On Non-degeneracy of Solutions to $SU(3)$ Toda System

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March 31, 2010

Abstract

We prove that the solution to the $SU(3)$ Toda system

$$\begin{cases} \Delta u + 2e^u - e^v = 0 & \text{in } \mathbb{R}^2, \\ \Delta v - e^u + 2e^v = 0 & \text{in } \mathbb{R}^2, \\ \displaystyle\int_{\mathbb{R}^2} e^u < \infty, \quad \int_{\mathbb{R}^2} e^v < \infty, \end{cases}$$

is *nondegenerate*, i.e., the kernel of the linearized operator is exactly eight-dimensional.

1 Introduction

Of concern is the nondegeneracy of solutions of the following two-dimensional $SU(3)$ Toda system

$$\begin{cases} \Delta u + 2e^u - e^v = 0 & \text{in } \mathbb{R}^2, \\ \Delta v - e^u + 2e^v = 0 & \text{in } \mathbb{R}^2, \\ \displaystyle\int_{\mathbb{R}^2} e^u < \infty, \quad \int_{\mathbb{R}^2} e^v < \infty, \end{cases} \tag{1}$$

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where $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is the usual Euclidean Laplacian in $\mathbb{R}^2$. System (1) is a natural generalization of the Liouville equation

$$\Delta u + e^u = 0 \quad \text{in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} e^u < \infty. \tag{2}$$

The Liouville equation (2) and the Toda system (1) arise in many physical models and geometrical problems. In Chern-Simons theories, the Liouville equation is related to Abelian models, while the Toda system is related to Non-Abelian models. We refer to the books by Dunne [2], Yang [10] for physical backgrounds. The $SU(3)$ Chern-Simons model has been studied in many papers. We refer to Jost-Wang [4], Jost-Lin-Wang [3], Li-Li [6], Malchiodi-Ndiaye [8], Ohtsuka-Suzuki [9] and the references therein.

Using algebraic geometry results, Jost and Wang [5] classified all solutions to (1). More precisely, when $N = 2$, all solutions to (1) can be written as follows:

$$u(z) = \log \frac{4\left(a_1^2 a_2^2 + a_1^2|2z+c|^2 + a_2^2|z^2+2bz+bc-d|^2\right)}{\left(a_1^2 + a_2^2|z+b|^2 + |z^2+cz+d|^2\right)^2}, \tag{3}$$

$$v(z) = \log \frac{16 a_1^2 a_2^2\left(a_1^2 + a_2^2|z+b|^2 + |z^2+cz+d|^2\right)}{\left(a_1^2 a_2^2 + a_1^2|2z+c|^2 + a_2^2|z^2+2bz+bc-d|^2\right)^2}, \tag{4}$$

where $z = x_1 + ix_2 \in \mathbb{C}$, and $a_1 > 0$, $a_2 > 0$ are real numbers and $b = b_1 + ib_2 \in \mathbb{C}$, $c = c_1 + ic_2 \in \mathbb{C}$, $d = d_1 + id_2 \in \mathbb{C}$. Note that in the above representation there are eight parameters $(a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2) \in \mathbb{R}^8$.

In the study of the blow-up behaviors for solutions of $SU(3)-$Toda system, a crucial question is the *nondegeneracy* of solutions. More precisely, we need to study the elements in the kernel of the associated linearized operator, i.e. the following linear system

$$\begin{cases} \Delta\varphi_1 + 2e^u\varphi_1 - e^v\varphi_2 = 0 & \text{in } \mathbb{R}^2, \\ \Delta\varphi_2 - e^u\varphi_1 + 2e^v\varphi_2 = 0 & \text{in } \mathbb{R}^2. \end{cases} \tag{5}$$

Here (u, v) are solutions to (1) given by (3)-(4). Certainly there are at least eight-dimensional kernels, since any differentiation of (u, v) with respect to the eight parameters satisfies (5).

The following theorems shows that the converse is also true, which shows that the solution (u, v) is *nondegenerate*.

