Orbital stability of bound states of semi-classical nonlinear Schrödinger equations with critical nonlinearity

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We consider the orbital stability of single-spike bound states of semi-classical nonlinear Schrödinger equations with critical nonlinearity and a trap potential. Due to the effect of the trap potential, we derive the asymptotic expansion formulas and obtain the necessary conditions for orbital stability and instability of single-spike bound states. Our argument is applied to two-component systems of nonlinear Schrödinger equations with a common trap potential, cubic nonlinearity in two spatial dimensions. The orbital stability of bound states with spikes of these systems is investigated. Our results show the existence of stable spikes in two-dimensional Bose-Einstein condensates.

1 Introduction

The nonlinear Schrödinger equation (NLS) with a trap potential is central to the understanding of many physical phenomena. For example, it has become a well-known model referred to as the Gross-Pitaevskii equation governing the evolution of Bose-Einstein condensates (BEC) given by

$$-i\hbar\frac{\partial\psi}{\partial t}=\frac{\hbar^2}{2m}\triangle\psi-V_{trap}\psi-\mu|\psi|^2\psi\,, \tag{1.1}$$

for $x\in\mathbb{R}^N$, $N\leq 3$ and $t>0$, where $\psi=\psi(x,t)\in\mathbb{C}$ is the wavefunction of BEC, and $V_{trap}=V_{trap}(x)$ is the trap potential. Besides, $\hbar$ is Planck constant, m is the atom mass, and $\mu\sim 4\pi\frac{\hbar^2}{2m}a$, where a denotes the s-wave scattering length.

In BEC, spikes may occur when the s-wave scattering length is negative and large. Due to Feshbach resonance, the s-wave scattering length of a single condensate can be tuned over a very large range by adjusting the externally applied magnetic field. As the s-wave scattering length of a single condensate is negative and large enough, the interactions of atoms are strongly attractive and the associated condensate tends to increase its density at the centre of the trap potential in order to lower the interaction energy (cf. [25]). Under the effect of trap potentials, spikes of BEC are observable by physical experiments (cf. [6]) so there must have stability to assure spikes appearing in the condensate wavefunction (cf. [5]). In [19], stable bright solitons (spikes) of BEC can be observed by numerical simulations, provided that the strength of the

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trap potential exceeds a threshold value. Here we want to develop mathematical theorems to support the existence of stable spikes in BEC.

To get spikes in BEC, we may assume the s-wave scattering length a i.e. μ is negative and large. Setting $h^2 = \hbar^2/(-2m\mu)$, $V_{trap}(x) = h^2 V(x)$ and suitable time scale, the equation (1.1) with negative and large μ can be equivalent to a semi-classical NLS given by

$$-ih\frac{\partial \psi}{\partial t} = h^2 \Delta \psi - V\psi + |\psi|^2\psi, \quad x \in \mathbb{R}^N\,, t > 0\,, \tag{1.2}$$

where $0 < h \ll 1$ is a small parameter and $V = V(x)$ is a smooth nonnegative function. We may generalize the equation (1.2) to a NLS having the following form

$$-ih\frac{\partial \psi}{\partial t} = h^2 \Delta \psi - V\psi + |\psi|^{p-1}\psi, \quad x \in \mathbb{R}^N\,, t > 0\,, \tag{1.3}$$

with critical nonlinearity

$$p = 1 + \frac{4}{N}, \quad N \geq 1\,. \tag{1.4}$$

In particular, when $N = 2$, the equation (1.3) is exactly same as (1.2).

Bound states of (1.3) are of the form $\psi(x,t) = e^{i\lambda t/h}u(x)$, where u satisfies the following nonlinear elliptic equation

$$h^2 \Delta u - (V + \lambda)u + u^p = 0\,, \quad u \in H^1(\mathbb{R}^N)\,, \quad u > 0 \quad \text{in} \quad \mathbb{R}^N\,, \tag{1.5}$$

with zero Dirichlet boundary condition i.e. $u(x) \to 0$ as $|x| \to \infty$. When $h > 0$ is fixed and λ is sufficiently large, one may refer to [8] for the stability problem which is different from our conditions that $\lambda > 0$ is fixed and $0 < h \ll 1$ is a small parameter. The existence of single or multiple spike solutions of (1.5) was first established by Floer-Weinstein [7] in one dimensional case i.e. $N = 1$ and $1 < p < 5$, and later extended by Oh [21]-[22] to higher dimensional case i.e. $N \geq 2$ and $1 < p < \frac{N+2}{N-2}$ under the condition that the trap potential V has nondegenerate critical points. When the trap potential V becomes degenerate, there have been many works in recent years. One may refer to [3], [4], [12], [23], [24] [15], [29], [28], [31], [32], and the references therein.

The trap potential V may also play a crucial role on the orbital (dynamic) stability of single-spike bound states. As the trap potential V is switched off, it is well known that all bound states of the equation (1.3) with the condition (1.4) are orbitally unstable if the dimension $N = 2$ (cf. [34]). To stabilize bound states, one has to turn on the trap potential. However, in general, some nonzero trap potentials may still cause dynamic instability in BEC. For instance, one may find bending-wave instability of vortex ring dynamics under some nonzero trap potentials (cf. [14]). Consequently, to get the dynamic stability of single-spike bound states, we have to choose trap potentials properly. For suitable trap potentials, Oh [22], and Grillakis, Shatah, Strauss [10] proved that when $N = 1$, the single-spike bound state (concentrating at local nondegenerate minimum of the trap potential V) is stable if $1 < p < 1 + \frac{4}{N}$ and unstable if $p > 1 + \frac{4}{N}$. Generically, the case of $p = 1 + \frac{4}{N}$ is left open and referred to as critical case in the literature. In this paper, we give an affirmative answer for such a case by studying the orbital stability and instability of single-spike bound states when the trap potential V has nondegenerate critical points.

