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On solutions with polynomial growth to an autonomous nonlinear elliptic problem

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Abstract

We study the following nonlinear elliptic problem

$$-\Delta u = F^{'}(u) \text{ in } \mathbb{R}^n$$

where $F(u)$ is a periodic function. Moser (1986) showed that for any minimal and nonself-intersecting solution, there exist $\alpha \in \mathbb{R}^n$ and $C > 0$ such that

$$(*) \qquad |u - \alpha \cdot x| \leq C.$$

He also showed the existence of solutions with any prescribed $\alpha \in \mathbb{R}^n$. In this note, we first prove that any solution satisfying (*) with nonzero vector α must be one dimensional. Then we show that in $\mathbb{R}^2$, for any positive integer $d \geq 1$ there exists a solution with polynomial growth $|x|^d$.

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1 Introduction and Main Results

In search of analogue of Aubry-Mather theory for quasilinear partial differential equations in $\mathbb{R}^n$, Moser [6] studied the following equation

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \mathcal{F}_{p_i}(x, u, Du) - \mathcal{F}_u(x, u, Du) = 0 \tag{1.1}$$

which is the Euler-Lagrangian equation for the functional

$$\int_{\mathbb{R}^n} \mathcal{F}(x, u, Du) dx \tag{1.2}$$

where $\mathcal{F}$ is $1-$periodic in all variables $x_1, ..., x_n$ and u, elliptic and of quadratic growth in $p = Du$.

A solution $u(x)$ of (1.1) is called *minimal* if

$$\int_{\mathbb{R}^n} \Big[\mathcal{F}(x, u + \varphi, Du + D\varphi)dx - \mathcal{F}(x, u, Du)\Big] dx \geq 0, \ \forall \ \varphi \in C_0^\infty(\mathbb{R}^n). \tag{1.3}$$

A solution of (1.1) is said to be *without self intersections* or WSI if (i) for each $j \in \mathbb{Z}^n$ and $j_{n+1} \in \mathbb{Z}, u(x + j) - u(x) - j_{n+1}$ does not change sign for $x \in \mathbb{R}^n$, or (ii) for some $j \in \mathbb{Z}^n$ and $j_{n+1} \in \mathbb{Z}, u(x + j) \equiv u(x) + j_{n+1}$.

For minimal and WSI solutions to (1.1), Moser [6] showed: (1) There exists a unique vector $\alpha \in \mathbb{R}^n$, the so-called *rotation vector* and a constant C, such that

$$|u(x) - \alpha \cdot x| \leq C, \ \forall \ x \in \mathbb{R}^n. \tag{1.4}$$

(2) Conversely, for every vector $\alpha \in \mathbb{R}^n$ there exists a minimal solution u with rotation vector α and a constant C and satisfying (1.4).

Moser's paper [6] has received lots of attention in the literature. Among many results, we mention that Bangert [2] showed the existence of heteroclinic states under some gap conditions, and Rabinowitz and Stredulinsky [8, 9] developed variational gluing methods for mixed states of Allen-Cahn type equations. (See also [10] for non-autonomous case.) There is also a strong connection between Moser's problem and De Giorgi's conjecture. See Farina and Valdinoci [5]. For the latest developments, we refer to the survey paper by Rabinowitz [7] and the references therein.

In this note, we consider the autonomous Moser's problem, namely we study the following problem

$$-\Delta u = F^{'}(u) \text{ in } \mathbb{R}^n \tag{1.5}$$

where $F(u)$ is a smooth periodic function. A typical example is the so-called sine-Gordon nonlinearity $F(u) = 1 - \cos(u)$.

Our first result is a classification theorem on solutions to (1.5) satisfying (1.4).

Theorem 1.1 *Let $u \in C^2(\mathbb{R}^n)$ be a solution of (1.5). Assume that there exist a nonzero vector $\alpha \in \mathbb{R}^n$ and a constant $C > 0$ such that*

$$|u(x) - \alpha \cdot x| \leq C \quad \text{for} \quad \forall x \in \mathbb{R}^n. \tag{1.6}$$

Then there is a function $v \in C^2(\mathbb{R}^n)$ such that $u(x) = v(\alpha \cdot x)$.