Theorem 1.1. *Let (φ_1, φ_2) satisfy (5). Assume that*

$$|\varphi_1| \le C(1 + |x|)^\tau, \quad |\varphi_2| \le C(1 + |x|)^\tau \quad \textit{for} x \in \mathbb{R}^2 \textit{and some } \tau \in (0, 1).$$

Then $\binom{\varphi_1}{\varphi_2}$ *belongs to the following linear space*

$$\mathrm{span}\left\{\begin{pmatrix}\partial_{a_1}u\\ \partial_{a_1}v\end{pmatrix},\begin{pmatrix}\partial_{a_2}u\\ \partial_{a_2}v\end{pmatrix},\begin{pmatrix}\partial_{b_1}u\\ \partial_{b_1}v\end{pmatrix},\begin{pmatrix}\partial_{b_2}u\\ \partial_{b_2}v\end{pmatrix},\begin{pmatrix}\partial_{c_1}u\\ \partial_{c_1}v\end{pmatrix},\begin{pmatrix}\partial_{c_2}u\\ \partial_{c_2}v\end{pmatrix},\begin{pmatrix}\partial_{d_1}u\\ \partial_{d_1}v\end{pmatrix},\begin{pmatrix}\partial_{d_2}u\\ \partial_{d_2}v\end{pmatrix}\right\}.$$

In the case of nondegeneracy of solutions of Liouville equation (2), the problem becomes to a single linear equation

$$\Delta\varphi + e^u\varphi = 0 \quad \text{in } \mathbb{R}^2. \tag{6}$$

Using conformal transformations, one can assume that u is radially symmetric. Then one can use the separation of variables to obtain the nondegeneracy result. See Lemma 2.3 of Chen-Lin [1]. Here we are dealing with a system. Firstly, we can not find a conformal transformation to transform any solution (u, v) to radially symmetric solution. Second, even if (u, v) is radially symmetric, the new system is still too complicated to study. To overcome these difficulties, we employ the *invariants* of the system (1). Using the invariants of (1), we also obtain invariants for (5) and thus prove Theorem 1.1.

For convenience, the language of complex variable is used in this paper. We refer $\bar z = x_1 - ix_2$ to the usual conjugate of $z = x_1 + ix_2 \in \mathbb{C}$. In addition, $U_z := \partial_z U = \frac{1}{2}(\frac{\partial U}{\partial x_1} - i\frac{\partial U}{\partial x_2})$, $U_{\bar z} := \partial_{\bar z} U = \frac{1}{2}(\frac{\partial U}{\partial x_1} + i\frac{\partial U}{\partial x_2})$. Notation C is a generic constant which may be different from line to line.

We believe that our method may be used to deal with the general case $SU(N+1)$. The major problem is how to obtain higher-order invariants as in Section 2.

2 Invariants of (1)

In this section, we derive some invariants for (1). For more discussions, we refer to Section 5.5 of the book by Leznov-Saveliev [7].

Let us first define the following transformation in whole $\mathbb{C}$,

$$U(z,\bar z) = \frac{2u}{3} + \frac{v}{3} - \log 4, \quad V(z,\bar z) = \frac{u}{3} + \frac{2v}{3} - \log 4. \tag{7}$$

Note that $\Delta = 4\partial_{z\bar z}$. Then the Toda system (1) can be rewritten as

$$\begin{cases} U_{z\bar z} + e^{2U-V} = 0, & \text{in } \mathbb{C},\\ V_{z\bar z} + e^{2V-U} = 0 & \text{in } \mathbb{C},\\ \int_{\mathbb{R}^2} e^{2U-V} < \infty, \quad \int_{\mathbb{R}^2} e^{2V-U} < \infty. & \end{cases} \tag{8}$$

We prove now some preliminary lemmas.