In [21]-[22], u_h a single-spike bound state solution of (1.5) can be obtained, provided the trap potential V is of class $(V)_a$ and fulfill other conditions. Hereafter, we set u_h as a single-spike bound state constructed in [21]-[22] and satisfying (1.5). Of course, the trap potential V is also of class $(V)_a$ and fulfill other conditions in [21]-[22]. Hence $\psi_h(x,t) = e^{i\lambda t/h} u_h(x)$ may form an orbit of (1.3). From [10], the orbital stability of ψ_h's is defined as follows: For all $\epsilon > 0$, there exists $\delta > 0$ such that if $\|\psi_0 - u_h\|_{H^1} < \delta$ and ψ is a solution of (1.3) in some interval $[0, t_0)$ with $\psi|_{t=0} = \psi_0$, then $\psi(t,\cdot)$ can be extended to a solution in $0 \le t < \infty$ and

$$\sup_{0<t<\infty} \inf_{s\in\mathbb{R}} \|\psi(\cdot,t) - \psi_h(\cdot,s)\|_{H^1} < \epsilon\,.$$

Otherwise, the orbit ψ_h is called orbitally unstable. To check the orbital stability of ψ_h, we use the linearized operator defined by

$$L_h = h^2\triangle - (V+\lambda) + \ pu_h^{p-1}, \quad p = 1 + \frac{4}{N}\,. \tag{1.6}$$

Observe that u_h may depend on the variable λ. Moreover, we assume u_h to be nondegenerate due to [13]. Let $n(L_h)$ be the number of positive eigenvalues of L_h and

$$d(\lambda) = \int_{\mathbb{R}^N} \left[\frac{h^2}{2}|\nabla u_h|^2 + \frac{1}{2}(V+\lambda)u_h^2 - \frac{1}{p+1}u_h^{p+1}\right]. \tag{1.7}$$

Assume that d is nondegenerate, i.e, $d'' \neq 0$. Let $p(d'') = 1$ if $d'' > 0$ and $p(d'') = 0$ if $d'' < 0$. According to general theory of orbital stability of bound states (cf. [10],[11]), u_h is orbitally stable if $n(L_h) = p(d'')$, and orbitally unstable if $n(L_h) - p(d'')$ is odd (see page 309 of [11]).

It is remarkable that if $V \equiv C$ and $p = 1 + \frac{4}{N}$, then $d''(\lambda) = 0$ i.e. the function d becomes degenerate, where C is a positive constant. Consequently, we may assume the trap potential V has nondegenerate critical points in order to derive the asymptotic expansion formulas for the operator L_h and the function d as the parameter h goes to zero. These formulas are crucial to obtain the orbital stability and instability of single-spike bound states as follows:

Theorem 1.1. *Let N be a positive integer and $p = 1 + \frac{4}{N}$. For $0 < h < 1$, let u_h be a bound state of (1.3) concentrating at a nondegenerate critical point x_0 of the potential V such that $\Delta V(x_0) \neq 0$. Let m denote the number of negative eigenvalues of the matrix $(\nabla^2 V(x_0))$. Suppose the parameter h is sufficiently small. Then u_h is orbitally stable if x_0 is a nondegenerate local minimum point of the potential V. Furthermore, u_h is orbitally unstable if $m - \frac{1}{2}(1 + \frac{\Delta V(x_0)}{|\Delta V(x_0)|})$ is even.*

Another motivation of studying the equation (1.3) in the critical case may come from two-component systems of nonlinear Schrödinger equations which describe a double condensate i.e. a binary mixture of Bose-Einstein condenstaes (cf. [25]). To get stable spikes of a double condensate with two spatial dimensions, we study orbitally stable bound states with spikes of a two-component system of nonlinear Schrödinger equations given by

$$\begin{cases} -ih\frac{\partial\Phi}{\partial t} = h^2\triangle\Phi - V\Phi + |\Phi|^2\Phi + \beta|\Psi|^2\Phi\,, \\ -ih\frac{\partial\Psi}{\partial t} = h^2\triangle\Psi - V\Psi + |\Psi|^2\Psi + \beta|\Phi|^2\Psi\,, \end{cases} \tag{1.8}$$

for $x \in \mathbb{R}^2$ and $t > 0$, where $V = V(x)$ is a smooth nonnegative function, $\beta \in \mathbb{R}$ is a nonzero constant, and $0 < h \ll 1$ is a small parameter. Bound states of (1.8) are of the form $\Phi(x,t) = e^{i\lambda t/h} u(x)$ and $\Psi(x,t) = e^{i\lambda t/h} v(x)$, where (u,v) satisfies the following nonlinear elliptic system

$$\begin{cases} h^2 \triangle u - (V+\lambda)u + u^3 + \beta u v^2 = 0 & , \ x \in \mathbb{R}^2 \\ h^2 \triangle v - (V+\lambda)v + v^3 + \beta u^2 v = 0 & , \ x \in \mathbb{R}^2, \\ u(x), v(x) > 0, u, v \in H^1(\mathbb{R}^2). \end{cases} \tag{1.9}$$