In the above theorem, $\alpha \neq 0$ is necessary. In fact for Allen-Cahn or Sine-Gordon equations, there are bounded solutions with multiple transitions ([1, 3, 4]). Theorem 1.1 also holds when $-\Delta u = f(u)$ where f is periodic. Note that it can be directly shown that one dimensional solutions satisfying (1.6) have no self-intersection.

Theorem 1.1 has been proved by Farina and Valdinoci [5] under the minimality condition. Here we have removed the minimality assumption. Theorem 1.1 shows that unbounded solutions to (1.5) with linear growth are all one dimensional. Notice that $\alpha \cdot x$ is the simplest nonconstant harmonic function in $\mathbb{R}^n$. Based on this, J. Byeon and P. Rabinowitz [1] asked

Question: given any harmonic function, w, on $\mathbb{R}^n$, is there a solution, u, of (1.1) with $||u - w||_{L^\infty(\mathbb{R}^n)}$ bounded?

The following theorem answers the question partially.

Theorem 1.2 *Let $n = 2$ and $d \geq 2$. Assume that $F(u)$ is even. Let $\varphi(x, y)$ be the real part of the harmonic polynomial z^d. (Here $z = x + iy$.) Then there exists a solution to (1.5), enjoying the same symmetry as $\varphi(x, y)$ and satisfying*

$$|u(x, y) - \varphi(x, y)| \leq C(1 + |z|)^{\frac{3}{2}}. \tag{1.7}$$

Furthermore, for $d \geq 3$ we also have the following improved upper bound:

$$|u(x, y) - \varphi(x, y)| \leq C(1 + |z|)^{2 - \frac{d}{2}}. \tag{1.8}$$

Remark 1.1 *If $d \geq 4$, then $\frac{d}{2} \geq 2$. Thus for $d \geq 4$, we answered Byeon-Rabinowitz's question affirmatively, in the autonomous setting (1.5). Note also that for $d > 4$, we have better decay estimates. The key to obtain (1.8) is some oscillatory integral estimate (see (4.6) below). For $d = 2$ or 3, this estimate is not sufficient. We believe that the L^∞ bound should also hold for $d = 2, 3$.*

Remark 1.2 *Another interesting question is whether or not the evenness condition is necessary.*

In the rest of the paper, we prove Theorem 1.1 in Section 2, the estimate (1.7) of Theorem 1.2 in Section 3 and the better estimate (1.8) of Theorem 1.2 in Section 4 respectively.

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[1] Private discussion

2 Proof of Theorem 1.1

In this section, we prove Theorem 1.1 by the method of moving planes.

Without loss of generality, assume that $|\alpha| = 1$ and α is the x_n direction. We use the notation that $x = (x', x_n)$ where $x' \in \mathbb{R}^{n-1}$. For any unit vector e such that $e \cdot \alpha > 0$, we will prove that for every $t \geq 0$,

$$u(x + te) \geq u(x) \quad \text{for all} \quad x \in \mathbb{R}^n. \tag{2.1}$$

This then implies that $e \cdot \nabla u \geq 0$ in $\mathbb{R}^n$. By continuity, this also holds for e and $-e$, if $e \cdot \alpha = 0$, which then implies that $e \cdot \nabla u \equiv 0$ and that u depends only on $\alpha \cdot x$.

