

A toroidal tube solution of a nonlocal geometric problem

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Abstract

The Ohta-Kawasaki theory for block copolymer morphology and the Gierer-Meinhardt theory for morphogenesis in cell development both give rise to a nonlocal geometric problem. One seeks a set in \mathbb{R}^3 which satisfies an equation that links the mean curvature of the boundary of the set to the Newtonian potential of the set. An axisymmetric, torus shaped, tube like solution exists in \mathbb{R}^3 if the lone parameter of the problem is sufficiently large. A cross section of the torus is small and the distance from the center of the cross section to the axis of symmetry is large. The solution is stable in the class of axisymmetric sets in a particular sense.

Key words. torus, tube, axisymmetry, nonlocal geometric problem, mean curvature, Newtonian potential, Approximate solution, Lyapunov-Schmidt reduction.

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Abbreviated title. Toroidal tube solution.

1 Introduction

Ring shaped objects in space have been observed in many physical systems. Known as the vortex ring in fluid dynamics, it is a region of rotating fluid where the flow pattern takes on a toroidal shape [3]. In a quantum fluid, a vortex ring is formed by a loop of poloidal quantized flow pattern. It was detected in superfluid helium by Rayfield and Reif [27], and more recently in Bose-Einstein condensates by Anderson, *et al*, [2].

Our study is inspired by recent experimental findings in the block copolymer research. Block copolymers are soft condensed materials characterized by fluid-like disorder on the molecular scale and a high degree of order on a longer length scale. Various morphological phases have been observed as orderly outcomes of the microphase separation property [4]. In 2004, Pochan, *et al*, produced a morphological phase of toroidal supramolecule assemblies, using a triblock copolymer [26].

In this paper we show the existence of a torus shaped set as a solution of a nonlocal geometric problem. Given two parameters $m > 0$ and $\gamma > 0$, find a set E in \mathbb{R}^3 and a number λ such that the volume of E is m and on the boundary of E the equation

$$\mathcal{H}(\partial E) + \gamma \mathcal{N}(E) = \lambda \tag{1.1}$$

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holds. In (1.1) $\mathcal{H}(\partial E)$ stands for the mean curvature of ∂E (the boundary of E) and the nonlocal \mathcal{N} operator is given by the Newtonian potential

$$\mathcal{N}(E)(x) = \int_E \frac{1}{4\pi|x-y|} dy. \quad (1.2)$$

The unknown constant λ is the Lagrange multiplier associated with the constraint that the volume of E is m . If the problem is considered in \mathbb{R}^2 , then $\mathcal{H}(\partial E)$ is the curvature of the curve ∂E and \mathcal{N} takes the form

$$\mathcal{N}(E)(x) = \int_E \frac{1}{2\pi} \log \frac{1}{|x-y|} dy. \quad (1.3)$$

The equation (1.1) admits a variational structure. It is the Euler-Lagrange equation of the functional

$$\mathcal{J}(E) = \frac{1}{2}\mathcal{P}(E) + \frac{\gamma}{2} \int_E \mathcal{N}(E)(x) dx \quad (1.4)$$

under the constraint (1.6). Here $\mathcal{P}(E)$ stands for the perimeter of E , i.e. the area of ∂E .

There is an analogy to (1.1) on bounded domains. Let $D \subset \mathbb{R}^3$ be a bounded and smooth domain and $a \in (0, 1)$ and $\gamma > 0$ two parameters. We look for a subset E of D whose volume is a times the volume of D , and a number λ such that the equation

$$\mathcal{H}(\partial_D E) + \gamma(-\Delta)^{-1}(\chi_E - a) = \lambda \quad (1.5)$$

holds on $\partial_D E$, the part of the boundary of E that is in D . If ∂E meets ∂D , we require that they meet orthogonally. Here $(-\Delta)^{-1}$ is the inverse of $-\Delta$ and $-\Delta$ acts on functions with the zero Neumann boundary condition on ∂D .

The problem (1.5) may be deduced from the Ohta-Kawasaki theory for block copolymers [23] as a strong segregation limit. This connection was first observed by Nishiura and Ohnishi [22]. In [29] this limit was shown to be a Γ -limit [6, 20, 19, 16]; see the appendix for more details.

This problem is also connected to the Gierer-Meinhardt system with saturation. The Gierer-Meinhardt system is an activator-inhibitor type reaction diffusion PDE system. It was first introduced to model the head formation of hydra [13]. More generally it can be used to study morphogenesis in cell development [17, 18, 21]. It is a minimal model that provides a theoretical bridge between observations on the one hand and the deduction of the underlying molecular-genetic mechanisms on the other hand.

Due to the lack of a variational structure, the connection cannot be interpreted as a Γ -convergence property. However an asymptotic analysis can be employed to show formally that in a limiting case the activator of a steady state of the system is close to a characteristic function of a set E , which satisfies the same equation (1.5); see the appendix.

The equation (1.5) was solved completely in one-dimension where the domain is a bounded interval [29]. There are countably many solutions and every solution is a local minimizer of the energy functional associated with (1.5). Among these local minimizers, two or four of them are global minimizers. In higher dimensions many types of solutions to (1.5) have been found [30, 31, 33, 32, 34, 35, 24, 14, 28, 15, 5]. The global minimizer has not been completely identified [1]. There are several ways to associate dynamic models to the stationary problem (1.5). One case was studied in [10] where every one-dimensional initial state converges to one of the local minimizers found in [29].

There is a strong connection between (1.1) and (1.5). One may use solutions of (1.1) as building blocks to construct solutions of (1.5). For instance, in \mathbb{R}^2 , the unit disc is a solution of (1.1). This is because the curvature of the boundary of the disc, the unit circle, has constant curvature, and the Newtonian potential of the unit disc is a radially symmetric function and therefore a constant on the boundary of the disc.

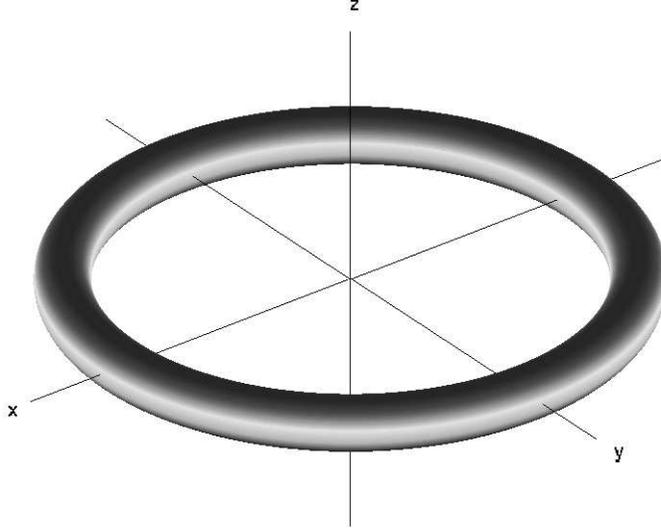


Figure 1: A toroidal tube like solution.

The sum of these two constants is again a constant and hence (1.1) is satisfied by the unit disc. Using this unit disc we found solutions of (1.5) with multiple components, each of which is close to a small disc (a scaled down version of the unit disc) [32]. When the number of the components is large, they form a hexagonal pattern. This gives a mathematical explanation for the cylindrical morphology phase in diblock copolymers. More discussion on the relationship between (1.1) and (1.5) is given in the last section.

In this paper we construct a solution which has the shape of a toroidal tube as depicted in Figure 1. It is a set obtained by rotating a small, approximately round disc in the yz -plane, far from the z -axis, about the z -axis.

The two parameters m and γ can be reduced to one. Without the loss of generality we assume that

$$m = 1. \quad (1.6)$$

The case $m \neq 1$ can be reduced to the case $m = 1$ by a change of space variable and a change of γ accordingly. Therefore the problem (1.1) has intrinsically only one parameter, which we take to be γ .

We define a function $f = f(\gamma)$ from $(0, \infty)$ to $(0, \infty)$ through its inverse

$$\gamma = \frac{2}{f^3 \log \frac{1}{2\pi^2 f^3}}. \quad (1.7)$$

Note that $f(\gamma) \rightarrow 0$ as $\gamma \rightarrow \infty$. Our main result is the following existence theorem.

Theorem 1.1 *When γ is large enough, (1.1) admits a solution of the unit volume which is close to a set enclosed by a torus. The torus is obtained by rotating a circle in the yz -plane about the z -axis. Let p_γ be the distance from the center of the circle to the z -axis and q_γ be the radius of the circle. Then $2\pi^2 p_\gamma q_\gamma^2 = 1$ and $\lim_{\gamma \rightarrow \infty} \frac{q_\gamma}{f(\gamma)} = 1$ and $\lim_{\gamma \rightarrow \infty} 2\pi^2 f^2(\gamma) p_\gamma = 1$.*

In the proof of the theorem we will see in what sense the solution is close to the region enclosed by the torus.

The theorem is proved by mathematically rigorous singular perturbation techniques developed in [33, 32, 34] for this type of nonlocal geometric problems. The proof consists of several steps. In Section 2 we

represent the nonlocal part of (1.1) by an integral operator whose kernel is a Green's function in the class of axisymmetric functions. The type of the singularity of this Green's function is determined in Lemma 2.1. In Section 3 a family of approximate solutions, sets bounded by tori, are studied. Lemma 3.2 shows how much error an approximate solution generates when plugged into (1.1). The value of \mathcal{J} at an approximate solution is estimated in Lemma 3.3. In Section 4 we specify a way to perturb each approximate solution and recast the problem (1.1) in terms of the perturbation. Lemma 4.1 contains the properties of the linearized operator at each approximate solution. In Section 5 we carry out a Lyapunov-Schmidt type argument to find a particular perturbation to each approximate solution so that the exact solution of (1.1) is in the family of the perturbed approximate solutions. In the last section we identify the exact solution in this family. The connections between the Ohta-Kawasaki theory, the Gierer-Meinhardt system, and (1.5) are explained in the appendix.

In this paper C denotes a constant independent of the parameters in the construction process. Its value may vary from place to place. Functions that are 2π -periodic are considered to be defined on the unit circle S^1 . For example $H^2(S^1)$ denotes the $W^{2,2}$ Sobolev space of 2π -periodic functions.

2 Green's function

Axisymmetric objects are best described by the cylindrical coordinates, i.e. $x = r \cos \sigma$, $y = r \sin \sigma$, and $z = z$, so that they are independent of the angle variable σ . The variables r and z are in

$$\mathbb{R}_+^2 = \{(r, z) : r > 0, z \in \mathbb{R}\}. \quad (2.1)$$

An axisymmetric object E in \mathbb{R}^3 is identified by its description in \mathbb{R}_+^2 , so for simplicity we view E as a subset in \mathbb{R}_+^2 . On such a set the \mathcal{N} operator takes the form

$$\mathcal{N}(E)(r, z) = \int_E G(r, z, s, t) ds dt \quad (2.2)$$

where the function G is our Green's function. By (1.2) G may be represented by

$$G(r, z, s, t) = \frac{s}{4\pi} \int_0^{2\pi} \frac{d\sigma}{\sqrt{r^2 + s^2 - 2rs \cos \sigma + (z - t)^2}}. \quad (2.3)$$

Moreover for each $(s, t) \in \mathbb{R}_+^2$ as a function of (r, z) , $G(\cdot, s, t)$ satisfies

$$-\Delta G(\cdot, s, t) - \frac{\partial}{r \partial r} G(\cdot, s, t) = \delta_{(s,t)} \quad \text{in } \mathbb{R}_+^2, \quad \frac{\partial}{\partial r} G(0, z, s, t) = 0 \quad \forall z \in \mathbb{R}. \quad (2.4)$$

Lemma 2.1 *For each $(s, t) \in \mathbb{R}_+^2$*

$$\begin{aligned} G(r, z, s, t) &= \frac{1}{2\pi} \log \frac{1}{|(r, z) - (s, t)|} + G_1(r, z, s, t) \\ &= \frac{1}{2\pi} \log \frac{1}{|(r, z) - (s, t)|} - \frac{r-s}{4\pi s} \log \frac{1}{|(r, z) - (s, t)|} + G_2(r, z, s, t) \\ &= \frac{1}{2\pi} \log \frac{1}{|(r, z) - (s, t)|} - \frac{r-s}{4\pi s} \log \frac{1}{|(r, z) - (s, t)|} + \frac{5(r-s)^2 - (z-t)^2}{32\pi s^2} \log \frac{1}{|(r, z) - (s, t)|} \\ &\quad + G_3(r, z, s, t) \end{aligned}$$

where, as functions of (r, z) , $G_1(\cdot, s, t) \in C_{loc}^\alpha(\mathbb{R}_+^2)$, $G_2(\cdot, s, t) \in C_{loc}^{1,\alpha}(\mathbb{R}_+^2)$, and $G_3(\cdot, s, t) \in C_{loc}^{2,\alpha}(\mathbb{R}_+^2)$ for all $\alpha \in (0, 1)$.