Note that in $\mathbb{R}^2$, the nonlinearity u^3, v^3 are critical nonlinearity by simple algebra $p = 3 = 1 + \frac{4}{N}$ with $N = 2$. The system (1.9) admits a bound state solution $(\frac{1}{\sqrt{1+\beta}} u_h, \frac{1}{\sqrt{1+\beta}} u_h)$, where $\beta > -1$ and u_h satisfies the equation (1.5). Generically, such a solution may not be either a unique positive solution or a ground state solution. Thus the stability problem is nontrivial. Here we want to get the orbital stability of such a solution using suitable trap potentials V's. To study the orbital stability of such a bound state solution, we set the linearized operator of (1.8) around $\left(\frac{1}{\sqrt{1+\beta}} u_h, \frac{1}{\sqrt{1+\beta}} u_h\right)$ given by

$$\mathbb{L}_h \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} h^2 \triangle \phi - (V+\lambda)\phi + \frac{3+\beta}{1+\beta} u_h^2 \phi + \frac{2\beta}{1+\beta} u_h^2 \psi \\ h^2 \triangle \psi - (V+\lambda)\psi + \frac{3+\beta}{1+\beta} u_h^2 \psi + \frac{2\beta}{1+\beta} u_h^2 \phi \end{pmatrix}. \tag{1.10}$$

Furthermore, we also need a function defined as follows:

$$\begin{aligned} d(\lambda_1, \lambda_2) = & \int_{\mathbb{R}^2} \frac{h^2}{2} |\nabla u_{h,\lambda_1,\lambda_2}|^2 + \frac{V(x)+\lambda+\lambda_1}{2} u_{h,\lambda_1,\lambda_2}^2 - \frac{1}{4} \int_{\mathbb{R}^2} u_{h,\lambda_1,\lambda_2}^4 \\ & + \int_{\mathbb{R}^2} \frac{h^2}{2} |\nabla v_{h,\lambda_1,\lambda_2}|^2 + \frac{V(x)+\lambda+\lambda_2}{2} v_{h,\lambda_1,\lambda_2}^2 - \frac{1}{4} \int_{\mathbb{R}^2} v_{h,\lambda_1,\lambda_2}^4 \\ & - \frac{\beta}{2} \int_{\mathbb{R}^2} u_{h,\lambda_1,\lambda_2}^2 v_{h,\lambda_1,\lambda_2}^2 , \end{aligned} \tag{1.11}$$

where $(u_{h,\lambda_1,\lambda_2}, v_{h,\lambda_1,\lambda_2})$ is the solution of

$$\begin{cases} h^2 \triangle u - (V+\lambda+\lambda_1)u + u^3 + \beta u v^2 = 0, & \text{in } \mathbb{R}^2 , \\ h^2 \triangle v - (V+\lambda+\lambda_2)v + v^3 + \beta u^2 v = 0, & \text{in } \mathbb{R}^2 , \end{cases} \tag{1.12}$$

such that $(u_{h,\lambda_1,\lambda_2}, v_{h,\lambda_1,\lambda_2}) \to \left(\frac{1}{\sqrt{1+\beta}} u_h, \frac{1}{\sqrt{1+\beta}} u_h\right)$ as $|\lambda_1| + |\lambda_2| \to 0$.

Suppose the solution $\left(\frac{1}{\sqrt{1+\beta}} u_h, \frac{1}{\sqrt{1+\beta}} u_h\right)$ is nondegenerate, i.e. the operator $\mathbb{L}_h$ has no zero eigenvalue. Let $n(\mathbb{L}_h)$ denote the positive eigenvalues of $\mathbb{L}_h$, and set p as the number of positive eigenvalues of the Hessian matrix $(\nabla^2 d(0,0))$. From [10]-[11], we know that the solution $\left(\frac{1}{\sqrt{1+\beta}} u_h, \frac{1}{\sqrt{1+\beta}} u_h\right)$ is orbitally stable if $n(\mathbb{L}_h) = p$, and orbitally unstable if $n(\mathbb{L}_h) - p$ is odd. The parameter β may affect the orbital stability of the solution $\left(\frac{1}{\sqrt{1+\beta}} u_h, \frac{1}{\sqrt{1+\beta}} u_h\right)$. Now we state our result as follows:

Theorem 1.2. *For $0 < h < 1$, let u_h be a single spike solution concentrating at a local minimum point of the function V. Suppose the parameter h is sufficiently small. Then $(\frac{1}{\sqrt{1+\beta}} u_h, \frac{1}{\sqrt{1+\beta}} u_h)$ is orbitally stable to (1.8) if $0 < \beta \neq 1$.*

Remark: The orbital instability of $(\frac{1}{\sqrt{1+\beta}}u_h, \frac{1}{\sqrt{1+\beta}}u_h)$ for $-1 < \beta < 0$ can also be investigated. However, the condition is quite complicated so we may omit it here. On the other hand, as $\beta = 1$, the system (1.9) may have infinitely many solutions with the form $(u, v) = (w, \eta w)$ for $\eta \neq 0$, where w is the solution of $h^2 \triangle w - (V + \lambda)w + (1 + \eta^2)w^3 = 0$ in $\mathbb{R}^2$. This may provide a reason to ignore the case $\beta = 1$ in Theorem 1.2.

For the existence of other bound states to the system (1.9), one may refer to [1], [2], [9], [16], [17], [20], [27], and the references therein. Our result here seems to be first in studying the orbital stability of (1.9) with a trapping potential.