For any $t > 0$, define $u^t(x) = u(x + te)$. First we note that, since $e_n = e \cdot \alpha > 0$, for t large, by (1.6),

$$u^t(x) \geq x_n + te_n - C \geq x_n + C \geq u(x).$$

Hence we can define

$$t_0 := \inf\{t : \forall s \geq t, (2.1) \ \text{ holds}\}.$$

Assuming that $t_0 > 0$, we will get a contradiction. First note that $u^{t_0} \geq u$ by continuity. It is impossible to have $u^{t_0} \equiv u$, because this would imply that u is t_0 periodic in the e direction, which contradicts (1.6). ($e \cdot \alpha > 0$ implies that u goes to infinity when x goes to infinity along the e direction.) Hence by the strong maximum principle we have

$$u^{t_0} > u. \tag{2.2}$$

By the definition of t_0, there exists $t_k < t_0$ such that

$$\inf_{\mathbb{R}^n}(u^{t_k} - u) < 0.$$

In particular, there exists $x_k \in \mathbb{R}^n$ such that

$$(u^{t_k} - u)(x_k) < 0. \tag{2.3}$$

Assume the period of $F(u)$ is T. By (1.6), we can take a constant a_k, which is a multiple of T such that

$$u_k(x) := u(x + x_k) - a_k$$

satisfies $|u_k(0)| \leq T$. (2.2) and (2.3) imply respectively that

$$u_k^{t_0} > u_k. \tag{2.4}$$

$$(u_k^{t_k} - u_k)(0) < 0. \tag{2.5}$$

Note that u_k still satisfies (1.6) with a larger constant $2C + T$, which is independent of k. By the elliptic regularity, u_k is uniformly bounded in $C^3(B_R(0))$ for any $R > 0$.

Hence we can take a subsequence of u_k such that u_k converges to u_∞ in $C^2(B_R(0))$ for any $R>0$. Letting $k\to+\infty$ in (2.4) and (2.5), we get

$$u_\infty^{t_0}\ge u_\infty,\quad u_\infty^{t_0}(0)=u_\infty(0).$$

By the strong maximum principle, $u_\infty^{t_0}\equiv u_\infty$. That is, u_∞ is t_0 periodic along the direction e. Since u_∞ satisfies (1.6), this is a contradiction and also finishes the proof of Theorem 1.1.

3 Proof of Theorem 1.2

In this section, we prove the existence of solutions satisfying estimate (1.7) in Theorem 1.2.

We denote $z=x+iy\in\mathbb{C}$. We also identify $z=re^{i\theta}$ with $(x,y)\in\mathbb{R}^2$. Let $d\ge 2$ be a positive integer and $\varphi(x,y)=Re(z^d)$. Denote G the rotation of order $2d$. Note that $\varphi(Gz)=-\varphi(z)$.

Let $D=\{-\frac{\pi}{2d}<\theta<\frac{\pi}{2d}\}$ be a nodal domain of φ. For every $R>0$, take $D_R=B_R(0)\cap D$ and u^R to be a minimizer of the functional

$$\int_{D_R}\frac{1}{2}|\nabla u|^2+F(u),$$

with the Dirichlet boundary condition $u=\varphi$ on ∂D_R.

First, the minimizer exists since $F(u)$ is a bounded periodic function. Second, we may assume that $u^R\ge 0$ in D_R since otherwise we may replace the minimizer with $|u^R|$ (noting that F is even and $F(|u|)=F(u)$). Since $F'(u)=0$, the strong maximum principle implies that $u_R>0$ in D_R. Once again by the oddness of $F'(u)$ and the fact that $F'(0)=0$, by rotational symmetry of $\frac{2\pi}{d}$, u^R can be extended to $B_R(0)$ and it satisfies the equation $-\Delta u=F^{'}(u)$ in $B_R(0)$. By construction, u^R has the same symmetry as φ, that is, $u^R(Gz)=-u^R(z)$ for $z\in B_R(0)$.[2] In particular, the nodal domain of u^R is the same with φ and $\{u^R=0\}$ is composed by $2d$ rays with the form $re^{i\frac{k\pi}{2d}}$ for $k=1,3,\cdots,4d-1$ and $r\in[0,R]$.

