COUNTEREXAMPLES TO LIN-NI’S CONJECTURE

LIPING WANG, JUNCHENG WEI, AND SHUSEN YAN

ABSTRACT. We consider the following nonlinear problem
\[
\begin{cases}
-\Delta u + \mu u = u^{\frac{N+2}{N-2}}, & u > 0 \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \mathbf{n}} = 0, & \text{on } \partial \Omega,
\end{cases}
\]
where \( \Omega \subset \mathbb{R}^N \) is a smooth and bounded domain, \( \mu > 0 \) and \( n \) denotes the outward unit normal vector. Lin and Ni ([31]) conjectured that for \( \mu \) small, all solutions are constants. We show that this conjecture is false for all dimensions in some (partially symmetric) nonconvex domains \( \Omega \).

1. Introduction

In this paper, we consider the nonlinear Neumann elliptic problem
\[
\begin{cases}
-\Delta u + \mu u - u^q = 0, & u > 0 \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \mathbf{n}} = 0, & \text{on } \partial \Omega,
\end{cases}
\]
where \( 1 < q < +\infty, \mu > 0, n \) denotes the outward unit normal vector and \( \Omega \) is a smooth and bounded domain in \( \mathbb{R}^N, N \geq 3 \).

Equation (1.1) arises in many branches of the applied science. For example, it can be viewed as a steady-state equation for the shadow system of the Gierer-Meinhardt system in biology pattern formation [20], [37], or for parabolic equations in chemotaxis, e.g. Keller-Segel model [32].

When \( q \) is subcritical, i.e. \( q < \frac{N+2}{N-2} \), Lin, Ni and Takagi [32] proved that the only solution, for small \( \mu \), is the constant one, whereas nonconstant solutions appear for large \( \mu \) [32] which blow up, as \( \mu \) goes to infinity, at one or several points. The least energy solution blows up at a boundary point which maximizes the mean curvature of the boundary [39], [40]. Higher energy solutions exist which blow up at one or several points, located on the boundary [14], [23], [29], [48], [27], in the interior of the domain [8], [13], or some of them on the boundary and others in the interior [25]. (A good review can be found in [37].)

In the critical case, for large \( \mu \), nonconstant solutions exist [1], [47]. As in the subcritical
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case the least energy solution blows up, as $\mu$ goes to infinity, at a unique point which maximizes the mean curvature of the boundary[3], [36]. Higher energy solutions have also been exhibited, blowing up at one [2], [48], [42], [22] or several separated boundary points[34], [31], [49], [50]. The question of interior blow-up is still open. However, in contrast with subcritical situation, at least one blow-up point has to lie on the boundary [35], [43]. Some priori estimates for those solutions are given in [22], [28].

As we mentioned above that in the case of small $\mu$, Lin, Ni and Takagi proved in the subcritical case that problem (1.1) admits only the trivial solution(i.e. $u \equiv \mu^{\frac{1}{p-1}}$). Based on this, Lin and Ni [31] asked:

**Lin-Ni’s Conjecture:** For $\mu$ small and $q = \frac{N+2}{N-2}$, problem (1.1) admits only the constant solution.

The above conjecture was studied by Adimurthi-Yadava [4], [5] and Budd-Knapp-Peletier [10] in the case $\Omega = B_R(0)$ and $u$ radial. Namely, they considered the following problem:

$$(1.2) \quad \begin{cases} 
\Delta u - \mu u + u^{\frac{N+2}{N-2}} = 0 \quad \text{in} \quad B_R(0), \\
u \text{ is radial, } \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial B_R(0).
\end{cases}$$

The following results were proved:

**Theorem A** ([4], [5], [6], [10]). For $\mu$ sufficiently small,

1. if $N = 3$ or $N \geq 7$, problem (1.2) admits only the constant solution;
2. if $N = 4, 5$ or $6$, problem (1.2) admits a nonconstant solution.

Theorem A reveals that Lin-Ni’s conjecture depends very sensitively on the dimension $N$. A natural question is: what about general dimensions? The proofs of Theorem A use radial symmetry to reduce the problem to an ODE boundary value problem. Consequently, they do not carry over to general domains. In the general three-dimensional domain case, M. Zhu [58] proved:

**Theorem B** ([58] [57]): The conjecture is true if $N = 3$ ($q = 5$) and $\Omega$ is convex.

Zhu’s proof relied strongly on a priori estimates while Wei and Xu [57] gave a direct proof of Theorem B, using only integration by parts.
In the case of $N = 5, q = \frac{7}{6},$ Rey and Wei [46] have proved that for any smooth bounded
domain $\Omega,$ problem (1.1) admits arbitrary numbers of positive solutions which blow up at
$K$ interior points for any $K \in N^*.$ Thus Lin-Ni’s conjecture is false in dimension five.

When $N \geq 7,$ O. Druet, Robert and Wei [16] proved the following result:

**Theorem C:** Suppose that $N \geq 7$ and $H(x) \neq 0$ for all $x \in \partial \Omega.$ Assume that there exists
$C > 0$ such that

$$(1.3) \quad \int_{\Omega} u^{\frac{N}{N-2}} \leq C.$$ 

Then for $\mu$ small, $u \equiv \text{constant}.$

The purpose of this paper is to give an **negative answer** to Lin-Ni’s conjecture in
all dimensions for some non-convex domain $\Omega.$ More precisely, we assume that $\Omega$ is a
smooth and bounded domain $\Omega$ satisfying the following properties: Let $y = (y', y'') \in
\mathbb{R}^2 \times \mathbb{R}^{N-2}, r = |y'|,$ then

(H1) $y \in \Omega$ if and only if $(y_1, y_2, y_3, \cdots, -y_i, \cdots, y_N) \in \Omega, \quad \forall \quad i = 3, \ldots, N$;

(H2) $(r \cos \theta, r \sin \theta, y'') \in \Omega$ if $(r, 0, y'') \in \Omega, \quad \forall \quad \theta \in (0, 2\pi)$;

(H3) Let $T := \partial \Omega \cap \{y_3 = \cdots = y_N = 0\}.$ There exists a connected component $\Gamma$ of
$T,$ such that $H(x) \equiv \gamma < 0, \quad \forall \quad x \in \Gamma,$ where $H(x)$ is the mean curvature of $\partial \Omega$
at $x \in \partial \Omega.$

Note that $\Gamma$ is a circle in the plane $y_3 = \cdots = y_N = 0.$ Without loss of generality, we
may assume that $\Gamma = \partial B_1(0) \cap \{y_3 = \cdots = y_N = 0\}. \quad \text{Thus} \quad \gamma = -1.$

For instance, the domains in Figure 1 satisfy (H1), (H2) and (H3). Note that $\Omega$ can be
simply connected.

Another example is the annulus: $\Omega = \{a < |x| < b\} \text{ with } 0 < a < b < +\infty.$

For normalization reason, we consider throughout the paper the equation

$$(1.4) \quad \begin{cases} -\Delta u + \mu u - \alpha_N u^{\frac{N+2}{N-2}} = 0, \quad u > 0 & \text{in } \Omega, \\
\frac{\partial u}{\partial n} = 0, \quad & \text{on } \partial \Omega,
\end{cases}$$

where $\alpha_N = N(N-2).$ The solutions are identical up to the multiplicative constant
$(\alpha_N)^{-\frac{N-2}{N+2}}.$

Our main result in this paper can be stated as follows:
Theorem 1.1. Suppose that $N \geq 3$ and $\Omega$ is a smooth and bounded domain satisfying $(H_1)$, $(H_2)$ and $(H_3)$. Let $\mu$ be any fixed positive number. Then problem (1.4) has infinitely many non-radial positive solutions, whose energy can be made arbitrarily large.

Since $\mu$ is a fixed positive number, we can make $\mu = 1$ by a suitable change of variable. Thus, throughout the paper, we consider

\[
(1.5) \quad \begin{cases} 
-\Delta u + u - \alpha_N u^{\frac{N+2}{N-2}} = 0, & \text{in } \Omega, \\
\frac{\partial u}{\partial n} = 0, & \text{on } \partial \Omega.
\end{cases}
\]

As far as we know, Theorem 1.1 seems to be the first result establishing the existence of infinitely many positive solutions with arbitrarily large energy. This is a new phenomena. For subcritical problems, by a compactness result of Gidas-Spruck [17], the energy of positive solutions remains uniformly bounded. So this kind of phenomena can only happen for critical exponent problems. On the other hand, the existence of infinitely many sign-changing radial solutions for another critical exponent problem with Dirichlet boundary condition has been studied by Cerami-Solimini-Struwe [12] for $N \geq 7$.

Theorem C and Theorem 1.1 suggest the following revised Lin-Ni’s conjecture for $N \geq 7$. 
Revised Lin-Ni Conjecture: Assume that \( N \geq 7 \) and \( H(x) > 0 \) for all \( x \in \partial \Omega \). Then all solutions to (1.4) are constants, for \( \mu \) sufficiently small.

We believe that the symmetric condition in Theorem 1.1 is technical. A more general result, as follows, should be true.

Conjecture: Assume that \( \min_{x \in \partial \Omega} H(x) < 0 \) and that the set \( \{ x \in \partial \Omega | H(x) = \min_{x \in \partial \Omega} H(x) \} \) is a smooth \( l \)-dimensional sub-manifold on \( \partial \Omega \), with \( 1 \leq l \leq N - 2 \). Then there are infinitely many positive solutions to (1.4).

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2. Outline of Proofs

We outline the main idea in the proof of Theorem 1.1. It is well-known that the functions

\[
U_{\lambda,a}(y) = \left( \frac{\lambda}{1 + \lambda^2 |y - a|^2} \right)^{\frac{N-2}{2}}, \quad \lambda > 0, \quad a \in \mathbb{R}^N
\]

are the only solutions to the problem

\[
-\Delta u = \alpha_N u^{\frac{N+2}{N-2}}, \quad u > 0 \quad \text{in} \quad \mathbb{R}^N.
\]

Let us fix a positive integer

\[
k \geq k_0
\]

where \( k_0 \) is large, to be determined later.

Integral estimates (see Appendix A) suggest to make the additional a priori assumption that \( \lambda \) behaves as the following

\[
\lambda = \frac{4}{\Lambda} k^{\frac{N-2}{N-3}} \quad \text{if} \quad N \geq 4,
\]

\[
\lambda \in \left[ C_1 e^{\frac{D_3}{2} k \ln k}, C_2 e^{\frac{D_2}{2} k \ln k} \right] \quad \text{if} \quad N = 3,
\]

where \( \delta \leq \Lambda \leq \frac{1}{\delta}, D_2, D_3 \) are some positive constants in Appendix A.2, \( C_1, C_2, \delta \) are positive and to be determined later.

Fix \( a \in \Gamma \subset \partial \Omega \). We introduce a boundary deformation which strengthens the boundary near \( a \). Without loss of generality, we may assume that \( a = 0 \) and after rotation and translation of the coordinate system we may assume that inward normal to \( \Gamma \) at \( a \) is...
the positive $x_N$-axis. Denote $x' = (x_1, \ldots, x_{N-1})$, $B'(\delta') = \{x \in \mathbb{R}^{N-1} : |x' < \delta'\}$, and $B(a, \delta) = \{x \in \mathbb{R}^N : |x - a| < \delta\}$.

Then, we can find a constant $\delta' > 0$ such that $\Gamma \cap B(a, \delta')$ can be represented by the graph of a smooth function $\rho_a(x') = \frac{1}{2} \sum_{i=1}^{N-1} k_i x_i^2 + O(|x'|^3)$, and

$$\Omega \cap B(a, \delta') = \{(x', x_N) \in B(a, \delta') : x_N > \rho_a(x')\}.$$  

(2.1)

Here $k_i, i = 1, \ldots, N-1$ are the principal curvatures at $a$. Furthermore, the average of the principal curvatures of $\Gamma$ at $a$ is the mean curvature $H(a) = \frac{1}{N-1} \sum_{i=1}^{N-1} k_i = -1$ because of (H$_3$). To avoid clumsy notations we drop the index $a$ in $\rho$.