Proof. Since $-\frac{r-s}{4\pi s} \log \frac{1}{|(r,z)-(s,t)|} \in C_{loc}^\alpha(\mathbb{R}_+^2)$ and $\frac{5(r-s)^2-(z-t)^2}{32\pi s^2} \log \frac{1}{|(r,z)-(s,t)|} \in C_{loc}^{1,\alpha}(\mathbb{R}_+^2)$, it suffices to show that $G_3(\cdot, s, t) \in C_{loc}^{2,\alpha}(\mathbb{R}_+^2)$. Compute

$$\begin{aligned}
& \left(-\Delta - \frac{1}{r\partial r} \right) \left[\frac{1}{2\pi} \log \frac{1}{|(r,z)-(s,t)|} - \frac{r-s}{4\pi s} \log \frac{1}{|(r,z)-(s,t)|} + \frac{5(r-s)^2-(z-t)^2}{32\pi s^2} \log \frac{1}{|(r,z)-(s,t)|} \right] \\
&= -\Delta \left(\frac{1}{2\pi} \log \frac{1}{|(r,z)-(s,t)|} \right) - \frac{1}{r\partial r} \left(\frac{1}{2\pi} \log \frac{1}{|(r,z)-(s,t)|} \right) \\
&\quad + \Delta \left(\frac{r-s}{4\pi s} \log \frac{1}{|(r,z)-(s,t)|} \right) + \frac{1}{r\partial r} \left(\frac{r-s}{4\pi s} \log \frac{1}{|(r,z)-(s,t)|} \right) \\
&\quad - \Delta \left(\frac{5(r-s)^2-(z-t)^2}{32\pi s^2} \log \frac{1}{|(r,z)-(s,t)|} \right) - \frac{1}{r\partial r} \left(\frac{5(r-s)^2-(z-t)^2}{32\pi s^2} \log \frac{1}{|(r,z)-(s,t)|} \right) \\
&= \delta_{(s,t)} + \frac{r-s}{2\pi r|(r,z)-(s,t)|^2} \\
&\quad - \frac{r-s}{2\pi s|(r,z)-(s,t)|^2} + \frac{1}{4\pi r s} \log \frac{1}{|(r,z)-(s,t)|} - \frac{(r-s)^2}{4\pi r s|(r,z)-(s,t)|^2} \\
&\quad - \frac{1}{4\pi s^2} \log \frac{1}{|(r,z)-(s,t)|} + \frac{5(r-s)^2-(z-t)^2}{8\pi s^2|(r,z)-(s,t)|^2} \\
&\quad - \frac{5(r-s)}{16\pi r s^2} \log \frac{1}{|(r,z)-(s,t)|} + \frac{5(r-s)^3-(z-t)^2(r-s)}{32\pi r s^2|(r,z)-(s,t)|^2}.
\end{aligned}$$

Note that the above is $\delta_{(s,t)}$ plus a $C_{loc}^\alpha(\mathbb{R}_+^2)$ function, since

$$\begin{aligned}
\frac{r-s}{2\pi r|(r,z)-(s,t)|^2} - \frac{r-s}{2\pi s|(r,z)-(s,t)|^2} - \frac{(r-s)^2}{4\pi r s|(r,z)-(s,t)|^2} + \frac{5(r-s)^2-(z-t)^2}{8\pi s^2|(r,z)-(s,t)|^2} &\in C_{loc}^{0,1}(\mathbb{R}_+^2) \\
\frac{1}{4\pi r s} \log \frac{1}{|(r,z)-(s,t)|} - \frac{1}{4\pi s^2} \log \frac{1}{|(r,z)-(s,t)|} &\in C_{loc}^\alpha(\mathbb{R}_+^2) \\
-\frac{5(r-s)}{16\pi r s^2} \log \frac{1}{|(r,z)-(s,t)|} &\in C_{loc}^\alpha(\mathbb{R}_+^2) \\
\frac{5(r-s)^3-(z-t)^2(r-s)}{32\pi r s^2|(r,z)-(s,t)|^2} &\in C_{loc}^{0,1}(\mathbb{R}_+^2).
\end{aligned}$$

Here $C_{loc}^{0,1}(\mathbb{R}_+^2)$ denotes the space of locally Lipschitz continuous functions on \mathbb{R}_+^2 . Because of (2.4), we deduce

$$(-\Delta - \frac{1}{r\partial r})G_3(\cdot, s, t) \in C_{loc}^\alpha(\mathbb{R}_+^2).$$

The elliptic regularity theory implies that $G_3(\cdot, s, t) \in C_{loc}^{2,\alpha}(\mathbb{R}_+^2)$. \square

Some useful properties of G , G_1 , G_2 and G_3 are listed below.

Lemma 2.2 *Denote the derivatives of $G = G(r, z, s, t)$ (or G_1, G_2, G_3) with respect to r, z, s , and t by D_1G, D_2G, D_3G and D_4G respectively.*

1. For $\lambda > 0$, $G(\lambda r, \lambda z, \lambda s, \lambda t) = G(r, z, s, t)$.
2. For $r \neq s$, $D_2G(r, 0, s, 0) = 0$.
3. For $r > 0$ and $s > 0$, $D_2G_3(r, 0, s, 0) = 0$.

Proof. Parts 1 and 2 follow directly from (2.3). To prove part 3, we first assume $r \neq s$ and deduce from $D_2(r, 0, s, 0)G = 0$ in part 2 that

$$\begin{aligned}
D_2G_3(r, 0, s, 0) &= -D_2 \Big|_{z=t=0} \left[\frac{1}{2\pi} \log \frac{1}{|(r, z) - (s, t)|} - \frac{r-s}{4\pi s} \log \frac{1}{|(r, z) - (s, t)|} \right. \\
&\quad \left. + \frac{5(r-s)^2 - (z-t)^2}{32\pi s^2} \log \frac{1}{|(r, z) - (s, t)|} \right] \\
&= - \left[-\frac{z-t}{2\pi|(r, z) - (s, t)|^2} + \frac{(r-s)(z-t)}{4\pi s|(r, z) - (s, t)|^2} \right. \\
&\quad \left. - \frac{(z-t)}{16\pi s^2} \log \frac{1}{|(r, z) - (s, t)|} - \frac{5(r-s)^2(z-t) - (z-t)^3}{32\pi s^2|(r, z) - (s, t)|^2} \right]_{z=t=0} \\
&= 0.
\end{aligned}$$

Since $G_3(\cdot, s, t) \in C_{loc}^{2,\alpha}(R_+^2)$, $D_2G_3(r, 0, s, 0) = 0$ remains valid even if $r = s$. \square

3 Approximate solutions

A perfect torus is obtained by rotating a circle of radius q about the z -axis. Due to the translation invariance of this problem in the z -direction, without the loss of generality the center of the circle is assumed to be in the xy -plane. Suppose that the distance of the center of the circle to the z -axis is p where $p > q > 0$. The region inside the torus is denoted by E_0 whose volume is $2\pi^2 pq^2$, and because of the constraint (1.6) we have

$$2\pi^2 pq^2 = 1. \quad (3.1)$$

The region E_0 is regarded as an approximate solution, parametrized by p . The next lemma estimates the effect of \mathcal{N} on E_0 .

Lemma 3.1 $\mathcal{N}(E_0)$ on E_0 is

$$\begin{aligned}
\mathcal{N}(E_0)(p + q\hat{h} \cos \theta, q\hat{h} \sin \theta) &= \frac{q^2}{2} \log \frac{1}{q} + q^2 \left[\pi G_3(p, 0, p, 0) + \frac{1}{4} - \frac{\hat{h}^2}{4} \right] \\
&\quad - \frac{q^3}{4p} \left(\log \frac{1}{q} \right) \hat{h} \cos \theta + \frac{q^3}{p} \left[\pi p D_1 G_3(p, 0, p, 0) \hat{h} + \frac{\hat{h}^3}{16} \right] \cos \theta + O\left(\frac{q^4}{p^2} \log \frac{p}{q}\right),
\end{aligned}$$

where $\hat{h} \in [0, 1]$.

Proof. In the Green's function $G(r, z, s, t)$ let

$$(r, z) = (p + h \cos \theta, h \sin \theta), \quad (s, t) = (p + \rho \cos \omega, \rho \sin \omega) \quad (3.2)$$

where $h, \rho \in [0, q]$, so both (r, z) and (s, t) are points in the the E_0 . Then by Lemma 2.1

$$\begin{aligned}
&\mathcal{N}(E_0)(p + h \cos \theta, h \sin \theta) \\
&= \int_0^q \int_0^{2\pi} G(p + h \cos \theta, h \sin \theta, p + \rho \cos \omega, \rho \sin \omega) \rho d\omega d\rho \\
&= \int_0^q \int_0^{2\pi} \frac{1}{2\pi} \log \frac{1}{\sqrt{h^2 + \rho^2 - 2h\rho \cos(\theta - \omega)}} \left[1 - \frac{h \cos \theta - \rho \cos \omega}{2(p + \rho \cos \omega)} \right. \\
&\quad \left. + \frac{5(h \cos \theta - \rho \cos \omega)^2 - (h \sin \theta - \rho \sin \omega)^2}{16(p + \rho \cos \omega)^2} \right] \rho d\omega d\rho
\end{aligned} \quad (3.3)$$

$$\begin{aligned}
& + \int_0^q \int_0^{2\pi} G_3(p + h \cos \theta, h \sin \theta, p + \rho \cos \omega, \rho \sin \omega) \rho \, d\omega d\rho \\
& = I + II
\end{aligned} \tag{3.4}$$

where I and II in (3.4) are given by the two integrals above.

Because of the scaling property

$$G_3(\lambda r, \lambda z, \lambda s, \lambda t) = G_3(r, z, s, t) - \frac{1}{2\pi} \log \frac{1}{\lambda} \left[1 - \frac{r-s}{2s} + \frac{5(r-s)^2 - (z-t)^2}{16s^2} \right] \tag{3.5}$$

which follows from Lemma 2.2 part 1 and Lemma 2.1, II becomes

$$\begin{aligned}
II & = -\frac{1}{2\pi} \log \frac{1}{p} \int_0^q \int_0^{2\pi} \left[1 - \frac{h \cos \theta - \rho \cos \omega}{2(p + \rho \cos \omega)} + \frac{5(h \cos \theta - \rho \cos \omega)^2 - (h \sin \theta - \rho \sin \omega)^2}{16(p + \rho \cos \omega)^2} \right] \rho \, d\omega d\rho \\
& + \int_0^q \int_0^{2\pi} G_3\left(1 + \frac{h}{p} \cos \theta, \frac{h}{p} \sin \theta, 1 + \frac{\rho}{p} \cos \omega, \frac{\rho}{p} \sin \omega\right) \rho \, d\omega d\rho \\
& = II_I + II_{II}
\end{aligned} \tag{3.6}$$

where II_I and II_{II} are given by the two integrals above.