Lemma 2.1. *We have, in whole $\mathbb{C}$, that*

$$U_{zz} + V_{zz} - U_z^2 - V_z^2 + U_z V_z \equiv 0,$$
$$U_{\bar{z}\bar{z}} + V_{\bar{z}\bar{z}} - U_{\bar{z}}^2 - V_{\bar{z}}^2 + U_{\bar{z}} V_{\bar{z}} \equiv 0.$$

Proof. The proof is a straightforward calculation. We only prove the first identity because the second one can be dealt with similarly.

Let

$$f(z, \bar{z}) = U_{zz} + V_{zz} - U_z^2 - V_z^2 + U_z V_z.$$

A straightforward computation and (8) show that in whole $\mathbb{C}$,

$$U_{zz\bar{z}} = -e^{2U-V}(2U_z - V_z), \qquad V_{zz\bar{z}} = -e^{2V-U}(2V_z - U_z),$$
$$(-U_z^2)_{\bar{z}} = 2U_z e^{2U-V}, \qquad (-V_z^2)_{\bar{z}} = 2V_z e^{2V-U},$$
$$(U_z V_z)_{\bar{z}} = -e^{2U-V} V_z - e^{2V-U} U_z.$$

Thus it holds that

$$f_{\bar{z}} \equiv 0, \quad \text{and thus} \quad f_{z\bar{z}} \equiv 0 \quad \text{in } \mathbb{C}.$$

Since f is smooth and goes to 0 at ∞ by (3)–(7), we have that, by Liouville's theorem,

$$f \equiv 0 \quad \text{in } \mathbb{C}.$$

The first identity is then concluded.

Simply exchanging $\bar{z}$ and z in the above proof leads to the second identity. The proof is then complete. □

Lemma 2.2. *We have*

$$U_{zzz} - 3U_z U_{zz} + U_z^3 \equiv 0, \qquad V_{zzz} - 3V_z V_{zz} + V_z^3 \equiv 0, \tag{9}$$
$$U_{\bar{z}\bar{z}\bar{z}} - 3U_{\bar{z}} U_{\bar{z}\bar{z}} + U_{\bar{z}}^3 \equiv 0, \qquad V_{\bar{z}\bar{z}\bar{z}} - 3V_{\bar{z}} V_{\bar{z}\bar{z}} + V_{\bar{z}}^3 \equiv 0. \tag{10}$$

Proof. Since the proofs of (9) and (10) are similar, we will only check the former. For convenience, we denote that

$$f_1(z, \bar{z}) = U_{zzz} - 3U_z U_{zz} + U_z^3, \quad f_2(z, \bar{z}) = V_{zzz} - 3V_z V_{zz} + V_z^3.$$

We **claim** that

$$f_{1,\bar{z}} \equiv 0, \quad f_{2,\bar{z}} \equiv 0.$$

In fact, a direct calculation gives that

$$U_{zzz\bar{z}} = -(e^{2U-V})_{zz} = -[e^{2U-V}(2U_z - V_z)]_z$$

$$= e^{2U-V}(-4U_z^2 + 4U_zV_z - V_z^2 - 2U_{zz} + V_{zz}),$$

$$\begin{aligned}-3(U_zU_{zz})_{\bar z} &= -3U_{z\bar z}U_{zz} - 3U_zU_{zz\bar z} = 3e^{2U-V}U_{zz} + 3e^{2U-V}U_z(2U_z - V_z)\\ &= e^{2U-V}(3U_{zz} + 6U_z^2 - 3U_zV_z),\end{aligned}$$

and

$$(U_z^3)_{\bar z} = 3U_z^2U_{z\bar z} = e^{2U-V}(-3U_z^2).$$

So we have

$$f_{1,\bar z} = e^{2U-V}(U_{zz} + V_{zz} - U_z^2 - V_z^2 + U_zV_z).$$

Then Lemma 2.1 implies that $f_{1,\bar z} \equiv 0$. Similarly we also have $f_{2,\bar z} \equiv 0$. The claim is proved.