On $\Gamma \cap B(a, \delta')$, the outward normal vector $n(x)$ is

$$n(x) = \frac{1}{\sqrt{1 + |\nabla \rho|^2}}(\nabla \rho, -1).$$

Let $2^* = \frac{2N}{N-2}$. For $N \geq 4$, using the transformation $u(y) \mapsto \varepsilon^{-\frac{N-2}{2}} u\left(\frac{y}{\varepsilon}\right)$, we find that (1.4) becomes

$$\begin{cases} -\Delta u + \varepsilon^2 u = \alpha_N u^{2^*-1}, & u > 0, \quad \text{in} \quad \Omega_\varepsilon, \\ \frac{\partial u}{\partial n} = 0, & \text{on} \quad \partial \Omega_\varepsilon, \end{cases}$$

(2.2)

where $\varepsilon = k^{-\frac{N-2}{2}}$, $\Omega_\varepsilon = \{y|\varepsilon y \in \Omega\}$.

We define $W_{\Lambda, \xi}$, $\xi = \frac{y}{\varepsilon}$ satisfying

$$\begin{cases} -\Delta W + \varepsilon^2 W = \alpha_N U^{2^*-1}_{\frac{\delta}{\varepsilon}}, & \text{in} \quad \Omega_\varepsilon, \\ \frac{\partial W}{\partial n} = 0, & \text{on} \quad \partial \Omega_\varepsilon. \end{cases}$$

(2.3)

Following [45], we derive the following asymptotic expansion of $W_{\Lambda, \xi}$:

$$W_{\Lambda, \xi}(y) = U_{\frac{\delta}{\varepsilon}}(\xi) - \varphi_{\Lambda, \xi}(y)$$

(2.4)

where

$$\varphi_{\Lambda, \xi}(y) = \varepsilon \Lambda^{\frac{4-N}{2}} \varphi_0\left(\frac{y - \xi}{\Lambda}\right) + O(\varepsilon^2 |\ln \varepsilon|^m)$$

with $m = 1$ for $N = 4$, $m = 0$ for $N \geq 5$ and $\varphi_0$ solving the following linear problem

$$\begin{cases} -\Delta \varphi_0 = 0, & \text{in} \quad \mathbb{R}_+^N = \{(x', x_N), x_N > 0\}, \\ \frac{\partial \varphi_0}{\partial n} = \frac{N-2}{2} \frac{\sum_{i=1}^{N-1} k_i x_i^2}{(1+|x'|^2)^{\frac{N}{2}}}, & \text{on} \quad \partial \mathbb{R}_+^N, \\ \varphi_0(x) \rightarrow 0, & \text{as} \quad |x| \rightarrow +\infty. \end{cases}$$

(2.5)
Furthermore, we have the following upper bound
\begin{equation}
|\varphi_{\lambda, \xi}(y)| \leq \frac{C_\varepsilon |\ln \varepsilon| n}{(1 + |y - \xi|)^{N-3}}, \quad y \in \Omega_\varepsilon
\end{equation}
where \( n = 1 \) for \( N = 4, 5 \) and \( n = 0 \) for \( N \geq 6 \), whence
\begin{equation}
|W_{\lambda, \xi}| + |\partial_\lambda W_{\lambda, \xi}| \leq C(U_{\lambda, \xi})^{1-\beta} \quad \text{in} \quad \Omega_\varepsilon,
\end{equation}
where \( \beta \) is a positive number which can be chosen to be zero as \( N \geq 6 \) and as small as desired as \( N = 4, 5 \). For simplicity we always denote \( \beta \) as any small number which can be made as small as desired.

For \( N = 3 \), we define
\begin{equation}
W_{\lambda, a}(x) = U_{\lambda, a}(x) - \frac{1}{\lambda^{\frac{1}{2}} |x - a|} (1 - e^{-|x - a|})
\end{equation}
as approximation.

Then \( W_{\lambda, a} \) satisfies
\begin{equation}
-\Delta W + W = 3U_0^\delta + \left( U_{\lambda, a} - \frac{1}{\lambda^{\frac{1}{2}} |x - a|} \right) \quad \text{in} \quad \Omega.
\end{equation}

Define
\[
H_s = \{ u : u \in H^1(\Omega_\varepsilon), u \text{ is even in } y_h, h = 2, \ldots, N, \\
u(r \cos \theta, r \sin \theta, y^\prime) = u(r \cos(\theta + \frac{2\pi j}{k}), r \sin(\theta + \frac{2\pi j}{k}), y^\prime), j = 1, \ldots, k - 1 \}
\]
if \( N \geq 4 \) and if \( N = 3 \) we replace \( \Omega_\varepsilon \) by \( \Omega_\lambda \) where \( \Omega_\lambda = \{ y | \lambda^{-1} y \in \Omega \} \).

For \( N \geq 4 \), let
\[
x_j = (\frac{1}{\varepsilon} \cos \frac{2(j - 1)\pi}{k}, \frac{1}{\varepsilon} \sin \frac{2(j - 1)\pi}{k}, 0), \quad j = 1, \ldots, k,
\]
where \( 0 \) is the zero vector in \( \mathbb{R}^{N-2} \) and
\[
W(y) = \sum_{j=1}^{k} W_{\lambda, x_j}.
\]

For \( N = 3 \), let
\[
x_j = (\lambda \cos \frac{2(j - 1)\pi}{k}, \lambda \sin \frac{2(j - 1)\pi}{k}, 0), \quad j = 1, \ldots, k,
\]
and
\[
W_{\lambda, x_j}(y) = \lambda^{-\frac{1}{2}} W_{\lambda, x_j}(\lambda^{-1} y), \quad W(y) = \sum_{j=1}^{k} W_{\lambda, x_j}(y).
\]
Theorem 1.1 is a direct consequence of the following result:

**Theorem 2.1.** Suppose that \( N \geq 3 \) and \( \Omega \) is a smooth and bounded domain satisfying \((H_1), (H_2)\) and \((H_3)\). Then there is an integer \( k_0 > 0 \), such that for any integer \( k \geq k_0 \), (2.2) has a solution \( u_k \) of the form

\[
u_k = W(y) + \omega_k,
\]
where \( \omega_k \in H_s \), and as \( k \to +\infty \), \( \|\omega\|_{L^\infty} \to 0 \).

We will use the techniques in the singularly perturbed elliptic problems to prove Theorem 2.1. In all the singularly perturbed problems, some small parameters are present either in the operator or in the nonlinearity or in the boundary condition. Here there is no parameter. Instead, we use \( k \), the number of the bubbles of the solutions, as the parameter in the construction of bubble solutions for (1.4). This is the new idea of this paper. This is motivated by recent paper of Wei-Yan[56] where they constructed infinitely many solutions to a prescribed curvature problem. The difference is that now the location of bubbles is given in the beginning.

The main difficulty in constructing solution with \( k \)-bubbles is that we need to obtain a better control of the error terms. Since the number of the bubbles is large, it is very hard to carry out the reduction procedure by using the standard norm. Noting that the maximum norm will not be affected by the number of the bubbles, we will carry out the reduction procedure in a space with weighted maximum norm. Similar weighted maximum norm has been used in [34],[44]–[46], [56]. But the estimates in the reduction procedure in this paper are much more complicated than those in [34],[44]–[46], because the number of the bubbles is large.

### 3. Finite-dimensional Reduction

In this section, we perform a finite-dimensional reduction.

Let

\[
\|u\|_* = \sup_y \left( \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{\frac{n-2}{2}}} \right)^{-1} |u(y)|,
\]
and
\( \|f\|_{s*} = \sup_y \left( \sum_{j=1}^{k} \frac{1}{(1 + |y-x_j|)^{N+2+\tau}} \right)^{-1} |f(y)|, \)

where we choose
\[
\begin{cases}
\tau > 1 & \text{and close to 1, } \quad N = 3, 5, 6, \ldots; \\
\tau > \frac{1}{2} & \text{and close to } \frac{1}{2}, \quad N = 4.
\end{cases}
\]

Let
\[ Y_i = \frac{\partial W_{\lambda, x_i}}{\partial \lambda}, \quad Z_i = -\Delta Y_i + \varepsilon^2 Y_i \quad \text{if } \quad N \geq 4, \]
and
\[ Y_i = \frac{\partial W_{\lambda, x_i}}{\partial \lambda}, \quad Z_i = -\Delta Y_i + \lambda^{-2} Y_i \quad \text{if } \quad N = 3. \]

We consider
\[
\begin{cases}
-\Delta \phi_k + \varepsilon^2 \phi_k - N(N+2)W^{2* - 2} \phi_k = h + c_1 \sum_{i=1}^{k} Z_i, & \text{in } \Omega_\varepsilon, \\
\frac{\partial \phi_k}{\partial n} = 0, & \text{on } \partial \Omega_\varepsilon, \\
\phi_k \in H_\varepsilon, & \\
< \sum_{i=1}^{k} Z_i, \phi_k > = 0
\end{cases}
\]
for some number \( c_1 \), where \( < u, v > = \int_{\Omega_\varepsilon} uv \) if \( N \geq 4 \) and if \( N = 3 \) we just replace \( \varepsilon \) by \( \lambda^{-1} \).

Let us remark that in general we also should include the translational derivatives of \( W \) in the right hand side of (3.3). However due the symmetry assumption \( \phi \in H_\varepsilon \), this part of kernel automatically disappears. This is the main reason for imposing the symmetries.

We recall the following result, whose proof is given in [46].

**Lemma 3.1.** Let \( f \) satisfy \( \|f\|_{s*} < \infty \) and \( u \) be the solution of
\[
-\Delta u + \varepsilon^2 u = f \quad \text{in } \Omega_\varepsilon, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial \Omega_\varepsilon.
\]

Then we have
\[
|u(x)| \leq C \int_{\Omega_\varepsilon} \frac{|f(y)|}{|x-y|^{N-2}} dy.
\]
The same result holds for \( N = 3 \) replacing \( \varepsilon \) by \( \lambda^{-1} \) and \( \Omega_\varepsilon \) by \( \Omega_\lambda \).

**Lemma 3.2.** Assume that \( \phi_k \) solves (3.3) for \( h = h_k \). If \( \|h_k\|_{s*} \) goes to zero as \( k \) goes to infinity, so does \( \|\phi_k\|_{s*} \).
Proof. We argue by contradiction. Suppose that there are \( k \to +\infty, h = h_k, \Lambda_k \in [\delta, \delta^{-1}], \)
and \( \phi_k \) solving (3.3) for \( h = h_k, \Lambda = \Lambda_k \), with \( \|h_k\|_* \to 0 \), and \( \|\phi_k\|_* \geq c' > 0 \). We may assume that \( \|\phi_k\|_* = 1 \). For simplicity, we drop the subscript \( k \).

