NEW SOLUTIONS FOR NONLINEAR SCHRÖDINGER EQUATIONS WITH CRITICAL NONLINEARITY

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ABSTRACT. We consider the following nonlinear Schrödinger equations in $\mathbb{R}^n$

$$\begin{cases} \varepsilon^2 \Delta u - V(r)u + u^p = 0 \text{ in } \mathbb{R}^n; \\ u > 0 \text{ in } \mathbb{R}^n \text{ and } u \in H^1(\mathbb{R}^n) \end{cases}$$

where $V(r)$ is a radially symmetric positive function. In [2], Ambrosetti, Malchiodi and Ni proved that if $M(r) = r^{n-1}(V(r))^{\frac{p+1}{p-1}-\frac{1}{2}}$ has a nondegenerate critical point $r_0 \neq 0$, then a layered solution concentrating near r_0 exists. In this paper, we show that if $p = \frac{n+2}{n-2}$ and the dimension $n = 3, 4$ or 5, another new type of solution exists: this solution has a layer near r_0 and a bubble at the origin.

1. INTRODUCTION

We consider *standing waves* for a nonlinear Schrödinger equation in $\mathbb{R}^n$ of the form

$$-i\varepsilon \frac{\partial \psi}{\partial t} = \varepsilon^2 \Delta \psi - Q(y)\psi + |\psi|^{p-1}\psi \tag{1.1}$$

where $p > 1$, namely solutions of the form $\psi(t, y) = \exp(i\lambda\varepsilon^{-1}t)u(y)$. Assuming that the amplitude $u(y)$ is positive and vanishes at infinity, we see that this ψ satisfies (1.1) if and only if u solves the nonlinear elliptic problem

$$\varepsilon^2 \Delta u - V(y)u + u^p = 0, \ u > 0, \quad u \in H^1(\mathbb{R}^n), \tag{1.2}$$

where $V(y) = Q(y) + \lambda$. In the rest of this paper we will assume that V is a smooth function with

$$\inf_{y \in \mathbb{R}^n} V(y) \ > \ 0 \ .$$

Considerable attention has been paid in recent years to the problem of construction of standing waves in the so-called *semi-classical limit* of (1.1) $\varepsilon \to 0$. In the pioneering work [18], Floer and Weinstein constructed positive solutions to this problem when $p = 3$ and $n = 1$ with concentration taking place near a given point y_0 with $V'(y_0) = 0$, $V''(y_0) \neq 0$,

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1991 *Mathematics Subject Classification.* Primary 35B40, 35B45; Secondary 35J40.

Key words and phrases. Bubbles, Nonlinear Schrödinger Equations, Layers.

being exponentially small in ε outside any neighborhood of y_0. More precisely, they established the existence of a solution u_ε such that

$$u_\varepsilon(y) \sim V(y_0)^{\frac{1}{p-1}} w(\, V(y_0)^{\frac{1}{2}} \varepsilon^{-1}(y-y_0)\,)$$

where w is the unique solution of

$$w'' - w + w^p = 0\,,\ u>0, \quad w'(0)=0, \quad w(\pm\infty)=0\,. \tag{1.3}$$

This result has been subsequently extended to higher dimensions to the construction of solutions exhibiting high concentration around one or more points of space under various assumptions on the potential and the nonlinearity by many authors. We refer the reader for instance to [1], [4], [8]-[16], [19, 22, 23, 27, 28, 31, 32, 33, 34]. In most of the papers, a subcriticality on p is assumed, namely $p<\frac{n+2}{n-2}$.

When $p=\frac{n+2}{n-2}$, there are very few results. Benci and Cerami [6] proved that if $\|\varepsilon^{-2}V(x)\|_{L^{\frac{n}{2}}(\mathbb{R}^n)}$ is small enough, there exists a solution. (That is to say that ε is large.) Concerning the existence of point concentrations (or bubbles) in the case of ε small, Cingolani and Pistoia [9] proved that if $n\geq 5, V(x)\in L^{\frac{n}{2}}(\mathbb{R}^n)$, then there are no single bubble solutions. Ding and Lin [12] considered (1.2) by adding a subcritical term, namely the following equation

$$\varepsilon^2\Delta u - V(x)u + P(x)u^q + K(x)u^{\frac{N+2}{N-2}} = 0, 1<q<\frac{N+2}{N-2}, u>0, u\in H^1(\mathbb{R}^n). \tag{1.4}$$

They proved the existence of one solution for ε small. Therefore, in general, it seems difficult to construct bubbling solutions for (1.2) with critical exponent.

In another direction, Ambrosetti, Malchiodi and Ni [2] have considered the case of $V=V(|y|)$, also treated in [5, 7], and constructed radial solutions $u_\varepsilon(|y|)$ exhibiting concentration on a sphere $|y|=r_0$ in the form

$$u_\varepsilon(r) \sim V(r_0)^{\frac{1}{p-1}} w(V(r_0)^{\frac{1}{2}}\varepsilon^{-1}(r-r_0))\ ,$$

under the assumption that $r_0>0$ is a non-degenerate (or topologically nontrivial) critical point of

$$M(r) = r^{n-1}V^{\sigma}(r) \tag{1.5}$$

where

$$\sigma = \frac{p+1}{p-1} - \frac{1}{2}, \tag{1.6}$$

and w is again the unique solution of (1.3).

In this paper, we show that under the same condition as in [2], another new type of solutions exists in the critical exponent case, i.e., $p = \frac{n+2}{n-2}$. This new type of solution is a superposition of the layered solutions constructed in [2] and a bubble at the origin. This kind of solutions only appears in the dimensions $n = 3, 4$ or 5. It represents a new phenomena for the critical exponent problems–the addition of a layered solution can produce a bubble in the interior.

More precisely, we consider the following singularly perturbed problem

$$\begin{cases} \varepsilon^2 \Delta u - V(|x|)u + n(n-2)u^{\frac{n+2}{n-2}} = 0, u > 0 \ \text{ in } \mathbb{R}^n, \\ u \in H^1(\mathbb{R}^n), u > 0 \end{cases} \tag{1.7}$$

where $V(|x|) > 0$ is such that

$$M(r) \text{ has a nondegenerate critical point } r_0 > 0. \tag{1.8}$$

It is well-known ([11]) that the functions

$$U_{a,\mu}(x) = \left(\frac{\mu}{1 + \mu^2 |x-a|^2} \right)^{\frac{n-2}{2}}, \ \mu > 0, \ a \in \mathbb{R}^n$$

are the only solutions to the problem

$$\Delta u + n(n-2)u^{\frac{n+2}{n-2}} = 0, \ u > 0 \text{ in } \mathbb{R}^n.$$

Our main result in this paper can be stated as follows:

Theorem 1.1. *Let $n = 3, 4,$ or 5. Suppose that $V(r)$ satisfies (1.8). Then for ε sufficiently small, problem (1.7) has solutions of the following form:*

$$\varepsilon^{\frac{n-2}{2}} U_{0,\mu_\varepsilon}(r) + u_\varepsilon(r) + o(1)$$

where $\mu_\varepsilon \sim e^{(d_n + o(1))/\varepsilon}$ for some positive constant $d_n > 0$ and u_ε is the layered solution constructed in [2].

The proof of Theorem 1.1 is divided into two steps:

Step 1: We show that the layered solutions u_ε constructed in [2] is *nondegenerate*, with spectrum gap $\geq C\varepsilon^2$.

Step 2: We add an interior bubble $\varepsilon^{\frac{n-2}{2}} U_{0,\mu}$ to u_ε. To obtain a real solution to (1.7), we adjust μ so that a small perturbation of $\varepsilon^{(n-2)/2} U_{0,\mu} + u_\varepsilon$ satisfies (1.7). This is done by a so-called "*Localized Energy Method*"-a combination of Liapunov-Schmidt finite dimensional reduction and variational methods-which has been used in [2]-[3] and successfully in many papers, see e.g. [20], [21], [25], [29], [30] and the references therein.

Throughout this paper, unless otherwise stated, the letter C will always denote various generic constants which are independent of ε, for ε sufficiently small. The notation $A_\varepsilon = O(B_\varepsilon)$ means that $|\frac{A_\varepsilon}{B_\varepsilon}| \le C$, while $A_\varepsilon = o(B_\varepsilon)$ means that $\lim_{\varepsilon\to 0}\frac{|A_\varepsilon|}{|B_\varepsilon|} = 0$.

Acknowledgment. The research of the first author (J.W.) is supported by an Earmarked Grant from RGC of Hong Kong.

2. Properties of Layered Solutions

In this section, we study the spectrum properties of the layered solutions constructed in [2].

Let u_ε be the layered solution constructed in [2]. A standard argument (similar to the proof of Claim 1 of Theorem 2.1 of [26]) shows that u_ε has a unique maximum point r_ε where $r_\varepsilon \to r_0$ with $M^{'}(r_0) = 0$. By a scaling argument, we may assume that $V(r_0) = 1$. Let $x = r_\varepsilon + \varepsilon y$ and $f(u) = n(n-2)u^{\frac{n+2}{n-2}}$.

Let w be the unique solution of

$$w^{''} - w + f(w) = 0 \text{ in } \mathbb{R}^1, w(0) = \max_{\mathbb{R}^1} w(y), w(y) \to 0 \text{ as } |y| \to +\infty. \tag{2.1}$$

The main result in this section is the following theorem, which contains elements of independent interest.