We introduce the rescaled variables \hat{h} and $\hat{\rho}$ by

$$h = q\hat{h}, \quad \rho = q\hat{\rho}. \tag{3.7}$$

Then I becomes

$$\begin{aligned}
I & = \frac{q^2}{2\pi} \log \frac{1}{q} \int_0^1 \int_0^{2\pi} \left[1 - \frac{\hat{h} \cos \theta - \hat{\rho} \cos \omega}{2(p/q + \hat{\rho} \cos \omega)} + \frac{5(\hat{h} \cos \theta - \hat{\rho} \cos \omega)^2 - (\hat{h} \sin \theta - \hat{\rho} \sin \omega)^2}{16(p/q + \hat{\rho} \cos \omega)^2} \right] \hat{\rho} \, d\omega d\hat{\rho} \\
& + \frac{q^2}{2\pi} \int_0^1 \int_0^{2\pi} \log \frac{1}{\sqrt{\hat{h}^2 + \hat{\rho}^2 - 2\hat{h}\hat{\rho} \cos(\theta - \omega)}} \left[1 - \frac{\hat{h} \cos \theta - \hat{\rho} \cos \omega}{2(p/q + \hat{\rho} \cos \omega)} \right. \\
& \left. + \frac{5(\hat{h} \cos \theta - \hat{\rho} \cos \omega)^2 - (\hat{h} \sin \theta - \hat{\rho} \sin \omega)^2}{16(p/q + \hat{\rho} \cos \omega)^2} \right] \hat{\rho} \, d\omega d\hat{\rho} \\
& = I_I + I_{II}
\end{aligned} \tag{3.8}$$

where I_I and I_{II} are the two integrals above.

Combining I_I and II_I we find

$$\begin{aligned}
I_I + II_I & = \frac{q^2}{2\pi} \log \frac{p}{q} \int_0^1 \int_0^{2\pi} \left[1 - \frac{\hat{h} \cos \theta - \hat{\rho} \cos \omega}{2(p/q + \hat{\rho} \cos \omega)} + \frac{5(\hat{h} \cos \theta - \hat{\rho} \cos \omega)^2 - (\hat{h} \sin \theta - \hat{\rho} \sin \omega)^2}{16(p/q + \hat{\rho} \cos \omega)^2} \right] \hat{\rho} \, d\omega d\hat{\rho} \\
& = \frac{q^2}{2\pi} \log \frac{p}{q} [A + B + C]
\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}
A & = \int_0^1 \int_0^{2\pi} \hat{\rho} \, d\omega d\hat{\rho} = \pi \\
B & = \int_0^1 \int_0^{2\pi} -\frac{\hat{h} \cos \theta - \hat{\rho} \cos \omega}{2(p/q + \hat{\rho} \cos \omega)} \hat{\rho} \, d\omega d\hat{\rho} \\
& = -\frac{q}{2p} \int_0^1 \int_0^{2\pi} \left(\hat{h} \cos \theta - \hat{\rho} \cos \omega + O\left(\frac{q}{p}\right) \right) \hat{\rho} \, d\omega d\hat{\rho} = -\frac{\pi q}{2p} \hat{h} \cos \theta + O\left(\frac{q^2}{p^2}\right) \\
C & = \int_0^1 \int_0^{2\pi} \frac{5(\hat{h} \cos \theta - \hat{\rho} \cos \omega)^2 - (\hat{h} \sin \theta - \hat{\rho} \sin \omega)^2}{16(p/q + \hat{\rho} \cos \omega)^2} \hat{\rho} \, d\omega d\hat{\rho} = O\left(\frac{q^2}{p^2}\right).
\end{aligned}$$

This shows

$$I_I + II_I = \frac{q^2}{2} \log \frac{p}{q} - \frac{q^3}{4p} \left(\log \frac{p}{q} \right) \hat{h} \cos \theta + O\left(\frac{q^4}{p^2} \log \frac{p}{q}\right) \quad (3.11)$$

The term II_I must be estimated carefully. We write

$$II_I = \frac{q^2}{2\pi} \int_0^1 \int_0^{2\pi} \log \frac{1}{\sqrt{\hat{h}^2 + \hat{\rho}^2 - 2\hat{h}\hat{\rho} \cos(\theta - \omega)}} \left[1 - \frac{\hat{h} \cos \theta - \hat{\rho} \cos \omega}{2(p/q + \hat{\rho} \cos \omega)} \right. \quad (3.12)$$

$$\left. + \frac{5(\hat{h} \cos \theta - \hat{\rho} \cos \omega)^2 - (\hat{h} \sin \theta - \hat{\rho} \sin \omega)^2}{16(p/q + \hat{\rho} \cos \omega)^2} \right] \hat{\rho} d\omega d\hat{\rho} \\ = \frac{q^2}{2\pi} [\tilde{A} + \tilde{B} + \tilde{C}] \quad (3.13)$$

where

$$\tilde{A} = \int_0^1 \int_0^{2\pi} \log \frac{1}{\sqrt{\hat{h}^2 + \hat{\rho}^2 - 2\hat{h}\hat{\rho} \cos(\theta - \omega)}} \hat{\rho} d\omega d\hat{\rho} \\ \tilde{B} = - \int_0^1 \int_0^{2\pi} \log \frac{1}{\sqrt{\hat{h}^2 + \hat{\rho}^2 - 2\hat{h}\hat{\rho} \cos(\theta - \omega)}} \frac{\hat{h} \cos \theta - \hat{\rho} \cos \omega}{2(p/q + \hat{\rho} \cos \omega)} \hat{\rho} d\omega d\hat{\rho} \\ \tilde{C} = \int_0^1 \int_0^{2\pi} \log \frac{1}{\sqrt{\hat{h}^2 + \hat{\rho}^2 - 2\hat{h}\hat{\rho} \cos(\theta - \omega)}} \frac{5(\hat{h} \cos \theta - \hat{\rho} \cos \omega)^2 - (\hat{h} \sin \theta - \hat{\rho} \sin \omega)^2}{16(p/q + \hat{\rho} \cos \omega)^2} \hat{\rho} d\omega d\hat{\rho} = O\left(\frac{q^2}{p^2}\right).$$

To study \tilde{A} and \tilde{B} we need the trigonometric series

$$\log \frac{1}{|1 - \beta e^{i\omega}|} = \sum_{n=1}^{\infty} \frac{\beta^n \cos n\omega}{n} \quad (3.14)$$

for $\beta \in [0, 1]$. One consequence of (3.14) is that

$$\int_0^{2\pi} \log \frac{1}{\sqrt{\hat{h}^2 + \hat{\rho}^2 - 2\hat{h}\hat{\rho} \cos(\theta - \omega)}} d\omega = \begin{cases} -2\pi \log \hat{h} & \text{if } \hat{\rho} < \hat{h} \\ -2\pi \log \hat{\rho} & \text{if } \hat{\rho} \geq \hat{h} \end{cases} \quad (3.15)$$

which implies that

$$\tilde{A} = -2\pi \int_0^{\hat{h}} (\log \tilde{h}) \hat{\rho} d\hat{\rho} - 2\pi \int_{\hat{h}}^1 (\log \tilde{\rho}) \hat{\rho} d\hat{\rho} = 2\pi \left(\frac{1}{4} - \frac{\hat{h}^2}{4} \right) \quad (3.16)$$

Regarding \tilde{B} we write

$$\tilde{B} = -\frac{q}{2p} \int_0^1 \int_0^{2\pi} \log \frac{1}{\sqrt{\hat{h}^2 + \hat{\rho}^2 - 2\hat{h}\hat{\rho} \cos(\theta - \omega)}} \hat{h} \cos \theta \hat{\rho} d\omega d\hat{\rho} \\ + \frac{q}{2p} \int_0^1 \int_0^{2\pi} \log \frac{1}{\sqrt{\hat{h}^2 + \hat{\rho}^2 - 2\hat{h}\hat{\rho} \cos(\theta - \omega)}} \hat{\rho} \cos \omega \hat{\rho} d\omega d\hat{\rho} \\ + O\left(\frac{q^2}{p^2}\right) \\ = \tilde{B}_1 + \tilde{B}_2 + O\left(\frac{q^2}{p^2}\right) \quad (3.17)$$

where \tilde{B}_1 and \tilde{B}_2 are given by the two integrals above.

Using (3.15) again, we derive

$$\tilde{B}_1 = -\frac{q}{2p} \left[\int_0^{\hat{h}} (-2\pi \log \hat{h}) \hat{h} \cos \theta \hat{\rho} d\hat{\rho} + \int_{\hat{h}}^1 (-2\pi \log \hat{\rho}) \hat{h} \cos \theta \hat{\rho} d\hat{\rho} \right] = \frac{\pi q}{p} \cos \theta \left(\frac{\hat{h}^3}{4} - \frac{\hat{h}}{4} \right) \quad (3.18)$$

For \tilde{B}_2 we note that

$$\begin{aligned} \tilde{B}_2 &= \frac{q}{2p} \int_0^1 \int_0^{2\pi} \log \frac{1}{\sqrt{\hat{h}^2 + \hat{\rho}^2 - 2\hat{h}\hat{\rho} \cos(\theta - \omega)}} \hat{\rho} \cos \omega \hat{\rho} d\omega d\hat{\rho} \\ &= \frac{q}{2p} \int_0^1 \int_0^{2\pi} \log \frac{1}{\sqrt{\hat{h}^2 + \hat{\rho}^2 - 2\hat{h}\hat{\rho} \cos(\theta - \omega)}} \hat{\rho} \cos(\theta - \omega) \cos \theta \hat{\rho} d\omega d\hat{\rho} \\ &\quad + \frac{q}{2p} \int_0^1 \int_0^{2\pi} \log \frac{1}{\sqrt{\hat{h}^2 + \hat{\rho}^2 - 2\hat{h}\hat{\rho} \cos(\theta - \omega)}} \hat{\rho} \sin(\theta - \omega) \sin \theta \hat{\rho} d\omega d\hat{\rho} \\ &= \frac{q}{2p} \int_0^1 \int_0^{2\pi} \log \frac{1}{\sqrt{\hat{h}^2 + \hat{\rho}^2 - 2\hat{h}\hat{\rho} \cos(\theta - \omega)}} \hat{\rho} \cos(\theta - \omega) \cos \theta \hat{\rho} d\omega d\hat{\rho}. \end{aligned}$$

To reach the last line, we have deduced from the absence of $\sin n\omega$ terms in (3.14) that

$$\frac{q}{2p} \int_0^1 \int_0^{2\pi} \log \frac{1}{\sqrt{\hat{h}^2 + \hat{\rho}^2 - 2\hat{h}\hat{\rho} \cos(\theta - \omega)}} \hat{\rho} \sin(\theta - \omega) \sin \theta \hat{\rho} d\omega d\hat{\rho} = 0.$$

Another consequence of (3.14) is

$$\begin{aligned} &\int_0^{2\pi} \log \frac{1}{\sqrt{\hat{h}^2 + \hat{\rho}^2 - 2\hat{h}\hat{\rho} \cos(\theta - \omega)}} \cos(\theta - \omega) d\omega \\ &= \int_0^{2\pi} \log \frac{1}{\sqrt{1 + \beta^2 - 2\beta \cos \omega}} \cos \omega d\omega \\ &= \int_0^{2\pi} \beta \cos^2 \omega d\omega = \pi\beta \quad \text{where } \beta = \begin{cases} \frac{\hat{\rho}}{\hat{h}} & \text{if } \hat{\rho} < \hat{h} \\ \frac{\hat{h}}{\hat{\rho}} & \text{if } \hat{\rho} \geq \hat{h} \end{cases} \end{aligned} \quad (3.19)$$

