_\delta$. We now claim $\mathbf{P}_\varepsilon \in \wedge_{\frac{\delta}{2}}$.

We first obtain an upper bound for $K_\varepsilon(\mathbf{P}_\varepsilon)$. Take $P_i = P_i^0, i = 1, ..., K$, we have

$$
\begin{aligned}
K_\varepsilon(\mathbf{P}) &\le \varepsilon^{-\frac{4}{p-1}} K[\frac{1}{2} - \frac{1}{p+1}] \int_S u_0^{p+1} \\
&- \varepsilon^{-\frac{4}{p-1}} \frac{1}{8\pi} c_3^2 e^{-\pi\frac{10}{\varepsilon K}} [\varepsilon D_1(\varepsilon) + \varepsilon D_2(\varepsilon) + O(\varepsilon^2)] \\
&- \frac{1}{2}(1+o(1))\gamma_1 \varepsilon^{-\frac{4}{p-1}} \sum_{i\ne j} e^{-\pi\frac{10}{K}}.
\end{aligned} \tag{5.14}
$$

Suppose now that $\mathbf{P}_\varepsilon = (P_{\varepsilon,1}, ..., P_{\varepsilon,K}) \notin \Lambda_\delta$, we then have $\varphi_K(\mathbf{P}_\varepsilon) \geq \frac{20}{K} - \frac{\delta}{2}$. Hence $\min(10 - P_{\varepsilon,i}, 10 + P_{\varepsilon,i}) \geq \frac{10}{K} - \frac{\delta}{4}, |P_{\varepsilon,k} - P_{\varepsilon,l}| \geq \frac{20}{K} - \frac{\delta}{2}$ for $i, k, l = 1, ..., K, k \neq l$. Since $D_1(\varepsilon) > 0, D_2(\varepsilon) > 0$, by (5.9), we have

$$\begin{aligned} K_\varepsilon(\mathbf{P}) \geq \varepsilon^{-\frac{4}{p-1}} K[\frac{1}{2} - \frac{1}{p+1}] \int_S u_0^{p+1} \\ - \varepsilon^{-\frac{4}{p-1}} \frac{1}{8\pi} c_3^2 e^{-\frac{\pi}{\varepsilon}(\frac{10}{K} - \frac{\delta}{4})} [\varepsilon D_1(\varepsilon) + \varepsilon D_2(\varepsilon) + O(\varepsilon^2)] \\ - \frac{1}{2}(1 + o(1))\gamma_1 \varepsilon^{-\frac{4}{p-1}} \sum_{i \neq j} e^{-\frac{\pi}{2\varepsilon}(\frac{20}{K} - \frac{\delta}{2})} \end{aligned} \tag{5.15}$$

Comparing (5.14) and (5.15), we obtain a contradiction.

The previous arguments actually show that as $\varepsilon \to 0$, $\varphi_K(\mathbf{P}_\varepsilon) \to \max_{\mathbf{P} \in \Lambda_\delta} \varphi_K(\mathbf{P}) = \frac{20}{K}$. Moreover, it is easy to see that after a permutation, we have $\mathbf{P}_\varepsilon \to (P_1^0, ..., P_K^0)$ as $\varepsilon \to 0$.

□

At the end of this section, we can now prove Theorem 1.3.

Proof of Theorem 1.3:

Let $\mathbf{P}_\varepsilon$ be the point obtained in Lemma 5.10. By Lemma 5.10, $\mathbf{P}_\varepsilon$ is a critical point of $K_\varepsilon(\mathbf{P})$.

By Lemma 5.8, $u_{\varepsilon,\mathbf{P}_\varepsilon} + v_{\varepsilon,\mathbf{P}_\varepsilon}$ is a critical point of J_ε. Let $u_\varepsilon = u_{\varepsilon,\mathbf{P}_\varepsilon} + v_{\varepsilon,\mathbf{P}_\varepsilon}$. It is easy to see that $u_\varepsilon \geq 0$ satisfies (1.1). By Maximum Principle, $u_\varepsilon > 0$ in Ω_ε. Furthermore, it is easy to see that u_ε satisfies the properties of Theorem 1.3.

□

We believe that similar results are still true for dimension large than 2. Our approaches here depend on the result of the kernel space of the linearized operators. Another possible approaches to deal with this problem are variational methods which can avoid the study of the kernel space. In conlusion, the effect of the domain shape on the existence of solutions are quite subtle and need more general studies.

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DEPARTMENT OF MATHEMATICS, THE CHINESE UNIVERSITY OF HONG KONG, SHATIN, HONG KONG

E-mail address: wei@math.cuhk.edu.hk

INSTITUTE OF MATHEMATICS, ACADEMIA SINICA, BEIJING, CHINA

E-mail address: lqzhang@math03.math.ac.cn