According to Lemma 3.1, we have

\[
|\phi(y)| \leq C \int_{\Omega_t} \frac{1}{|z - y|^{N-2}} W^{2}\phi(z) \, dz
\]

\[+ C \int_{\Omega_t} \frac{1}{|z - y|^{N-2}} (|h(z)| + |c_1 \sum_{i=1}^{k} Z_i(z)|) \, dz \]

(3.4)

Using Lemma B.3, there is a strictly positive number \( \theta \) such that

\[
\left| \int_{\Omega_t} \frac{1}{|z - y|^{N-2}} W^{2}\phi(z) \, dz \right|
\]

\[\leq C\|\phi\|_{*} \left( \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{\frac{N+2}{2}+\theta}} + o(1) \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{\frac{N+2}{2}+\theta}} \right). \]

(3.5)

It follows from Lemma B.2 that

\[
\left| \int_{\Omega_t} \frac{1}{|z - y|^{N-2}} h(z) \, dz \right|
\]

\[\leq C\|h\|_{**} \int_{\mathbb{R}^N} \frac{1}{|z - y|^{N-2}} \sum_{j=1}^{k} \frac{1}{(1 + |z - x_j|)^{\frac{N+2}{2}+\theta}} \, dz \]

\[\leq C\|h\|_{**} \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{\frac{N+2}{2}+\theta}}, \]

and

\[
\left| \int_{\Omega_t} \frac{1}{|z - y|^{N-2}} \sum_{i=1}^{k} Z_i(z) \, dz \right|
\]

\[\leq C \sum_{i=1}^{k} \int_{\mathbb{R}^N} \frac{1}{|z - y|^{N-2}} \frac{1}{(1 + |z - x_i|)^{N+2}} \, dz \]

\[\leq C \sum_{i=1}^{k} \frac{1}{(1 + |y - x_i|)^{\frac{N+2}{2}+\theta}}. \]

(3.7)
Next, we estimate $c_1$. Multiplying (3.3) by $Z_1$ and integrating, we see that $c_1$ satisfies

$$
\langle \sum_{i=1}^{k} Z_i, Z_1 \rangle c_1 = \langle -\Delta \phi + \varepsilon^2 \phi - N(N + 2)W^{2\varepsilon - 2}\phi, Z_1 \rangle - \langle h, Z_1 \rangle.
$$

(3.8)

It follows from Lemma B.1 that

$$
|\langle h, Z_1 \rangle| \leq C \|h\| \int_{\mathbb{R}^N} \frac{1}{(1 + |z - x_j|)^{N-2}} \sum_{j=1}^{k} \frac{1}{(1 + |z - x_j|)^{N/2 + \tau}} dz
$$

$$
\leq C \|h\|_*.
$$

On the other hand,

$$
\langle -\Delta \phi + \varepsilon^2 \phi - N(N + 2)W^{2\varepsilon - 2}\phi, Z_1 \rangle
$$

$$
= \langle -\Delta Z_1 + \varepsilon^2 Z_1 - N(N + 2)W^{2\varepsilon - 2}Z_1, \phi \rangle
$$

$$
= N(N + 2) \langle (V^{2\varepsilon - 2} - W^{2\varepsilon - 2})Z_1, \phi \rangle + \varepsilon^2 \langle Z_1, \phi \rangle.
$$

(3.9)

$$
|\langle Z_1, \phi \rangle| \leq \|\phi\| \int_{\mathbb{R}^N} \frac{1}{(1 + |z - x_j|)^{N+2}} \sum_{i=1}^{k} \frac{1}{(1 + |z - x_i|)^{N/2 + \tau}}
$$

$$
\leq C \|\phi\|_*.
$$

(3.10)
Similar to the proof of Lemma B.3, we obtain for $N \geq 6,$

\[
(3.11) \quad \left| \left( \frac{U_{x_1}^{2} - W_{x_1}^{2}}{x_1} \right) Z_1, \phi \right| 
\]

\[
\leq C \| \phi \| \star \int_{\Omega_e} \frac{1}{(1 + |z - x_1|)^{N+2}} \sum_{i=2}^{k} \frac{1}{(1 + |z - x_i|)^{4}} \sum_{j=1}^{k} \frac{1}{(1 + |z - x_j|)^{\frac{N-2+\tau}{2}} + 6} dz 
\]

\[
\leq C \| \phi \| \star \int_{\Omega_e} \frac{1}{(1 + |z - x_1|)^{N+2}} \sum_{i=2}^{k} \frac{1}{(1 + |z - x_i|)^{4}} dz 
\]

\[
+ C \| \phi \| \star \int_{\Omega_e} \frac{1}{(1 + |z - x_1|)^{N+2}} \sum_{i=2}^{k} \frac{1}{(1 + |z - x_i|)^{4+\frac{N-2+\tau}{2}}} dz 
\]

\[
+ C \| \phi \| \star \int_{\Omega_e} \frac{1}{(1 + |z - x_1|)^{N+2}} \sum_{i=2}^{k} \sum_{i \neq j} \frac{1}{(1 + |z - x_i|)^{4}} \frac{1}{(1 + |z - x_j|)^{\frac{N-2+\tau}{2}}} dz 
\]

\[
\leq C \| \phi \| \star \sum_{j=2}^{k} \frac{1}{|x_1 - x_j|^4} 
\]

\[
+ C \| \phi \| \star \int_{\Omega_e} \frac{1}{(1 + |z - x_1|)^{N+2}} \sum_{j=2}^{k} \frac{1}{(1 + |z - x_j|)^{3+\frac{N-2+\tau}{2} - \beta}} \sum_{i \neq j} \frac{1}{(|x_j - x_i|)^{1+\beta}} dz 
\]

\[
\leq C \| \phi \| \star (\varepsilon k)^4 + C \| \phi \| \star \sum_{j=2}^{k} \frac{1}{|x_1 - x_j|^2} 
\]

\[
= o(1) \| \phi \| \star. 
\]
For $N = 4, 5$, $\frac{4}{N-2} > 1$, then similar in Appendix B.3

\begin{equation}
\left| \langle (U_{x_1}^{2^* - 2} - W^{2^* - 2})Z_1, \phi \rangle \right|
\leq C \int_{\Omega_c} U_{x_1}^{2^* - 3} \sum_{j=2}^k U_{x_j}^{|Z_1 \phi|} + C \int_{\Omega_c} \left( \sum_{j=2}^k U_{x_j}^{|Z_1 \phi|} \right)^{\frac{4}{N-2}} \\\frac{1}{(1 + |z - x_j|)^{(N-2)(1-\beta)}}
\leq C \|\phi\|_* \int_{\Omega_c} \left( 1 + |z - x_1| \right)^{(6-N+N+2)(1-\beta)} \sum_{j=2}^k \left( 1 + |z - x_j| \right)^{(N-2)(1-\beta)} \frac{1}{(1 + |z - x_i|)^{\frac{N-2}{2} + \tau}}
+ C \|\phi\|_* \int_{\Omega_c} \left( 1 + |z - x_1| \right)^{(N+2)(1-\beta)} \sum_{j=2}^k \left( 1 + |z - x_j| \right)^{(1-\beta)} \frac{1}{(1 + |z - x_i|)^{\frac{N-2}{2} + \tau}}
= o(1) \|\phi\|_*.
\end{equation}

But there is a constant $\bar{c} > 0$,

$$\langle \sum_{i=1}^k Z_i, Z_1 \rangle = \bar{c} + o(1).$$

Thus we obtain that

$$c_1 = o(\|\phi\|_*) + O(\|h\|_{**}).$$

So,

\begin{equation}
\|\phi\|_* \leq \left( o(1) + \|h_k\|_{**} + \sum_{j=1}^k \frac{1}{(1 + |y - x_j|)^{\frac{N-2}{2} + \tau}} \right).
\end{equation}

Since $\|\phi\|_* = 1$, we obtain from (3.13) that there is $R > 0$, such that

\begin{equation}
\|\phi(y)\|_{B_R(x_i)} \geq c_0 > 0,
\end{equation}

for some $i$. But $\tilde{\phi}(y) = \phi(y - x_i)$ converges uniformly in any compact set of $\mathbb{R}^N_i$ to a solution $u$ of

\begin{equation}
\Delta u + N(N + 2)U_{x_1}^{2^* - 2}u = 0
\end{equation}
for some $\Lambda \in [\delta, \delta^{-1}]$, and $u$ is perpendicular to the kernel of (3.15). So, $u = 0$. This is a contradiction to (3.14).

For $N = 3$, the proof is very similar and the key estimate is given in Appendix B.3. Here we don’t repeat the proof.

\[ \square \]

From Lemma 3.2, using the same argument as in the proof of Proposition 4.1 in [34], Proposition 3.1 in [46], we can prove the following result:

**Proposition 3.3.** There exists $k_0 > 0$ and a constant $C > 0$, independent of $k$, such that for all $k \geq k_0$ and all $h \in L^\infty(\Omega_\varepsilon)$, problem (3.3) has a unique solution $\phi \equiv L_k(h)$. Besides,

\begin{equation}
\|L_k(h)\|_* \leq C\|h\|_{**}, \quad |c_1| \leq C\|h\|_{**}.
\end{equation}

Moreover, the map $L_k(h)$ is $C^1$ with respect to $\Lambda$ for $N \geq 4$ and with respect to $\lambda$ for $N = 3$.

Now, we consider

\begin{equation}
\begin{cases}
-\Delta(W + \phi) + \varepsilon^2(W + \phi) = \alpha_N(W + \phi)^{2^*-1} + c_1 \sum_{i=1}^k Z_i, & \text{in } \Omega_\varepsilon, \\
\frac{\partial \phi}{\partial n} = 0, & \text{on } \partial \Omega_\varepsilon, \\
\phi \in H_s, & \\
< \sum_{i=1}^k Z_i, \phi > = 0
\end{cases}
\end{equation}

if $N \geq 4$ and if $N = 3$ we just replace $\varepsilon$ by $\lambda^{-1}$.

We have

**Proposition 3.4.** There is an integer $k_0 > 0$, such that for each $k \geq k_0$, $\delta \leq \Lambda \leq \delta^{-1}$, where $\delta$ is a fixed small constant, (3.17) has a unique solution $\phi$, satisfying

\[ \|\phi\|_* \leq \begin{cases}
Ck^{-\frac{N^2}{2(N-3)}} + \frac{N}{N-3} + \eta, & \text{if } N \geq 5; \\
Ck^{-2+\eta}, & \text{if } N = 4; \\
C\lambda^{-\frac{3}{2} + \eta}, & \text{if } N = 3.
\end{cases} \]

Moreover, $\Lambda \rightarrow \phi(\Lambda)$ (or $\lambda \rightarrow \phi(\lambda)$) is $C^1$. 
Rewrite (3.17) as

\[
\begin{cases}
-\Delta \phi + \varepsilon^2 \phi - N(N + 2)W^{2^* - 2}\phi = N(\phi) + l_k + c_1 \sum_{i=1}^k Z_i, & \text{in } \Omega_\varepsilon, \\
\frac{\partial \phi}{\partial n} = 0, & \text{on } \partial \Omega_\varepsilon, \\
\phi \in H_s, \\
< \sum_{i=1}^k Z_i, \phi > = 0
\end{cases}
\]

(3.18)

where

\[
N(\phi) = \alpha_N \left( (W + \phi)^{2^* - 1} - W^{2^* - 1} - (2^* - 1)W^{2^* - 2}\phi \right),
\]

and

\[
l_k = \begin{cases}
\alpha_N \left( W^{2^* - 1} - \sum_{j=1}^k U_j^{2^* - 1} \right), & \text{if } N \geq 4; \\
3 \left( W^5 - \sum_{j=1}^k U_j^5 \right) + \lambda^{-2} \sum_{j=1}^k \left( U_{1,x_j} - \frac{1}{|y - x_j|} \right) & \text{if } N = 3.
\end{cases}
\]

In order to use the contraction mapping theorem to prove that (3.18), is uniquely solvable in the set that \( \|\phi\|_* \) is small, we need to estimate \( N(\phi) \) and \( l_k \).