Therefore

$$\tilde{B}_2 = \frac{\pi q}{2p} \left[\int_0^{\hat{h}} \frac{\hat{\rho}}{\hat{h}} \hat{\rho}^2 \cos \theta d\hat{\rho} + \int_{\hat{h}}^1 \frac{\hat{h}}{\hat{\rho}} \hat{\rho}^2 \cos \theta d\hat{\rho} \right] = \frac{\pi q}{2p} \cos \theta \left(\frac{\hat{h}}{2} - \frac{\hat{h}^3}{4} \right) \quad (3.20)$$

Following (3.18) and (3.20) we deduce

$$\tilde{B} = \frac{\pi q}{8p} \hat{h}^3 \cos \theta + O\left(\frac{q^2}{p^2}\right). \quad (3.21)$$

By (3.13), (3.16), and (3.21) we find

$$I_{II} = q^2 \left(\frac{1}{4} - \frac{\hat{h}^2}{4} \right) + \frac{q^3}{16p} \hat{h}^3 \cos \theta + O\left(\frac{q^4}{p^2}\right). \quad (3.22)$$

The estimation of II_{II} is straightforward. Because $G_3(\cdot, \cdot, s, t)$ is in $C^{2,\alpha}(\mathbb{R}_+^2)$ and $D_2G_3(1, 0, 1, 0) = 0$ according to Lemma 2.2 part 3,

$$\begin{aligned}
II_{II} &= q^2 \int_0^1 \int_0^{2\pi} G_3\left(1 + \frac{q}{p}\hat{h}\cos\theta, \frac{q}{p}\hat{h}\sin\theta, 1 + \frac{q}{p}\hat{\rho}\cos\omega, \frac{q}{p}\hat{\rho}\sin\omega\right)\hat{\rho}d\omega d\hat{\rho} \\
&= q^2 \int_0^1 \int_0^{2\pi} \left[G_3(1, 0, 1, 0) + D_1G_3(1, 0, 1, 0)\frac{q}{p}\hat{h}\cos\theta + D_2G_3(1, 0, 1, 0)\frac{q}{p}\hat{h}\sin\theta \right. \\
&\quad \left. + D_3G_3(1, 0, 1, 0)\frac{q}{p}\hat{\rho}\cos\omega + D_4G_3(1, 0, 1, 0)\frac{q}{p}\hat{\rho}\sin\omega + O\left(\frac{q^2}{p^2}\right) \right] \hat{\rho}d\omega d\hat{\rho} \\
&= q^2 \int_0^1 \int_0^{2\pi} \left[G_3(1, 0, 1, 0) + D_1G_3(1, 0, 1, 0)\frac{q}{p}\hat{h}\cos\theta \right] \hat{\rho}d\omega d\hat{\rho} + O\left(\frac{q^4}{p^2}\right) \\
&= \pi q^2 G_3(1, 0, 1, 0) + \frac{\pi q^3}{p} D_1G_3(1, 0, 1, 0)\hat{h}\cos\theta + O\left(\frac{q^4}{p^2}\right). \tag{3.23}
\end{aligned}$$

We now combine (3.11), (3.22), and (3.23) to conclude that

$$\begin{aligned}
\mathcal{N}(E_0)(p + h\cos\theta, h\sin\theta) &= \frac{q^2}{2} \log \frac{p}{q} + q^2 \left[\frac{1}{4} - \frac{\hat{h}^2}{4} + \pi G_3(1, 0, 1, 0) \right] \\
&\quad - \frac{q^3}{4p} \left(\log \frac{p}{q} \right) \hat{h}\cos\theta + \frac{q^3}{p} \left[\frac{\hat{h}^3}{16} + \pi D_1G_3(1, 0, 1, 0)\hat{h} \right] \cos\theta \\
&\quad + O\left(\frac{q^4}{p^2} \log \frac{p}{q}\right). \tag{3.24}
\end{aligned}$$

Since

$$G_3(p, 0, p, 0) = G_3(1, 0, 1, 0) - \frac{1}{2\pi} \log \frac{1}{p} \quad \text{and} \quad pD_1G_3(p, 0, p, 0) = D_1G_3(1, 0, 1, 0) + \frac{1}{4\pi} \log \frac{1}{p}$$

by (3.5), the lemma follows from (3.24). \square

We insert E_0 into the left side of the equation $\mathcal{H}(\partial E) + \gamma\mathcal{N}(E) = \lambda$ and see how much it deviates from a constant.

Lemma 3.2 *On ∂E_0*

$$\begin{aligned}
&\left(\mathcal{H}(\partial E_0) + \gamma\mathcal{N}(E_0) \right)(p + q\cos\theta, q\sin\theta) \\
&= \frac{1}{2q} + \frac{\cos\theta}{2p} + O\left(\frac{q}{p^2}\right) + \gamma \left\{ \frac{q^2}{2} \log \frac{1}{q} + \pi q^2 G_3(p, 0, p, 0) \right. \\
&\quad \left. - \frac{q^3}{4p} \left(\log \frac{1}{q} \right) \cos\theta + \frac{q^3}{p} \left[\pi p D_1G_3(p, 0, p, 0) + \frac{1}{16} \right] \cos\theta + O\left(\frac{q^4}{p^2} \log \frac{p}{q}\right) \right\}.
\end{aligned}$$

Proof. The mean curvature of the torus ∂T_j at $(r, z) = (p + q\cos\theta, q\sin\theta)$ is given by

$$\mathcal{H}(\partial T_j)(p + q\cos\theta, q\sin\theta) = \frac{1}{2} \left(\frac{1}{q} + \frac{\cos\theta}{p + q\cos\theta} \right) = \frac{1}{2q} + \frac{\cos\theta}{2p} + O\left(\frac{q}{p^2}\right). \tag{3.25}$$

Lemma 3.2 follows from (3.25) and Lemma 3.1. \square

We proceed to estimate $\mathcal{J}(E_0)$.

Lemma 3.3

$$\mathcal{J}(E_0) = 2\pi^2 pq + \frac{\gamma}{2} \left(\pi^2 pq^4 \log \frac{1}{q} + \frac{\pi^2 pq^4}{4} + 2\pi^3 pq^4 G_1(p, 0, p, 0) \right) + O(\gamma q^5 \log \frac{p}{q})$$

Proof. The perimeter of E_0 is

$$\mathcal{P}(E_0) = \sum_{j=1}^2 4\pi^2 pq, \quad (3.26)$$

which is the area of ∂E_0 .

The nonlocal part of $\mathcal{J}(E_0)$ is

$$\begin{aligned} & \int_{E_0} \int_{E_0} \frac{dxdy}{4\pi|x-y|} \\ &= 2\pi \int_0^{2\pi} \int_0^q \int_0^{2\pi} \int_0^q G(p+h\cos\theta, h\sin\theta, p+\rho\cos\omega, \rho\sin\omega)(p+h\cos\theta)\rho h d\rho d\omega dh d\theta. \end{aligned} \quad (3.27)$$

Let us write

$$G(r, z, s, t) = \frac{1}{2\pi} \log \frac{1}{|(r, z) - (s, t)|} + G_1(r, z, s, t). \quad (3.28)$$

Note that

$$\int_0^{2\pi} \log \sqrt{h^2 + \rho^2 - 2h\rho \cos(\theta - \omega)} d\omega = \begin{cases} 2\pi \log h & \text{if } \rho < h \\ 2\pi \log \rho & \text{if } \rho \geq h \end{cases}$$

by (3.15). Then

$$\begin{aligned} & \int_0^{2\pi} \int_0^q \frac{1}{2\pi} \log \frac{1}{|(h\cos\theta - \rho\cos\omega, h\sin\theta - \rho\sin\omega)|} \rho d\rho d\omega \\ &= \int_0^h \frac{\rho}{2\pi} 2\pi \log \frac{1}{h} d\rho + \int_h^q \frac{\rho}{2\pi} 2\pi \log \frac{1}{\rho} d\rho = \frac{q^2}{2} \log \frac{1}{q} + \frac{q^2}{4} - \frac{h^2}{4} \end{aligned}$$

Consequently

$$\begin{aligned} & 2\pi \int_0^{2\pi} \int_0^q \int_0^{2\pi} \int_0^q \frac{1}{2\pi} \log \frac{1}{|(h\cos\theta - \rho\cos\omega, h\sin\theta - \rho\sin\omega)|} (p+h\cos\theta)\rho h d\rho d\omega dh d\theta \\ &= \pi^2 pq^4 \log \frac{1}{q} + \frac{\pi^2}{4} pq^4. \end{aligned} \quad (3.29)$$

It is easy to see that

$$\begin{aligned} & 2\pi \int_0^{2\pi} \int_0^q \int_0^{2\pi} \int_0^q G_1(p+h\cos\theta, h\sin\theta, p+\rho\cos\omega, \rho\sin\omega)(p+h\cos\theta)\rho h d\rho d\omega dh d\theta \\ &= 2\pi^3 pq^4 G_1(p, 0, p, 0) + O(q^5 \log \frac{p}{q}). \end{aligned} \quad (3.30)$$

From (3.26), (3.29) and (3.30) one finds $\mathcal{J}(E_0)$. \square

At this point we can guess what the sizes of p and q should be for E_0 to be a good approximate solution. We want the right side of the estimate in Lemma 3.2 to be as close to a constant as possible. Therefore the $\cos\theta$ dependent terms should cancel out. Note that these terms are

$$\frac{\cos\theta}{2p} + \gamma \left\{ -\frac{q^3}{4p} \left(\log \frac{1}{q} \right) \cos\theta + \frac{q^3}{p} \left[\pi p D_1 G_3(p, 0, p, 0) + \frac{1}{16} \right] \cos\theta \right\}.$$

As we know $pD_1G_3(p, 0, p, 0) = D_1G_3(1, 0, 1, 0) + \frac{1}{4\pi} \log \frac{1}{p}$ in the proof of Lemma 3.1, the above is

$$\frac{\cos \theta}{2p} + \gamma \left\{ -\frac{q^3}{4p} \left(\log \frac{p}{q} \right) \cos \theta + \frac{q^3}{p} \left[\pi D_1G_3(1, 0, 1, 0) + \frac{1}{16} \right] \cos \theta \right\}.$$

The leading order of this quantity is

$$\frac{\cos \theta}{2p} - \frac{\gamma q^3}{4p} \left(\log \frac{p}{q} \right) \cos \theta$$

So to be a good approximate solution, E_0 should satisfy

$$\frac{1}{2p} - \frac{\gamma q^3}{4p} \log \frac{p}{q} \approx 0. \quad (3.31)$$

By (3.1), (3.31) becomes

$$\gamma \approx \frac{2}{3q^3 \log \frac{1}{q}}.$$

Comparing this to the definition (1.7) of the function f , we find that for E_0 to be a good approximate solution

$$q \approx f(\gamma). \quad (3.32)$$

Motivated by (3.32) from now on we require that the cross section radius q of our approximate solution satisfy

$$q \in \left[\frac{f(\gamma)}{2}, 2f(\gamma) \right] \quad (3.33)$$

Consequently $p = \frac{1}{2\pi^2 q^2}$ satisfies

$$p \in \left[\frac{1}{8\pi^2 f^2(\gamma)}, \frac{2}{\pi^2 f^2(\gamma)} \right]. \quad (3.34)$$

Note that

$$\text{as } \gamma \rightarrow \infty, \quad q \rightarrow 0 \text{ and } p \rightarrow \infty. \quad (3.35)$$

4 Perturbation to E_0

The exact solution will be found near an approximate solution E_0 . For this reason we need a way to perturb E_0 . Let $u = u(\theta)$ be a 2π -periodic positive function and define a set

$$E = \bigcup_{\theta \in [0, 2\pi]} \{(p + h \cos \theta, h \sin \theta) : h \in [0, u(\theta)]\} \quad (4.1)$$

in \mathbb{R}_+^2 . Corresponding to E there is a axisymmetric set in \mathbb{R}^3 , which we again denote by E . If $u(\theta) = q$ for all $\theta \in [0, 2\pi]$, E becomes our torus E_0 considered before. When $u(\theta)$ is close to q for all $\theta \in [0, 2\pi]$, E is a perturbation of E_0 .