The argument of Theorem 1.2 is applicable to study another two-component system of nonlinear Schrödinger equations having symbiotic bright solitons (cf. [18] and [26]) given by

$$\begin{cases} -ih\frac{\partial \Phi}{\partial t} = h^2 \triangle \Phi - V\Phi - |\Phi|^2\Phi + \beta|\Psi|^2\Phi\,, \\ -ih\frac{\partial \Psi}{\partial t} = h^2 \triangle \Psi - V\Psi - |\Psi|^2\Psi + \beta|\Phi|^2\Psi\,, \end{cases} \tag{1.13}$$

for $x \in \mathbb{R}^2$ and $t > 0$, where $V = V(x)$ is a smooth nonnegative function, $\beta \in \mathbb{R}$ is a nonzero constant, and $0 < h \ll 1$ is a small parameter. It is remarkable that the coefficients of the terms $|\Phi|^2\Phi$ and $|\Psi|^2\Psi$ of the system (1.13) have opposite sign to those of the system (1.8). As for the system (1.9), bound states of (1.13) are of the form $\Phi(x,t) = e^{i\lambda t/h}u(x)$ and $\Psi(x,t) = e^{i\lambda t/h}v(x)$, where (u, v) satisfies the following nonlinear elliptic system

$$\begin{cases} h^2 \triangle u - (V + \lambda)u - u^3 + \beta uv^2 = 0 \quad , \ x \in \mathbb{R}^2 \\ h^2 \triangle v - (V + \lambda)v - v^3 + \beta u^2 v = 0 \quad , \ x \in \mathbb{R}^2, \\ u(x), v(x) > 0, u, v \in H^1(\mathbb{R}^2). \end{cases} \tag{1.14}$$

It is easy to check that the system (1.14) has a solution $(\frac{1}{\sqrt{\beta-1}}u_h, \frac{1}{\sqrt{\beta-1}}u_h)$ for $\beta > 1$. As for Theorem 1.2, we may have

Corollary 1.3. *For $0 < h < 1$, let u_h be a single spike solution concentrating at a local minimum point of the function V. Suppose the parameter h is sufficiently small. Then $(\frac{1}{\sqrt{\beta-1}}u_h, \frac{1}{\sqrt{\beta-1}}u_h)$ is orbitally stable to (1.13) if $\beta > 1$.*

The proof of Corollary 1.3 is quite similar to that of Theorem 1.2 so we may neglect the detail proof here.

The rest of this paper is organized as follows: In Section 2, we figure out the properties of u_h and the spectrum of the linearized operator L_h as the parameter h goes to zero. Then we state the proof of Theorem 1.1 in Section 3. Finally, we provide the proof of Theorem 1.2 in Section 4.

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2 Properties of u_h

In this section, we study the properties of u_h which is a single spike solution concentrated at a nondegerentate critical point x_0 of $V(x)$. Let x_h be the unique local maximum point of u_h. So $x_h \to x_0$. Let us recall the following results of Grossi [13].

Lemma 2.1. *(1)* $x_h = x_0 + \mathrm{o}(h)$ *;*

(2) u_h *is unique and nondegenerate, i.e.* L_h *has no zero eigenvalue.*

Proof. (1) follows from Lemma 5.4 of [13], while (2) follows from Theorem 1.1 of [13]. □

We need the following two lemmas. The first one is asymptotic behavior of u_h:

Lemma 2.2.

$$u_h(x_h + hy) = (V(x_h)+\lambda)^{\frac{1}{p-1}}\, w(\sqrt{V(x_h)+\lambda}\,y) + h^2\phi_0 + \mathrm{o}(h^2), \tag{2.1}$$

where w *is the unique positive solution of*

$$\triangle w - w + w^p = 0, w(0) = \max_{y\in\mathbb{R}^N} w(y), w > 0 \quad in \quad \mathbb{R}^N, w \to 0 \ as \ |y| \to +\infty, \tag{2.2}$$

ϕ_0 *satisfies*

$$\triangle\phi_0 - (V(x_h)+\lambda)\phi_0 + pw_{x_h}^{p-1}\phi_0 - \frac{1}{2}\sum_{i,j} V_{ij}(x_0) y_i y_j w_{x_h} = 0\,, \tag{2.3}$$

with

$$w_{x_h}(y) := (V(x_h)+\lambda)^{\frac{1}{p-1}}\, w(\sqrt{V(x_h)+\lambda}\,y)\,, \tag{2.4}$$

Proof. Note that for fixed s, $w_s(y)$ satisfies

$$\triangle w_s - (V(s)+\lambda)w_s + w_s^p = 0. \tag{2.5}$$

Let $\phi_h(y) = u_h(x_h+hy) - w_{x_h}(y)$. Then $|\phi_h| \to 0$ uniformly and ϕ_h satisfies

$$\triangle\phi_h - (V(x_h+hy)+\lambda)\phi_h + pw_{x_h}^{p-1}\phi_h + N(\phi_h) - (V(x_h+hy) - V(x_h))w_{x_h} = 0, \tag{2.6}$$

where $N(\phi_h) = (w_{x_h}+\phi_h)^p - w_{x_h}^p - pw_{x_h}^{p-1}\phi_h$. Note that $\nabla\phi_h(0) = 0$ and

$$\begin{aligned}(V(x_h+hy) - V(x_h)) =& (\nabla V(x_h))hy + \frac{1}{2}\sum_{i,j} V_{ij}(x_h)h^2 y_i y_j + \mathrm{O}(h^3|y|^3)\\ =& \mathrm{o}(h^2)|y| + \frac{1}{2}\sum_{i,j} V_{ij}(x_0)h^2 y_i y_j + \mathrm{o}(h^2|y|^2).\end{aligned} \tag{2.7}$$

Here we have used Lemma 2.1(1).

Now we claim that $|\phi_h| \le ch^2$. In fact, suppose not. We may assume that $|\phi_h|_{L^\infty} h^{-2} \to \infty$. Let $\widetilde{\phi}_h = \frac{\phi_h}{|\phi_h|_{L^\infty}}$. Then $\widetilde{\phi}_h$ satisfies

$$\triangle\widetilde{\phi}_h - (V+\lambda)\widetilde{\phi}_h + pw_{x_h}^{p-1}\widetilde{\phi}_h + \frac{N(\phi_h)}{|\phi_h|_{L^\infty}} - \frac{(V(x_h+hy) - V(x_h))w_{x_h}}{|\phi_h|_{L^\infty}} = 0. \tag{2.8}$$

Note that by (2.7),

$$\frac{|V(x_h+hy)-V(x_h)||w_{x_h}|}{|\phi_h|_{L^\infty}} \le \frac{h^2|y|^2|w_{x_h}|}{|\phi_h|_{L^\infty}} \le \mathrm{o}(1)|y|^2|w_{x_h}|\,, \tag{2.9}$$

for $|y| \ge 1$. Let y_h be the global maximum point of $\widetilde{\phi}_h$ i.e. $\widetilde{\phi}_h(y_h) = \max_y \frac{\phi_h(y)}{|\phi_h|_{L^\infty}} = 1$. Then by (2.8)−(2.9) and the Maximum Principle, we have $|y_h| \le C$. Here we have used the fact that $V \ge 0$ and $\lambda > 0$.