For any $r\in(0,R)$, let φ^r be the solution of

$$\begin{cases}\Delta\varphi^r=0, & \text{in } B_r,\\ \varphi^r=u^R, & \text{on } \partial B_r.\end{cases}$$

Since u^R has the same symmetry as φ, by the uniqueness of the solution to the above problem, φ^r has the same symmetry as φ, and $\{\varphi^r=0\}$ is composed by $2d$ rays of the form $re^{i\frac{k\pi}{2d}}$ for $d=1,3,\cdots,4d-1$ and $r\in[0,r]$. This implies that $\varphi^r=u^R$ on ∂D_r and φ^r is also the harmonic extension of u^R from ∂D_r to D^r.

[2] Another method to get u^R is to find a minimizer of $\int_{B_R}\frac{1}{2}|\nabla u|^2+F(u)$ in the invariant class $\{u=\varphi\ \text{ on }\ \partial B_R,\ \text{ and }\ u(Gz)=-u(z)\}$. u^R can be proved to satisfy $-\Delta u=F^{'}(u)$ in $B_R(0)$ by the heat flow method. Note that because $F(u)$ is even, the invariant class is positively invariant by the heat flow.

Lemma 3.1 *There exists a constant C, independent of r and R, such that*

$$\int_{B_r(0)} |\nabla\varphi^r - \nabla u^R|^2 \le Cr^2. \tag{3.1}$$

Since we expect u^R grows like $|z|^d$ and $|\nabla u^R|$ grows like $|z|^{d-1}$ with $d \ge 2$, this estimate implies that u^R and φ^r are close to each other (after a rescaling) at large scale. Below we will use this inequality to estimate the error $u^R - \varphi^r$.

Proof. By the minimality of u^R, we have

$$\int_{D_r} \frac{1}{2}|\nabla u^R|^2 + F(u^R) \le \int_{D_r} \frac{1}{2}|\nabla\varphi^r|^2 + F(\varphi^r)$$

which implies

$$\int_{D_r} \left[|\nabla u^R|^2 - |\nabla\varphi^r|^2\right] \le Cr^2 \tag{3.2}$$

since F is a bounded periodic function.

On the other hand, an integration by parts using the fact that $u^R = \varphi^r$ on ∂D_r shows that

$$\int_{D_r} |\nabla\varphi^r - \nabla u^R|^2 = \int_{D_r} |\nabla u^R|^2 - |\nabla\varphi^r|^2.$$

Substituting the above equality into the inequality (3.2), we get (3.1).

Lemma 3.2 *There exists a constant C, independent of r and R, such that for all $0 < r < R$,*

$$\sup_{B_{r/2}(0)} |\varphi^r - u^R| \le Cr^{3/2}.$$

Proof. We will assume that r is large enough. Let $\bar{u}^r(z) := \frac{1}{r^d}u^R(rz)$ and $\bar{\varphi}^r(z) := \frac{1}{r^d}\varphi^r(rz)$ for $z \in B_1(0)$. By (3.1),

$$\int_{B_1(0)} |\nabla\bar{\varphi}^r - \nabla\bar{u}^r|^2 \le Cr^{2-2d}.$$

Since $\bar{u}^r = \bar{\varphi}^r$ on $\partial B_1(0)$, by the Poincare inequality,

$$\int_{B_1(0)} |\bar{\varphi}^r - \bar{u}^r|^2 \le Cr^{2-2d}. \tag{3.3}$$

Note that

$$|\Delta(\bar{\varphi}^r - \bar{u}^r)| = |r^{2-d}F'(r^d\bar{u}^r)| \le Cr^{2-d}. \tag{3.4}$$

Take a $r_0 \in (3/4, 1)$ such that

$$\int_{\partial B_{r_0}(0)} |\bar{\varphi}^r - \bar{u}^r|^2 \le 8Cr^{2-2d}$$

which is possible because of (3.3).