Once the set E is described by u , $\mathcal{J}(E)$ can be expressed in terms of u :

$$\mathcal{J}(E) = \pi \int_0^{2\pi} (p + u(\theta) \cos \theta) \sqrt{(u'(\theta))^2 + (u(\theta))^2} d\theta + \gamma \pi \int_E \mathcal{N}(E)(r, z) r dr dz \quad (4.2)$$

where the first term gives the area of ∂E and the second term is the nonlocal part of $\mathcal{J}(E)$.

It is often more convenient to introduce another variable v in place of u . The constraint (1.6) requires

$$\begin{aligned}
1 &= 2\pi \int_E r dr dz \\
&= 2\pi \int_0^{2\pi} \int_0^{u(\theta)} (p + \rho \cos \theta) \rho d\rho d\theta \\
&= 2\pi \int_0^{2\pi} \left(\frac{pu^2(\theta)}{2} + \frac{u^3(\theta) \cos \theta}{3} \right) d\theta.
\end{aligned} \tag{4.3}$$

We set

$$v(\theta) = \frac{pu^2(\theta)}{2} + \frac{u^3(\theta) \cos \theta}{3}. \tag{4.4}$$

The set E is now specified by the function v . With respect to v by (4.3) the constraint (1.6) takes the form

$$\int_0^{2\pi} v(\theta) d\theta = \frac{1}{2\pi}, \tag{4.5}$$

which is simpler than (4.3). Consequently \mathcal{J} becomes a functional of v : $\mathcal{J} = \mathcal{J}(v)$. The variation of \mathcal{J} is written as

$$\mathcal{H}(v) + \gamma \mathcal{N}(v) \tag{4.6}$$

where $\mathcal{H}(v)$ is the variation, with respect to v , of the first term of \mathcal{J} and $\gamma \mathcal{N}(v)$ is the variation of the second term of \mathcal{J} . More precisely let $\phi = \phi(\theta)$ be a 2π -periodic function satisfying $\int_0^{2\pi} \phi(\theta) d\theta = 0$. Then

$$\mathcal{J}'(v)(\phi) = \left. \frac{d\mathcal{J}(v + \epsilon\phi)}{d\epsilon} \right|_{\epsilon=0} = 2\pi \int_0^{2\pi} \left(\mathcal{H}(v) + \gamma \mathcal{N}(v) \right) \phi d\theta. \tag{4.7}$$

Both $\mathcal{H}(v)$ and $\mathcal{N}(v)$ are functions of θ .

The reader may have noted that we used the same \mathcal{H} to denote the mean curvature operator of the curve ∂E earlier, and an operator on a 2π -periodic function v now. This abuse of notation is deliberate. $\mathcal{H}(\partial E)$ and $\mathcal{H}(v)$ stand for the same mean curvature of the boundary of the set E described by v . Similarly $\mathcal{N}(E)$ and $\mathcal{N}(v)$ are the same Newtonian potential of the set E described by v . The equation (1.1) now becomes

$$\mathcal{H}(v) + \gamma \mathcal{N}(v) = \lambda. \tag{4.8}$$

Since the same set E can be described by both the variable u and the variable v , to avoid confusion we will call u the radius variable of the set E and v the volume variable of the set E . The approximate solution E_0 is described by the radius variable

$$u(\theta) = q. \tag{4.9}$$

We denote the corresponding volume variable for E_0 by ψ , given by

$$\psi(\theta) = \frac{pq^2}{2} + \frac{q^3 \cos \theta}{3}. \tag{4.10}$$

Note that unlike radius variable $u(\theta) = q$, the volume variable ψ is not a constant function.

Note that if we integrate (4.8) with respect to θ , then

$$\lambda = \frac{1}{2\pi} \int_0^{2\pi} \left(\mathcal{H}(v) + \gamma \mathcal{N}(v) \right) d\theta. \tag{4.11}$$

Introduce the notion of average

$$\overline{\mathcal{H}(v)} = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{H}(v) d\theta, \quad \overline{\mathcal{N}(v)} = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{N}(v) d\theta. \quad (4.12)$$

Also introduce the operator \mathcal{S} by

$$\mathcal{S}(v) = \mathcal{H}(v) + \gamma \mathcal{N}(v) - \overline{\mathcal{H}(v)} - \gamma \overline{\mathcal{N}(v)}. \quad (4.13)$$

Then (4.8) becomes

$$\mathcal{S}(v) = 0. \quad (4.14)$$

For \mathcal{S} to be meaningful, we must specify its domain. Recall $H^2(S^1)$, the $W^{2,2}$ Sobolev space of 2π -periodic functions. Let

$$\mathcal{X} = \{v \in H^2(S^1) : \int_0^{2\pi} v(\theta) d\theta = \frac{1}{2\pi}\} \quad (4.15)$$

be the metric space equipped with the $H^2(S^1)$ norm. Note that $\psi \in \mathcal{X}$. Our \mathcal{S} is defined in a sufficiently small neighborhood \mathcal{O} of ψ in \mathcal{X} . We often write $v = \psi + \phi$ for $v \in \mathcal{O}$. Then ϕ is in the linear subspace

$$\mathcal{X}' = \{\phi \in H^2(S^1) : \int_0^{2\pi} \phi(\theta) d\theta = 0\}. \quad (4.16)$$

Let $\mathcal{O}' = \mathcal{O} - \psi$. Then

$$\phi \in \mathcal{O}' \subset \mathcal{X}'. \quad (4.17)$$

The target space of the operator \mathcal{S} is

$$\mathcal{Y} = \{g \in L^2(S^1) : \int_0^{2\pi} g(\theta) d\theta = 0\}. \quad (4.18)$$

In summary

$$\mathcal{S} : \mathcal{O} \subset \mathcal{X} \rightarrow \mathcal{Y}. \quad (4.19)$$

Next we analyze $\mathcal{S}'(\psi)$ where \mathcal{S}' stands for the Fréchet derivative of \mathcal{S} . At ψ , $\mathcal{S}'(\psi)$ is a linear operator from \mathcal{X}' to \mathcal{Y} :

$$\mathcal{S}'(\psi) : \mathcal{X}' \rightarrow \mathcal{Y}. \quad (4.20)$$

This operator will be written as the sum of a major part and a minor part denoted by \mathcal{M} and $\widetilde{\mathcal{M}}$ respectively.

To identify \mathcal{M} we treat the two parts of \mathcal{S} , $\mathcal{H}(v) - \overline{\mathcal{H}(v)}$ and $\mathcal{N}(v) - \overline{\mathcal{N}(v)}$, separately. The first part of \mathcal{S} comes from the perimeter part of \mathcal{J} which is

$$\pi \int_0^{2\pi} (p + u \cos \theta) \sqrt{(u')^2 + u^2} d\theta \quad (4.21)$$

when the radius variable u is used. This is to be converted to an integral with respect to the volume variable v . Let

$$A(u, \theta) = \frac{pu^2}{2} + \frac{u^3 \cos \theta}{3}. \quad (4.22)$$

Then

$$v(\theta) = A(u(\theta), \theta). \quad (4.23)$$

Denote the inverse of A , as a function of u with θ held fixed, by $B = B(v, \theta)$ such that

$$u(\theta) = B(v(\theta), \theta). \quad (4.24)$$

Define the Lagrangian L from $\mathbb{R} \times \mathbb{R}_+ \times S^1$ to \mathbb{R} , with respect to the volume variable, by

$$L(\dot{v}, v, \theta) = (p + u \cos \theta) \sqrt{\dot{u}^2 + u^2}, \quad (4.25)$$

where (\dot{u}, u, θ) and (\dot{v}, v, θ) transform to each other according the rules

$$\begin{pmatrix} \dot{u} \\ u \\ \theta \end{pmatrix} \rightarrow \begin{pmatrix} \dot{v} \\ v \\ \theta \end{pmatrix} = \begin{pmatrix} \dot{u}A_u(u, \theta) + A_\theta(u, \theta) \\ A(u, \theta) \\ \theta \end{pmatrix} \quad (4.26)$$

and

$$\begin{pmatrix} \dot{v} \\ v \\ \theta \end{pmatrix} \rightarrow \begin{pmatrix} \dot{u} \\ u \\ \theta \end{pmatrix} = \begin{pmatrix} \dot{v}B_v(v, \theta) + B_\theta(v, \theta) \\ B(v, \theta) \\ \theta \end{pmatrix}. \quad (4.27)$$

Following (4.25) and (4.27) we find that as $p \rightarrow \infty$,

$$p^{-1/2}L(\dot{v}, v, \theta) \rightarrow \sqrt{\frac{\dot{v}^2}{2v} + 2v} \quad (4.28)$$

in $C_{loc}^\infty(\mathbb{R} \times \mathbb{R}_+ \times S^1)$, and

$$\psi(\theta) \rightarrow \frac{1}{4\pi^2} \quad (4.29)$$

in $C^\infty(S^1)$. The second derivative of the functional

$$v \rightarrow \int_0^{2\pi} \sqrt{\frac{(v'(\theta))^2}{2v(\theta)} + 2v(\theta)} d\theta \quad (4.30)$$

at $v = \frac{1}{4\pi^2}$ is the linear operator

$$\phi \in \mathcal{X}' \rightarrow -2^{5/2}\pi^3(\phi'' + \phi) \in \mathcal{Y}. \quad (4.31)$$

This gives us the first part of \mathcal{M} :

$$\phi \in \mathcal{X}' \rightarrow -2^{5/2}\pi^3 p^{1/2}(\phi'' + \phi) \in \mathcal{Y}. \quad (4.32)$$

The second part of \mathcal{M} comes from the Fréchet derivative of \mathcal{N} . Let $u = B(\psi + \epsilon\phi)$ and compute

$$\begin{aligned} & \left. \frac{\partial \mathcal{N}(B(\psi + \epsilon\phi))}{\partial \epsilon} \right|_{\epsilon=0} \\ &= \int_0^{2\pi} G(p + q \cos \theta, q \sin \theta, p + q \cos \omega, p + q \sin \omega) q D_1 B(\psi(\omega), \omega) \phi(\omega) d\omega \\ & \quad + \phi(\theta) D_1 B(\psi(\theta), \theta) \int_{E_0} \left(D_1 G(p + q \cos \theta, q \sin \theta, s, t), D_2 G(p + q \cos \theta, q \sin \theta, s, t) \right) \cdot e^{i\theta} ds dt. \end{aligned} \quad (4.33)$$

Some smaller terms are ignored at this point. First note that

$$D_1 B(\psi(\theta), \theta) = \frac{1}{pq + q^2 \cos \theta}. \quad (4.34)$$

We will replace $D_1 B(\psi)$ by $\frac{1}{pq}$. Second we replace $G(r, z, s, t)$ by $\frac{1}{2\pi} \log \frac{1}{|(r,z)-(s,t)|}$ because of Lemma 2.1. Then (4.33) is replaced by

$$\phi \rightarrow \frac{1}{p} \int_0^{2\pi} \frac{1}{2\pi} \log \frac{1}{q|e^{i\theta} - e^{i\omega}|} \phi(\omega) d\omega - \frac{\phi(\theta)}{2\pi pq} \int_0^{2\pi} \int_0^q \frac{qe^{i\theta} - \rho e^{i\omega}}{|qe^{i\theta} - \rho e^{i\omega}|^2} \cdot e^{i\theta} \rho d\rho d\omega. \quad (4.35)$$