By usual elliptic regularity theory, we may take a subsequence $\widetilde{\phi}_h \to \overline{\phi}_0$, where $\overline{\phi}_0$ satisfies

$$\triangle\overline{\phi}_0 - (V(x_0)+\lambda)\overline{\phi}_0 + pw_{x_0}^{p-1}\overline{\phi}_0 = 0,\ \nabla\overline{\phi}_0(0) = 0. \tag{2.10}$$

Since $\nabla\overline{\phi}_0(0) = 0$, we see that $\overline{\phi}_0 = \sum_{j=1}^N c_j \frac{\partial w_{x_0}}{\partial y_j}$, and hence $c_j = 0$. Consequently $\overline{\phi}_0 \equiv 0$. This may contradict to the fact that $1 = \widetilde{\phi}_h(y_h) \to \overline{\phi}_0(y_0)$ for some y_0. Therefore $|\phi_h| \le ch^2$. Now we let $\phi_h = h^2\phi_0 + h^2\overline{\phi}_h$. Then as for previous argument, we may have $\overline{\phi}_h = \mathrm{o}(1)$ and complete the proof of Lemma 2.2. □

As in Proposition 3.6 of [15], one may get two lemmas as follows:

Lemma 2.3. *For each $s \in \mathbb{R}^N$, the map*

$$L_s\phi := \triangle\phi - (V(s)+\lambda)\phi + pw_s^{p-1}\phi \tag{2.11}$$

is invertible from $K_s^\perp$ to $C_s^\perp$, where

$$K_s^\perp = \left\{\phi \in H^2(\mathbb{R}^N)\,\middle|\,\int_{\mathbb{R}^N}\phi\frac{\partial w_s}{\partial y_j}(y)\,dy = 0\,, j = 1,\cdots,N\right\} \subset H^2(\mathbb{R}^N),$$
$$C_s^\perp = \left\{\phi \in L^2(\mathbb{R}^N)\,\middle|\,\int_{\mathbb{R}^N}\phi\frac{\partial w_s}{\partial y_j}(y)\,dy = 0\,, j = 1,\cdots,N\right\} \subset L^2(\mathbb{R}^N).$$

Lemma 2.4. *The map*

$$L_0\phi := \triangle\phi - (V(x_0)+\lambda)\phi + pw_{x_0}^{p-1}\phi \tag{2.12}$$

admits the following eigenvalues

$$\lambda_1 > 0\,,\ \lambda_2 = \ldots = \lambda_{N+1} = 0\,,\ \lambda_{N+2} < 0\,,$$

where the kernel of L_0 is spanned by $\frac{\partial w_{x_0}}{\partial y_j}$, $j = 1, \ldots, N$.

Our main result in this section is the following

Theorem 2.5. *The eigenvalue problem*

$$L_h\psi_h = \lambda_h\psi_h \tag{2.13}$$

admits eigenvalues

$$\lambda_{h,1} > \lambda_{h,2} > \ldots > \lambda_{h,N+1} > \lambda_{h,N+2}, \tag{2.14}$$

satisfying as $h \to 0$, $\lambda_{h,1} \to \lambda_1 > 0$, $\lambda_{h,N+2} \to \lambda_{N+2} < 0$ and

$$\frac{\lambda_{h,j}}{h^2} \to c_0\nu_{j-1},\ \ j = 2, \ldots, N+1, \tag{2.15}$$

where c_0 is a negative constant and ν_j's are eigenvalues of the Hessian matrix $(\nabla^2V(x_0))$.

Proof. We follow the proofs given in Section 5 of [33]. Assume that $\|\psi_h\|_{L^2} = 1$. It is easy to see that for eigenvalues $\lambda_h \in [\frac{1}{2}\lambda_{N+2}, \frac{1}{2}\lambda_1]$, as $h \downarrow 0$, $\lambda_h \to \lambda_j$ for some j, where λ_j's are given in Lemma 2.4. Now we focus on the case $\lambda_{h,j} \to 0$, i.e. $\lambda_h \to 0$ as $h \downarrow 0$. Then the corresponding eigenfunctions can be written as

$$\psi_h(x_h + hy) = \sum_{j=1}^{N} c_j \frac{\partial w_{x_h}}{\partial y_j}(y) + \psi_h^{\perp}(y), \tag{2.16}$$

where $\int_{\mathbb{R}^N} \frac{\partial w_{x_h}}{\partial y_j}\psi_h^{\perp}(y)\, dy = 0$, $j = 1, 2, ..., N$. Hence by (2.13) and (2.16), $\psi_h^{\perp}$ may satisfy

$$\begin{aligned} &\triangle\psi_h^{\perp} - (V(x_h + hy) + \lambda)\psi_h^{\perp} + pw_{x_h}^{p-1}(y)\psi_h^{\perp} + p(u_h^{p-1}(x_h + hy) - w_{x_h}^{p-1}(y))\psi_h^{\perp} \\ &+ \sum_j c_j L_h \frac{\partial w_{x_h}}{\partial y_j} = \lambda_h \left(\sum_j c_j \frac{\partial w_{x_h}}{\partial y_j} + \psi_h^{\perp} \right) \end{aligned} \tag{2.17}$$