Theorem 2.1. *Consider the following eigenvalue problem*

$$\varepsilon^2\Delta\psi - V(r)\psi + f^{'}(u_\varepsilon)\psi = \lambda\psi, \psi \in H^1(\mathbb{R}^n). \tag{2.2}$$

If $(\bar\psi_\varepsilon, \bar\lambda_\varepsilon)$, $\bar\psi_\varepsilon \ne 0$, is a solution of the above problem, then we have

$$|\bar\lambda_\varepsilon| \ge C\varepsilon^2. \tag{2.3}$$

To prove Theorem 2.1, we first need some asymptotic behavior of the layered solution u_ε. Let

$$\tilde u_\varepsilon(y) = V(r_\varepsilon)^{-\frac{1}{p-1}} u_\varepsilon\big(\varepsilon_1 y + r_\varepsilon\big), \quad \varepsilon_1 = \frac{\varepsilon}{\sqrt{V(r_\varepsilon)}},$$

and

$$\Delta_\varepsilon' u = u'' + \frac{\varepsilon_1(n-1)}{r_\varepsilon + \varepsilon_1 y} u', \text{ where } y \in I_\varepsilon = \left[-\frac{r_\varepsilon}{\varepsilon_1}, +\infty\right). \tag{2.4}$$

We have

Lemma 2.2. *(1)* $r_\varepsilon = r_0 + o(\varepsilon)$*;*

(2) $\tilde{u}_\varepsilon(y) = w(y) + \varepsilon_1 \phi_1(y) + \varepsilon_1^2 \phi_2(y) + o(\varepsilon^2)$*, where* $\phi_1(y)$ *is the unique (odd) solution of*

$$\phi_1'' - \phi_1 + f'(w)\phi_1 + \frac{n-1}{r_0} w' - V'(r_0) y w(y) = 0, \phi_1'(0) = 0, \tag{2.5}$$

and $\phi_2(y)$ *is the unique (even) solution of*

(2.6)

$$\phi_2'' - \phi_2 + f'(w)\phi_2 - \frac{n-1}{r_0^2} w' y - \frac{1}{2} V''(r_0) y^2 w(y) + \frac{n-1}{r_0} \phi_1' - V'(r_0) y \phi_1 + \frac{1}{2} f''(w) \phi_1^2 = 0.$$

Remark: Since $M'(r_0) = 0, V(r_0) = 1$, we obtain that $\sigma V'(r_0) = -\frac{n-1}{r_0}$. Hence $\int_{\mathbb{R}} [\frac{n-1}{r_0} w' - V'(r_0) y w(y)] w' = 0$ and the problem (2.5) admits a unique odd solution with $\phi_1'(0) = 0$.

Proof: Let

$$V_\varepsilon(r) = \frac{V(r)}{V(r_\varepsilon)}.$$

Then, $V_\varepsilon(r_\varepsilon) = 1$, and $\tilde{u}_\varepsilon$ satisfies

$$\tilde{u}_\varepsilon'' + \frac{\varepsilon_1(n-1)}{r_\varepsilon + \varepsilon_1 y} \tilde{u}_\varepsilon' - V_\varepsilon(\varepsilon_1 y + r_\varepsilon)\tilde{u}_\varepsilon + \tilde{u}_\varepsilon^p = 0.$$

Let $\tilde{u}_\varepsilon(y) = w(y) + \varepsilon_1 \phi_\varepsilon(y)$. Then $\phi_\varepsilon(y)$ satisfies

$$\phi_\varepsilon'' + \frac{\varepsilon_1(n-1)}{r_\varepsilon + \varepsilon_1 y} \phi_\varepsilon' - \phi_\varepsilon + f'(w)\phi_\varepsilon + \frac{\varepsilon_1}{2} f''(w)\phi_\varepsilon^2 + o(\varepsilon |f''(w)||\phi_\varepsilon|^2)$$

$$= -\frac{(n-1)}{r_\varepsilon + \varepsilon_1 y} w' + \frac{\varepsilon_1(n-1)}{r_\varepsilon^2} y w' + V_\varepsilon'(r_\varepsilon) y w + \frac{\varepsilon_1}{2} V_\varepsilon''(r_\varepsilon) y^2 w + O(\varepsilon^2 (1+|y|)^3 w). \tag{2.7}$$

We first claim that $\|\phi_\varepsilon\|_{L^\infty} < +\infty$. In fact, suppose not. We may assume that $\max_{y \in I_\varepsilon} \phi_\varepsilon(y) \to +\infty$. Note that $\tilde{u}_\varepsilon(y) \to w(y)$ uniformly $\mathbb{R}$ as $\varepsilon \to 0$. So, $\varepsilon_1 \|\phi_\varepsilon\|_{L^\infty} \to 0$ as $\varepsilon \to 0$. We set $\hat{\phi}_\varepsilon = \frac{\phi_\varepsilon}{\|\phi_\varepsilon\|_{L^\infty}}$ and $\phi(y_\varepsilon) = \max_{y \in I_\varepsilon} \phi_\varepsilon(y)$. Then $|y_\varepsilon| \le C$. (If not, we evaluate (2.7) at $y = y_\varepsilon$. Using the fact that $\phi_\varepsilon''(y_\varepsilon) \le 0, \phi_\varepsilon'(y_\varepsilon) = 0$ and that $f'(u_\varepsilon(y_\varepsilon)) < \frac{1}{2}$, we arrive at $\phi_\varepsilon(y_\varepsilon) \le C$, which contradicts with our assumption.) Since $|\hat{\phi}_\varepsilon| \le 1$, standard regularity argument shows that $\hat{\phi}_\varepsilon \to \phi_0$ as $\varepsilon \to 0$ in $C^2_{loc}(\mathbb{R})$ and ϕ_0 satisfies $\phi_0'' - \phi_0 + f'(w)\phi_0 = 0$. So $\phi_0 = cw'$ for some constant c. Since $u_\varepsilon'(r_\varepsilon) = 0, w'(0) = 0$, we have $\phi_\varepsilon'(0) = 0$, which implies $cw''(0) = 0$ and hence $c = 0$. On the other hand $\hat{\phi}_\varepsilon(y_\varepsilon) = 1 \to \phi_0(y_0)$ as $y_\varepsilon \to y_0$. A contradiction.

Now we multiply (2.7) by w' and integrate over $\mathbb{R}$, we have that

$$-\int_{\mathbb{R}} \frac{(n-1)}{r_\varepsilon + \varepsilon_1 y}(w')^2 + V_\varepsilon'(r_\varepsilon)\int_{\mathbb{R}} yww' = \int_{\mathbb{R}} \frac{\varepsilon_1(n-1)}{r_\varepsilon + \varepsilon_1 y}\phi_\varepsilon' w' + \frac{\varepsilon_1}{2}\int_{\mathbb{R}} f''(w)\phi_\varepsilon^2 w' + o(\varepsilon), \tag{2.8}$$

which, together with $\int_{\mathbb{R}} |w'|^2 y = 0$, implies that

$$-\frac{n-1}{r_\varepsilon}\int_{\mathbb{R}} (w')^2 - \frac{1}{2}V_\varepsilon'(r_\varepsilon)\int_{\mathbb{R}} w^2 = \int_{\mathbb{R}} \frac{\varepsilon_1(n-1)}{r_\varepsilon + \varepsilon_1 y}\phi_\varepsilon' w' + \frac{\varepsilon_1}{2}\int_{\mathbb{R}} f''(w)\phi_\varepsilon^2 w' + o(\varepsilon). \tag{2.9}$$

Since $\int_{\mathbb{R}}(w')^2 = \frac{p-1}{p+3}\int_{\mathbb{R}} w^2 = \frac{1}{2\sigma}\int_{\mathbb{R}} w^2$ and $M'(r) = r^{n-1}V^{\sigma-1}(\frac{(n-1)}{r}V(r) + \sigma V'(r))$, we deduce from (2.9) the following rough estimate

$$M'(r_\varepsilon) = O(\varepsilon), \tag{2.10}$$

and thus,

$$r_\varepsilon = r_0 + O(\varepsilon),$$

since $M''(r_0) \ne 0$.

Since $\phi_\varepsilon = O(1)$, $\phi_\varepsilon \to \phi_1$ as $\varepsilon \to 0$, where ϕ_1 satisfies the equation in (2.5). Since $\phi_\varepsilon'(0) = 0$, we deduce $\phi_1'(0) = 0$. Moreover, noting that $\frac{n-1}{r_0}w' - V'(r_0)yw$ is an odd function, we know that ϕ_1 is an odd function. We then set $\phi_\varepsilon = \phi_1(y) + \varepsilon_1\phi_{\varepsilon,2}$. The same argument as before gives $\phi_{\varepsilon,2} = O(1)$. Because ϕ_1 is odd, we see

$$\int_{\mathbb{R}} \phi_1' w' = 0, \quad \int_{\mathbb{R}} f''(w)\phi_1^2 w' = 0.$$

Using (2.9) again, we obtain that $M'(r_\varepsilon) = o(\varepsilon), r_\varepsilon = r_0 + o(\varepsilon)$. This proves (1).

Since $r_\varepsilon = r_0 + o(\varepsilon)$, we see that $\frac{1}{r_\varepsilon} = \frac{1}{r_0} + o(\varepsilon), V'(r_\varepsilon) = V'(r_0) + o(\varepsilon)$. Similar argument proves that $\phi_{\varepsilon,2} \to \phi_2$ where ϕ_2 satisfies (2.6). This proved (2). □

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1: Let us assume that there exists $(\bar{\psi}_\varepsilon, \bar{\lambda}_\varepsilon)$ satisfying (2.2) such that $\bar{\lambda}_\varepsilon = o(\varepsilon^2)$. We shall derive a contradiction.

Let

$$\psi_\varepsilon(y) = \bar{\psi}_\varepsilon\Big(\varepsilon_1 y + r_\varepsilon\Big), \quad \varepsilon_1 = \frac{\varepsilon}{\sqrt{V(r_\varepsilon)}},$$

and

$$\lambda_\varepsilon = \frac{1}{V(r_\varepsilon)}\bar{\lambda}_\varepsilon.$$

Then

$$-\Delta'_{\varepsilon_1}\psi_\varepsilon - V_\varepsilon(\varepsilon_1 y + r_\varepsilon)\psi_\varepsilon + f'(\tilde{u}_\varepsilon)\psi_\varepsilon = \lambda_\varepsilon\psi_\varepsilon,$$

and $\lambda_\varepsilon = o(\varepsilon^2)$.