The double integral in (4.35) may be evaluated

$$\int_0^{2\pi} \int_0^q \frac{qe^{i\theta} - \rho e^{i\omega}}{|qe^{i\theta} - \rho e^{i\omega}|^2} \cdot e^{i\theta} \rho d\rho d\omega = q\pi. \quad (4.36)$$

Then (4.35) becomes

$$\phi \rightarrow \frac{1}{2\pi p} \log \frac{1}{q} \int_0^{2\pi} \phi(\omega) d\omega + \frac{1}{p} \int_0^{2\pi} \frac{1}{2\pi} \log \frac{1}{|e^{i\theta} - e^{i\omega}|} \phi(\omega) d\omega - \frac{\phi(\theta)}{2p} \quad (4.37)$$

Since our ϕ is in \mathcal{X}' where $\int_0^{2\pi} \phi(\omega) d\omega = 0$, the first term in (4.37) vanishes and (4.37) becomes

$$\phi \in \mathcal{X}' \rightarrow \frac{1}{p} \int_0^{2\pi} \frac{1}{2\pi} \log \frac{1}{|e^{i\theta} - e^{i\omega}|} \phi(\omega) d\omega - \frac{\phi(\theta)}{2p} \in \mathcal{Y} \quad (4.38)$$

Combining (4.32) and (4.38) we find that $\mathcal{H}'(\psi) - \overline{\mathcal{H}'(\psi)}$ is dominated by

$$\phi \in \mathcal{X}' \rightarrow -2^{5/2} \pi^3 p^{1/2} (\phi'' + \phi) \in \mathcal{Y}, \quad (4.39)$$

and $\gamma \mathcal{N}'(\psi) - \overline{\gamma \mathcal{N}'(\psi)}$ is dominated by

$$\phi \in \mathcal{X}' \rightarrow \frac{\gamma}{p} \int_0^{2\pi} \frac{1}{2\pi} \log \frac{1}{|e^{i\theta} - e^{i\omega}|} \phi(\omega) d\omega - \frac{\gamma \phi(\theta)}{2p} \in \mathcal{Y}. \quad (4.40)$$

However $p^{1/2} \gg \frac{\gamma}{p}$ by (1.7) and (3.34). The operator (4.40) is negligible compared to (4.39). We have identified \mathcal{M} :

$$\phi \in \mathcal{X}' \rightarrow \mathcal{M}(\phi) = -2^{5/2} \pi^3 p^{1/2} (\phi'' + \phi) \in \mathcal{Y}. \quad (4.41)$$

The rest of $\mathcal{S}'(\psi)$ is denoted by $\widetilde{\mathcal{M}}$ so that

$$\mathcal{S}'(\psi) = \mathcal{M} + \widetilde{\mathcal{M}}. \quad (4.42)$$

The eigenpairs of \mathcal{M} are

$$2^{5/2} \pi^3 p^{1/2} (n^2 - 1); \cos n\theta, \sin n\theta; n = 1, 2, 3, \dots \quad (4.43)$$

Note that the functions in \mathcal{X}' have zero average, so the $n = 0$ mode is not included in (4.43). All eigenvalues in (4.43) are positive except for the mode $n = 1$ where 0 is an eigenvalue and $\cos \theta$ and $\sin \theta$ are the corresponding eigenfunctions. To proceed we remove $\cos \theta$ and $\sin \theta$ from \mathcal{X}' and set

$$\mathcal{X}'_* = \{\phi \in \mathcal{X}' : \phi \perp \cos \theta, \phi \perp \sin \theta\}, \quad \mathcal{Y}_* = \{\phi \in \mathcal{Y} : \phi \perp \cos \theta, \phi \perp \sin \theta\}. \quad (4.44)$$

Let Π be the orthogonal projection operator

$$\Pi : \mathcal{Y} \rightarrow \mathcal{Y}_*, \quad (4.45)$$

and consider the operator

$$\Pi \mathcal{S}'(\psi) : \mathcal{X}'_* \rightarrow \mathcal{Y}_*. \quad (4.46)$$

The following lemma summarizes the properties of $\Pi \mathcal{S}'(\psi) : \mathcal{X}'_* \rightarrow \mathcal{Y}_*$, whose proof which we omit is similar to that of Lemma 3.1 [33].

Lemma 4.1 1. *There exists $C > 0$ independent of p, q and γ such that*

$$\|\phi\|_{L^2} \leq Cp^{-1/2} \|\Pi\mathcal{S}'(\psi)\phi\|_{L^2}$$

for all $\phi \in \mathcal{X}'_$. Moreover the operator $\Pi\mathcal{S}'(\psi)$ is one-to-one and onto from \mathcal{X}'_* to \mathcal{Y}_* .*

2. *There exists $C > 0$ independent of p, q and γ such that for all $\phi \in \mathcal{X}'_*$*

$$\|\phi\|_{L^2}^2 \leq Cp^{-1/2} \langle \Pi\mathcal{S}'(\psi)\phi, \phi \rangle.$$

The second Fréchet derivative of \mathcal{S} is estimated in the next lemma. The proof is similar to that of Lemma 3.2 [33] which we again omit.

Lemma 4.2 *There exists $C > 0$ independent of p, q and γ such that*

$$\|\mathcal{S}''(v)(\phi_1, \phi_2)\|_{L^2} \leq Cp^{1/2} \|u\|_{H^2} \|v\|_{H^2}$$

for all v near ψ , in the sense $\|v - \psi\|_{H^2} < \frac{1}{2}\|\psi\|_{H^2}$, and all $\phi_1, \phi_2 \in \mathcal{X}'$.

5 Reduction

The nonlinear operator \mathcal{S} maps from a neighborhood of ψ in \mathcal{X} to \mathcal{Y} . In this section it will be proved that, for each $p \in [\frac{1}{8\pi^2 f^2(\gamma)}, \frac{2}{\pi^2 f^2(\gamma)}]$, a function $\varphi \in \mathcal{X}'_*$ exists such that

$$\mathcal{S}(\psi + \varphi)(\theta) = A_1 \cos \theta + A_2 \sin \theta \tag{5.1}$$

for some $A_1, A_2 \in \mathbb{R}$. The equation (5.1) may be written as

$$\Pi\mathcal{S}(\psi + \varphi) = 0 \tag{5.2}$$

where Π is given in (4.45). In the next section we will find a particular p , say p_γ , such that at $p = p_\gamma$, $A_1 = A_2 = 0$. This means that by finding φ one reduces the original infinite-dimensional problem (1.1) to a one-dimensional problem of finding p_γ in the interval $[\frac{1}{8\pi^2 f^2(\gamma)}, \frac{2}{\pi^2 f^2(\gamma)}]$.

Lemma 5.1 *There exists $\varphi \in \mathcal{X}'_*$ such that for every $p \in [\frac{1}{8\pi^2 f^2(\gamma)}, \frac{2}{\pi^2 f^2(\gamma)}]$, φ solves (5.2) and $\|\varphi\|_{H^2} \leq Mf^6(\gamma)$ where M is a sufficiently large constant independent of γ and p .*

Proof. Expand $\mathcal{S}(\psi + \phi)$ as

$$\mathcal{S}(\psi + \phi) = \mathcal{S}(\psi) + \mathcal{S}'(\psi)(\phi) + \mathcal{R}(\phi) \tag{5.3}$$

where \mathcal{R} is a higher order term defined by (5.3). Rewrite (5.2) in a fixed point form:

$$\phi = -(\Pi\mathcal{S}'(\psi))^{-1}(\Pi\mathcal{S}(\psi) + \Pi\mathcal{R}(\phi)). \tag{5.4}$$

To use the Contraction Mapping Principle in the fixed point setting (5.4), let

$$\mathcal{T}(\phi) = -(\Pi\mathcal{S}'(\psi))^{-1}(\Pi\mathcal{S}(\psi) + \Pi\mathcal{R}(\phi)) \tag{5.5}$$

be an operator defined on

$$D(\mathcal{T}) = \{\phi \in \mathcal{X}'_* : \|\phi\|_{H^2} \leq Mf^6(\gamma)\} \tag{5.6}$$

where the constant M is sufficiently large and will be made more precise later. Note that when γ is sufficiently large, every $\phi \in D(\mathcal{T})$ is very small compared to ψ , and hence \mathcal{T} is well-defined.

Lemma 3.2, (3.33), and (3.34) assert that

$$\begin{aligned} \mathcal{H}(\psi) + \gamma \mathcal{N}(\psi) &= \frac{1}{2q} + \frac{\cos \theta}{2p} + \gamma \left[\frac{q^2}{2} \log \frac{1}{q} + \pi q^2 G_3(p, 0, p, 0) \right. \\ &\quad \left. - \frac{q^3}{4p} \left(\log \frac{1}{q} \right) \cos \theta + \frac{q^3}{p} \left(\pi p D_1 G_3(p, 0, p, 0) + \frac{1}{16} \right) \cos \theta \right] + O(f^5(\gamma)) \end{aligned} \quad (5.7)$$

Taking the average of (5.7) we find

$$\overline{\mathcal{H}(\psi)} + \gamma \overline{\mathcal{N}(\psi)} = \frac{1}{2q} + \gamma \left[\frac{q^2}{2} \log \frac{1}{q} + \pi q^2 G_3(p, 0, p, 0) \right] + O(f^5(\gamma)). \quad (5.8)$$

Subtract (5.8) from (5.7) to obtain

$$\mathcal{S}(\psi) = \frac{\cos \theta}{2p} + \gamma \left[-\frac{q^3}{4p} \left(\log \frac{1}{q} \right) \cos \theta + \frac{q^3}{p} \left(\pi p D_1 G_3(p, 0, p, 0) + \frac{1}{16} \right) \cos \theta \right] + O(f^5(\gamma)) \quad (5.9)$$

holds uniformly in θ . After one applies Π , the $\cos \theta$ terms vanish and

$$\|\Pi \mathcal{S}(\psi)\|_{L^2} = O(f^5(\gamma)). \quad (5.10)$$

From Lemma 4.1 and (3.34) we deduce that

$$\|(\Pi \mathcal{S}'(\psi))^{-1}\| \leq C f(\gamma), \quad (5.11)$$

where $\|\cdot\|$ stands for the norm of an operator. and consequently

$$\|(\Pi \mathcal{S}'(\psi))^{-1} \Pi \mathcal{S}(\psi)\|_{H^2} \leq C_1 f^6(\gamma). \quad (5.12)$$

Lemma 4.2 and (3.34) imply that

$$\|\mathcal{S}''(v)(\phi_1, \phi_2)\|_{L^2} \leq C f^{-1}(\gamma) \|\phi_1\|_{H^2} \|\phi_2\|_{H^2}. \quad (5.13)$$

Therefore

$$\|\mathcal{R}(\phi)\|_{L^2} \leq C f^{-1}(\gamma) \|\phi\|_{H^2}^2 \quad (5.14)$$

and consequently

$$\|(\Pi \mathcal{S}'(\psi))^{-1} \Pi \mathcal{R}(\phi)\|_{H^2} \leq C_2 \|\phi\|_{H^2}^2. \quad (5.15)$$

Using (5.5), (5.12), (5.6), and (5.15) we find that

$$\|\mathcal{T}(\phi)\|_{H^2} \leq C_1 f^6(\gamma) + C_2 M^2 f^{12}(\gamma) \leq M f^6(\gamma)$$

if $M = 2C_1$ and γ is sufficiently large. Therefore \mathcal{T} is a map from $D(\mathcal{T})$ into itself.