Using (2.1) and (2.7) of Lemma 2.2, we have

$$\begin{aligned} L_h \frac{\partial w_{x_h}}{\partial y_j} =& \triangle \left(\frac{\partial w_{x_h}}{\partial y_j} \right) - (V(x_h + hy) + \lambda) \frac{\partial w_{x_h}}{\partial y_j} + pu_h^{p-1}(x_h + hy) \frac{\partial w_{x_h}}{\partial y_j} \\ =& (V(x_h) - V(x_h + hy)) \frac{\partial w_{x_h}}{\partial y_j} + p(u_h^{p-1}(x_h + hy) - w_{x_h}^{p-1}(y)) \frac{\partial w_{x_h}}{\partial y_j} \\ =& O(h^2). \end{aligned} \tag{2.18}$$

From Lemma 2.3, the map $L_{x_h} = \triangle - (V(x_h) + \lambda) + pw_{x_h}^{p-1}$ is invertible in the space $K_{x_h}^{\perp}$. Thus by (2.1), (2.7) and (2.18), (2.17) may give

$$||\psi_h^{\perp}||_{H^2} \le c(h^2 + |\lambda_h|) \sum_j |c_j|\,. \tag{2.19}$$

Now we set $z_j(y) = \frac{\partial w_{x_h}}{\partial y_j}(y)$ for $j = 1, \cdots, N$. Then multiplying (2.17) by z_k and integrating over $\mathbb{R}^N$, it is obvious that

$$\int_{\mathbb{R}^N} (L_h \psi_h^{\perp})\, z_k\, dy + \sum_j c_j \int_{\mathbb{R}^N} \left(L_h \frac{\partial w_{x_h}}{\partial y_j} \right) z_k\, dy = \lambda_h \left(\sum_j c_j \int_{\mathbb{R}^N} z_j z_k \right) dy\,. \tag{2.20}$$

Here we have used the fact that $\psi_h^{\perp} \in K_{x_h}^{\perp}$. Using (2.18), (2.19), $\lambda_h = \mathrm{o}(1)$ and integration by parts, we obtain

$$\int_{\mathbb{R}^N} (L_h \psi_h^{\perp})\, z_k = \int_{\mathbb{R}^N} \psi_h^{\perp} L_h z_k = \mathrm{o}(h^2)\,, \tag{2.21}$$

and

$$\begin{aligned} \int_{\mathbb{R}^N} (L_h z_j)\, z_k =& \int_{\mathbb{R}^N} (V(x_h) - V(x_h + hy)) z_j z_k + p \int_{\mathbb{R}^N} \left(u_h^{p-1} - w_{x_h}^{p-1} \right) z_j z_k \\ :=& I_1 + I_2, \end{aligned} \tag{2.22}$$

where

$$\begin{aligned} I_1 =& \int_{\mathbb{R}^N} (V(x_h) - V(x_h + hy)) z_j z_k \\ =& \mathrm{o}(h^2) - \frac{h^2}{2} \sum_{l,m} V_{lm}(x_h) \int_{\mathbb{R}^N} y_l y_m z_j z_k \\ =& - \frac{h^2}{2} V_{jk}(x_h) \int_{\mathbb{R}^N} y_j z_j y_k z_k \,. \end{aligned} \tag{2.23}$$

Here we have used (2.7) to get (2.23). For I_2, we use Lemma 2.2:

$$\begin{aligned} I_2 =& p(p-1) h^2 \int_{\mathbb{R}^N} \phi_0 w_{x_h}^{p-2} z_j z_k + \mathrm{o}(h^2) \\ =& - h^2 \int_{\mathbb{R}^N} L_0 \left(\frac{\partial^2 w}{\partial y_j \partial y_k} \right) \phi_0 + \mathrm{o}(h^2) \\ =& - h^2 \int_{\mathbb{R}^N} (L_0 \phi_0) \frac{\partial^2 w}{\partial y_j \partial y_k} + \mathrm{o}(h^2) \\ =& - \frac{h^2}{2} \sum_{l,m} V_{lm}(x_h) \int_{\mathbb{R}^N} y_l y_m w_{x_h} \frac{\partial^2 w}{\partial y_j \partial y_k} + \mathrm{o}(h^2) \\ =& \frac{h^2}{2} \sum_{l,m} V_{lm}(x_h) \int_{\mathbb{R}^N} \frac{\partial}{\partial y_j} (y_l y_m w_{x_h}) z_k + \mathrm{o}(h^2) \\ =& \frac{h^2}{2} V_{jk}(x_h) \int_{\mathbb{R}^N} y_j y_k z_j z_k - \frac{h^2}{2} V_{jk}(x_h) \int_{\mathbb{R}^N} w_{x_h}^2 + \mathrm{o}(h^2) \,. \end{aligned} \tag{2.24}$$

Here we have used the following identity:

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{\partial}{\partial y_j} (y_l y_m w_{x_h}) z_k &= \int_{\mathbb{R}^N} \delta_{jl} y_m w_{x_h} z_k + \int_{\mathbb{R}^N} \delta_{jm} y_l w_{x_h} z_k + \int_{\mathbb{R}^N} y_l y_m z_j z_k \\ &= \frac{1}{2} \int_{\mathbb{R}^N} \delta_{jl} y_m \frac{\partial}{\partial y_k} (w_{x_h}^2) + \frac{1}{2} \int_{\mathbb{R}^N} \delta_{jm} y_l \frac{\partial}{\partial y_k} (w_{x_h}^2) + \int_{\mathbb{R}^N} y_l y_m z_j z_k \\ &= -(\delta_{jl} \delta_{km} + \delta_{jm} \delta_{lk}) \frac{1}{2} \int_{\mathbb{R}^N} w_{x_h}^2 + \int_{\mathbb{R}^N} y_l y_m z_j z_k . \end{aligned}$$