Take the decomposition $\bar{\varphi}^r - \bar{u}^r = h + g$, where h is harmonic in $B_{r_0}(0)$ and $h = \bar{\varphi}^r - \bar{u}^r$ on $\partial B_{r_0}(0)$. By the mean value property of harmonic functions, we have

$$\sup_{B_{5/8}(0)} |h| \leq Cr^{1-d}.$$

Since $g = 0$ on $\partial B_{r_0}(0)$ and

$$|\Delta g| \leq Cr^{2-d} = -\Delta(\frac{Cr^{2-d}}{4}(r_0^2 - |z|^2),$$

comparison principle implies

$$\sup_{B_{5/8}(0)} |g| \leq Cr^{2-d}.$$

Combining these two we obtain

$$\sup_{B_{5/8}(0)} |\bar{\varphi}^r - \bar{u}^r| \leq Cr^{2-d}.$$

Combining with (3.4), by elliptic estimates we see

$$\sup_{B_{9/16}(0)} |\nabla(\bar{\varphi}^r - \bar{u}^r)| \leq Cr^{2-d}. \tag{3.5}$$

By (3.3),

$$|\{|\bar{\varphi}^r - \bar{u}^r| > r^{3/2-d}\} \cap B_{9/16}(0)| \leq Cr^{-1}.$$

In particular, for any ball $B_{Mr^{-1/2}}(x) \subset B_{9/16}(0)$ where M is a large constant, there exists $y \in B_{Mr^{-1/2}}(x) \cap \{|\bar{\varphi}^r - \bar{u}^r| < r^{3/2-d}\}$. Integrating along the segment from y to x and using (3.5), we get

$$|\bar{\varphi}^r(x) - \bar{u}^r(x)| \leq Cr^{3/2-d} \quad \text{for any} \quad x \in B_{1/2}(0).$$

Rescaling back we can finish the proof.

Lemma 3.3 *There exists a constant C, independent of r and R, such that for any $r \in (0, R/2)$,*

$$\sup_{B_{r/2}(0)} |\nabla^2 \varphi^r - \nabla^2 \varphi^{2r}| \leq \frac{C}{r}. \tag{3.6}$$

Proof. By (3.1) we get

$$\int_{B_r(0)} |\nabla \varphi^r - \nabla \varphi^{2r}|^2 \leq Cr^2.$$

Since both φ^r and φ^{2r} are harmonic, by interior gradient estimates we obtain the claim.

Lemma 3.4 *For any $r \in (0, R)$,*

$$\sup_{B_{r/2}(0)} |\varphi - u^R| \leq C r^{3/2}.$$

Proof. Take an i_0 such that $R/2 < 2^{i_0} r \leq R$. Checking the proof of the previous lemma we see

$$\sup_{B_{2^{i_0-1}r}(0)} |\nabla^2 \varphi^{2^{i_0} r} - \nabla^2 \varphi| \leq \frac{C}{2^{i_0} r}.$$

Adding this and (3.6) from $i = 1$ to $i = i_0$ we get

$$\sup_{B_{r/2}(0)} |\nabla^2 \varphi^r - \nabla^2 \varphi| \leq \frac{C}{r}. \tag{3.7}$$

Since for each r, φ^r has the same symmetry as φ and it is harmonic (recall that the degree of φ, $d \geq 2$), we have

$$\varphi^r(0) = \varphi(0) = 0, \quad \nabla \varphi^r(0) = \nabla \varphi(0) = 0.$$

Integrating (3.7) twice we obtain,

$$\sup_{B_{r/2}(0)} |\varphi^r - \varphi| \leq C r. \tag{3.8}$$

This combined with Lemma 3.2 implies the required claim.

A direct corollary of this lemma is the uniform boundedness of u^R on any compact set. Hence we can take the limit $u_\infty := \lim_{R \to +\infty} u^R$ which is a solution of (1.5) on the entire $\mathbb{R}^2$, enjoying the same symmetry as φ, $\{u_\infty > 0\} = \{\varphi > 0\}$, and satisfies

$$|u_\infty(x, y) - \varphi(x, y)| \leq C(|x| + |y|)^{3/2}.$$

In particular, u_∞ is unbounded and grows at least quadratically.

This proves Theorem 1.2.