By direct computations, we have

$$\begin{aligned}
& \Delta'_{\varepsilon_1} w' - V_\varepsilon(\varepsilon_1 y + r_\varepsilon)w' + f'(\tilde{u}_\varepsilon)w' \\
= \;& w''' + \frac{\varepsilon_1(n-1)}{r_\varepsilon + \varepsilon_1 y}w'' - V_\varepsilon(r_\varepsilon + \varepsilon_1 y)w' + f'(\tilde{u}_\varepsilon)w' \\
= \;& \varepsilon_1[\frac{(n-1)}{r_0}w'' - V'(r_0)yw' + f''(w)\phi_1 w'] \\
& +\varepsilon_1^2[-\frac{n-1}{r_\varepsilon^2}yw'' - \frac{1}{2}V''(r_0)y^2w' + \frac{1}{2}f'''(w)\phi_1^2 w' + f''(w)\phi_2 w'] + o(\varepsilon^2).
\end{aligned}$$

Now we let ψ_1 be the unique even function of

$$\psi_1'' - \psi_1 + f'(w)\psi_1 = -\frac{(n-1)}{r_0}w'' + V'(r_0)yw' - f''(w)\phi_1 w'. \tag{2.11}$$

Note that

$$\psi_1 = \phi_1' + \hat{\psi}_1 \tag{2.12}$$

where $\hat{\psi}_1$ is the unique even solution of $\hat{\psi}_1'' - \hat{\psi}_1 + f'(w)\hat{\psi}_1 + V'(r_0)w = 0$, which is given by

$$\hat{\psi}_1(y) = -V'(r_0)(\frac{1}{p-1}w + \frac{1}{2}yw') \tag{2.13}$$

Set

$$\Psi(y) = w'(y) + \varepsilon_1\psi_1(y). \tag{2.14}$$

For later use, we need to compute

$$
\begin{aligned}
(2.15)\quad & \Delta'_{\varepsilon_1}\Psi - V_\varepsilon(\varepsilon_1 y + r_\varepsilon)\Psi + f'(\tilde{u}_\varepsilon)\Psi \\
= \;& \Delta'_{\varepsilon_1} w' - V_\varepsilon(\varepsilon_1 y + r_\varepsilon) w' + f'(\tilde{u}_\varepsilon) w' + \varepsilon_1[\Delta'\psi_1 - V_\varepsilon(\varepsilon_1 y + r_\varepsilon)\psi_1 + f'(\tilde{u}_\varepsilon)\psi_1] \\
= \;& \varepsilon_1^2[-\frac{n-1}{r_0^2} y w'' - \frac{1}{2} V''(r_0) y^2 w'] \\
+ \;& \varepsilon_1^2[\frac{n-1}{r_0}\psi_1' - V'(r_0) y \psi_1] + \varepsilon_1^2[f''(w)\psi_1\phi_1 + \frac{1}{2} f'''(w)\phi_1^2 w' + f''(w)\phi_2 w'] + o(\varepsilon^2).
\end{aligned}
$$

We decompose

$$
\psi_\varepsilon = c_\varepsilon \Psi(y) + \psi_\varepsilon^\perp, \quad \int_{I_\varepsilon} \Psi(y)\psi_\varepsilon^\perp = 0, \tag{2.16}
$$

where $\psi_\varepsilon^\perp$ satisfies
(2.17)

$$
-\Delta'_{\varepsilon_1}\psi_\varepsilon^\perp + V_\varepsilon(\varepsilon_1 y + r_\varepsilon)\psi_\varepsilon^\perp - f'(\tilde{u}_\varepsilon)\psi_\varepsilon^\perp = c_\varepsilon\big(\Delta'_{\varepsilon_1}\Psi - V_\varepsilon(\varepsilon_1 y + r_\varepsilon)\Psi + f'(\tilde{u}_\varepsilon)\Psi\big) - \lambda_\varepsilon \psi_\varepsilon \text{ in } I_\varepsilon,
$$

and

$$
\int_{I_\varepsilon} \Psi(y)\psi_\varepsilon^\perp = 0.
$$

Then by the same argument as in of [2], we have that

$$
|\psi_\varepsilon^\perp| \le C\varepsilon^2 |c_\varepsilon|. \tag{2.18}
$$

Now we multiply (2.17) by w', integrate over I_ε and use (2.18) to obtain

$$
c_\varepsilon \int_{I_\varepsilon} (\Delta'_{\varepsilon_1}\Psi - V_\varepsilon(\varepsilon_1 y + r_\varepsilon)\Psi + f'(\tilde{u}_\varepsilon)\Psi) w' = \lambda_\varepsilon c_\varepsilon (\int_{I_\varepsilon} \Psi w') + O(\varepsilon^3 |c_\varepsilon|). \tag{2.19}
$$

We now analyze both sides of (2.19). The right hand side is relatively easy to understand since

$$
\int_{I_\varepsilon} \Psi w' = \int_{\mathbb{R}} (w')^2 + O(\varepsilon). \tag{2.20}
$$

We use (2.15) to compute

$$
\begin{aligned}
(2.21)\quad \varepsilon_1^{-2}\int_{I_\varepsilon} (\Delta'_{\varepsilon_1}\Psi - V_\varepsilon(\varepsilon_1 y + r_\varepsilon)\Psi + f'(u_\varepsilon)\Psi) w' &= \int_{\mathbb{R}} w'[-\frac{n-1}{r_0^2} y w'' - \frac{1}{2} V''(r_0) y^2 w'] \\
&+ \int_{\mathbb{R}} w'[\frac{n-1}{r_0}\psi_1' - V'(r_0) y \psi_1] \\
&+ \int_{\mathbb{R}} w'[f''(w)\psi_1\phi_1 + \frac{1}{2} f'''(w)\phi_1^2 w' + f''(w)\phi_2 w'] + o(1).
\end{aligned}
$$

Note that $(w'')'' - w'' + f'(w)w'' + f''(w)(w')^2 = 0$. We derive that

$$\int_{\mathbb{R}} f''(w)(w')^2\phi_2 = -\int_{\mathbb{R}}[(w'')'' - w'' + f'(w)w'']\phi_2$$

$$= -\int_{\mathbb{R}}[\phi_2'' - \phi_2 + f'(w)\phi_2]w''$$

$$= \int_{\mathbb{R}}[-\frac{n-1}{r_0^2}w'y - \frac{1}{2}V''(r_0)y^2w]w'' + \int_{\mathbb{R}}[\frac{n-1}{r_0}\phi_1'w'' + V'(r_0)(y\phi_1' + \phi_1)w'] + \int_{\mathbb{R}}[\frac{1}{2}f''(w)\phi_1^2]w'' \tag{2.22}$$

Substituting (2.22) into (2.21) and using (2.12), we obtain

$$\varepsilon_1^{-2}\int_{I_\varepsilon}(\Delta_{\varepsilon_1}'\Psi - V_\varepsilon(\varepsilon_1 y + r_\varepsilon)\Psi + f'(\tilde{u}_\varepsilon)\Psi)w' = \frac{n-1}{r_0^2}\int_{\mathbb{R}}(w')^2 - \frac{1}{2}V''(r_0)\int_{\mathbb{R}}w^2 \tag{2.23}$$

$$+ \int_{\mathbb{R}}[f''(w)\phi_1 w'\hat{\psi}_1 + V'(r_0)\phi_1 w' - \frac{n-1}{r_0}w''\hat{\psi}_1 - V'(r_0)yw'\hat{\psi}_1] + o(1).$$

Using the equation for $\hat{\psi}_1$, we obtain $(\hat{\psi}_1')'' - \hat{\psi}_1' + f'(w)\hat{\psi}_1' + f''(w)w'\hat{\psi}_1 + V'(r_0)w' = 0$. Hence

$$\int_{\mathbb{R}}[f''(w)\phi_1 w'\hat{\psi}_1 + V'(r_0)\phi_1 w' - \frac{n-1}{r_0}w''\hat{\psi}_1 - V'(r_0)yw'\hat{\psi}_1]$$
$$= \int_{\mathbb{R}}[\frac{n-1}{r_0}w'\hat{\psi}_1' - V'(r_0)yw\hat{\psi}_1' - \frac{n-1}{r_0}w''\hat{\psi}_1 - V'(r_0)yw'\hat{\psi}_1]$$
$$= -\frac{2(n-1)}{r_0}\int_{\mathbb{R}}w''\hat{\psi}_1 + V'(r_0)\int_{\mathbb{R}}w\hat{\psi}_1$$
$$= \frac{2(n-1)}{r_0}V'(r_0)\int_{\mathbb{R}}w''(\frac{1}{p-1}w + \frac{1}{2}yw') + (V'(r_0))^2\int_{\mathbb{R}}w(-\frac{1}{p-1}w - \frac{1}{2}yw')$$
$$= \frac{2(n-1)}{r_0}V'(r_0)\int_{\mathbb{R}}(-\frac{1}{p-1} - \frac{1}{4})|w'|^2 + (V'(r_0))^2\int_{\mathbb{R}}(-\frac{1}{p-1} + \frac{1}{4})w^2$$
$$= \frac{1}{2}(V'(r_0))^2\int_{\mathbb{R}}w^2,$$

since $\int_{\mathbb{R}}(w')^2 = \frac{p-1}{p+3}\int_{\mathbb{R}}w^2 = \frac{1}{2\sigma}\int_{\mathbb{R}}w^2$ and $\sigma V'(r_0) = -\frac{(n-1)}{r_0}V(r_0) = -\frac{(n-1)}{r_0}$.

Substituting the above identity to (2.23), we obtain

$$\varepsilon_1^{-2}\int_{I_\varepsilon}(\Delta_{\varepsilon'}'\Psi - V_\varepsilon(\varepsilon_1 y + r_\varepsilon)\Psi + f'(u_\varepsilon)\Psi)w' \tag{2.24}$$

$$= \frac{n-1}{r_0^2}\int_{\mathbb{R}}(w')^2 + \frac{1}{2}(V'(r_0))^2\int_{\mathbb{R}}w^2 - \frac{1}{2}V''(r_0)\int_{\mathbb{R}}w^2 + o(1).$$

Thus, we obtain

$$\varepsilon_1^{-2}\int_{I_\varepsilon}(\Delta'_{\varepsilon_1}\Psi - V_\varepsilon(\varepsilon_1 y + r_\varepsilon)\Psi + f'(u_\varepsilon))w' = B_0 M''(r_0) + o(1) \tag{2.25}$$

where

$$\begin{aligned} B_0 &= -r_0^{1-n}\int_{\mathbb{R}} w^2, \\ M''(r_0) &= (n-1)(n-2)r_0^{n-3} + 2\sigma(n-1)r_0^{n-2}V'(r_0) + r_0^{n-1}(\sigma(\sigma-1)(V'(r_0))^2 + \sigma V''(r_0)) \\ &= r_0^{n-1}[(n-1)(n-2)r_0^{-2} - \sigma(\sigma+1)(V'(r_0))^2 + \sigma V''(r_0) \\ &= r_0^{n-1}[-(n-1)r_0^{-2} - \sigma(V'(r_0))^2 + \sigma V''(r_0)]. \end{aligned}$$

Here we have used the fact that $\sigma V'(r_0) = -(n-1)r_0^{-1}$.