**Lemma 3.5.** If \( N \geq 3 \), then

\[
\|N(\phi)\|_* \leq C \|\phi\|_*^{\min(2^* - 1, 2)}.
\]

**Proof.** We have

\[
|N(\phi)| \leq \begin{cases}
C|\phi|^{2^* - 1}, & N \geq 6; \\
CW \frac{N-2}{2}\phi^2, & N = 3, 4, 5.
\end{cases}
\]

Firstly, we consider \( N \geq 6 \). We have

\[
|N(\phi)| \leq C\|\phi\|_*^{2^* - 1} \left( \sum_{j=1}^k \frac{1}{(1 + |y - x_j|)^{\frac{N-2}{2} + \tau}} \right)^{2^* - 1} 
\]

(3.19)

\[
\leq C\|\phi\|_*^{2^* - 1} \sum_{j=1}^k \frac{1}{(1 + |y - x_j|)^{\frac{N-2}{2} + \tau}} \left( \sum_{j=1}^k \frac{1}{(1 + |y - x_j|)^{\tau}} \right)^{\frac{4}{N-2}}
\]
where we use the inequality
\[
\sum_{j=1}^{k} a_j b_j \leq \left( \sum_{j=1}^{k} a_j^p \right)^{\frac{1}{p}} \left( \sum_{j=1}^{k} b_j^q \right)^{\frac{1}{q}}, \quad \frac{1}{p} + \frac{1}{q} = 1, a_j, b_j \geq 0, j = 1, \ldots, k.
\]

Define
\[
\Omega_j = \{ y = (y', y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : \frac{y_j'}{|y'|} \cdot \frac{x_j}{|x_j|} \geq \cos \frac{\pi}{k} \}.
\]
Without loss of generality, we assume \( y \in \Omega_1 \). Then,
\[
|y - x_j| \geq |y - x_1|, \quad \forall y \in \Omega_1.
\]
If \( |y - x_1| \leq \frac{1}{2} |x_1 - x_j| \),
\[
|y - x_j| \geq |x_j - x_1| - |y - x_1| \geq \frac{1}{2} |x_1 - x_j|.
\]
But if \( |y - x_1| \geq \frac{1}{2} |x_1 - x_j| \),
\[
|y - x_j| \geq |y - x_1| \geq \frac{1}{2} |x_1 - x_j|, \quad \forall y \in \Omega_1.
\]
Thus,
\[
|y - x_j| \geq \frac{1}{2} |x_1 - x_j|, \quad \forall y \in \Omega_1, \ j = 2, \ldots, k.
\]
Hence
\[
\sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^\tau} \leq C + \sum_{j=2}^{k} \frac{C}{(1 + |x_1 - x_j|)^\tau} \leq C
\]
since \( \tau > 1 \).

For \( N = 5 \), similarly we have
\[
|N(\phi)| \leq C \|\phi\|^2 \left( \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{3(1-\beta)}} \right)^{\frac{1}{3}} \left( \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{\frac{3}{2} + \tau}} \right)^{\frac{2}{3}}
\]
\[
\leq C \|\phi\|^2 \left( \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{\frac{3}{2} + \tau}} \right)^{\frac{2}{3}}
\]
\[
\leq C \|\phi\|^2 \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{\frac{3}{2} + \tau}} \left( \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{\tau}} \right)^{\frac{4}{3}}
\]
\[
\leq C \|\phi\|^2 \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{\frac{3}{2} + \tau}}
\]

(3.20)
since \( \tau > 1 \) and is close to \( 1 \), \( \beta \) can be made as small as desired.

It remains the case \( N = 3, 4 \). For \( N = 4 \), similarly

\[
|N(\phi)| \leq C \|\phi\|_2^2 \left( \sum_{j=1}^k \frac{1}{(1 + |y - x_j|)^{2(1-\beta)}} \right) \left( \sum_{j=1}^k \frac{1}{(1 + |y - x_j|)^{1+\tau}} \right)^2
\]

\[
\leq C \|\phi\|_2^2 \left( \sum_{j=1}^k \frac{1}{(1 + |y - x_j|)^{1+\tau}} \right)^3
\]

\[
\leq C \|\phi\|_2^2 \sum_{j=1}^k \frac{1}{(1 + |y - x_j|)^{3+\tau}} \left( \sum_{j=1}^k \frac{1}{(1 + |y - x_j|)^{1+\tau}} \right)^2
\]

\[
\leq C \|\phi\|_2^2 \sum_{j=1}^k \frac{1}{(1 + |y - x_j|)^{3+\tau}} \left( C + \sum_{j=2}^k \frac{C}{|x_1 - x_j|^\tau} \right)^2
\]

\[
\leq C \|\phi\|_2^2 \sum_{j=1}^k \frac{1}{(1 + |y - x_j|)^{3+\tau}}
\]

since \( \tau > \frac{1}{2} \) and is close to \( \frac{1}{2} \), \( \varepsilon = k^{-2} \),

\[
\sum_{j=2}^k \frac{C}{(|x_1 - x_j|)^\tau} \leq Ck\varepsilon = C.
\]

Now we prove the case \( N = 3 \). Without loss of generality, we may assume that \( y \in \Omega_1 \).

Then

\[
\sum_{j=2}^k \frac{1}{(1 + |y - x_j|)} \leq \frac{C}{(1 + |y - x_1|)^{1-\beta}} \sum_{j=2}^k \frac{1}{(1 + |y - x_j|)^\beta}
\]

\[
\leq \sum_{j=2}^k \frac{C}{|x_1 - x_j|^\beta \left( \frac{1}{(1 + |y - x_j|)^{1-\beta}} + \frac{1}{(1 + |y - x_1|)^{1-\beta}} \right)}
\]

\[
\leq \frac{C}{(1 + |y - x_1|)^{1-\beta}}.
\]

since \( \lambda \in [C_1 e^{\frac{D_2}{2} k \ln k}, C_2 e^{\frac{D_2}{2} k \ln k}] \).

Similarly,

\[
\sum_{j=2}^k \frac{1}{(1 + |y - x_j|)^{\frac{1}{2} + \tau}} \leq \frac{C}{(1 + |y - x_1|)^{\frac{1}{2} + \tau - \beta}}.
\]
Thus
\[ |N(\phi)| \leq \|\phi\|_{\ast}^2 \frac{C}{(1 + |y - x_1|)^{3 + 1 + 2\varepsilon - 5\beta}} \]
\[ \leq \|\phi\|_{\ast}^2 \frac{C}{(1 + |y - x_1|)^{\frac{5}{2} + \varepsilon}}, \quad y \in \Omega_1 \]

since $\beta$ can be made as small as desired.

Thus
\[ \|N(\phi)\|_{\ast\ast} \leq C\|\phi\|_{\ast}^{\min(2^\varepsilon - 1, 2)}. \]

Next, we estimate $l_k$.

**Lemma 3.6.**
\[
\|l_k\|_{\ast\ast} \leq \begin{cases} 
Ck^{-\frac{N+4}{2(N-2)} + \frac{N}{2} + \eta}, & \text{if } N \geq 5; \\
Ck^{-2+\eta}, & \text{if } N = 4; \\
C\lambda^{-\frac{3}{2} + \eta} & \text{if } N = 3
\end{cases}
\]

where $\eta$ can be chosen to be zero if $N \geq 6$ and if $N = 3, 4, 5$ $\eta$ is a very small positive number.

**Proof.** Recall
\[ \Omega_j = \{y = (y', y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : \frac{y_j'}{|y'|} \cdot \frac{x_j}{|x_j|} \geq \cos \frac{\pi}{k} \}. \]

From the symmetry, we can assume that $y \in \Omega_1$. Then,
\[ |y - x_j| \geq |y - x_1|, \quad \forall y \in \Omega_1. \]

Thus, for $N \geq 4$,
\[
|l_k| \leq C \frac{1}{(1 + |y - x_1|)^{3(1-\beta)}} \sum_{j=2}^{k} \frac{1}{(1 + |y - x_j|)^{(N-2)(1-\beta)}} \\
+ C \left( \sum_{j=2}^{k} \frac{1}{(1 + |y - x_j|)^{(N-2)(1-\beta)}} \right)^{2^\varepsilon - 1}.
\]

(3.21)
and for \( N = 3 \),
\[
|l_k| \leq C\left( \frac{1}{(1 + |y - x_1|)^4} \sum_{j=2}^{k} \frac{1}{1 + |y - x_j|} \right) + C \left( \sum_{j=2}^{k} \frac{1}{1 + |y - x_j|} \right)^5 \\
+ \lambda^{-2} \left| \sum_{i=1}^{k} \left( \frac{1}{(1 + |y - x_i|)^{\frac{1}{2} + \tau}} - \frac{1}{|y - x_i|} \right) \right|.
\]
(3.22)

Using Lemma B.1, we obtain
\[
\leq C \left( \frac{1}{(1 + |y - x_1|)^{\frac{N+2}{2} + \tau}} + \frac{1}{(1 + |y - x_j|)^{\frac{N+2}{2} + \tau}} \right) \frac{1}{|x_j - x_1|^\frac{(N+2)(1-2\beta)}{2} - \tau} \\
\leq C \frac{1}{(1 + |y - x_1|)^{\frac{N+2}{2} + \tau}} |x_j - x_1|^{\frac{(N+2)(1-2\beta)}{2} - \tau}, \quad j > 1.
\]
(3.23)

Since \( \tau \) is close to 1 and \( \beta \) can be made as small as desired, we see \( \frac{(N+2)(1-2\beta)}{2} - \tau > 1 \). Thus
\[
\leq C \frac{1}{(1 + |y - x_1|)^{\frac{N+2}{2} + \tau}} (k\xi)^{\frac{(N+2)(1-2\beta)}{2} - \tau}.
\]
(3.24)

On the other hand, for \( y \in \Omega_1 \), using Lemma B.1 again,
\[
\leq C \frac{1}{|x_j - x_1|^\frac{(N-2)(1-2\beta)}{2} + \frac{N-2}{N+2} \tau} \left( \frac{1}{(1 + |y - x_1|)^{\frac{(N-2)(1-\beta)}{2}}} + \frac{1}{(1 + |y - x_j|)^{\frac{(N-2)(1-\beta)}{2}}} \right) \\
\leq C \frac{1}{|x_j - x_1|^\frac{(N-2)(1-2\beta)}{2} + \frac{N-2}{N+2} \tau} (1 + |y - x_1|)^{\frac{N-2}{2} + \frac{N-2}{N+2} \tau}.
\]

If \( N \geq 5 \), \( \tau = 1 + \eta \) and \( \eta > 0 \) is small, then \( \frac{(N-2)(1-2\beta)}{2} + \frac{N-2}{N+2} \tau > 1 \). Thus
\[
\sum_{j=2}^{k} \frac{1}{(1 + |y - x_j|)^{(N-2)(1-\beta)}} \leq C(k\varepsilon)^{\frac{(N-2)(1-2\beta)}{2}} \frac{1}{(1 + |y - x_1|)^{N-2 + \frac{N-2}{2}}}.
\]

which, gives for \( y \in \Omega_1 \)

\[
\left( \sum_{j=2}^{k} \frac{1}{(1 + |y - x_j|)^{(N-2)(1-\beta)}} \right)^{2^* - 1} \leq C(k\varepsilon)^{\frac{(N-2)(1-2\beta)}{2}} \frac{1}{(1 + |y - x_1|)^{N-2 + \frac{N-2}{2}}}.
\]

Thus, we have proved for \( N \geq 5 \),

\[
\|l_k\|_{\ast^*} \leq C(k\varepsilon)^{\frac{(N-2)(1-2\beta)}{2}} = Ck^{-\frac{N+2}{2N} - \frac{1}{N-3} + \frac{N-2}{2N}}.
\]

If \( N = 4 \), by the same computation we get

\[
\sum_{j=2}^{k} \frac{1}{(1 + |y - x_j|)^2} \leq C \sum_{j=2}^{k} \frac{|x_1 - x_j|^{1-2\beta}}{(1 + |y - x_1|)^{1+\frac{1}{2}}} \leq Ck^{-1+2\beta} \leq Ck^{-1+4\beta+\frac{2}{7}}, \quad y \in \Omega_1.
\]

Hence

\[
\left( \sum_{j=2}^{k} \frac{1}{(1 + |y - x_j|)^2} \right)^{\frac{3}{2}} \leq \sum_{i=1}^{k} \frac{Ck^{-3+12\beta+2\tau}}{(1 + |y - x_i|)^{3+\tau}}.
\]

Now we choose \( \tau \) such that \( \tau = \frac{1}{2} + \eta, \eta \) can be small enough, then

\[
\left( \sum_{j=2}^{k} \frac{1}{(1 + |y - x_j|)^2} \right)^{\frac{3}{2}} \leq \sum_{i=1}^{k} \frac{Ck^{-2+\eta}}{(1 + |y - x_i|)^{3+\tau}}
\]

where \( \eta = 12\beta + 2\eta \) can be small enough.