Remark 3.1 *By [1], there exists a second solution u of (1.5) satisfying the symmetry $u(Gz) = -u(z)$, which is bounded in $\mathbb{R}^2$. For example, if $F(u) = 1 + \cos u$, we can construct a solution such that $-\pi < u < \pi$ in $\mathbb{R}^2$. In fact, in this case, if we modify $F(u)$ outside $[-\pi, \pi]$ to get a standard double-well potential, it becomes exactly the problem studied in [1, 3]. The bounded solution produced by this method takes values in $(-\pi, \pi)$ and it is still the solution of the original problem (1.5).*

4 Proof of the Improvement Estimate (1.8)

Let u be the solution constructed in the previous section. Written in the exponential polar coordinate $(r, \theta) = (e^t, \theta)$, u satisfies

$$\partial_t^2 u + \partial_\theta^2 u + e^{2t} f(u) = 0.$$

Let $v(t,\theta) = e^{-dt}u(t,\theta)$. Then $v(t,\theta)$ satisfies

$$\partial_t^2 v + 2d\partial_t v + d^2 v + \partial_\theta^2 v + e^{(2-d)t} f(e^{dt}v) = 0. \tag{4.1}$$

By the error bound established in the previous section, for $t \geq 0$,

$$|v(t,\theta) - \cos(d\theta)| \leq Ce^{(3/2-d)t}. \tag{4.2}$$

By interior gradient estimates, for any $\varepsilon > 0$ there exists a constant C such that for any ball $B_1(t,\theta) \subset \mathbb{R}\times\mathbb{S}^1$ (with respect to the product metric on $\mathbb{R}\times\mathbb{S}^1$) and $u \in C^2(B_1(t,\theta))$,

$$\sup_{B_{1/2}(t,\theta)} |\partial_\theta u| + |\partial_t u| \leq \varepsilon \sup_{B_1(t,\theta)} |\partial_t^2 u + \partial_\theta^2 u| + \frac{C}{\varepsilon} \sup_{B_1(t,\theta)} |u|. \tag{4.3}$$

Since $|e^{(2-d)t} f(e^{dt}v)| \leq Ce^{(2-d)t}$, applying (4.3) to $v - \cos(d\theta)$ with $\varepsilon = e^{-\frac{t}{4}}$ we get a constant C such that for all $t \geq 0$,

$$|\partial_\theta(v(t,\theta) - \cos(d\theta))| + |\partial_t v(t,\theta)| \leq Ce^{(\frac{7}{4}-d)t}. \tag{4.4}$$

Differentiating (4.1) in t we get

$$\partial_t^2\partial_t v + 2d\partial_t\partial_t v + d^2\partial_t v + \partial_\theta^2\partial_t v + e^{2t} f'(e^{dt}v)\partial_t v = 0.$$

By the bound on $\partial_t v$, we have $|e^{2t} f'(e^{dt}v)\partial_t v| \leq Ce^{(\frac{7}{4}+2-d)t}$. By taking $\varepsilon = e^{-t}$ in (4.3) we obtain

$$|\partial_\theta\partial_t v(t,\theta)| + |\partial_t^2 v(t,\theta)| \leq Ce^{(\frac{7}{4}+1-d)t}.$$

Substituting this and (4.2), (4.4) into (4.1), we get

$$|\partial_\theta^2(v(t,\theta) - \cos(d\theta))| \leq Ce^{(\frac{7}{4}+1-d)t}. \tag{4.5}$$

If $d \geq 3$, this gives the exponential convergence of v to $\cos(d\theta)$ in $C^2(\mathbb{S}^1)$.

Below we assume that $d \geq 3$.

Let $v(t,\theta) = \sum_{j\geq 0} c_j(t)\cos(j\theta)$ be the Fourier decomposition of $v(t,\cdot)$. Note that because v is even in θ, there are only terms $\cos(j\theta)$ appearing in this decomposition. Moreover, by our construction,

$$\sum_{j\geq 0} c_j(t)\cos(j\theta + \frac{j\pi}{d}) = v(t,\theta + \frac{\pi}{d}) = -v(t,\theta) = -\sum_{j\geq 0} c_j(t)\cos(j\theta),$$

so $c_j(t) = 0$ if there is no nonnegative integer k such that $j = (2k+1)d$. In particular,

$$c_j(t) \equiv 0 \quad \text{for} \quad j < d.$$

Hence below we concentrate on those $c_j(t)$ with $j = d$ and $j \geq 3d$.