Substituting (2.25) into (2.19) we obtain

$$B_0 c_\varepsilon M''(r_0)\varepsilon_1^2 + o(\varepsilon^2 c_\varepsilon) = \lambda_\varepsilon c_\varepsilon\Big(\int_{\mathbb{R}}(w')^2 + O(\varepsilon)\Big), \tag{2.26}$$

which, together with $\lambda_\varepsilon = o(\varepsilon^2)$, implies that

$$c_\varepsilon = 0. \tag{2.27}$$

This forces $\psi_\varepsilon^\perp \equiv 0$, by (2.18). Thus, $\psi_\varepsilon \equiv 0$. A contradiction.

This completes the proof of Theorem 2.1.

□

From Theorem 2.1, we deduce the following corollary

Corollary 2.3. *The linearized operator*

$$\varepsilon^2\Delta\psi - V(r)\psi + f'(u_\varepsilon)\psi$$

is an invertible operator from $H_r^2(\mathbb{R}^n) = H^2(\mathbb{R}^2)\cap\{u = u(r)\}$ to $L_r^2(\mathbb{R}^n) = L^2(\mathbb{R}^n)\cap\{u = u(r)\}$. Furthermore, we have

$$\|\phi\|_{L^\infty(\mathbb{R}^n)} \le C\varepsilon^{-\beta}\|\varepsilon^2\Delta\phi - V\phi + f'(u_\varepsilon)\phi\|_{L^\infty(\mathbb{R}^n)} \tag{2.28}$$

where $\beta > 4$ is some fixed constant.

Proof: From Theorem 2.1, we have that

$$\|\phi\|_{L^2(\mathbb{R}^n)} \le C\varepsilon^{-2}\|\varepsilon^2\Delta\phi - V\phi + f'(u_\varepsilon)\phi\|_{L^2(\mathbb{R}^n)}. \tag{2.29}$$

But

$$\|\Delta\phi\|_{L^2(\mathbb{R}^n)} \le \varepsilon^{-2}\|\varepsilon^2\Delta\phi - V\phi + f'(u_\varepsilon)\phi\|_{L^2(\mathbb{R}^n)} + C\varepsilon^{-2}\|\phi\|_{L^2(\mathbb{R}^n)}.$$

Hence

$$\|\phi\|_{H^2(\mathbb{R}^n)} \le C\|\Delta\phi\|_{L^2(\mathbb{R}^n)} + C\|\phi\|_{L^2(\mathbb{R}^n)} \le C\varepsilon^{-4}\|\varepsilon^2\Delta\phi - V\phi + f'(u_\varepsilon)\phi\|_{L^2(\mathbb{R}^n)}. \tag{2.30}$$

By a bootstrapping argument, we obtain (2.28). □

3. Energy and Error Estimates

Let u_ε be the layered solutions constructed in [2]. In this section, we are going to add an interior bubble to u_ε and compute the induced energy and errors.

We first study the asymptotic behavior of $u_\varepsilon(r)$ for $r < r_0$. To this end, we consider the following function

$$\varepsilon^2\Delta h_{\varepsilon,R} - V(r)h_{\varepsilon,R} = 0 \text{ in } B_R, h_{\varepsilon,R} = 1 \text{ on } \partial B_R. \tag{3.1}$$

By the following transformation, $h_{\varepsilon,R} = e^{-\frac{\psi_{\varepsilon,R}}{\varepsilon}}$, we see that $\psi_{\varepsilon,R}$ satisfies

$$\varepsilon\Delta\psi_{\varepsilon,R} - |\nabla\psi_{\varepsilon,R}|^2 + V(r) = 0 \text{ in } B_R, \psi_{\varepsilon,R} = 0 \text{ on } \partial B_R. \tag{3.2}$$

The solutions to the limiting problem $|\nabla\psi|^2 = V(r)$ are $c\pm\rho(r)$ where $\rho(r) = \int_0^r \sqrt{V(s)}ds$. By the method of viscosity solutions ([24]), we conclude that as $\varepsilon \to 0$, $\psi_{\varepsilon,R}(r) \to \rho(R) - \rho(r)$.

Since $\varepsilon^2\Delta u_\varepsilon - V(r)u_\varepsilon < 0$ and $u_\varepsilon(r_\varepsilon) \ge c_0 > 0$, it is easy to see that $u_\varepsilon \ge Ch_{\varepsilon,r_\varepsilon}$. On the other hand, u_ε satisfies $\varepsilon^2\Delta u_\varepsilon - (1-\delta)^2 u_\varepsilon \ge 0$ for $r < r_0(1-\delta)$. Thus $u_\varepsilon \le Ch_{\frac{\varepsilon}{1-\delta}, r_0(1-\delta)}$ for any δ small. Therefore we have for $r < r_0$, $-\varepsilon \log u_\varepsilon \to \rho(r_0) - \rho(r)$. That is we have

Lemma 3.1. *As $\varepsilon \to 0$, we have $-\varepsilon \log u_\varepsilon \to \rho(r_0) - \rho(r)$ for any $r < r_0$. Moreover, the convergence is uniform when $r \le r_0(1-\delta)$.*

Recall that $U_\mu := U_{0,\mu}$ is the unique family of radial solutions of $\Delta U + n(n-2)U^{\frac{n+2}{n-2}} = 0, U > 0$ in $\mathbb{R}^n$. Now we choose μ such that

$$\mu \in \Lambda := \left(e^{\frac{a_n-\delta}{\varepsilon}}, e^{\frac{a_n+\delta}{\varepsilon}}\right), \tag{3.3}$$

where $a_3 = a_5 = 2\rho(r_0), a_4 = \rho(r_0)$, and $0 < \delta < \frac{1}{100}\rho(r_0)$ is a fixed small number. Then by the following scaling $u(x) = (\mu\varepsilon)^{\frac{n-2}{2}}\hat{u}(y), x = \frac{1}{\mu}y$, problem (1.7) becomes the following

$$\mathcal{S}[u] := \Delta u - (\mu\varepsilon)^{-2}V(\frac{|y|}{\mu})u + n(n-2)u_+^{\frac{n+2}{n-2}} = 0. \tag{3.4}$$

Solutions to (1.7) can be found as critical points of the following energy functional

$$J_\varepsilon[u] = \frac{1}{2}\int_{\mathbb{R}^n} \left(|\nabla u|^2 + (\mu\varepsilon)^{-2} V(\frac{|y|}{\mu}) u^2\right) - \frac{(n-2)^2}{2}\int_{\mathbb{R}^n} u_+^{\frac{2n}{n-2}}. \tag{3.5}$$

We consider a linear problem which can be viewed as a projection of U_μ:

$$\Delta V_\mu - \varepsilon^{-2} V(|x|) V_\mu + n(n-2) U_\mu^{\frac{n+2}{n-2}} = 0 \text{ in } \mathbb{R}^n. \tag{3.6}$$

Define

$$W_1 := \mu^{-\frac{n-2}{2}} V_\mu(\frac{|y|}{\mu}), \quad W_2 := (\mu\varepsilon)^{-\frac{n-2}{2}} u_\varepsilon(\frac{|y|}{\mu}), \; W := W_1 + W_2. \tag{3.7}$$

By the Maximum Principle, it holds

$$0 < W_1 \le U_{0,1}. \tag{3.8}$$

We first need to analyze the projection $V_\mu(r)$. This will depend on the dimension.

When $n = 3$, we write V_μ as

$$V_\mu(|x|) = U_\mu(|x|) - \frac{1}{\mu^{1/2}}\frac{1}{|x|}\left(1 - \psi_\varepsilon(|x|)\right) + \varphi_{\mu,\varepsilon}(|x|),$$

where $\psi_\varepsilon(r)$ satisfies

$$\psi_\varepsilon^{''} - \varepsilon^{-2} V(r)\psi_\varepsilon = 0, \psi_\varepsilon(0) = 1, \psi_\varepsilon(\infty) = 0. \tag{3.9}$$

It is easy to see that

$$-\varepsilon\log(\psi_\varepsilon) \to \; \rho(r), \; e^{-\frac{C_1 r}{\varepsilon}} \le \psi_\varepsilon \le e^{-\frac{C_2 r}{\varepsilon}}, \tag{3.10}$$

where C_1 and C_2 are two fixed constants. Another property we need is that $\varepsilon\psi_\varepsilon'(0) \to -\sqrt{V(0)}$.

Then $\varphi_{\mu,\varepsilon}(|x|)$ satisfies

$$\Delta\varphi_{\mu,\varepsilon} - \varepsilon^{-2} V(|x|)\varphi_{\mu,\varepsilon} + \varepsilon^{-2} V(|x|)(U_\mu - \frac{\mu^{-1/2}}{|x|}) = 0 \text{ in } \mathbb{R}^n. \tag{3.11}$$

By comparison principle, it is easy to see that (see e.g. estimate (2.10) of [29])

$$|\varphi_{\mu,\varepsilon}(|x|)| \le C\big(\frac{1}{\varepsilon^2\mu^{3/2}(1+\mu|x|)}\big). \tag{3.12}$$

Hence for $n = 3$ we obtain the following estimates for W_1

$$W_1(y) = U_{0,1}(y) - \frac{1}{|y|}(1 - \psi_\varepsilon(\frac{|y|}{\mu})) + O(\frac{1}{\varepsilon^2\mu^2(1+|y|)}). \tag{3.13}$$

When $n = 4$ or 5, we write

$$W_1(y) = U_{0,1}(y) - (\mu\varepsilon)^{-2}\varphi_\mu(y), \tag{3.14}$$

where $\varphi_\mu(y)$ satisfies

$$\Delta\varphi_\mu - (\mu\varepsilon)^{-2}V(\frac{|y|}{\mu})\varphi_\mu + V(\frac{|y|}{\mu})(\frac{1}{1+|y|^2})^{\frac{n-2}{2}} = 0 \text{ in } \mathbb{R}^n. \tag{3.15}$$

The study of the asymptotic behavior of φ_μ is given by the following lemma, whose proof is provided in Appendix A.