For \( N = 3 \), noting \( \lambda \in [C_1e^{\frac{2\alpha k}{2}k\ln k}, C_2e^{\frac{2\alpha k}{2}k\ln k}] \), by the similar computation we can get

\[
\frac{1}{(1 + |y - x_1|)^4} \sum_{j=2}^{k} \frac{1}{1 + |y - x_j|} + C \left( \sum_{j=2}^{k} \frac{1}{1 + |y - x_j|} \right)^5 \leq C\lambda^{-\frac{5}{2}+\tau+\beta}
\]
and
\[
\lambda^{-2} \sum_{i=1}^{k} \frac{1}{(1 + |y - x_i|)^{\frac{4}{2} + \tau}} \left| \sum_{i=1}^{k} \left( \frac{1}{(1 + |y - x_i|^2)^{\frac{1}{2}}} - \frac{1}{|y - x_i|} \right) \right| \leq C \lambda^{-\frac{5}{2} + \tau + \beta}.
\]

Now, we are ready to prove Proposition 3.4.

**Proof of Proposition 3.4.** Let us recall that
\[
\varepsilon = k^{-\frac{N-4}{3}}
\]
and
\[
\lambda \in \left[ C_1 e^{\frac{D_1 k \ln k}{2}}, C_2 e^{\frac{D_2 k \ln k}{2}} \right].
\]

Let
\[
E_N = \{ u : u \in C(\Omega_\varepsilon), \| u \|_* \leq C k^{-\frac{N+2}{2(N-3)} + \frac{\tau}{N-3} + \eta + \sigma}, \int_{\Omega_\varepsilon} \sum_{i=1}^{k} Z_i \phi = 0 \}
\]
if $N \geq 5$.

\[
E_4 = \{ u : u \in C(\Omega_\varepsilon), \| u \|_* \leq C k^{-2 + \eta + \sigma}, \int_{\Omega_\varepsilon} \sum_{i=1}^{k} Z_i \phi = 0 \}
\]
and
\[
E_3 = \{ u : u \in C(\Omega_\lambda), \| u \|_* \leq C \lambda^{-\frac{3}{2} + \eta + \sigma}, \int_{\Omega_\lambda} \sum_{i=1}^{k} Z_i \phi = 0 \}
\]
where $\sigma$ is a fixed positive small constant. Then, (3.18) is equivalent to
\[
\phi = A(\phi) =: L(N(\phi)) + L(l_k).
\]

We will prove that $A$ is a contraction map from $E_N$ to $E_N$.

In fact, if $N \geq 6$, 

\[
\| \phi \|_* \leq C \| N(\phi) \|_{**} + C \| l_k \|_{**}
\]
\[
\leq C \| \phi \|_*^{\frac{2^* - 1}{2} + C k^{-\frac{N+2}{2(N-3)} + \frac{\tau}{N-3} + \eta}}
\]
\[
\leq C k^{-\frac{N+2}{2(N-3)} + \frac{\tau}{N-3} + \eta + \sigma} \frac{N+2}{2(N-3)} + C k^{-\frac{N+2}{2(N-3)} + \frac{\tau}{N-3} + \eta}
\]
\[
= C \left( 1 + k^{-\frac{N+2}{2(N-3)} + \frac{\tau}{N-3} + \eta + \sigma} \right) \frac{4}{N-2 + \sigma} \right) k^{-\frac{N+2}{2(N-3)} + \frac{\tau}{N-3} + \eta}
\]
\[
< k^{-\frac{N+2}{2(N-3)} + \frac{\tau}{N-3} + \eta + \sigma},
\]

(3.25)
since we can take $\sigma > 0$ small. Thus, $A$ maps $E_N$ to $E_N$.

On the other hand,

$$\|A(\phi_1) - A(\phi_2)\|_* = \|L(N(\phi_1)) - L(N(\phi_2))\|_* \leq C\|N(\phi_1) - N(\phi_2)\|_{**}.$$

If $N \geq 6$, then

$$|N'(t)| \leq C|t|^{2^*-2}.$$

As a result,

$$|N(\phi_1) - N(\phi_2)| \leq C(|\phi_1|^{2^*-2} + |\phi_2|^{2^*-2})|\phi_1 - \phi_2|$$

$$\leq C(\|\phi_1\|_*^{2^*-2} + \|\phi_2\|_*^{2^*-2})\|\phi_1 - \phi_2\|_* \left(\sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{\frac{N-2}{2} + r}}\right)^{2^*-1}.$$

As before, we have

$$\left(\sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{\frac{N-2}{2} + r}}\right)^{2^*-1} \leq C\sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{\frac{N+2}{2} + r}}.$$

So,

$$\|A(\phi_1) - A(\phi_2)\|_* \leq C\|N(\phi_1) - N(\phi_2)\|_{**}$$

$$\leq C(\|\phi_1\|_*^{2^*-2} + \|\phi_2\|_*^{2^*-2})\|\phi_1 - \phi_2\|_*$$

$$\leq Ck^{-\frac{N+2}{3(N-3)} + \frac{r}{3} + \beta + \sigma} \|\phi_1 - \phi_2\|_*$$

$$\leq \frac{1}{2}\|\phi_1 - \phi_2\|_*.$$

Thus, $A$ is a contraction map.

It follows from the contraction mapping theorem that there is a unique $\phi \in E$, such that

$$\phi = A(\phi).$$

Moreover, it follows from (3.25) that

$$\|\phi\|_* \leq C k^{-\frac{N+2}{3(N-3)} + \frac{r}{3} + \eta}.$$
Now we prove the case $N = 3, 4, 5$. Similarly, we have

$$\|N(\phi)\|_* \leq \begin{cases} 
C(1 + k^{-\frac{3}{2}} + \frac{\eta}{2} + 2\xi)k^{-\frac{\eta}{2} + \frac{\eta}{3} + \eta}, & N = 5, \\
C(1 + k^{-2 + \frac{3}{2}} + \frac{\eta}{2} + \xi)k^{-\frac{\eta}{2} + \xi}, & N = 4, \\
C(1 + \lambda^{-\frac{3}{2}} + \frac{\eta}{2} + 2\xi)\lambda^{-\frac{\eta}{2} + \frac{\eta}{3} + \eta}, & N = 3,
\end{cases}$$

and

$$\|N(\phi_1) - N(\phi_2)\|_* \leq \begin{cases} 
Ck^{-\frac{3}{2}} + \frac{\eta}{2} + 2\xi\|\phi_1 - \phi_2\|_*, & N = 5, \\
Ck^{-2 + \frac{3}{2}} + 2\xi\|\phi_1 - \phi_2\|_*, & N = 4, \\
C\lambda^{-\frac{3}{2}} + \frac{\eta}{2} + 2\xi\|\phi_1 - \phi_2\|_*, & N = 3.
\end{cases}$$

\[
\square
\]

4. Proof of Theorem 2.1

Let

$$F(\Lambda) = I(W + \phi), \quad \text{if } N \geq 4$$

and

$$F(\lambda) = I(W + \phi), \quad \text{if } N = 3$$

where $\phi$ is the function obtained in Proposition 3.4, and for $N \geq 4$

$$I(u) = \frac{1}{2} \int_{\Omega_\varepsilon} (|Du|^2 + \varepsilon^2 u^2) - \frac{(N - 2)^2}{2} \int_{\Omega_\varepsilon} |u|^{2\varepsilon}$$

and for $N = 3$,

$$I(u) = \frac{1}{2} \int_{\Omega_\lambda} (|Du|^2 + \lambda^{-2} u^2) - \frac{(N - 2)^2}{2} \int_{\Omega_\lambda} |u|^{2\varepsilon}.$$

Using the symmetry, we can check that if $\Lambda$ (or $\lambda$) is a critical point of $F(\Lambda)$ (or $F(\lambda)$), then $W + \phi$ is a solution of (1.5).

**Proposition 4.1.** For $N \geq 4$, we have

$$F(\Lambda) = k \left( A_0 + A_1 \Lambda k^{-\frac{N-2}{N-3}} - A_2 \Lambda^{N-2} k^{-\frac{N-2}{N-3}} + o(k^{-\frac{N-2}{N-3}}) \right),$$

where the constant $A_i, i = 0, 1, 2$ are strictly positive numbers, which are given in Lemma A.1.

For $N = 3$, we have

$$F(\lambda) = k \left( D_1 + D_2 \lambda^{-1} \ln \lambda - D_3 \lambda^{-1} \ln k + D_4 \lambda^{-1} + o(\lambda^{-1}) \right),$$
where the constants $D_i, i = 1, 2, 3, 4$ are strictly positive numbers which are given in Lemma A.2.

Proof. First we prove the case $N \geq 6$. Since

$$\langle I'(W), \phi \rangle = 0,$$

there is $t \in (0, 1)$ such that

$$F(\Lambda) = I(W) + \frac{1}{2} D^2 I(W + t\phi)(\phi, \phi)$$

$$= I(W) + \int_{\Omega} \left( |D\phi|^2 + \varepsilon^2 \phi^2 - N(N + 2)(W + t\phi)^{2^*-2}\phi^2 \right)$$

$$= I(W) - N(N + 2) \int_{\Omega} \left( (W + t\phi)^{2^*-2} - W^{2^*-2} \right) \phi^2$$

$$+ \int_{\Omega} \left( N(\phi) + l_k \right) \phi$$

$$= I(W) + O \left( \int_{\Omega} \left( |\phi|^{2^*} + |N(\phi)||\phi| + |l_k||\phi| \right) \right).$$

But

$$\int_{\Omega} \left( |N(\phi)||\phi| + |l_k||\phi| \right)$$

$$\leq C \left( \|N(\phi)\|_{**} + \|l_k\|_{**} \right) \|\phi\|_{**} \int_{\Omega} \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{\frac{N+2}{2} + r}} \sum_{i=1}^{k} \frac{1}{(1 + |y - x_i|)^{\frac{N+2}{2} + r}}.$$

Using Lemma B.1
\[
\sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{N/2 + \tau}} \geq \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{N/2 + \tau}} + \sum_{i \neq j}^{k} \frac{1}{(1 + |y - x_j|)^{N/2 + \tau}} \leq \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{N + 2\tau}} + C \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{N + \tau}} \sum_{i=2}^{k} \frac{1}{|x_i - x_1|^{\tau}} \leq C \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{N + \tau}},
\]

since \(\tau > 1\). Thus, we obtain

\[
\int_{\Omega_t} (|N(\phi)|\|\phi\| + \|l_k\|\|\phi\|) \leq C k \left(\|N(\phi)\|_{*} + \|l_k\|_{*}\right)\|\phi\|_{*} \leq C k^{1 - \frac{N}{N - 2} + \frac{2}{N - 3} + 2\eta}
\]

where \(\eta\) can be chosen to be zero.