Combining (2.23) and (2.24), we have

$$I_1 + I_2 = -\frac{h^2}{2} V_{jk}(x_h) \int_{\mathbb{R}^N} w_{x_0}^2 + \mathrm{o}(h^2) \tag{2.25}$$

Substituting (2.21) and (2.25) into (2.20), we may obtain $\frac{\lambda_j}{h^2} \to c_0 \nu_j$ for $j = 1, \cdots, N$, where $c_0 = -\frac{\int_{\mathbb{R}^N} w_{x_0}^2 \, dy}{\int_{\mathbb{R}^N} z_j^2 \, dy}$ is a negative constant. The rest of the proof follows from a perturbation result, similar to page 1473-1474 of [33]. We may omit the details here. $\square$

From Theorem 2.5, we may deduce that

Theorem 2.6. *u_h is smooth in λ. Moreover, let $R_h = \frac{\partial u_h}{\partial \lambda}$. Then R_h satisfies*

$$L_h R_h - u_h = 0\,, \tag{2.26}$$

and

$$R_h = \sum_{j=1}^{N} c_j^h z_j + R_0 + \mathrm{o}(1)\,, \tag{2.27}$$

where $R_0 = \frac{\partial}{\partial \lambda} w_{x_h} = (V(x_h)+\lambda)^{-1}\left(\frac{1}{p-1} w_{x_h} + \frac{1}{2} y \cdot \nabla w_{x_h}\right)$ and $|c_j^h| = \mathrm{O}(1)$ for $j = 1, \cdots, N$.

Proof. Since u_h is unique and L_h is invertible, it is easy to see that u_h is smooth in λ and R_h satisfies (2.26). Now we decompose R_h as

$$R_h = \sum_{j=1}^{N} c_j^h z_j + R_0 + \overline{R}_h,$$

where $\int_{\mathbb{R}^N} z_j \overline{R}_h = 0,\ j = 1, .., N$. Then $\overline{R}_h$ satisfies

$$L_h \overline{R}_h - u_h + L_h R_0 + \sum_{j=1}^{N} c_j^h L_h z_j = 0. \tag{2.28}$$

As for the proof of Theorem 2.5, we have

$$\|\overline{R}_h\|_{H^2} \le c\left(|c_j^h| h^2 + \|L_h R_0 - u_h\|_{L^2}\right). \tag{2.29}$$

From (2.4) and (2.5), it is easy to check $R_0 = \frac{\partial}{\partial \lambda} w_{x_h} = (V(x_h)+\lambda)^{-1}\left(\frac{1}{p-1} w_{x_h} + \frac{1}{2} y \cdot \nabla w_{x_h}\right)$ and $L_{x_h} R_0 = w_{x_h}$ by differentiating (2.5) with respect to λ. Hence

$$L_h R_0 - u_h = p(u_h^{p-1}(x_h + hy) - w_{x_h}^{p-1}(y))R_0 - (V(x_h + hy) - V(x_h))\, R_0 + w_{x_h} - u_h\,.$$

Consequently, by Lemma 2.2 and (2.7), we obtain

$$\|L_h R_0 - u_h\|_{L^2} = \mathrm{O}(h^2)\,, \tag{2.30}$$

and then by (2.29),

$$\|\overline{R}_h\|_{H^2} = (1 + |c_j^h|)\mathrm{O}(h^2)\,. \tag{2.31}$$

To estimate c_j^h's, we may multiply (2.28) by z_k and integrate over $\mathbb{R}^N$. Then

$$\sum_{j=1}^{N} c_j^h \int_{\mathbb{R}^N} (L_h z_j) z_k + \int_{\mathbb{R}^N} (L_h R_0 - u_h) z_k + \int_{\mathbb{R}^N} (L_h \overline{R}_h) z_k = 0\,. \tag{2.32}$$

Hence by (2.22) and (2.25), (2.32) may imply

$$|c_j^h| \le \frac{C}{h^2}\left(\left|\int_{\mathbb{R}^N} (L_h R_0 - u_h) z_k\right| + \left|\int_{\mathbb{R}^N} (L_h \overline{R}_h) z_k\right|\right), \tag{2.33}$$

Using integration by parts and (2.18), we have

$$\int_{\mathbb{R}^N} (L_h \overline{R}_h) z_k = \int_{\mathbb{R}^N} (L_h z_k) \overline{R}_h = \|\overline{R}_h\|_{L^2}\, \mathrm{O}(h^2)\,. \tag{2.34}$$

Therefore by (2.30), (2.31), (2.33) and (2.34),we may obtain $|c_j^h| = \mathrm{O}(1)$ and complete the proof. $\square$

3 Proof of Theorem 1.1

Let $p = 1 + \frac{4}{N}$. By Theorem 2.5, L_h has $m+1$ positive eigenvalues and no zero eigenvalue, where m is the number of negative eigenvalues of the matrix $(\nabla^2 V(x_0))$. Let us now compute $d''(\lambda)$.