Next we show that \mathcal{T} is a contraction. Let $\phi_1, \phi_2 \in D(\mathcal{T})$. Note that

$$\mathcal{T}(\phi_1) - \mathcal{T}(\phi_2) = (\Pi \mathcal{S}'(\psi))^{-1} (-\Pi)(\mathcal{R}(\phi_1) - \mathcal{R}(\phi_2)). \quad (5.16)$$

Because

$$\mathcal{R}(\phi_1) - \mathcal{R}(\phi_2) = \mathcal{S}(\psi + \phi_1) - \mathcal{S}(\psi + \phi_2) - \mathcal{S}'(\psi)(\phi_1 - \phi_2),$$

we deduce, with the help of (5.13) and (5.6), that

$$\begin{aligned}
\|\mathcal{R}(\phi_1) - \mathcal{R}(\phi_2)\|_{L^2} &\leq \|\mathcal{S}'(\psi + \phi_2)(\phi_1 - \phi_2) - \mathcal{S}'(\psi)(\phi_1 - \phi_2)\|_{L^2} + Cf^{-1}(\gamma)\|\phi_1 - \phi_2\|_{H^2}^2 \\
&\leq Cf^{-1}(\gamma)\|\phi_2\|_{H^2}\|\phi_1 - \phi_2\|_{H^2} + Cf^{-1}(\gamma)\|\phi_1 - \phi_2\|_{H^2}^2 \\
&\leq Cf^{-1}(\gamma)(\|\phi_1\|_{H^2} + \|\phi_2\|_{H^2})\|\phi_1 - \phi_2\|_{H^2} \\
&\leq CMf^5(\gamma)\|\phi_1 - \phi_2\|_{H^2}.
\end{aligned}$$

Then Lemma 4.1 implies that

$$\|\mathcal{T}(\phi_1) - \mathcal{T}(\phi_2)\|_{H^2} \leq CMf^5(\gamma)\|\phi_1 - \phi_2\|_{H^2}. \quad (5.17)$$

Therefore \mathcal{T} is a contraction mapping from $D(\mathcal{T})$ to itself, when γ is sufficiently large. There is a unique fixed point in $D(\mathcal{T})$, which we denote by φ . \square

6 Minimization

We prove Theorem 1.1 in this section. From now on we emphasize the dependence on p . The approximate solution $\psi = \psi(\theta)$ is now denoted by $\psi = \psi(\theta, p)$. By Lemma 5.1 for every $p \in [\frac{1}{8\pi^2 f^2(\gamma)}, \frac{2}{\pi^2 f^2(\gamma)}]$ there exists $\varphi = \varphi(\theta, p) \in \mathcal{X}'_*$ such that $\Pi\mathcal{S}(\psi(\cdot, p) + \varphi(\cdot, p)) = 0$, i.e. (5.1) holds. In this section we find a particular p denoted by p_γ such that $\mathcal{S}(\psi(\cdot, p_\gamma) + \varphi(\cdot, p_\gamma)) = 0$.

Let us denote the set specified by the function $\psi(\cdot, p) + \varphi(\cdot, p)$ by $E_{\varphi(\cdot, p)}$. This notation is consistent with our earlier notation E_0 , the set characterized by $\psi(\cdot, p)$.

Lemma 6.1 $\mathcal{J}(E_{\varphi(\cdot, p)}) = \mathcal{J}(E_0) + O(f^{11}(\gamma))$.

Proof. Expanding $\mathcal{J}(E_\varphi)$ yields

$$\mathcal{J}(E_\varphi) = \mathcal{J}(E_0) + 2\pi \int_0^{2\pi} \mathcal{S}(\psi)\varphi d\theta + \frac{2\pi}{2} \int_0^{2\pi} \mathcal{S}'(\psi)(\varphi)\varphi d\theta + O(f^{17}(\gamma)). \quad (6.1)$$

The error term in (6.1) is obtained by Lemma 4.2. and the fact $\|\varphi\|_{H^2} = O(f^6(\gamma))$.

On the other hand $\Pi\mathcal{S}(\psi + \varphi) = 0$ implies that

$$\Pi(\mathcal{S}(\psi) + \mathcal{S}'(\psi)(\varphi) + \mathcal{R}(\varphi)) = 0$$

where \mathcal{R} is given in (5.3). Multiply the last equation by φ and integrate to derive, again with the help of Lemma 4.2,

$$\int_0^{2\pi} \mathcal{S}(\psi)\varphi d\theta + \sum \int_0^{2\pi} \mathcal{S}'(\psi)(\varphi)\varphi d\theta_k = O(f^{17}(\gamma)).$$

Rewrite (6.1) as

$$\mathcal{J}(E_\varphi) = \mathcal{J}(E_0) + \frac{2\pi}{2} \int_0^{2\pi} \mathcal{S}(\psi)\varphi d\theta + O(f^{17}(\gamma)).$$

Note that (5.9) which follows from Lemma 3.2 and the facts $\varphi \perp \cos \theta$ and $\|\varphi\|_{H^2} = O(f^6(\gamma))$ imply that

$$\int_0^{2\pi} \mathcal{S}(\psi)\varphi d\theta = O(f^{11}(\gamma)) \quad (6.2)$$

Therefore

$$\mathcal{J}(E_\varphi) = \mathcal{J}(E_0) + O(f^{11}(\gamma)). \quad (6.3)$$

This proves the lemma. \square

Lemma 3.3 shows that

$$\mathcal{J}(E_0) = 2\pi^2 pq + \frac{\gamma}{2} \left(\pi^2 pq^4 \log \frac{1}{q} + \frac{\pi^2 pq^4}{4} + 2\pi^3 pq^4 G_1(p, 0, p, 0) \right) + O(\gamma q^5 \log \frac{p}{q}). \quad (6.4)$$

Sometimes it is convenient to consider rescaled versions of p and q . Introduce P and Q so that

$$p = \frac{P}{2\pi^2 f^2(\gamma)}, \quad q = f(\gamma)Q, \quad j = 1, 2. \quad (6.5)$$

By (3.34) and (3.33)

$$P \in [\frac{1}{4}, 4], \quad Q \in [\frac{1}{2}, 2]. \quad (6.6)$$

The constraint $2\pi^2 pq^2 = 1$ implies that

$$PQ^2 = 1. \quad (6.7)$$

In terms of P and Q , $\mathcal{J}(E_0)$ is given by

Lemma 6.2

$$\mathcal{J}(E_0) = \frac{1}{f(\gamma)} \left(PQ + \frac{PQ^4}{2} \right) + \gamma f^2(\gamma) \left(\frac{PQ^4}{4} \log \frac{1}{Q} + \frac{PQ^4}{16} + \frac{\pi PQ^4}{2} G_1(P, 0, P, 0) \right) + O(f^2(\gamma)).$$

Proof. Use (6.4), (3.33), (3.34), and the formulae

$$G_1(\lambda r, \lambda z, \lambda s, \lambda t) = G_1(r, z, s, t) - \frac{1}{2\pi} \log \frac{1}{\lambda} \quad (6.8)$$

which follows from Lemma 2.2 part 1 and Lemma 2.1.

Lemma 6.3 *There exists $p_\gamma \in [\frac{1}{8\pi^2 f^2(\gamma)}, \frac{2}{\pi^2 f^2(\gamma)}]$ such that when $p = p_\gamma$, $\mathcal{S}(\varphi(\cdot, p_\gamma)) = 0$. Moreover $\lim_{\gamma \rightarrow \infty} 2\pi^2 f^2(\gamma) p_\gamma = 1$ and $\lim_{\gamma \rightarrow \infty} \frac{q_\gamma}{f(\gamma)} = 1$.*

Proof. By Lemmas 6.2, 6.1, and the constraint (6.7)

$$\begin{aligned} \mathcal{J}(E_\varphi) &= \frac{1}{f(\gamma)} \left(PQ + \frac{PQ^4}{2} \right) + \gamma f^2(\gamma) \left(\frac{PQ^4}{4} \log \frac{1}{Q} + \frac{PQ^4}{16} + \frac{\pi PQ^4}{2} G_1(P, 0, P, 0) \right) + O(f^2(\gamma)) \\ &= \frac{1}{f(\gamma)} \left(PQ + \frac{PQ^4}{2} \right) + O(\gamma f^2(\gamma)) \\ &= \frac{1}{f(\gamma)} \left(\sqrt{P} + \frac{1}{2P} \right) + O(\gamma f^2(\gamma)). \end{aligned} \quad (6.9)$$

We view $\mathcal{J}(E_{\varphi(\cdot, p)})$ as a function of p , or P . Let $P_\gamma \in (\frac{1}{4}, 4)$ such that at $P = P_\gamma$, $\mathcal{J}(E_{\varphi(\cdot, p)})$ is minimized. Since $\frac{1}{f(\gamma)} \gg \gamma f^2(\gamma)$ and $\sqrt{P} + \frac{1}{2P}$ is minimized at $P = 1$, we conclude that $\lim_{\gamma \rightarrow \infty} P_\gamma = 1$. Let

$$p_\gamma = \frac{P_\gamma}{2\pi^2 f^2(\gamma)}, \quad q_\gamma = \sqrt{\frac{1}{2\pi^2 p_\gamma}}. \quad (6.10)$$

As a function of p , $\mathcal{J}(E_{\varphi(\cdot,p)})$ is minimized at $p = p_\gamma$ and

$$\lim_{\gamma \rightarrow \infty} 2\pi^2 f^2(\gamma) p_\gamma = 1, \quad \lim_{\gamma \rightarrow \infty} \frac{q_\gamma}{f(\gamma)} = 1. \quad (6.11)$$

It follows from (6.11) that when γ is large, p_γ is in $(\frac{1}{8\pi^2 f^2(\gamma)}, \frac{2}{\pi^2 f^2(\gamma)})$. Therefore

$$\left. \frac{d\mathcal{J}(E_{\varphi(\cdot,p)})}{dp} \right|_{p=p_\gamma} = 0. \quad (6.12)$$

By (6.12) and the invariance of the problem in the z -direction, $\mathcal{S}(\varphi(\cdot, p_\gamma)) = 0$. The proof of the last assertion, which we omit, is similar to that of [33, Lemma 4.4]. \square

We have found the solution $E_{\varphi(\cdot, p_\gamma)}$ and completed the proof of Theorem 1.1. The fact that our solution set $E_{\varphi(\cdot, p_\gamma)}$ is close to the region E_0 bounded by an exact torus comes from the estimate $\|\varphi\|_{H^2} = O(f^6(\gamma))$ in Lemma 5.1. This quantity is much smaller than the size of $\psi(\theta) = \frac{pq^2}{2} + \frac{q^3 \cos \theta}{3}$ which is of order 1.

7 Discussion

The stability of the tube solution in \mathbb{R}^3 is not discussed in this paper. However within the class of axisymmetric sets, the solution we found is stable in some sense. In this class every set is obtained by rotating a two-dimensional subset of \mathbb{R}_+^2 about the z -axis, and the stability is reduced to the stability of a 2-dimensional problem in the function space \mathcal{X} .

In Section 5 we constructed a family of the sets $E_{\varphi(\cdot, p)} \subset \mathbb{R}_+^2$ identified by $\psi(\cdot, p) + \varphi(\cdot, p)$ that satisfy $\Pi\mathcal{S}(\psi(\cdot, p) + \varphi(\cdot, p)) = 0$. Now we shift each $E_{\varphi(\cdot, p)}$ in z -direction by an amount z . When z is close to 0, each shifted set can still be described by a volume variable, which we denote by $v = v(\theta, p, z)$. The corresponding radius variable is denoted by $u = u(\theta, p, z)$, so the shifted set is

$$\bigcup_{\theta \in [0, 2\pi]} \{(p + h \cos \theta, h \sin \theta) : h \in [0, u(\theta, p, z)]\}. \quad (7.1)$$

The radius variable $u(\cdot, p, z)$ may be determined from $u(\cdot, p, 0)$ implicitly through

$$u(\theta, p, z)e^{i\theta} = u(\eta, p, 0)e^{i\eta} + (0, h), \quad (7.2)$$

and $u(\cdot, p, 0)$ is found from $u(\theta, p, 0) = B(v(\theta, p, 0), \theta)$ where $v(\cdot, p, 0) = \psi(\cdot, p) + \varphi(\cdot, p)$.