Thus we know $f_{1,z\bar z} \equiv 0$ and so does $f_{2,z\bar z}$. Since $f_1 \to 0$ and $f_2 \to 0$ as $|z| \to \infty$, again by Liouville's theorem, we get (9). This concludes the proof. $\square$

3 Proof of the main theorem

In what follows, we discuss the kernel of the corresponding linearized operator of (8), which is equivalent to (5). All functions and equations discussed here are defined in the whole plane $\mathbb{C}$. Let ϕ, ψ be functions satisfy

$$\phi_{z\bar z} + e^{2U-V}(2\phi - \psi) = 0, \quad \psi_{z\bar z} + e^{2V-U}(2\psi - \phi) = 0. \tag{11}$$

We prove the following proposition, which gives the proof of Theorem 1.1.

Proposition 3.1. *Let (ϕ, ψ) satisfy (11). Assume that*

$$|\phi| \le C(1+|z|)^\tau, \quad |\psi| \le C(1+|z|)^\tau \quad \text{for some } \tau \in (0,1). \tag{12}$$

Then $\binom{\phi}{\psi}$ belongs to the following linear space

$$\operatorname{span}\left\{\begin{pmatrix}\partial_{a_1}U\\ \partial_{a_1}V\end{pmatrix}, \begin{pmatrix}\partial_{a_2}U\\ \partial_{a_2}V\end{pmatrix}, \begin{pmatrix}\partial_{b_1}U\\ \partial_{b_1}V\end{pmatrix}, \begin{pmatrix}\partial_{b_2}U\\ \partial_{b_2}V\end{pmatrix}, \begin{pmatrix}\partial_{c_1}U\\ \partial_{c_1}V\end{pmatrix}, \begin{pmatrix}\partial_{c_2}U\\ \partial_{c_2}V\end{pmatrix}, \begin{pmatrix}\partial_{d_1}U\\ \partial_{d_1}V\end{pmatrix}, \begin{pmatrix}\partial_{d_2}U\\ \partial_{d_2}V\end{pmatrix}\right\}.$$

Remark 3.2. *Under the assumption (12), we know that all the derivatives of ϕ and ψ approach to 0 as $|z|$ goes to ∞. Indeed, if we define that, for $x \in \mathbb{R}^2$,*

$$\tilde\phi(x) = \frac{1}{8\pi}\int_{\mathbb{R}^2} \log|x-y| e^{2U(y)-V(y)}[2\phi(y) - \psi(y)]\mathrm{d}y,$$

then $|\tilde{\phi}| \le C\log(1+|x|)$ and $\Delta(\phi - \tilde{\phi}) = 0$. Therefore, $\phi = \tilde{\phi} + C$. The potential theory implies that ϕ's derivatives vanish at infinity. So do the derivatives of ψ.

Lemma 3.3. *Under the assumption of Proposition 3.1, it holds that*

$$\begin{aligned}
&\phi_{zz} + \psi_{zz} - 2U_z\phi_z - 2V_z\psi_z + U_z\psi_z + V_z\phi_z \equiv 0,\\
&\phi_{\bar{z}\bar{z}} + \psi_{\bar{z}\bar{z}} - 2U_{\bar{z}}\phi_{\bar{z}} - 2V_{\bar{z}}\psi_{\bar{z}} + U_{\bar{z}}\psi_{\bar{z}} + V_{\bar{z}}\phi_{\bar{z}} \equiv 0.
\end{aligned}$$