Lemma 3.2. *(a) When $n = 4$, we have*

$$C_1(\ln(\varepsilon\mu)^2 - \ln(1+|y|^2) + O(1)) \le \varphi_\mu(|y|) \le C_2(\ln(\varepsilon\mu)^2 - \ln(1+|y|^2) + O(1)) \quad \textit{for } |y| \le \theta\varepsilon\mu \tag{3.16}$$

where θ is a small but fixed number.

(b) When $n = 5$, we have

$$0 < \varphi_\mu(y) \le \frac{C}{1+|y|} \tag{3.17}$$

(c) It holds that

$$V_\mu(|x|) \le C_1\varepsilon^{-2}\mu^{-\frac{n+2}{2}}, \qquad \textit{for } |x| > \delta. \tag{3.18}$$

Next we define two Sobolev norms. (See [17] and [30].) Let

$$\|\phi\|_* = \sup_{y\in\mathbb{R}^n}(1+|y|)^{(n-2)/2+\delta}|\phi(y)|, \ \|f\|_{**} = \sup_{y\in\mathbb{R}^n}((1+|y|)^{(n+2)/2+\delta}|f(y)|). \tag{3.19}$$

Finally we state the following important lemma on the error estimates. The proof of it is delayed to Appendix B.

Lemma 3.3. *(a) It holds that*

$$\|\mathcal{S}[W]\|_{**} \le C\mu^{-\frac{\beta_n+2\delta}{2}} \tag{3.20}$$

where $\beta_n = 1$ when $n = 3$, and $\beta_n = 2$ when $n \ge 4$, and

$$J_\varepsilon(W_1) = (n-2)\int_{\mathbb{R}^n} U_{0,1}^{\frac{2n}{n-2}} + \begin{cases} \frac{A_3+o(1)}{\mu\varepsilon}, & \textit{when } n = 3, \\ \frac{(A_4+o(1))\log\mu}{\mu^2\varepsilon^2}, & \textit{when } n = 4, \\ \frac{A_5+o(1)}{\mu^2\varepsilon^2}, & \textit{when } n = 5, \end{cases} \tag{3.21}$$

where A_i is some positive constant, $i = 3, 4, 5$.

(b) For $n = 3$, we have

$$J_\varepsilon[W_1 + W_2] = J_\varepsilon[W_2] + \int_{\mathbb{R}^3} U_{0,1}^6 + (A_3 + o(1))\varepsilon^{-1}\mu^{-1} - 3\varepsilon^{-1/2}\int_{|x|\le\delta} u_\varepsilon V_\mu^5 + O(e^{-c_1/\varepsilon}\mu^{-1}). \tag{3.22}$$

(c) For $n \geq 4$, we obtain

$$J_\varepsilon\big(W_1+W_2\big) = J_\varepsilon(W_2)+(n-2)\int_{R^n} U_{0,1}^{2n/(n-2)} + \frac{(A_n+o(1))(\log\mu)^m}{\mu^2\varepsilon^2} - n(n-2)\varepsilon^{-(n-2)/2}\int_{|x|\leq\delta} u_\varepsilon V_\mu^{\frac{n+2}{n-2}} + O(e^{-c_1/\varepsilon}\mu^{-2}) \tag{3.23}$$

where $m=1$ for $n=4$ and $m=0$ for $n=5$.

Remark: A scaling argument shows that $J_\varepsilon[W_2]$ is independent of μ: in fact,

$$J_\varepsilon[W_2] = \varepsilon^{-n}\left[\frac{\varepsilon^2}{2}\int_{\mathbb{R}^n}|\nabla u_\varepsilon|^2 + \frac{1}{2}\int_{\mathbb{R}^n} V u_\varepsilon^2 - \frac{(n-2)^2}{2}\int_{\mathbb{R}^n} u_\varepsilon^{\frac{2n}{n-2}}\right] \tag{3.24}$$

4. Finite-dimensional Reduction

In this section, we perform a finite-dimensional reduction procedure which is similar to that of [29] and [30].

Let $z=\frac{n-2}{2}U_{0,1}+y\nabla U_{0,1}$. Let $\eta(r)$ be a smooth cut-off function such that $\eta(r)=1$ for $r<\frac{1}{20}r_0\mu$ and $\eta(r)=0$ for $r>\frac{1}{10}r_0\mu$, $|D\eta|\leq C\mu^{-1}$, and $|D^2\eta|\leq C\mu^{-2}$. Set $Z=\Delta(z\eta)$. We first consider the following linear problem: Given $h=h(r)$, find a pair (ϕ,c) satisfying

(4.1)

$$\Delta\phi-(\mu\varepsilon)^{-2}V\big(\frac{|y|}{\mu}\big)\phi+f'(W)\phi=h+cZ,\ \text{in }\mathbb{R}^n,\ \phi=\phi(|y|),\ \phi\in H^1(\mathbb{R}^n),\ \int_{\mathbb{R}^n}\phi Z=0.$$

We have the following a priori estimates.

Lemma 4.1. *Let (ϕ,c) satisfy (4.1). Then for ε sufficiently small, there holds $\|\phi\|_*\leq C\varepsilon^{-\beta}\|h\|_{**}$, where $\beta>0$ is given in Corollary 2.3.*

Proof: Our main idea is to decompose problem (4.1) into two problems: one in the bubble region and another one in the layer region. The constant $C\varepsilon^{-\beta}$ comes from consideration in the layer region.

We argue by contradiction. Suppose there exists a sequence of $\mu_k,\varepsilon_k,\phi_k,h_k$ such that $\|\phi_k\|_*=1$, $\|h_k\|_{**}=o(\varepsilon^\beta)$. For the sake of simplicity, we omit the dependence of k.

We first estimate the constant c in (4.1). Multiplying (4.1) by $z\eta$ and integrating by parts, we obtain

$$c\int_{\mathbb{R}^n} zZ\eta = \int_{\mathbb{R}^n}\big(\Delta\phi+f'(W)\phi\big)z\eta-(\mu\varepsilon)^{-2}\int_{\mathbb{R}^n} V\big(\frac{|y|}{\mu}\big)\phi z\eta+O(\|h\|_{**})=o(\varepsilon^\beta), \tag{4.2}$$

since

$$\int_{\mathbb{R}^n} \big(\Delta\phi + f'(W)\phi\big) z\eta = \int_{\mathbb{R}^n} \big(\Delta(z\eta) + f'(W)z\eta\big)\phi$$
$$= \int_{\mathbb{R}^n} \Big(z\Delta\eta + 2DzD\eta + \big(f'(W) - f'(U_{0,1})\big)z\eta\Big)\phi = o(\varepsilon^\beta).$$

But

$$\int_{\mathbb{R}^n} zZ\eta = \int_{\mathbb{R}^n} |\nabla z|^2 + o(1).$$

So, we obtain

$$c = o(\varepsilon^\beta).$$

Let ψ_1 be such that $\Delta\psi_1 - \varepsilon^{-2}V(r)\psi_1 = 0$ in $B_{\frac{3}{10}r_0}$ and $\psi_1(r) = 1$ on $\partial B_{\frac{3}{10}r_0}$. We consider the new function $\phi_1 = \phi - d\psi_1(\mu^{-1}|y|)$ where $d = \phi(\frac{3r_0\mu}{10})$. Since $\|\phi\|_* = 1$, we derive that $|d| = O(\mu^{-(n-2)/2-\delta})$. Then ϕ_1 satisfies

$$\Delta\phi_1 - (\mu\varepsilon)^{-2}V\big(\frac{|y|}{\mu}\big)\phi_1 + f'(W_1)\phi_1 = \tilde{g} + cZ, \int_{\mathbb{R}^n} \phi_1 Z = d\int_{\mathbb{R}^n} \psi_1 Z, \phi_1(\frac{3}{10}r_0\mu) = 0, \tag{4.3}$$

where $\tilde{g}$ satisfies

$$\tilde{g} = h + df'(W)\psi_1 + (f'(W_1) - f'(W))\phi_1. \tag{4.4}$$

Note that for $|y| \le \frac{3}{10}r_0\mu$,

$$|f'(W_1) - f'(W)||\phi_1| \le C\|\phi_1\|_*(1+|y|)^{-(\frac{n-2}{2}+\delta)}(CW_2^{\frac{4}{n-2}} + f''(W_1)W_2)$$
$$\le C\|\phi_1\|_*(1+|y|)^{-(\frac{n-2}{2}+\delta)}\Big((\mu\varepsilon)^{-2}u_\varepsilon^{4/(n-2)}\big(\frac{|y|}{\mu}\big) + (\mu\varepsilon)^{-(n-2)/2}u_\varepsilon\big(\frac{|y|}{\mu}\big)\frac{1}{(1+|y|)^{6-n}}\Big)$$
$$\le Ce^{-\delta/\varepsilon}(1+|y|)^{-((n+2)/2+\delta)}\Big(1 + \frac{1}{\mu^{(n-2)/2}(1+|y|)^{4-n}}\Big) \le Ce^{-\delta/\varepsilon}(1+|y|)^{-((n+2)/2+\delta)}.$$

Similarly, we can show that for $|y| \le \frac{3}{10}r_0\mu$,

$$|d||W^{\frac{4}{n-2}}||\psi_1(\mu^{-1}y)| \le Ce^{-\delta/\varepsilon}(1+|y|)^{-((n+2)/2+\delta)}.$$