On the other hand,

\[
\int_{\Omega_t} |\phi|^2 \leq C \|\phi\|_{*}^2 \int_{\Omega_t} \left(\sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{N/2 + \tau}} \right)^{2\tau}.
\]

But using Lemma B.1, if \(y \in \Omega_1\),

\[
\sum_{j=2}^{k} \frac{1}{(1 + |y - x_j|)^{N/2 + \tau}} \leq \sum_{j=2}^{k} \frac{1}{(1 + |y - x_1|)^{N/4 + \frac{1}{2}\tau}} \leq C \frac{1}{(1 + |y - x_1|)^{N/2 + \frac{1}{2}\eta}} \sum_{j=2}^{k} \frac{1}{|x_j - x_1|^{\tau - \frac{1}{2}\eta}} \leq C \frac{1}{(1 + |y - x_1|)^{N/2 + \frac{1}{2}\eta}},
\]

and then

\[
\left(\sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{N/2 + \tau}}\right)^{2\tau} \leq \frac{C}{(1 + |y - x_1|)^{N + 2\tau - \frac{1}{2}\eta}}, \quad y \in \Omega_1.
\]
Thus
\[
\int_{\Omega_c} \left( \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{N/2 + \tau}} \right)^{2^*} \leq Ck.
\]
So, we have proved
\[
\int_{\Omega_c} \phi^2 \leq Ck\|\phi\|_2^{2^*} \leq Ck^{1-2^*(\frac{N}{N+2} - \frac{1}{N+3} - \eta)}.
\]
For \( N = 3, 4, 5 \)
\[
F = I(W) + O \left( W^{\frac{N-N}{N-2}} |\phi|^3 + |N(\phi)||\phi| + |l_k(\phi)||\phi| \right).
\]
Similarly, for \( N = 5 \),
\[
\int_{\Omega_c} \left( |N(\phi)||\phi| + |l_k||\phi| \right) \leq Ck^{1-\frac{7}{2} + \tau + 2\eta}.
\]
For \( y \in \Omega_1 \), similar as in the proof of Lemma B.3,
\[
\int_{\Omega_c} W^{\frac{3}{2}} |\phi|^3 \leq Ck \int_{\Omega_c} \frac{1}{(1 + |y - x_1|)^{\frac{3}{2} - \delta + 3\tau - \beta}} \|\phi\|_3^3
= Ck k^{-\frac{3}{2} + \frac{3\delta}{2} + 3\eta}
= Ck o(k^{-\frac{3}{2}})
\]
since \( \tau \) is close to 1 and \( \eta \) can be chosen small enough.

If \( N = 4 \), similar as in the proof of Lemma B.3
\[
\int_{\Omega_c} |W||\phi|^3 \leq Ck \|\phi\|^3 \int_{\Omega_c} \frac{1}{(1 + |y - x_1|)^{3+3\tau}}
= Ck \times k^{-6+3\eta}
\]
and
\[
\int_{\Omega_c} \left( |N(\phi)||\phi| + |l_k||\phi| \right) \leq Ck^{1-4+2\eta}.
\]
For \( N = 3 \), similar as in the proof of Lemma B.3, we get
\[
\int_{\Omega_3} W^3 |\phi|^3 \leq Ck \|\phi\|_3^3 \int_{\Omega_3} \frac{1}{(1 + |y - x_1|)^{(3 + \tau)\times 3}}
= Ck \times \lambda^{3(-\frac{3}{2} + \eta)}
\]
and
\[
\int_{\Omega_3} \left( |N(\phi)||\phi| + |l_k||\phi| \right) \leq Ck \lambda^{-3+2\eta}.
\]
**Proof of Theorem 2.1:** We prove that $F$ has a critical point.

For $N \geq 4$, the function

$$A_1 \Lambda - A_2 \Lambda^{N-2}$$

has a maximum point at $\Lambda_0 = \left( \frac{A_1}{A_2(N-2)} \right)^{\frac{1}{N-3}}$. Thus, $F(\Lambda)$ attains its maximum in the interior of $[\delta, \delta^{-1}]$ if $\delta > 0$ is small. As a result, $F(\Lambda)$ has a critical point in $[\delta, \delta^{-1}]$.

For $N = 3$, the function

$$D_2 \frac{\ln \lambda}{\lambda} - D_3 \frac{k \ln k + D_4}{\lambda}$$

has a maximum point at $\lambda_0 = e^{\frac{D_4}{D_2} k \ln k + \frac{D_4 - D_2}{D_2}}$. So, we can prove that $F(\lambda)$ has a critical point in $[C_1 e^{\frac{D_4}{D_2} k \ln k}, C_2 e^{\frac{D_4 - D_2}{D_2} k \ln k}]$, if we choose

$$C_1 = \frac{1}{2} e^{\frac{D_4 - D_2}{D_2}}, \quad C_2 = 2 e^{\frac{D_4 - D_2}{D_2}}.$$

\[ \square \]

**APPENDIX A. ENERGY EXPANSION**

In all of the appendixes, we always assume that for $N \geq 4$,

$$x_j = \left( \frac{1}{\varepsilon} \cos \frac{2(j - 1)\pi}{k}, \frac{1}{\varepsilon} \sin \frac{2(j - 1)\pi}{k}, 0 \right), \quad j = 1, \ldots, k,$$

where 0 is the zero vector in $\mathbb{R}^{N-2}$ and

$$\varepsilon = k^{-\frac{N-2}{2}},$$

and for $N = 3$,

$$x_j = (\lambda \cos \frac{2(j - 1)\pi}{k}, \lambda \sin \frac{2(j - 1)\pi}{k}, 0), \quad j = 1, \ldots, k,$$

where

$$\lambda \in [C_1 e^{\frac{D_4}{D_2} k \ln k}, C_2 e^{\frac{D_4 - D_2}{D_2} k \ln k}].$$

First we compute the energy for $N \geq 4$. Recall

$$I(u) = \frac{1}{2} \int_{\Omega_\varepsilon} (|Du|^2 + \varepsilon^2 |u|^2) - N(N + 2) \int_{\Omega_\varepsilon} |u|^{2^*},$$

and

$$U_{\frac{1}{x_1}, x_2}(y) = \frac{\left( \frac{1}{N} \right)^{N-2}}{(1 + \frac{1}{|x_1|} |y - x_1|^2)^{\frac{N-2}{2}}}.$$
and
\[ W(y) = \sum_{j=1}^{k} W_{\Lambda,x_j}(y). \]

**Proposition A.1.** For \( N \geq 4 \), we have
\[ I(W) = k \left( A_0 + A_1 \varepsilon - A_2 \varepsilon^{N-2} + o(\varepsilon) \right), \]
where \( A_i, i = 0,1,2 \), is some positive constant.

**Proof.** By using the symmetry, we have
\[
\int_{\Omega} |DW|^2 + \varepsilon^2 W^2 = \alpha_N \sum_{j=1}^{k} \sum_{i=1}^{k} \int_{\Omega} U^{2\varepsilon^{-1} - 1}_{x,x_j} \left( U^{2\varepsilon^{-1}}_{x,x_j} - \varphi_{\Lambda,x_j} \right)
\]
\[ = k \alpha_N \left( \int_{\Omega} U^{2\varepsilon^{-1}}_{x_1} - \int_{\Omega} U^{2\varepsilon^{-1}}_{x_1} \varphi_{\Lambda,x_1} + \sum_{i=2}^{k} \int_{\Omega} U^{2\varepsilon^{-1}}_{x_1} U^{2\varepsilon^{-1}}_{x_i} - \sum_{i=2}^{k} \int_{\Omega} U^{2\varepsilon^{-1}}_{x_1} \varphi_{\Lambda,x_i} \right). \]

Next we compute term by term.
\[
\int_{\Omega} U^{2\varepsilon^{-1}}_{x_1} = \int_{\mathbb{R}^N_+} U^{2\varepsilon^{-1}}_{x_1}(y_1, y_N) \frac{\rho(y_1)}{\varepsilon} + O(\varepsilon^{2-\beta})
\]
\[ = \int_{\mathbb{R}^N_+} U^{2\varepsilon^{-1}}_{x_1}(y_1, y_N) \int_{\mathbb{R}^N_+} \frac{\partial U^{2\varepsilon^{-1}}_{x_1}(y_1, y_N)}{\partial y_N} \frac{\rho(y_1)}{\varepsilon} + O(\varepsilon^{2-\beta})
\]
\[ = \int_{\mathbb{R}^N_+} U^{2\varepsilon^{-1}}_{x_1} + \frac{\Lambda \varepsilon}{2} \int_{\partial \mathbb{R}^N_+} |y_1|^2 + O(\varepsilon^{2-\beta}). \]

\[
\alpha_N \int_{\Omega} U^{2\varepsilon^{-1}}_{x_1} \varphi_{\Lambda,x_1} = \Lambda \varepsilon \alpha_N \int_{\mathbb{R}^N_+} U^{2\varepsilon^{-1}}_{x_1} \varphi_0 + O(\varepsilon^{2-\beta})
\]
\[ = \Lambda \varepsilon \int_{\mathbb{R}^N_+} (-\Delta U_{1,0} \varphi_0 + U_{1,0} \varphi_0) + O(\varepsilon^{2-\beta})
\]
\[ = \Lambda \varepsilon \int_{\partial \mathbb{R}^N_+} (-\frac{\partial \varphi_0}{\partial y_N} U_{1,0}) + O(\varepsilon^{2-\beta})
\]
\[ = \frac{(N-2) \Lambda \varepsilon}{2} \int_{\partial \mathbb{R}^N_+} \frac{|y_1|^2}{(1 + |y_1|^2)^{N-1}} + O(\varepsilon^{2-\beta}). \]

\[
\int_{\Omega} U^{2\varepsilon^{-1}}_{x_1} U^{2\varepsilon^{-1}}_{x_i} = \frac{B_0 \Lambda^{N-2}}{|x_1 - x_i|^{N-2}} + O\left( \frac{1}{|x_1 - x_i|^{N-2} + \varepsilon^{2-\beta}} \right)
\]
where \( \sigma \) is a strictly small positive number.
Let
\[ \Omega_j = \{ y = (y', y'') \in \Omega_\varepsilon : \left| \frac{y'}{|y'|} \cdot \frac{x_j}{|x_j|} \right| \geq \cos \frac{\pi}{k} \}. \]

Then,
\[ |y - x_i| \geq |y - x_j|, \quad \forall y \in \Omega_j. \]

\[ \left| \int_{\Omega_j} U^{2^*-1}_{\frac{1}{k}x_1} \varphi_{\Lambda,x_1} \right| \leq C \left( \int_{\Omega_j} U^{2^*-1}_{\frac{1}{k}x_1} |\varphi_{\Lambda,x_1}| + \int_{\Omega_j} U^{2^*-1}_{\frac{1}{k}x_1} |\varphi_{\Lambda,x_1}| + \sum_{j \neq i,j \neq i} \int_{\Omega_j} U^{2^*-1}_{\frac{1}{k}x_1} |\varphi_{\Lambda,x_1}| \right) \]
\[ \leq C \int_{\partial R_N} \frac{1}{(1 + |y|)^{N+2}} \frac{1}{(1 + |x_1 - x_i|)^{N-3}} \frac{1}{|x_1 - x_i|^{N-3}} + \sum_{j \neq i} |x_i - x_j|^{N-2} \]
\[ \leq C \varepsilon |\ln \varepsilon|^m + C \varepsilon^2 |\ln \varepsilon|^m \]

since \( k = \varepsilon^{-\frac{N-2}{N-3}} \).

\[ \alpha_N \frac{2^*}{2} \int_{\Omega_\varepsilon} W^{2^*} = \alpha_N \frac{2^*}{2} \int_{\Omega_\varepsilon} \left( \sum_{j=1}^{k} \left( U_{\frac{1}{k}x_j} - \varphi_{\Lambda,x_j} \right) \right)^{2^*} \]
\[ = \alpha_N k \left( \int_{\Omega_\varepsilon} U^{2^*-1}_{\frac{1}{k}x_1} + 2^* \int_{\Omega_\varepsilon} \sum_{i=2}^{k} U^{2^*-1}_{\frac{1}{k}x_1} U_{\frac{1}{k}x_1} - 2^* \int_{\Omega_\varepsilon} U^{2^*-1}_{\frac{1}{k}x_1} \varphi_{\Lambda,x_1} \right) \]
\[ + kO \left( \int_{\Omega_\varepsilon} U^{2^*-2}_{\frac{1}{k}x_1} \left( \sum_{i=2}^{k} U_{\frac{1}{k}x_1} \right)^2 + \int_{\Omega_\varepsilon} U^{2^*-2}_{\frac{1}{k}x_1} \varphi_{\Lambda,x_1}^2 + \int_{\Omega_\varepsilon} U^{2^*-1}_{\frac{1}{k}x_1} \sum_{i=2}^{k} |\varphi_{\Lambda,x_1}| \right). \]