Proof. The proof is similar as that of Lemma 2.1, using Remark 3.2. □

Lemma 3.4. *Under the assumption of Proposition 3.1, we have*

$$\begin{aligned}
&\phi_{zzz} - 3\phi_{zz}U_z - 3\phi_z U_{zz} + 3U_z^2\phi_z \equiv 0,\\
&\phi_{\bar{z}\bar{z}\bar{z}} - 3\phi_{\bar{z}\bar{z}}U_{\bar{z}} - 3\phi_{\bar{z}}U_{\bar{z}\bar{z}} + 3U_{\bar{z}}^2\phi_{\bar{z}} \equiv 0,\\
&\psi_{zzz} - 3\psi_{zz}V_z - 3\psi_z V_{zz} + 3V_z^2\psi_z \equiv 0,\\
&\psi_{\bar{z}\bar{z}\bar{z}} - 3\psi_{\bar{z}\bar{z}}V_{\bar{z}} - 3\psi_{\bar{z}}V_{\bar{z}\bar{z}} + 3V_{\bar{z}}^2\psi_{\bar{z}} \equiv 0.
\end{aligned}$$

Proof. We only check the first one since the others are similar. By direct computation, we get, using (8),

$$\begin{aligned}
\phi_{zzz\bar{z}} &= -[e^{2U-V}(2\phi - \psi)]_{zz}\\
&= -[e^{2U-V}(2U_z - V_z)(2\phi - \psi)]_z - [e^{2U-V}(2\phi_z - \psi_z)]_z\\
&= -e^{2U-V}(2U_z - V_z)^2(2\phi - \psi) - e^{2U-V}(2U_{zz} - V_{zz})(2\phi - \psi)\\
&\quad - e^{2U-V}2(2U_z - V_z)(2\phi_z - \psi_z) - e^{2U-V}(2\phi_{zz} - \psi_{zz})\\
&= e^{2U-V}(-8U_z^2\phi + 4U_z^2\psi + 8U_zV_z\phi - 4U_zV_z\psi - 2V_z^2\phi + V_z^2\psi - 4U_{zz}\phi\\
&\qquad + 2U_{zz}\psi + 2V_{zz}\phi - V_{zz}\psi - 8U_z\phi_z + 4U_z\psi_z + 4V_z\phi_z - 2V_z\psi_z\\
&\qquad - 2\phi_{zz} + \psi_{zz}),
\end{aligned}$$

$$\begin{aligned}
-3(\phi_{zz}U_z)_{\bar{z}} &= 3[e^{2U-V}(2\phi - \psi)]_z U_z + 3e^{2U-V}\phi_{zz}\\
&= e^{2U-V}(12U_z^2\phi - 6U_z^2\psi - 6U_zV_z\phi + 3U_zV_z\psi + 6U_z\phi_z\\
&\qquad - 3U_z\psi_z + 3\phi_{zz}),
\end{aligned}$$

$$\begin{aligned}
-3(\phi_z U_{zz})_{\bar{z}} &= e^{2U-V}(6U_{zz}\phi - 3U_{zz}\psi + 6U_z\phi_z - 3V_z\phi_z),\\
3(U_z^2\phi_z)_{\bar{z}} &= e^{2U-V}(-6U_z\phi_z - 6U_z^2\phi + 3U_z^2\psi).
\end{aligned}$$

So it holds that

$$\begin{aligned}&(\phi_{zzz}-3\phi_{zz}U_z-3\phi_zU_{zz}+3U_z^2\phi_z)_{\bar z}\\ =\ &e^{2U-V}[(U_{zz}+V_{zz}-U_z^2-V_z^2+U_zV_z)(2\phi-\psi)]\\ &\quad+e^{2U-V}(\phi_{zz}+\psi_{zz}-2U_z\phi_z-2V_z\psi_z+U_z\psi_z+V_z\phi_z).\end{aligned}$$

Then Lemma 2.1, Lemma 3.3 and Remark 3.2 yield that

$$\phi_{zzz}-3\phi_{zz}U_z-3\phi_zU_{zz}+3U_z^2\phi_z\equiv 0.$$

The proof is completed. □

Proof of Proposition 3.1. Let $\phi_1=e^{-U}\phi$. Since we have easily that

$$\phi_{zzz}-3\phi_{zz}U_z-3\phi_zU_{zz}+3U_z^2\phi_z=e^U\left[\phi_{1,zzz}+(U_{zzz}-3U_zU_{zz}+U_z^3)\phi_1\right],$$

using Lemma 2.2 and Lemma 3.4, we have $\phi_{1,zzz}\equiv 0$. Similarly, it also holds that $\phi_{1,\bar z\bar z\bar z}\equiv 0$. This implies that

$$\phi_1=\sum_{k,\ell=0}^{2}\alpha_{k\ell}z^k\bar z^\ell \quad (\text{with all } \alpha_{k\ell}\in\mathbb{C}). \tag{13}$$