Moreover, since

$$|Z| = |\Delta z\eta + 2DzD\eta + \Delta\eta z| \le C(1+|y|)^{-n} + C\mu^{-1}(1+|y|)^{n-1} + C\mu^{-2}(1+|y|)^{n-2},$$

we obtain that for $|y| \le \frac{3}{10} r_0 \mu$,

$$|cZ| = o(\varepsilon^\beta)(1+|y|)^{-n}.$$

Thus we have

$$\sup_{|y|\le \frac{3}{10}\mu r_0} (1+|y|)^{(n+2)/2+\delta}\big(|\tilde g| + c|Z|\big) \le C\varepsilon^\beta. \tag{4.5}$$

Observe also that the Green's function of $\Delta G - (\mu\varepsilon)^{-2} V(\frac{|y|}{\mu})G + \delta_y = 0$ in $B_{\frac{3}{10}\mu r_0}$, $G = 0$ on $\partial B_{\frac{3}{10}\mu r_0}$ has the following decay property:

$$G(x,y) \le \frac{C}{|x-y|^{n-2}}. \tag{4.6}$$

Then as in the proof of Proposition 3.1 of [29], we have that

$$\sup_{|y|\le \frac{3}{10}\mu r_0} (1+|y|)^{(n-2)/2+\delta}|\phi_1| \le C\varepsilon^\beta, \tag{4.7}$$

which implies that

$$\sup_{|y|\le \frac{1}{5}\mu r_0} (1+|y|)^{(n-2)/2+\delta}|\phi| \le C\varepsilon^\beta + \sup_{|y|\le \frac{1}{5}\mu r_0} (1+|y|)^{(n-2)/2+\delta}|d||\psi_1(\mu^{-1}y)| \le C\varepsilon^\beta, \tag{4.8}$$

since $|d| \le C\mu^{-(n-2)/2-\delta}$, and $|\psi_1(y)| \le e^{-\delta/\varepsilon}$ if $|y| \le \frac{1}{5} r_0$.

Similarly, we let ψ_2 be such that $\Delta\psi_2 - \varepsilon^{-2}V(r)\psi_2 = 0, r > \frac{1}{10}r_0, \psi_2(\frac{1}{10}r_0) = 1$, and consider the function $\phi_2 = \phi - d_2\psi_2(\mu^{-1}y)$ with $d_2 = \phi\big(\frac{1}{10}r_0\mu\big) \sim \mu^{-(n-2)/2-\delta}$. Then in the x coordinate, we have

$$\varepsilon^2\Delta\phi_2 - V(|x|)\phi_2 + f'(u_\varepsilon)\phi_2 = \tilde g_2 \tag{4.9}$$

where

$$\tilde g_2(|x|) = (\mu\varepsilon)^2 h(\mu|x|) + d_2 f'(u_\varepsilon)\psi_2 + (f'(\varepsilon^{(n-2)/2}V_\mu(|x|) + u_\varepsilon) - f'(u_\varepsilon))\phi.$$

For $|y| \ge \frac{1}{10} r_0\mu$, or $|x| \ge \frac{1}{10}r_0$, we have

$$|(\mu\varepsilon)^2 h(\mu|x|)| = o(\varepsilon^\beta)(\mu\varepsilon)^2\mu^{-(n+2)/2-\delta} = o(\varepsilon^{2+\beta})\mu^{-(n-2)/2-\delta},$$

$$|d_2 f'(u_\varepsilon)\psi_2| \le C|u_\varepsilon|^{4/(n-2)}|\psi_2|\mu^{-(n-2)/2-\delta} \le Ce^{-\delta/\varepsilon}\mu^{-(n-2)/2-\delta},$$

and

$$|(f'(\varepsilon^{(n-2)/2}V_\mu(|x|) + u_\varepsilon) - f'(u_\varepsilon))\phi|$$

$$\leq C\mu^{-(n-2)/2-\delta}\big(|\varepsilon^{(n-2)/2}V_\mu(|x|)|^{4/(n-2)}+|f''(u_\varepsilon)||\varepsilon^{(n-2)/2}V_\mu(|x|)|\big)$$
$$\leq C\mu^{-(n-2)/2-\delta}(\varepsilon^2\mu^{-2}+\varepsilon^{(n-2)/2}\mu^{-(n-2)/2})=o(\varepsilon^{2+\beta})\mu^{-(n-2)/2-\delta}.$$

Thus, we have proved that

$$\|\tilde g_2\|_{L^\infty(\mathbb{R}^n\setminus B_{\frac{1}{10}r_0})}\leq o(\varepsilon^{2+\beta})\mu^{-(n-2)/2-\delta}.$$

By Corollary 2.3, we have that for $|x|\geq \frac{1}{10}r_0$,

$$|\phi_2(|x|)|\leq C\varepsilon^{-\beta}\|\tilde g_2\|_{L^\infty(\mathbb{R}^n\setminus B_{\frac{1}{10}r_0})}\leq o(1)\mu^{-(n-2)/2-\delta}. \tag{4.10}$$

Thus

$$\sup_{\frac{1}{5}\mu r_0\leq|y|\leq 10\mu r_0}|\phi|(1+|y|)^{(n-2)/2+\delta}\leq C\mu^{(n-2)/2+\delta}\sup_{\frac{1}{5}\mu r_0\leq|y|}\big(|\phi_2|+d_2|\psi_2(\mu^{-1}|y|)\big)\leq o(1). \tag{4.11}$$

Finally, we estimate $\sup_{|y|\geq 10\mu r_0}|\phi|(1+|y|)^{(n-2)/2+\delta}$.

Recall the ϕ satisfies

$$\Delta\phi-(\mu\varepsilon)^{-2}V\big(\frac{|y|}{\mu}\big)\phi+f'(W)\phi=h,\quad |y|\geq 10\mu r_0.$$

We have

$$|h(|y|)|=o(\varepsilon^\beta)(1+|y|)^{-(n+2)/2-\delta}.$$

On the other hand, for $|y|\geq 10\mu r_0$,

$$|f'(W)|\leq C(\varepsilon\mu)^{-2}u_\varepsilon^{4/(n-2)}\big(\frac{|y|}{\mu}\big)+C\mu^{-2}V_\mu^{4/(n-2)}\big(\frac{|y|}{\mu}\big)=o(\mu^{-2}\varepsilon^{-2}).$$

Thus, $|\phi|$ is a subsolution of

$$\Delta v-\frac12(\mu\varepsilon)^{-2}V\big(\frac{|y|}{\mu}\big)v=o(\varepsilon^\beta)(1+|y|)^{-(n+2)/2-\delta}.$$

But

$$\Delta\frac{1}{|y|^{(n-2)/2+\delta}}-\frac12(\mu\varepsilon)^{-2}V\big(\frac{|y|}{\mu}\big)\frac{1}{|y|^{(n-2)/2+\delta}}\leq -c_0\frac{1}{|y|^{(n+2)/2+\delta}}.$$

Noting that $|\phi(|y|)|\leq o(1)\mu^{-(n-2)/2-\delta}$ if $|y|=10r_0\mu$, by the comparison theorem, we obtain

$$|\phi(|y|)| \le o(1)\frac{1}{|y|^{(n-2)/2+\delta}}, \quad |y| \ge 10r_0\mu,$$

which implies

$$\sup_{|y|\ge 10\mu r_0} |y|^{(n-2)/2+\delta}|\phi(|y|)| = o(1). \tag{4.12}$$

Combining (4.8), (4.11) and (4.12), we obtain $\|\phi\|_* = o(1)$. This is a contradiction. $\square$

By Lemma 3.3, Lemma 4.1 and a contraction mapping principle similar to those of Proposition 3.2 of [29], we derive the following main reduction lemma:

Lemma 4.2. *There exists $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$, the following problem*

$$\Delta(W+\phi) - (\mu\varepsilon)^{-2}V(\frac{|y|}{\mu})(W+\phi) + f(W+\phi) = c_\mu Z \text{ in } \mathbb{R}^n, \quad \phi \in H^1(\mathbb{R}^n), \int_{\mathbb{R}^n} \phi Z = 0 \tag{4.13}$$

has a unique solution (ϕ_μ, c_μ). Moreover the map $\mu \to \phi_\mu$ is C^1 and the following estimates hold

$$\|\phi_\mu\|_* \le C\mu^{-\frac{\beta_n+2\delta}{2}}, \tag{4.14}$$

where $\delta > 0$ is some fixed constant.

Now we define

$$M(\mu) = J_\varepsilon[W+\phi_\mu] - J_\varepsilon[W_2] - (n-2)\int_{\mathbb{R}^n} U_{0,1}^{\frac{2n}{n-2}} \tag{4.15}$$

Note that by the remark following Lemma 3.3, $J_\varepsilon[W_2]$ is independent of μ. Then we also have (see Proposition 3.3 of [29])

Lemma 4.3. *If $\mu = \mu_\varepsilon$ is a critical point of $M(\mu)$ in μ, then $w_\varepsilon = W + \phi_{\mu_\varepsilon}$ is a solution to (3.4).*

Thus we are reduced to finding a critical point of $M(\mu)$.

5. Proof of Theorem 1.1

We first prove Theorem 1.1 when $n = 3$. To this end, we need to expand $M(\mu)$: using Lemma 3.3 and Lemma 4.2, we deduce that

$$\begin{aligned} M(\mu) &= J_\varepsilon[W + \phi_\mu] - J_\varepsilon[W_2] - \int_{\mathbb{R}^3} U_{0,1}^6 \\ &= J_\varepsilon[W] + \int_{\mathbb{R}^n} S_\mu[W]\phi_\mu + O(\|\phi_\mu\|_*^2) - J_\varepsilon[W_2] - (n-2)\int_{\mathbb{R}^n} U_{0,1}^{\frac{2n}{n-2}}. \end{aligned}$$

We use (3.22) to derive that

$$M(\mu) = \big(A_3 + o(1)\big)\varepsilon^{-1}\mu^{-1} - 3\varepsilon^{-1/2}\int_{|x|<\delta} u_\varepsilon V_\mu^5 + O\big(\mu^{-(1+2\delta)} + e^{-c_1/\varepsilon}\mu^{-1}\big) \tag{5.1}$$

By Lemma 3.1, we obtain

$$e^{-(\rho(r_0)+\delta)/\varepsilon}\mu^{-1/2} \le \int_{|x|<\delta} u_\varepsilon V_\mu^5 \le e^{-(\rho(r_0)-\delta)/\varepsilon}\mu^{-1/2}. \tag{5.2}$$

Thus the function $M(\mu)$ attains its minimum at $\mu_\varepsilon \sim e^{\frac{2\rho(r_0)+o(1)}{\varepsilon}} \in \Lambda$. This minimum point must be a critical point of $M(\mu)$ since it stays inside the interior of Λ.