Note that for \( y \in \Omega_1, |y - x_i| \geq \frac{1}{2} |x_i - x_1| \). Thus
\[ \sum_{i=2}^{k} U_{\frac{1}{k}x_i} \leq C \sum_{i=2}^{k} \frac{1}{(1 + |y - x_1|)^{N-3} |x_1 - x_i|^{N-1}} \]
\[ \leq \frac{1}{(1 + |y - x_1|)^{N-3}} \sum_{i=2}^{k} \frac{1}{|x_1 - x_i|^{N-1}}. \]

Thus
\[ \int_{\Omega_\varepsilon} U^{2^*-2}_{\frac{1}{k}x_1} \left( \sum_{i=2}^{k} U_{\frac{1}{k}x_i} \right)^2 = O \left( (\varepsilon k)^{N-1} \right). \]
\[
\left| \int_{\Omega_{\varepsilon}} U_{\frac{2\varepsilon^2}{N^2}} x_1 \varphi_{\Lambda, x_1} \right| \leq C \int_{\partial \Omega_{\varepsilon}} \frac{1}{(1 + |y|)^{4}} \left( 1 + |y| \right)^{2N-6} \varepsilon^2 |\ln \varepsilon|^{2m} = O(\varepsilon^{2-\beta})
\]
and
\[
\int_{\Omega_{\varepsilon}} U_{\frac{2\varepsilon^2}{N^2}} x_1 \sum_{i=2}^{k} |\varphi_{\Lambda, x_i}| = O\left( \sum_{i=2}^{k} \frac{\varepsilon |\ln \varepsilon|^{m}}{|x_1 - x_i|^{N-3}} \right).
\]
Since
\[
|x_j - x_1| = 2|x_1| \sin \frac{2(j-1)\pi}{k}, \quad j = 2, \ldots, k,
\]
we have
\[
\sum_{j=2}^{k} \frac{1}{|x_j - x_1|^{N-2}} = \frac{1}{(2|x_1|)^{N-2}} \sum_{j=2}^{k} \left( \frac{1}{(\frac{(j-1)}{k})^{N-2}} + \frac{1}{(\frac{N}{k})^{N-2}} \right), \quad \text{if } k \text{ is even;}
\]
\[
\sum_{j=2}^{k} \left( \frac{1}{(\frac{(j-1)}{k})^{N-2}} + \frac{1}{(\frac{N}{k})^{N-2}} \right), \quad \text{if } k \text{ is odd.}
\]
But
\[
0 < c' \leq \frac{\sin \left( \frac{(j-1)\pi}{k} \right)}{\frac{(j-1)\pi}{k}} \leq c'', \quad j = 2, \ldots, \left[ \frac{k}{2} \right].
\]
So, there is a constant \( B_4 > 0 \), such that
\[
\sum_{j=2}^{k} \frac{1}{|x_j - x_1|^{N-2}} = B_4 (\varepsilon k)^{N-2} + O(\varepsilon^{N-2} k).
\]
Using \( \varepsilon = k^{-\frac{N-2}{N-3}} \) and the equality
\[
\int_{\partial \Omega_{\varepsilon}} U_{1,0}^{2\varepsilon} |y|^2 dy = \frac{N-3}{2(N-1)} \int_{\partial \Omega_{\varepsilon}} \left( \varepsilon k \right)^{N-2} \frac{|y|^2}{(1 + |y|^2)^{N-1}} dy
\]
we obtain
\[
I(W) = k(N-2) \left( \int_{\mathbb{R}_+^{N}} U_{1,0}^{\frac{2N-2}{N-3}} + \frac{(N-2)A}{N-3} \right) \int_{\partial \Omega_{\varepsilon}} U_{1,0}^{\frac{2N-2}{N-3}} |y|^2 - \frac{NB_4 A^{N-2} \varepsilon}{2} + o(\varepsilon) \right).
\]
\( \square \)
Recall that for $N = 3$,

$$W_{\lambda,a}(x) = U_{\lambda,a}(x) - \frac{1}{\lambda^\frac{1}{2}|x - a|}(1 - e^{-|x - a|}),$$

$$a_j = (\cos \frac{2(j - 1)\pi}{k}, \sin \frac{2(j - 1)\pi}{k}, 0), \quad j = 1, \ldots, k,$$

and

$$I(W) = \frac{1}{2} \int_\Omega \left( |\nabla \sum_{j=1}^k W_{\lambda,a_j}|^2 + \left( \sum_{j=1}^k W_{\lambda,a_j} \right)^2 \right) - \frac{1}{2} \int_\Omega \left( \sum_{j=1}^k W_{\lambda,a_j} \right)^6.$$

**Proposition A.2.** For $N = 3$, we have

$$I(W) = k \left(D_1 + D_2 \lambda^{-1} \ln \lambda - D_3 \lambda^{-1} k \ln k + D_4 \lambda^{-1} + o(\lambda^{-1})\right),$$

where $D_i$, $i = 1, 2, 3, 4$, is some positive constant.

**Proof.** According to [42], [55] we get for any $i$,

$$I(W_{\lambda,a_i}) = \frac{\sqrt{3}}{4} \pi \frac{3}{4} + \frac{2 \pi \frac{3}{4}}{\sqrt{\lambda b}} \ln \lambda + \frac{E_1}{\lambda} + O\left(\frac{1}{\lambda^2}\right)$$

where $E_1$ is some positive number.

Next, we compute $I(W)$.

$$I(W) = \sum_{j=1}^k I(W_{\lambda,a_j}) + \sum_{i>j} \int_{\partial \Omega} \frac{\partial W_{\lambda,a_i}}{\partial n} W_{\lambda,a_j}$$

$$\quad + \sum_{i>j} \int_{\Omega} \left(U_{\lambda,a_i} W_{\lambda,a_j} - \frac{1}{\lambda^\frac{1}{2}|x - a_i|} W_{\lambda,a_j}\right)$$

$$\quad - 3 \sum_{i\neq j} \int_{\Omega} U_{\lambda,a_i}^5 W_{\lambda,a_j} + 3 \sum_{i\neq j} \int_{\Omega} (U_{\lambda,a_i}^5 - W_{\lambda,a_i}^5) W_{\lambda,a_j}$$

$$\quad - \frac{1}{2} \int_{\Omega} \left( \sum_{j=1}^k W_{\lambda,a_j} \right)^6 - \sum_{j=1}^k W_{\lambda,a_j}^6 - 6 \sum_{j=1}^k W_{\lambda,a_j} W_{\lambda,a_j} \right).$$

Since

$$W_{\lambda,a_j} = \frac{1}{\lambda^\frac{1}{2}|x - a_j|} e^{-|x - a_j|} + O\left(\frac{1}{\lambda^\frac{3}{2}|a_i - a_j|^3}\right), \quad y \in \Omega \setminus B_{\frac{1}{2}|a_i - a_j|}(a_j),$$

it is easy to check that

$$\frac{1}{2} \int_{\Omega} \left( \sum_{j=1}^k W_{\lambda,a_j} \right)^6 - \sum_{j=1}^k W_{\lambda,a_j}^6 - 6 \sum_{i\neq j} W_{\lambda,a_i} W_{\lambda,a_j} \right) = O(\lambda^{-1-\sigma})$$
and for \(i \neq j\),

\[
\int_{\Omega} U_{\lambda, a_i}^5 W_{\lambda, a_j} = \int_{B_{3|a_i - a_j|}(a_j)} U_{\lambda, a_i}^5 W_{\lambda, a_j} + o(\lambda^{-2})
\]

\[
= \frac{B_3}{|a_i - a_j|} + o(\lambda^{-2})
\]

where \(B_3 > 0\) is a constant.

On the other hand, let \(\phi_{\lambda, a_i} = \frac{1 - e^{-|x - a_i|}}{\lambda^{\frac{3}{2}|x - a_i|}}\).

\[
\left| \int_{\Omega} (U_{\lambda, a_i}^5 - W_{\lambda, a_i}^5) W_{\lambda, a_j} \right| \leq C \int_{\Omega} U_{\lambda, a_i}^4 \phi_{\lambda, a_i} W_{\lambda, a_j} + o(\lambda^{-2})
\]

\[
= O\left(\frac{1}{\lambda^{\frac{3}{2}|a_i - a_j|}} \int_{B_{3|a_i - a_j|}(a_j)} U_{\lambda, a_i}^4 \phi_{\lambda, a_i} + o(\lambda^{-2}) \right)
\]

\[
= O\left(\frac{1}{\lambda^{\frac{3}{2}|a_i - a_j|}}\right).
\]

So

\[
I(W) = k I(W_{\lambda, x_1}) + \sum_{i \neq j} \int_{\partial \Omega} \frac{\partial W_{\lambda, a_i}}{\partial n} W_{\lambda, a_j}
\]

\[
+ \sum_{i \neq j} \int_{\Omega} (U_{\lambda, a_i} - \frac{1}{\lambda^{\frac{3}{2}|x - a_i|}}) W_{\lambda, a_j}
\]

\[
- \frac{B_3 k \ln k}{\lambda} + o(\lambda^{-1})
\]

since \(\lambda \in [C_1 e^{B_2 k \ln k}, C_2 e^{B_2 k \ln k}]\).

Obviously,

\[
\left| \int_{\Omega} (U_{\lambda, a_i} - \frac{1}{\lambda^{\frac{3}{2}|x - a_i|}}) W_{\lambda, a_j} \right|
\]

\[
\leq \frac{C}{\lambda^{\frac{3}{2}|a_i - a_j|}} \int_{\Omega \setminus B_{2|a_i - a_j|}(a_j)} \frac{1}{|x - a_j|}
\]

\[
+ \frac{C}{\lambda^{\frac{3}{2}}} \int_{B_{2|a_i - a_j|}(a_j)} \frac{1}{|x - a_i|}
\]

\[
= O\left(\frac{1}{\lambda^{\frac{3}{2}|a_i - a_j|}}\right).
\]
Similarly,

\[
\left| \int_{\partial \Omega} \frac{\partial W_{\lambda, a_i}}{\partial n} W_{\lambda, a_j} \right| \leq C \int_{\partial \Omega} \frac{|W'_{\lambda, a_i}| |x - a_i| W_{\lambda, a_j}}{\lambda^2 |a_i - a_j|^2 + \lambda^{1+\sigma} |a_i - a_j|^{1+\sigma}}.
\]

Thus we finish the computation of \( I(W) \).

\[\square\]

**Appendix B. Basic Estimates**

For each fixed \( i \) and \( j, \ i \neq j \), consider the following function

\[(2.6) \quad g_{ij}(y) = \frac{1}{(1 + |y - x_j|)^{2\alpha}} \frac{1}{(1 + |y - x_i|)^{2\beta}},\]

where \( \alpha \geq 1 \) and \( \beta \geq 1 \) are two constants. The following two lemmas are well-known and we omit the proof. Reader may refer Appendix B in [56].