Since ϕ_1 is real, it must hold that

$$\alpha_{00},\ \alpha_{11},\ \alpha_{22}\in\mathbb{R} \quad\text{and}\quad \alpha_{01}=\bar\alpha_{10},\ \alpha_{02}=\bar\alpha_{20},\ \alpha_{12}=\bar\alpha_{21}.$$

On the other hand, denote that $\psi_1=e^{-V}\psi$. Similarly we can also obtain that

$$\psi_1=\sum_{k,\ell=0}^{2}\beta_{k\ell}z^k\bar z^\ell \quad (\text{with all } \beta_{k\ell}\in\mathbb{C}), \tag{14}$$

where $\beta_{k\ell}$ satisfy

$$\beta_{00},\ \beta_{11},\ \beta_{22}\in\mathbb{R} \quad\text{and}\quad \beta_{01}=\bar\beta_{10},\ \beta_{02}=\bar\beta_{20},\ \beta_{12}=\bar\beta_{21}.$$

Rewriting (11) in the term of ϕ_1 and ψ_1, we have

$$\phi_{1,z\bar z}+U_{\bar z}\phi_{1,z}+U_z\phi_{1,\bar z}+(e^{2U-V}+U_zU_{\bar z})\phi_1-e^U\psi_1=0, \tag{15}$$

$$\psi_{1,z\bar z}+V_{\bar z}\psi_{1,z}+V_z\psi_{1,\bar z}+(e^{2V-U}+V_zV_{\bar z})\psi_1-e^V\phi_1=0. \tag{16}$$

Substituting (13), (14) into (15) and using *Mathematica*, we find that

$$\beta_{00}=\frac{\alpha_{11}a_1^2+\alpha_{00}a_2^2-\alpha_{10}a_2^2b-\alpha_{01}a_2^2\bar b+\alpha_{11}a_2^2|b|^2+\alpha_{00}|c|^2-\alpha_{10}\bar c d-\alpha_{01}c\bar d+\alpha_{11}|d|^2}{2^{2/3}\sqrt[3]{a_1^2a_2^2}},$$

$$
\begin{aligned}
\beta_{11} &= \frac{2\sqrt[3]{2}(\alpha_{22}a_1^2 + \alpha_{00} + a_2^2\alpha_{22}|b|^2 - \alpha_{20}d - \alpha_{02}\bar{d} + \alpha_{22}|d|^2)}{\sqrt[3]{a_1^2a_2^2}},\\
\beta_{22} &= \frac{\alpha_{22}a_2^2 + \alpha_{11} - \alpha_{21}c - \alpha_{12}\bar{c} + \alpha_{22}|c|^2}{2^{2/3}\sqrt[3]{a_1^2a_2^2}},\\
\beta_{01} &= \frac{\sqrt[3]{2}(\alpha_{12}a_1^2 - \alpha_{02}a_2^2\bar{b} + \alpha_{12}a_2^2|b|^2 + \alpha_{00}c - \alpha_{10}d - \alpha_{02}c\bar{d} + \alpha_{12}|d|^2)}{\sqrt[3]{a_1^2a_2^2}},\\
\beta_{02} &= -\frac{\alpha_{02}a_2^2 - \alpha_{12}ba_2^2 - \alpha_{01}c + \alpha_{02}|c|^2 + \alpha_{11}d - \alpha_{12}\bar{c}d}{2^{2/3}\sqrt[3]{a_1^2a_2^2}},\\
\beta_{12} &= \frac{\sqrt[3]{2}(\alpha_{22}ba_2^2 + \alpha_{01} - \alpha_{02}\bar{c} - \alpha_{21}d + \alpha_{22}\bar{c}d)}{\sqrt[3]{a_1^2a_2^2}},\\
\beta_{10} &= \bar{\beta}_{01}, \qquad \beta_{20} = \bar{\beta}_{02}, \qquad \beta_{21} = \bar{\beta}_{12}.
\end{aligned}
$$