When $n = 4$ or 5, we have

$$M(\mu) = \frac{(A_n + o(1))(\log\mu)^m}{\mu^2\varepsilon^2} - n(n-2)\int_{|x|\le\delta} u_\varepsilon V_\mu^{\frac{n+2}{n-2}} + O\big(e^{-c_1/\varepsilon}\mu^{-2}\big) \tag{5.3}$$

where

$$C_1 e^{-(\rho(r_0)+\delta)/\varepsilon}\mu^{-(n-2)/2} \le \int_{|x|\le\delta} u_\varepsilon V_\mu^{\frac{n+2}{n-2}} \le C_2 e^{-(\rho(r_0)-\delta)/\varepsilon}\mu^{-(n-2)/2} \tag{5.4}$$

Using (5.4), we find that the function $M(\mu)$ has a critical point $\bar{\mu}_\varepsilon \in [e^{\frac{2\rho(r_0)-\delta}{(6-n)\varepsilon}}, e^{\frac{2\rho(r_0)+\delta}{(6-n)\varepsilon}}] \subset \Lambda$. Thus, the reduced energy functional $M(\mu)$ also has a critical point $\mu_\varepsilon \sim e^{\frac{2+O(\delta)}{(6-n)\varepsilon}}$.

By Lemma 4.3, $W_1 + W_2 + \phi_{\mu_\varepsilon}$ is a solution to (3.4). Let $u_\varepsilon(x) = W_1(\mu_\varepsilon r) + W_2(\mu_\varepsilon r) + \phi_{\mu_\varepsilon}(\mu_\varepsilon r)$. Then u_ε satisfies all the properties of Theorem 1.1.

□

Appendix A: Proof of Lemma 3.2

Proof of (3.16):

When $n = 4$, we study φ_μ in $\{|y| \le \varepsilon\mu\}$. To this end, we let $\varphi_4(|y|)$ be the unique solution of

$$\Delta\varphi_4 + U_{0,1} = 0 \ \text{ for } \ |y| < \varepsilon\mu, \ \varphi_4(\mu) = \varphi_\mu(\varepsilon\mu). \tag{5.5}$$

Note that from (3.8) it holds that

$$0 \le \varphi_\mu(\varepsilon\mu) \le (\mu\varepsilon)^2 U_{0,1}(\varepsilon\mu) \le C.$$

Let $\xi(|y|) = \frac{1}{4}\ln(\varepsilon\mu)^2 - \frac{1}{4}\ln(1+|y|^2)$. Direct calculations show that

$$\Delta\xi + U_{0,1} = -\frac{1}{(1+|y|^2)^2}.$$

Let ψ be the unique solution of

$$\Delta\psi = -\frac{1}{(1+|y|^2)^2} \quad \text{in } \mathbb{R}^n, \ \psi(y) \to 0 \text{ as } |y| \to +\infty.$$

Then, we have $\psi(|y|) \le C(\log(1+|y|))(1+|y|)^{-2}$. It is easy to check that $\varphi_4 - \xi - \psi$ is a harmonic function in $B_{\varepsilon\mu}$, which is bounded on $|y| = \varepsilon\mu$. So we have proved that

$$\varphi_4(y) = \frac{1}{4}\ln(\varepsilon\mu)^2 - \frac{1}{4}\ln(1+|y|^2) + O(1), \quad |y| \le \varepsilon\mu.$$

Since φ_μ satisfies

$$\Delta\varphi_\mu + CU_{0,1} \ge 0,$$

where $C > 0$ is some constant, by the comparison theorem, we obtain

$$\varphi_\mu(|y|) \le C\varphi_4(|y|), \quad |y| \le \varepsilon\mu. \tag{5.6}$$

Next, we show that if we choose $\alpha > 0$ small enough, $\alpha\varphi_4$ is a sub-solution to equation (3.15) on the ball $\{|y| < \theta\varepsilon\mu\}$, where $\theta > 0$ is a small constant. In fact, we have

$$\begin{aligned} &\alpha\Delta\varphi_4 - (\mu\varepsilon)^{-2} V(\frac{|y|}{\mu})\alpha\varphi_4 + V(\frac{|y|}{\mu})\frac{1}{1+|y|^2} \\ &\ge (c_0 - \alpha)U_{0,1} - (\mu\varepsilon)^{-2} C\alpha\varphi_4 \\ &\ge (\mu\varepsilon)^{-2}\Big(\frac{(c_0-\alpha)\varepsilon^2\mu^2}{1+|y|^2} - C\alpha\big(\frac{1}{4}\ln(\varepsilon\mu)^2 - \frac{1}{4}\ln(1+|y|^2) + C'\big)\Big), \end{aligned} \tag{5.7}$$

where $c_0 > 0$, $C > 0$ and $C' > 0$ are some constants.

Denote

$$g(t) = \frac{(c_0-\alpha)(\varepsilon\mu)^2}{1+t^2} - C\alpha\big(\frac{1}{4}\ln(\varepsilon\mu)^2 - \frac{1}{4}\ln(1+t^2) + C'\big).$$

Then, $g'(t) < 0$ in $(0, t_\mu)$ with $t_\mu = \sqrt{\frac{(c_0-\alpha)}{C\alpha}}\varepsilon\mu\big(1+o(1)\big)$. But if $\theta > 0$ is small, then

$$g\big(\theta\varepsilon\mu\big) \geq \frac{c_0-\alpha}{\theta^2} + C\alpha\ln\theta - C'' > 0.$$

So, $g(t) > 0$ if $t \in [0, \theta\varepsilon\mu]$.

Using (5.7), we see that in if $|y| \leq \theta\varepsilon\mu$,

$$\alpha\Delta\varphi_4 - (\mu\varepsilon)^{-2}V(\frac{|y|}{\mu})\alpha\varphi_4 + V(\frac{|x|}{\mu})(\frac{1}{1+|y|^2})^{\frac{n-2}{2}} > 0.$$

So, $\alpha\varphi_4$ is a sub-solution of (3.15).

Since $\varphi_4(\theta\varepsilon\mu) = -\frac{1}{2}\ln\theta + O(1)$, we can find a constant $C_\theta > 0$, such that $\alpha\varphi_4(\theta\varepsilon\mu) \leq C_\theta$. As a result, we deduce that

$$\varphi_\mu(\theta\varepsilon\mu) \geq 0 \geq \alpha\varphi_4(\theta\varepsilon\mu) - C_\theta.$$

But $\alpha\varphi_4 - C_\theta$ is also a sub-solution of (3.15). By the comparison theorem, we obtain

$$\varphi_\mu(|y|) \geq \alpha\varphi_4(|y|) - C_\theta, \quad |y| \leq \theta\varepsilon\mu. \tag{5.8}$$

Combining (5.6) and (5.8), we obtain (3.16).

□

Proof of (3.17): When $n = 5$, similar to estimates in Lemma A.1 of [30], we obtain (3.17). Furthermore, $\varphi_\mu(y) \to \varphi_5$ where φ_5 is the unique solution satisfying

$$\Delta\varphi_5 + V(0)U_{0,1} = 0 \text{ in } \mathbb{R}^n,\ \varphi_5 \to 0 \text{ as } |y| \to +\infty.$$

□

Proof of (3.18): To prove (3.18), we first let

$$\xi_\mu(|x|) = V_\mu(|x|) - U_{0,\mu}(|x|) + \frac{1}{\mu^{(n-2)/2}|x|^{n-2}}.$$

Then, ξ_μ satisfies

$$\Delta\xi_\mu - \varepsilon^{-2}V(|x|)\xi_\mu + \varepsilon^{-2}V(|x|)\Big(U_{0,\mu}(|x|) - \frac{1}{\mu^{(n-2)/2}|x|^{n-2}}\Big) = 0.$$

For $|x| \ge \delta > 0$,

$$U_{0,\mu}(|x|) - \frac{1}{\mu^{(n-2)/2}|x|^{n-2}} = O\Big(\frac{\mu^{(n-2)/2}}{(\mu|x|)^n}\Big).$$

Using the equation that $\bar\xi(|y|) := \xi_\varepsilon\big(\mu^{-1}|y|\big)$ satisfies, we can prove

$$|\xi(|x|)| \le \frac{C\mu^{(n-2)/2}}{\varepsilon^2\mu^2(1+\mu|x|)^{n-2}}.$$

So, (3.18) follows.

$\square$

Appendix B: Proof of Lemma 3.3

Proof of (3.20): It is easy to see that

(5.9)
$$|\mathcal{S}[W]| = n(n-2)\left|(W_1+W_2)_+^{\frac{n+2}{n-2}} - W_2^{\frac{n+2}{n-2}} - U_{0,1}^{\frac{n+2}{n-2}}\right| \le CW_1^{\frac{4}{n-2}}W_2 + CW_2^{\frac{4}{n-2}}W_1 + CU_{0,1}^{\frac{4}{n-2}}|W_1 - U_{0,1}|.$$

The most difficult term in (5.9) is $W_2^{\frac{4}{n-2}}W_1$. When $n=3$ and $|y| < \delta\mu$, i.e., $|x| < \delta$, we have

$$W_2^4W_1 \le (\varepsilon\mu)^{-2}W_1u_\varepsilon^4 \le (\varepsilon\mu)^{-2}(1+|y|)^{-1}e^{-4(\rho(r_0)-2\delta)/\varepsilon} \le \mu^{-3/2+3\delta}(1+|y|)^{-5/2},$$

since $\mu^{-2} \sim e^{-4\rho(r_0)/\varepsilon}$.