Multiplying (4.1) by $\cos(j\theta)$ and integrating, we get the equation for $c_j(t)$

$$\partial_t^2 c_j + 2d\partial_t c_j + (d^2 - j^2)c_j + e^{(2-d)t}\left(\int_0^{2\pi} f(e^{dt}v)\cos(j\theta)d\theta\right) = 0.$$

Denote $g_j(t) = e^{(2-d)t}\left(\int_0^{2\pi} f(e^{dt}v)\cos(j\theta)d\theta\right)$. Since $\cos(d\theta)$ has only non-degenerate critical points and $v(t,\theta) \to \cos(d\theta)$ in $C^2(\mathbb{S}^1)$ as $t \to +\infty$ (cf. (4.4) and (4.5)), for t large, $v(t,\cdot)$ has only non-degenerate critical points. By the oscillatory integral estimate ([Section 8.1, [11]]) we get a constant C_j such that

$$\int_0^{2\pi} f(e^{dt}v)\cos(j\theta)d\theta = O(e^{-\frac{dt}{2}}), \quad |g_j(t)| \leq C_j e^{(2-\frac{3d}{2})t}. \tag{4.6}$$

For $t \geq 0$, we have the representation formula

$$c_j(t) = A_j e^{-(d+j)t} + B_j e^{-(d-j)t} + e^{-(d-j)t}\int_t^{+\infty} e^{(d-j)s}\int_0^s e^{(d+j)(\tau-s)}g_j(\tau)d\tau ds. \tag{4.7}$$

Substituting (4.6) into this and integrating directly, we see the last integral is bounded by $\frac{C_j}{j^2}e^{(2-\frac{3d}{2})t}$. In particular, for $j = d$,

$$|c_d(t) - B_d| \leq Ce^{(2-\frac{3d}{2})t}. \tag{4.8}$$

Here, by (4.2), $B_d = 1$.

It remains to estimate $v^{\parallel} := v - c_d(t)\cos(d\theta)$. First note that for $j > d$, $|c_j(t)| \leq Ce^{(3/2-d)t}$ by (4.2). Hence we must have $B_j = 0$. Next we have

Lemma 4.1 *For t large, when measured in $L^\infty(\mathbb{S}^1)$,*

$$v^\flat := \sum_{j>d} e^{-(d-j)t}\int_t^{+\infty} e^{(d-j)s}\int_0^s e^{(d+j)(\tau-s)}g_j(\tau)\cos(j\theta)d\tau ds = O(e^{(2-\frac{3d}{2})t}).$$

Proof. Direct calculations give, for $t + \tau < 2s$,

$$\sum_{j>d} e^{-(d-j)t}e^{(d-j)s}e^{(d+j)(\tau-s)}\cos(j\theta) = e^{t-(2d+2)s+(2d+1)\tau}\frac{\cos(d+1)\theta - e^{t+\tau-2s}\cos d\theta}{1 - 2e^{t+\tau-2s}\cos\theta + e^{2(t+\tau-2s)}}.$$

Using this kernel, $v^\flat$ can be written as

$$v^\flat(t,\theta) = \int_t^{+\infty}\int_0^s\int_0^{2\pi} e^{t-(2d+2)s+(2d+1)\tau}\frac{\cos(d+1)\theta - e^{t+\tau-2s}\cos d\theta}{1 - 2e^{t+\tau-2s}\cos\theta + e^{2(t+\tau-2s)}}g(\tau,\theta)d\theta d\tau ds,$$

where $g(\tau,\theta) = e^{(2-d)\tau}f(e^{d\tau}v(\tau,\theta))$.