Finally we insert all the above quantities into (16) again and thus obtain another relation

$$
\begin{aligned}
\alpha_{22} = {} & \frac{-\alpha_{11}a_1^2 + \alpha_{21}ca_1^2 + \alpha_{12}\bar{c}a_1^2 - \alpha_{00}a_2^2 + \alpha_{10}a_2^2b + \alpha_{01}a_2^2\bar{b} - \alpha_{11}a_2^2|b|^2}{a_1^2a_2^2 + |c|^2a_1^2 + a_2^2|b|^2|c|^2 - a_2^2\bar{b}\bar{c}d - a_2^2bc\bar{d} + a_2^2|d|^2}\\
& + \frac{-\alpha_{20}a_2^2bc - \alpha_{02}a_2^2\bar{b}\bar{c} + \alpha_{21}a_2^2|b|^2c + \alpha_{12}a_2^2|b|^2\bar{c} + \alpha_{20}a_2^2d + \alpha_{02}a_2^2\bar{d}}{a_1^2a_2^2 + |c|^2a_1^2 + a_2^2|b|^2|c|^2 - a_2^2\bar{b}\bar{c}d - a_2^2bc\bar{d} + a_2^2|d|^2}\\
& + \frac{-\alpha_{21}a_2^2\bar{b}d - \alpha_{12}a_2^2b\bar{d}}{a_1^2a_2^2 + |c|^2a_1^2 + a_2^2|b|^2|c|^2 - a_2^2\bar{b}\bar{c}d - a_2^2bc\bar{d} + a_2^2|d|^2},
\end{aligned}
$$

from which we know that ϕ_1 and ψ_1 actually depend on 8 real parameters rather than formally 9. Therefore, the dimension of the space $\{(\phi, \psi)\}$ is exactly 8. Since it is known that

$$
\begin{pmatrix}\partial_{a_1}U\\ \partial_{a_1}V\end{pmatrix}, \begin{pmatrix}\partial_{a_2}U\\ \partial_{a_2}V\end{pmatrix}, \begin{pmatrix}\partial_{b_1}U\\ \partial_{b_1}V\end{pmatrix}, \begin{pmatrix}\partial_{b_2}U\\ \partial_{b_2}V\end{pmatrix}, \begin{pmatrix}\partial_{c_1}U\\ \partial_{c_1}V\end{pmatrix}, \begin{pmatrix}\partial_{c_2}U\\ \partial_{c_2}V\end{pmatrix}, \begin{pmatrix}\partial_{d_1}U\\ \partial_{d_1}V\end{pmatrix}, \begin{pmatrix}\partial_{d_2}U\\ \partial_{d_2}V\end{pmatrix}
$$

are linearly independent and satisfy (11), we then complete the proof of Proposition 3.1. □

Finally let

$$
\varphi_1 = 2\phi - \psi, \qquad \varphi_2 = 2\psi - \phi,
$$

where ϕ, ψ satisfy (11). It is easy to check that φ_1, φ_2 satisfy (5). Thus Theorem 1.1 is equivalent to Proposition 3.1. Theorem 1.1 is concluded.

Acknowledgments: The research of the first and second author is partially supported by a research Grant from GRF of Hong Kong and a Focused Research Scheme of CUHK. The third author is supported in part by NSFC

No. 10971067 and Shanghai project 09XD1401600. Part of this work was done while the third author was visiting CUHK, he thanks the department of Mathematics for its warm hospitality. We thank Prof. Changshou Lin for useful discussions.

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