**Lemma B.1.** For any constant \( 0 \leq \sigma \leq \min(\alpha, \beta) \), there is a constant \( C > 0 \), such that

\[
g_{ij}(y) \leq \frac{C}{|x_i - x_j|^{\sigma}} \left( \frac{1}{(1 + |y - x_j|)^{\alpha + \beta - \sigma}} + \frac{1}{(1 + |y - x_i|)^{\alpha + \beta - \sigma}} \right).
\]

**Lemma B.2.** For any constant \( 0 < \sigma < N - 2 \), there is a constant \( C > 0 \), such that

\[
\int_{\mathbb{R}^N} \frac{1}{|y - z|^{N - 2}} \frac{1}{(1 + |z|)^{2+\sigma}} dz \leq \frac{C}{(1 + |y|)^{\sigma}}.
\]

Let's recall that

\[
\varepsilon = k^{-\frac{N-2}{N-3}} \quad \text{if} \quad N \geq 4, \quad \lambda \in [C_1 e^{\frac{B_2}{2} k \ln k}, C_2 e^{\frac{B_3}{2} k \ln k}] \quad \text{if} \quad N = 3.
\]

and

\[
W = \begin{cases} 
\sum_{j=1}^{k} W_{\lambda, x_j} & \text{if} \quad N \geq 4, \quad \text{and} \quad W_{\lambda, x_j} \leq C U_{1, x_j}^{1-\beta} \\
\sum_{j=1}^{k} W_{\lambda, x_j} & \text{if} \quad N = 3, \quad \text{and} \quad W_{\lambda, x_j} \leq C U_{1, x_j},
\end{cases}
\]

where \( \beta \) is a positive number which can be chosen to be zero as \( N \geq 6 \) and as small as desired as \( N = 4, 5 \).
Lemma B.3. Suppose that
\[
\begin{align*}
\tau &> 1 & \text{and close to } 1, & \quad N = 3, 5, 6, \ldots; \\
\tau &> \frac{1}{2} & \text{and close to } \frac{1}{2}, & \quad N = 4.
\end{align*}
\]
Then there is a small \( \theta > 0 \), such that
\[
\int_{\mathbb{R}^N} \frac{1}{|y - z|^{N-2}} W_{\frac{4}{N-2}}(z) \sum_{j=1}^{k} \frac{1}{(1 + |z - x_j|)^{\frac{N-2}{2} + \tau}} \, dz
\]
\[
\leq C \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{\frac{N-2}{2} + \tau + \theta}} + o(1) \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{\frac{N-2}{2} + \tau}},
\]
where \( o(1) \to 0 \) as \( k \to +\infty \).

Proof. Firstly, we consider \( N \geq 6 \). Then \( \frac{4}{N-2} \leq 1 \). Thus
\[
W_{\frac{4}{N-2}}(z) \leq \sum_{i=1}^{k} \frac{1}{(1 + |z - x_i|)^4}.
\]
So, we obtain
\[
\int_{\mathbb{R}^N} \frac{1}{|y - z|^{N-2}} W_{\frac{4}{N-2}}(z) \sum_{j=1}^{k} \frac{1}{(1 + |z - x_j|)^{\frac{N-2}{2} + \tau}} \, dz
\]
\[
\leq \sum_{j=1}^{k} \int_{\mathbb{R}^N} \frac{1}{|y - z|^{N-2}} \frac{1}{(1 + |z - x_j|)^4 (1 + |y - x_j|)^{\frac{N-2}{2} + \tau + \theta}} \, dz
\]
\[
+ \sum_{j=1}^{k} \sum_{i \neq j} \int_{\mathbb{R}^N} \frac{1}{|y - z|^{N-2}} \frac{1}{(1 + |z - x_i|)^4 (1 + |z - x_j|)^{\frac{N-2}{2} + \tau}} \, dz.
\]
By Lemma B.2, if \( \theta > 0 \) is so small that \( \frac{N-2}{2} + \tau + \theta < N - 2 \), then
\[
\int_{\mathbb{R}^N} \frac{1}{|y - z|^{N-2}} \frac{1}{(1 + |z - x_j|)^{4 + \frac{N-2}{2} + \tau}} \, dz
\]
\[
\leq \int_{\mathbb{R}^N} \frac{1}{|y - z|^{N-2}} \frac{1}{(1 + |z - x_j|)^{2 + \frac{N-2}{2} + \tau + \theta}} \, dz \leq \frac{C}{(1 + |y - x_j|)^{\frac{N-2}{2} + \tau + \theta}}.
\]
On the other hand, it follows from Lemmas B.1 and B.2 that for \( i \neq j \),
\[
\int_{\mathbb{R}^N} \frac{1}{|y - z|^{N-2}} \left( \frac{1}{(1 + |z - x_i|)^4} \right) \frac{1}{(1 + |z - x_j|)^{\frac{N-2+\tau}{2}}} \, dz
\]
\[
\leq \frac{C}{|x_i - x_j|^2} \int_{\mathbb{R}^N} \frac{1}{|y - z|^{N-2}} \left( \frac{1}{(1 + |z - x_i|)^4 + \frac{N-2+\tau}{2}} \right) \frac{1}{(1 + |z - x_j|)^{\frac{N-2+\tau}{2}}} \, dz
\]
\[
\leq \frac{C}{|x_i - x_j|^2} \left( \frac{1}{(1 + |y - x_i|)^{\frac{N-2+\tau}{2}}} + \frac{1}{(1 + |y - x_j|)^{\frac{N-2+\tau}{2}}} \right).
\]

Noting that
\[
\sum_{j \neq i} \frac{1}{|x_i - x_j|^2} \leq C \varepsilon^2 k^2 \sum_{j=1}^{k} \frac{1}{j^2} = O(\varepsilon^{\frac{N-\tau}{2}}),
\]
we obtain
\[
\sum_{j=1}^{k} \sum_{i \neq j} \int_{\mathbb{R}^N} \frac{1}{|y - z|^{N-2}} \left( \frac{1}{(1 + |z - x_i|)^4} \right) \frac{1}{(1 + |z - x_j|)^{\frac{N-2+\tau}{2}}} \, dz
\]
\[
= o(1) \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{\frac{N-2+\tau}{2}}}.
\]

Suppose now that \( N = 5 \). Recall that
\[
\Omega_j = \left\{ y = (y', y'') \in \Omega_\varepsilon : \left\langle \frac{y'}{|y'|}, \frac{x_j}{|x_j|} \right\rangle \geq \cos \frac{\pi}{k} \right\}.
\]

For \( z \in \Omega_1 \), we have \( |z - x_j| \geq |z - x_1| \). Using Lemma B.1, we obtain
\[
\sum_{j=2}^{k} \frac{1}{(1 + |z - x_j|)^{3(1-\beta)}} \leq \frac{1}{(1 + |z - x_1|)^{2}} \sum_{j=2}^{k} \frac{1}{(1 + |z - x_j|)^{1-3\beta}}
\]
\[
\leq \frac{C}{(1 + |z - x_1|)^2} \sum_{j=2}^{k} \frac{1}{|x_j - x_1|^{1-3\beta}} = O(\varepsilon^{\frac{1}{3}-3\beta}) \frac{1}{(1 + |z - x_1|)^2}
\]
since
\[
\sum_{j=2}^{k} \frac{1}{|x_j - x_1|^{1-3\beta}} \leq C(\varepsilon k)^{1-3\beta} \sum_{j=2}^{k} \frac{1}{j^{1-3\beta}} = O(\varepsilon^{1-3\beta} k) = O(\varepsilon^{\frac{1}{3}-3\beta}).
\]

Thus,
\[ W^\frac{1}{t}(z) \leq \frac{\varepsilon^{\frac{t}{u} - 4\beta}}{(1 + |z - x_1|)^{\frac{1}{t}}} . \]

As a result, for \( z \in \Omega_1 \), using Lemma B.1 again, we find

\[
W^\frac{1}{t}(z) \sum_{j=1}^{k} \frac{1}{(1 + |z - x_j|)^{\frac{2}{2} + \frac{r}{2}}} \\
\leq \frac{C}{(1 + |z - x_1|)^{\frac{2}{2} + \frac{r}{2}}} + \frac{\varepsilon^{\frac{t}{u} - 4\beta}}{(1 + |z - x_1|)^{\frac{2}{2} + \frac{r}{2}}} \sum_{j=2}^{k} \frac{1}{|x_j - x_1|^{\frac{1}{t}}} \\
\leq \frac{C}{(1 + |z - x_1|)^{\frac{2}{2} + \frac{r}{2}}} + (k\varepsilon^{\frac{t}{u} - 4\beta}) \frac{C}{(1 + |z - x_1|)^{\frac{2}{2} + \frac{r}{2}}} \\
\leq \frac{C}{(1 + |z - x_1|)^{\frac{2}{2} + \frac{r}{2}}} ,
\]

since

\[-\frac{2}{3} + \frac{1}{3} + \frac{4}{9} - 4\beta > 0.\]

So, we obtain

\[
\int_{\Omega_1} \frac{1}{|y - z|^3} W^\frac{1}{t}(z) \sum_{j=1}^{k} \frac{1}{(1 + |z - x_j|)^{\frac{2}{2} + \frac{r}{2}}} dz \\
\leq \int_{\Omega_1} \frac{1}{|y - z|^3} \frac{C}{(1 + |z - x_1|)^{\frac{2}{2} + \frac{r}{2}}} dz \leq \frac{C}{(1 + |y - x_1|)^{\frac{2}{2} + \frac{r}{2}}} ,
\]

which gives

\[
\int_{\Omega_1} \frac{1}{|y - z|^3} W^\frac{1}{t}(z) \sum_{j=1}^{k} \frac{1}{(1 + |z - x_j|)^{\frac{2}{2} + \frac{r}{2}}} dz = \sum_{i=1}^{k} \int_{\Omega_1} \frac{1}{|y - z|^3} W^\frac{1}{t}(z) \sum_{j=1}^{k} \frac{1}{(1 + |z - x_j|)^{\frac{2}{2} + \frac{r}{2}}} dz \\
\leq \sum_{i=1}^{k} \frac{C}{(1 + |y - x_i|)^{\frac{2}{2} + \frac{r}{2}}} .
\]
Similarly, we have for \( N = 4, z \in \Omega_1, \)
\[
\sum_{j=2}^{k} \frac{1}{(1 + |z - x_j|)^{2(1-\beta)}} \leq C \frac{1}{(1 + |z - x_1|)^{\frac{3}{2}-2\beta}}
\]
and
\[
W^2(z) \sum_{j=1}^{k} \frac{1}{(1 + |z - x_j|)^{1+\tau}} \leq C \frac{1}{(1 + |z - x_1|)^{2+1+\tau+\frac{1}{2}-4\beta}}
\]
which gives
\[
\int_{\Omega} \frac{1}{|y - z|^2} W^2(z) \sum_{j=1}^{k} \frac{1}{(1 + |z - x_j|)^{1+\tau}} \, dz = \sum_{i=1}^{k} \int_{\Omega_i} \frac{1}{|y - z|^2} W^2(z) \sum_{j=1}^{k} \frac{1}{(1 + |z - x_j|)^{1+\tau}} \, dz \leq \sum_{i=1}^{k} \frac{C}{(1 + |y - x_i|)^{\frac{3}{2}-4\beta+1+\tau}}.
\]
For \( N = 3, z \in \Omega_1, \) since \( k^n \lambda^{-\alpha} = o(1) \) for any \( n > 0 \) and \( \alpha > 0 \) as \( k \to +\infty, \)
\[
\sum_{j=2}^{k} \frac{1}{(1 + |z - x_j|)^{1-\alpha}} \leq C \frac{1}{(1 + |z - x_1|)^{1-\alpha}}
\]
and
\[
W^4(z) \sum_{j=1}^{k} \frac{1}{(1 + |z - x_j|)^{\frac{3}{2}+\tau}} \leq C \frac{1}{(1 + |z - x_1|)^{2+\frac{1}{2}+\tau+2-5\alpha}}
\]
which gives
\[
\int_{\Omega} \frac{1}{|y - z|^4} W^4(z) \sum_{j=1}^{k} \frac{1}{(1 + |z - x_j|)^{\frac{3}{2}+\tau}} \, dz = \sum_{i=1}^{k} \int_{\Omega_i} \frac{1}{|y - z|^4} W^4(z) \sum_{j=1}^{k} \frac{1}{(1 + |z - x_j|)^{\frac{3}{2}+\tau}} \, dz \leq \sum_{i=1}^{k} \frac{C}{(1 + |y - x_i|)^{\frac{3}{2}+\tau-5\alpha+2}}.
\]
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Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong
E-mail address: lpwang@math.cuhk.edu.hk

Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong
E-mail address: wei@math.cuhk.edu.hk

School of Mathematics, Statistics and Computer Science, The University of New England, Armidale, NSW 2351, Australia
E-mail address: syan@turing.une.edu.au