As (p, z) varies near $(p_\gamma, 0)$, $v(\cdot, p, z)$ forms a two dimensional manifold in \mathcal{X} . At each $v(\cdot, p, z)$, the space $\mathcal{X}'_* + v(\cdot, p, z)$ is transversal to the manifold and has co-dimension 2. By Lemma 4.1 part 2, Lemma 4.2 and the fact $\Pi\mathcal{S}(v(\cdot, p, z)) = 0$, one can show that $v(\cdot, p, z)$ is a non-degenerate local minimum of \mathcal{J} in $\mathcal{X}'_* + v(\cdot, p, z)$. According to Section 6, $v(\cdot, p_\gamma, 0) = \psi(\cdot, p_\gamma) + \varphi(\cdot, p_\gamma)$ is a minimum of \mathcal{J} in the 2-dimensional manifold labeled by (p, z) . Hence in this sense $\psi(\cdot, p_\gamma) + \varphi(\cdot, p_\gamma)$ is a local minimum of \mathcal{J} , and is stable. Note that since $\mathcal{J}(\psi(\cdot, p_\gamma) + \varphi(\cdot, p_\gamma)) = \mathcal{J}(v(\cdot, p_\gamma, z))$ for all z close to 0, $\psi(\cdot, p_\gamma) + \varphi(\cdot, p_\gamma)$ is a degenerate local minimum of \mathcal{J} due to the translation invariance in the z -direction.

In the introduction we mentioned the unit disc solution to (1.1) in \mathbb{R}^2 and its connection to (1.5). There are a few other solutions.

In \mathbb{R}^2 there exists a ring shaped solution of the form $\{x \in \mathbb{R}^2 : 0 < R_1 < |x| < R_2\}$, if γ is greater than a threshold value γ_0 ; see Kang and Ren [14]. The inner and outer radii R_1 and R_2 of the ring solution depend on γ . There is another threshold $\gamma_1 > \gamma_0$, such that the ring solution is unstable if $\gamma \in (\gamma_0, \gamma_1)$ and stable if $\gamma > \gamma_1$. This ring solution to (1.1) may be used to construct a solution to (1.5) with multiple, small rings, or a solution to (1.5) with a mix of small rings and small discs [15].

When γ is slightly greater than 12, there is another solution to (1.1) in \mathbb{R}^2 of an oval shape. Corresponding to this oval solution, there exist two solutions to (1.5), each of which is a small set of an oval shape, if the domain D does not possess certain symmetry [35]. The two small oval sets center at the same point, but align along two orthogonal directions. The two directions are determined by the shape of D .

In \mathbb{R}^3 the unit ball is a solution to (1.1). On any bounded domain there exist solutions with multiple components each of which is close to a small ball [34]. The small balls form a body centered cubic pattern, consistent with the spherical morphological phase of diblock copolymers.

There is also a shell shaped solution of the form $\{x \in \mathbb{R}^3 : 0 < R_1 < |x| < R_2\}$ if γ is large enough [28]. However this shell solution is always unstable.

The toroidal solution found in this paper is a neighborhood around the circle

$$x^2 + y^2 = p^2, \quad z = 0 \tag{7.3}$$

in \mathbb{R}^3 enclosed by a tube like surface. An interesting point is that the circle is not a geodesic and does not have zero curvature. Also the boundary of our solution is not a constant mean curvature surface. This is in contrast to a result of Mazzeo and Pacard [25]. They showed that if (M, g) is a Riemannian manifold and Γ is a closed, non-degenerate geodesic in M , then most geodesic tubes of sufficiently small radii around Γ can be perturbed into constant mean curvature surfaces. Some kind of converse is also true. If a small tube of any curve Γ is a constant mean curvature surface with the second fundamental form bounded then the curve must be a geodesic. This result differs from ours in two aspects: first they need a metric and their curve is a geodesic. Second they do not have the nonlocal term. In our case, we constructed a tube around a circle (which is not a geodesic under the Euclidean metric) with the help of the nonlocal interaction term. Our tube does not have a constant mean curvature. Instead it satisfies (1.1). It is unclear in our case if the circle can be replaced by any other curve in \mathbb{R}^3 .

A Appendix

The Ohta-Kawasaki free energy functional for a diblock copolymer is a Landau type integral with a unique long range interaction term. The density field of A -monomers is given by a function u on a bounded domain $D \subset \mathbb{R}^n$ ($n = 1, 2$, or 3) and the density of B -monomers is given by $1 - u$. The free energy of a diblock copolymer is

$$\mathcal{I}_D(u) = \int_D \left[\frac{\epsilon^2}{2} |\nabla u|^2 + W(u) + \frac{\epsilon\gamma}{2} |(-\Delta)^{-1/2}(u - a)|^2 \right] dx \tag{A.1}$$

with three parameters $\epsilon > 0$, $\gamma > 0$, and $a \in (0, 1)$, where u is in

$$\{u \in H^1(D) : \frac{1}{|D|} \int_D u(x) dx = a\}. \tag{A.2}$$

The function W is a balanced double well potential such as $W(u) = \frac{1}{4}u^2(1 - u)^2$. In (A.1) $(-\Delta)^{-1/2}$ is a long range interaction term. It is the square root of the inverse of the operator $-\Delta$. The $-\Delta$ operator acts on functions with the zero Neumann boundary condition.

One can show as in [29], that as $\epsilon \rightarrow 0$, the functional $\epsilon^{-1}\mathcal{I}_D$ converges, in the sense of the Γ -convergence theory [6, 20, 19, 16], to a limiting functional \mathcal{J}_D . The functional \mathcal{J}_D is defined on sets whose Lebesgue measure is a times the Lebesgue measure of D . and takes the form

$$\mathcal{J}_D(E) = \frac{1}{n-1} \mathcal{P}_D(E) + \frac{\gamma}{2} \int_D |(-\Delta)^{-1/2}(\chi_E - a)|^2 dx. \tag{A.3}$$

Here $\mathcal{P}_D(E)$ stands for the size of the part of ∂E which is inside D , and χ_E is the characteristic function of E . The Euler-Lagrange equation of \mathcal{J}_D is exactly (1.5).

We mentioned in the introduction that the problem (1.5) is also connected to the Gierer-Meinhardt system in developmental morphogenesis. The equilibrium state of the system is

$$\epsilon^2 \Delta u - u + \frac{u^p}{(1 + \kappa u^p)v^q} = 0 \text{ in } D; \quad d\Delta v - v + \frac{u^r}{v^s} = 0 \text{ in } D; \quad \frac{\partial u}{\partial \nu} \Big|_{\partial D} = 0; \quad \frac{\partial v}{\partial \nu} \Big|_{\partial D} = 0, \quad (\text{A.4})$$

where p, q, r , and s are positive parameters and they satisfy $0 < \frac{p-1}{q} < \frac{r}{s+1}$. Here the constant κ is the saturation parameter. The case $\kappa > 0$ is considered. Then (A.4) is known as the GM system with saturation. We derive a formal singular limit of (A.4) as $\epsilon \rightarrow 0$. It seems possible that one may carry out a rigorous deduction using techniques developed by del Pino, Kowalczyk and Wei [9, 8].

The diffusion coefficient ϵ^2 of the variable u must be small. The diffusion coefficient of v must be large in the sense that $d = \frac{d_0}{\epsilon}$ where $d_0 > 0$ is independent of ϵ . In a slightly different parameter range, del Pino discovered a radially symmetric solution whose u component has one transition layer [7]. Sakamoto and Suzuki found a radially symmetric solution with one transition layer in exactly the same parameter range as here [36]. The u component of their solution tends to a constant multiple of a characteristic function of a ball as $\epsilon \rightarrow 0$. More information on the Gierer-Meinhardt type PDE systems may be found in [11, 12].

The nonlinearity in the first equation of (A.4) is denoted by

$$f(u, v) = -u + \frac{u^p}{(1 + \kappa u^p)v^q}. \quad (\text{A.5})$$

It has a cubic shape with respect to u . For each $v > 0$, there exist three zeros of $f(u, v)$ as a function of u . There is a particular value v_0 such that at $v = v_0$, $f(\cdot, v_0)$ becomes a balanced cubic nonlinearity, in the sense $\int_0^z f(u, v_0) du = 0$. Here z is the largest zero of $f(\cdot, v_0)$.

A subset E of D is expected so that $u(x)$ of a solution (u, v) to (A.4) is close to z if x is in E and close to 0 if x is in $D \setminus E$. The boundary of E in D is a collection of curves which we denote by Γ . The value of u changes abruptly across Γ . Away from Γ we take $u \approx z\chi_E$ where χ_E is the characteristic function of E . The shape of u near Γ is more complicated. Let $Q(\xi, s)$ be the traveling wave solution of the problem

$$Q_{\xi\xi} + c(s)Q_{\xi} + f(Q, s) = 0, \quad \xi \in \mathbb{R}. \quad (\text{A.6})$$

In (A.6), s is a parameter. As ξ tends to $-\infty$, we require that $Q(\xi, s)$ tend to 0, and as ξ tends to ∞ , we require that $Q(\xi, s)$ tend to the largest zero of $f(\cdot, s)$. The constant $c(s)$ is the velocity of the traveling wave. It is unknown and must be determined from the equation (A.6).

Let $d(x)$ be the signed distance function from a point x to Γ . The sign of $d(x, \tau)$ is positive if x is in E and negative if x is in $D \setminus E$. We assume that approximately

$$u(x) \approx Q\left(\frac{d(x)}{\epsilon}, s\right) \quad (\text{A.7})$$

where s is constant, discussed later. Insert this Q into the first equation in (A.4) to find

$$Q_{\xi\xi} |\nabla d|^2 + \epsilon Q_{\xi} \Delta d + f(Q, s) = 0.$$

It is known that on the set Γ , $|\nabla d| = 1$ and $\Delta d(x) = H(x)$ where $H(x)$ is the mean curvature of Γ at x , viewed from E . Therefore we obtain

$$Q_{\xi\xi} + \epsilon H Q_{\xi} + f(Q, s) = 0. \quad (\text{A.8})$$

Comparing (A.8) to (A.6) we deduce

$$c(s) = \epsilon H. \quad (\text{A.9})$$

Now we discuss $c(s)$. On the boundary Γ , s must be equal to $v(x)$. However unlike $u(x)$, $v(x)$ changes slowly in x . Asymptotically we have the expansion $v(x) \approx v_0 + \epsilon w(x)$, so that $c(s) = c(v) \approx c(v_0) + \epsilon c'(v_0)w$. Since v_0 is the point where $f(\cdot, v_0)$ is balanced, $c(v_0) = 0$. Hence (A.9) implies that

$$H = c'(v_0)w. \quad (\text{A.10})$$

It remains to find an equation for w . In the second equation in (A.4) we deduce

$$\frac{d_0}{\epsilon} \Delta(v_0 + \epsilon - (v_0 + \epsilon w) + \frac{u^r}{(v_0 + \epsilon w)^s} = 0.$$

As $\epsilon \rightarrow 0$, we find

$$d_0 \Delta w - v_0 + \frac{z^r}{v_0^s} \chi_E = 0. \quad (\text{A.11})$$

The equations (A.10) and (A.11) form a system for the boundary Γ :

$$H = c'(0)w \text{ on } \Gamma, \quad d_0 \Delta w - v_0 + \frac{z^r}{v_0^s} \chi_E = 0 \text{ in } D. \quad (\text{A.12})$$

Note that the Neumann boundary condition for v implies the same boundary condition for w and hence

$$\int_D (-v_0 + \frac{z^r}{v_0^s} \chi_E) dx = 0.$$

Therefore $|E| = \frac{v_0^{s+1}}{z^r} |D|$. Define

$$a = \frac{v_0^{s+1}}{z^r}, \quad \gamma = -\frac{c'(v_0)z^r}{d_0 v_0^s}, \quad (\text{A.13})$$

to turn (A.12) to (1.5).

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