For $2r_0\mu \ge |y| \ge \delta\mu$, we use (3.18) to obtain

$$W_2^4W_1 \le (\varepsilon\mu)^{-2}W_1u_\varepsilon^4 \le C(\varepsilon\mu)^{-2}\varepsilon^{-2}\mu^{-5/2} \le C\mu^{-1}(1+|y|)^{-5/2}.$$

For $|y| \geq 2r_0\mu$,

$$u_\varepsilon^4(|x|) \leq Ce^{-c'|x|/\varepsilon} \leq C|x|^{-5/2} \leq C\mu^{5/2}(1+|y|)^{-5/2}.$$

Using (3.18) again, we obtain that for $|y| \geq 2r_0\mu$,

$$W_2^4 W_1 \leq (\varepsilon\mu)^{-2} W_1 u_\varepsilon^4 \leq C(\varepsilon\mu)^{-2}\varepsilon^{-2}\mu^{-5/2}u_\varepsilon^4 \leq \mu^{-1}(1+|y|)^{-5/2}.$$

Thus when $n=3$ we have

$$\|W_2^4 W_1\|_{**} < \mu^{-(1-\delta)}. \tag{5.10}$$

Similarly, when $n \geq 4$, we use (3.13), (3.14) and (3.18) to obtain

$$\|W_2^{\frac{4}{n-2}} W_1\|_{**} < \mu^{-(\beta_n-\delta)}. \tag{5.11}$$

The other terms in (5.9) can be estimated by straightforward computations. We omit the details. $\square$

Proof of (3.21): Observe that

$$J_\varepsilon[W_1] = -\frac{(n-2)^2}{2}\int_{\mathbb{R}^n}(U_{0,1}+W_1-U_{0,1})^{\frac{2n}{n-2}} + \frac{n(n-2)}{2}\int_{\mathbb{R}^n} U_{0,1}^{\frac{n+2}{n-2}}(U_{0,1}+W_1-U_{0,1})$$

By (3.13) when $n=3$

$$\int_{\mathbb{R}^n} U_{0,1}^{\frac{n+2}{n-2}}(W_1-U_{0,1}) = \int_{\mathbb{R}^n} U_{0,1}^5\frac{1}{|y|}(1-\psi_\varepsilon(\frac{|y|}{\mu})) + O(\mu^{-2})$$

$$= -\frac{A_3+o(1)}{\mu\varepsilon}, \tag{5.12}$$

where $A_3 = \sqrt{V'(0)}\int_{\mathbb{R}^3} U_{0,1}^5 > 0$.

When $n=4$, we have

$$J_\varepsilon[W_1] = -\frac{(n-2)^2}{2}\int_{\mathbb{R}^n} W_1^{\frac{2n}{n-2}} + \frac{n(n-2)}{2}\int_{\mathbb{R}^n} U_{0,1}^{\frac{n+2}{n-2}} W_1$$

$$= -\frac{(n-2)^2}{2}\int_{\mathbb{R}^n\cap\{|y|\leq\theta\varepsilon\mu\}} W_1^{\frac{2n}{n-2}} + \frac{n(n-2)}{2}\int_{\mathbb{R}^n\cap\{|y|\leq\theta\varepsilon\mu\}} U_{0,1}^{\frac{n+2}{n-2}} W_1 + O((\varepsilon\mu)^{-4})$$

$$= (n-2)\int_{\mathbb{R}^n} U_{0,1}^{\frac{2n}{n-2}} + (\mu\varepsilon)^{-2}\frac{n(n-2)}{2}\int_{\mathbb{R}^n\cap\{|y|\leq\theta\varepsilon\mu\}} U_{0,1}^{\frac{n+2}{n-2}}\varphi_\mu + O(\mu^{-5/2}).$$

By (3.16), using

$$\int_{\mathbb{R}^n} U_{0,1}^{\frac{n+2}{n-2}}\ln(1+|y|) \leq C,$$

we deduce that

$$C_1 \ln(\varepsilon\mu) \le \int_{\mathbb{R}^n\cap\{|y|\le\theta\varepsilon\mu\}} U_{0,1}^{\frac{n+2}{n-2}}\varphi_\mu \le C_2\ln(\varepsilon\mu). \tag{5.13}$$

Hence we may write

$$\int_{\mathbb{R}^n\cap\{|y|\le\theta\varepsilon\mu\}} U_{0,1}^{\frac{n+2}{n-2}}\varphi_\mu = (A_4+o(1))\ln\mu, \tag{5.14}$$

where A_4 is a constant bounded from below and above.

When $n=5$, we have

$$\begin{aligned} J_\varepsilon[W_1] &= -\frac{(n-2)^2}{2}\int_{\mathbb{R}^n} W_1^{\frac{2n}{n-2}} + \frac{n(n-2)}{2}\int_{\mathbb{R}^n} U_{0,1}^{\frac{n+2}{n-2}}W_1 \\ &= -\frac{(n-2)^2}{2}\int_{\mathbb{R}^n\cap\{|y|\le\mu\}} W_1^{\frac{2n}{n-2}} + \frac{n(n-2)}{2}\int_{\mathbb{R}^n\cap\{|y|\le\mu\}} U_{0,1}^{\frac{n+2}{n-2}}W_1 + O(\mu^{-5/2}) \\ &= (n-2)\int_{\mathbb{R}^n} U_{0,1}^{\frac{2n}{n-2}} + (\mu\varepsilon)^{-2}\frac{n(n-2)}{2}\int_{\mathbb{R}^n} U_{0,1}^{\frac{n+2}{n-2}}\varphi_\mu + O(\mu^{-5/2}), \end{aligned}$$

$$= (n-2)\int_{\mathbb{R}^n} U_{0,1}^{\frac{2n}{n-2}} - (A_5+o(1))(\mu\varepsilon)^{-2} \tag{5.15}$$

where $A_5 = \frac{n(n-2)}{2}\int_{\mathbb{R}^n} U_{0,1}^{\frac{n+2}{n-2}}\varphi_5 > 0$.

So, (3.21) follows from estimates (5.12), (5.14) and (5.15). □

Proof of (3.22) and (3.23): Since u_ε is a solution to (1.7), we have $J_\varepsilon(W_1+W_2) = J_\varepsilon(W_1)+J_\varepsilon(W_2)-I_\varepsilon$, where

$$\begin{aligned} I_\varepsilon &= \tfrac{(n-2)^2}{2}\int_{\mathbb{R}^n}\Big((W_1+W_2)^{\frac{2n}{n-2}} - W_1^{\frac{2n}{n-2}} - W_2^{\frac{2n}{n-2}} - \tfrac{2n}{n-2}W_2^{\frac{n+2}{n-2}}W_1\Big) \\ &= \tfrac{(n-2)^2}{2}\int_{\mathbb{R}^n}\Big((W_1+W_2)^{\frac{2n}{n-2}} - W_1^{\frac{2n}{n-2}} - W_2^{\frac{2n}{n-2}} - \tfrac{2n}{n-2}W_2^{\frac{n+2}{n-2}}W_1 - \tfrac{2n}{n-2}W_2W_1^{\frac{n+2}{n-2}}\Big) \\ &\quad + n(n-2)\int_{\mathbb{R}^n} W_2W_1^{\frac{n+2}{n-2}} \\ &=: I_{\varepsilon,1}+I_{\varepsilon,2} \end{aligned}$$

where

$$I_{\varepsilon,2} = n(n-2)\int_{\mathbb{R}^n} W_2W_1^{\frac{n+2}{n-2}} = n(n-2)\varepsilon^{-(n-2)/2}\int_{\mathbb{R}^n} V_\mu^{\frac{n+2}{n-2}}u_\varepsilon$$

$$= n(n-2)\varepsilon^{-(n-2)/2}\int_{|x|\le\delta} u_\varepsilon V_\mu^{\frac{n+2}{n-2}} + O\Big(\varepsilon^{-(n-2)/2}\varepsilon\mu^{-\frac{n+2}{2}}\Big). \tag{5.16}$$

We use (3.18) to estimate $I_{\varepsilon,1}$ as follows

$$|I_{\varepsilon,1}| \leq C \int_{\mathbb{R}^n} (W_2^{\frac{4}{n-2}} W_1^2 + W_1^{\frac{4}{n-2}} W_2^2), \tag{5.17}$$

where

$$\int_{\mathbb{R}^n} W_2^{\frac{4}{n-2}} W_1^2 = \varepsilon^{-2} \int_{\mathbb{R}^n} V_\mu^2 u_\varepsilon^{\frac{4}{n-2}} = \varepsilon^{-2} \int_{|x|<2\delta} V_\mu^2 u_\varepsilon^{\frac{4}{n-2}} + \varepsilon^{-2} \int_{|x|>2\delta} V_\mu^2 u_\varepsilon^{\frac{4}{n-2}}$$

$$\leq Ce^{-c_1/\varepsilon} \mu^{-2} + C\varepsilon^{-3} \mu^{-(n+2)}. \tag{5.18}$$

Similarly, if $n = 3$,

$$\int_{\mathbb{R}^n} W_1^{\frac{4}{n-2}} W_2^2 = \varepsilon^{-(n-2)} \int_{\mathbb{R}^n} V_\mu^{\frac{4}{n-2}} u_\varepsilon^2 = \varepsilon^{-(n-2)} \int_{|x|<2\delta} V_\mu^{\frac{4}{n-2}} u_\varepsilon^2 + \varepsilon^{-(n-2)} \int_{|x|>2\delta} V_\mu^{\frac{4}{n-2}} u_\varepsilon^2$$

$$\leq Ce^{-c_1/\varepsilon} \mu^{-1}. \tag{5.19}$$

If $n = 4, 5$,

$$\int_{\mathbb{R}^n} W_1^{\frac{4}{n-2}} W_2^2 \leq Ce^{-c_1/\varepsilon} \mu^{-2}. \tag{5.20}$$

Substituting (5.16)–(5.20) into I_ε, we obtain (3.22) and (3.23). $\square$

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