Note that

$$\frac{\cos(d+1)\theta - e^{t+\tau-2s}\cos d\theta}{1-2e^{t+\tau-2s}\cos\theta + e^{2(t+\tau-2s)}}$$

is uniformly bounded in $C^3(\mathbb{S}^1)$ when $t+\tau-2s \le 0$. Hence by the oscillatory integral estimate ([11]),

$$\int_0^{2\pi} \frac{\cos(d+1)\theta - e^{t+\tau-2s}\cos d\theta}{1-2e^{t+\tau-2s}\cos\theta + e^{2(t+\tau-2s)}} g(\tau,\theta)d\theta = O(e^{(2-\frac{3d}{2})\tau}).$$

Substituting this into the above representation formula of $v^\flat$ we finish the proof.

Lemma 4.2 *For t large, when measured in $L^\infty(\mathbb{S}^1)$,*

$$v^\sharp := v^\parallel - v^\flat = O(e^{-\frac{d-1+\sqrt{17d^2-2d+1}}{2}t}).$$

Proof. Direct calculations show that

$$(\partial_t^2 + 2d\partial_t + \partial_\theta^2 + d^2)v^\flat + e^{(2-d)t}f(e^{dt}v(t,\theta)) = 0.$$

Hence

$$(\partial_t^2 + 2d\partial_t + \partial_\theta^2 + d^2)v^\sharp = 0.$$

Multiplying by $v^\sharp$ and integrating on $\mathbb{S}^1$, we get

$$\frac{d^2}{dt^2}\int_0^{2\pi}(v^\sharp)^2 d\theta + (2d-2)\frac{d}{dt}\int_0^{2\pi}(v^\sharp)^2 d\theta + 2d^2\int_0^{2\pi}(v^\sharp)^2 d\theta - 2\int_0^{2\pi}(\partial_\theta v^\sharp)^2 d\theta = 0.$$

By our construction, for any $t \ge 0$, $v^\sharp$ is orthogonal to $\cos(j\theta), \sin(j\theta)$ for every $|j| \le 3d-1$. Hence

$$\int_0^{2\pi}(\partial_\theta v^\sharp)^2 d\theta \ge (3d)^2\int_0^{2\pi}(v^\sharp)^2 d\theta,$$

and

$$L\int_0^{2\pi}(v^\sharp)^2 d\theta := \frac{d^2}{dt^2}\int_0^{2\pi}(v^\sharp)^2 d\theta + (2d-2)\frac{d}{dt}\int_0^{2\pi}(v^\sharp)^2 d\theta - 16d^2\int_0^{2\pi}(v^\sharp)^2 d\theta \ge 0.$$

By (4.2), (4.8) and the previous lemma, we have the decay estimate

$$\int_0^{2\pi} v^\sharp(t,\theta)^2 d\theta \le Ce^{(3-2d)t}.$$

Let $\left(\int_0^{2\pi}(v^\sharp(0,\theta)^2 d\theta\right) e^{-(d-1+\sqrt{17d^2-2d+1})t}$ be a solution of $Lh = 0$ which has the same boundary value at $t=0$ and $t=+\infty$. By the comparison principle we get for any $t \ge 0$,

$$\int_0^{2\pi} v^\sharp(t,\theta)^2 d\theta \le (\int_0^{2\pi} v^\sharp(0)^2 d\theta)e^{-(d-1+\sqrt{17d^2-2d+1})t}.$$

Then by applying standard elliptic estimates to $v^\sharp$ we get its $L^\infty(\mathbb{S}^1)$ bound.

For $d \geq 2$,

$$\frac{d-1+\sqrt{17d^2-2d+1}}{2} \geq \frac{3d}{2} - 2.$$

Putting the above estimates together we see for every $d \geq 3$,

$$\sup |v(t,\theta) - \cos(d\theta)| \leq Ce^{-(\frac{3d}{2}-2)t}.$$

Coming back to u, we get a constant C such that for all $z \in \mathbb{C}$

$$|u(z) - \varphi(z)| \leq C(1+|z|)^{-(\frac{d}{2}-2)}$$

which proves (1.8).

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