From (1.7), it is easy to get

$$d'(\lambda) = \frac{1}{2}\int_{\mathbb{R}^N} u_h^2$$

and hence

$$d''(\lambda) = \int_{\mathbb{R}^N} \frac{\partial u_h}{\partial \lambda} u_h = \int_{\mathbb{R}^N} R_h u_h.$$

By direct computations,

$$L_h\left(\frac{1}{p-1}u_h + \frac{1}{2}hy\cdot\nabla u_h\right) = \frac{1}{2}hy\cdot\nabla V(x_h+hy)u_h + (V(x_h+hy)+\lambda)u_h. \tag{3.1}$$

Consequently,

$$\begin{aligned}(V(x_h)+\lambda)\int_{\mathbb{R}^N} R_h u_h &= \int_{\mathbb{R}^N} R_h(V(x_h)-V(x_h+hy))u_h + \int_{\mathbb{R}^N} R_h(V(x_h+hy)+\lambda)u_h \\ &= \int_{\mathbb{R}^N} R_h(V(x_h)-V(x_h+hy) - \frac{1}{2}hy\cdot\nabla V(x_h+hy))u_h \\ &\quad + \int_{\mathbb{R}^N} R_h L_h\left(\frac{1}{p-1}u_h + \frac{1}{2}hy\cdot\nabla u_h\right).\end{aligned} \tag{3.2}$$

Using integration by parts and (2.26), we may obtain

$$\begin{aligned}\int_{\mathbb{R}^N} R_h L_h\left(\frac{1}{p-1}u_h + \frac{1}{2}hy\cdot\nabla u_h\right) &= \int_{\mathbb{R}^N}(L_h R_h)\left(\frac{1}{p-1}u_h + \frac{1}{2}hy\cdot\nabla u_h\right) \\ &= \int_{\mathbb{R}^N} u_h\left(\frac{1}{p-1}u_h + \frac{1}{2}hy\cdot\nabla u_h\right) \\ &= \left(\frac{1}{p-1} - \frac{N}{4}\right)\int_{\mathbb{R}^N} u_h^2 = 0,\end{aligned} \tag{3.3}$$

since $p=1+\frac{4}{N}$. So by (2.7), (3.2), (3.3), Lemma 2.2 and Theorem 2.6, we have

$$
\begin{aligned}
d''(\lambda) =& \frac{1}{V(x_h)+\lambda}\int_{\mathbb{R}^N} R_h(V(x_h)-V(x_h+hy)-\frac{1}{2}hy\cdot\nabla V(x_h+hy))u_h \\
=& \frac{h^2}{V(x_h)+\lambda}\int_{\mathbb{R}^N} R_h\left[\sum_{i,j}-V_{ij}(x_h)y_iy_j\right]u_h+\mathrm{o}(h^2) \\
=& \frac{h^2}{V(x_h)+\lambda}\int_{\mathbb{R}^N}\left(\sum_l c_l^h z_l+R_0+\mathrm{o}(1)\right)\left(\sum_{i,j}-V_{ij}(x_h)y_iy_j\right)(w_{x_h}+\mathrm{O}(h^2))+\mathrm{o}(h^2) \\
=& -\frac{h^2}{V(x_h)+\lambda}\sum_i V_{ii}(x_h)\int_{\mathbb{R}^N} R_0 y_i^2 w_{x_h}+\mathrm{o}(h^2) \\
=& -\frac{h^2}{(V(x_h)+\lambda)^2}\sum_i V_{ii}(x_h)\int_{\mathbb{R}^N}\left(\frac{1}{p-1}w_{x_h}+\frac{1}{2}y\cdot\nabla w_{x_h}\right)y_i^2 w_{x_h}+\mathrm{o}(h^2) \\
=& -\frac{h^2}{(V(x_h)+\lambda)^2}\left(\frac{1}{p-1}-\frac{N+2}{4}\right)\sum_i V_{ii}(x_h)\int_{\mathbb{R}^N} y_i^2 w_{x_h}^2+\mathrm{o}(h^2) \\
=& \frac{h^2}{2(V(x_0)+\lambda)^2}\Delta V(x_0)\int_{\mathbb{R}^N} y_i^2 w_{x_0}^2+\mathrm{o}(h^2) \quad (\text{because } p=1+\frac{4}{N}). \qquad (3.4)
\end{aligned}
$$

If x_0 is a local minimum point, then $m=0$ and $n(L_h)=1$. Since the Hessian matrix $(\nabla^2 V(x_0))$ is positive definite, then

$$d''(\lambda)>0, \quad p(d''(\lambda))=1, \qquad (3.5)$$

which implies that u_h is orbitally unstable by the orbital stability criteria of [10]-[11].

If x_0 is not a local minimum, then $m\geq 1$ and $n(L_h)\geq 2$. In this case, by the formula (3.4), $p(d''(\lambda))=\frac{1}{2}(1+\frac{\Delta V(x_0)}{|\Delta V(x_0)|})$. By the instability criteria of [11], we conclude that u_h is orbitally unstable if $m-\frac{1}{2}(1+\frac{\Delta V(x_0)}{|\Delta V(x_0)|})$ is even. This completes the proof of Theorem 1.1.

4 Proof of Theorem 1.2

Let $\left(\frac{1}{\sqrt{1+\beta}}u_h, \frac{1}{\sqrt{1+\beta}}u_h\right)$ be a solution of (1.9). The linearized operator of (1.8) around $\left(\frac{1}{\sqrt{1+\beta}}u_h, \frac{1}{\sqrt{1+\beta}}u_h\right)$ is

$$\mathbb{L}_h\begin{pmatrix}\phi\\ \psi\end{pmatrix}=\begin{pmatrix} h^2\Delta\phi-(V(x)+\lambda)\phi+\frac{3+\beta}{1+\beta}u_h^2\phi+\frac{2\beta}{1+\beta}u_h^2\psi \\ h^2\Delta\psi-(V(x)+\lambda)\psi+\frac{3+\beta}{1+\beta}u_h^2\psi+\frac{2\beta}{1+\beta}u_h^2\phi\end{pmatrix}. \qquad (4.1)$$

We first define a sequence of numbers $\beta_j\in(-1,0)$: By Lemma 4.2 of [30], the eigenvalue problem

$$\Delta\psi-(V(x_0)+\lambda)\psi+\mu w_{x_0}^2\psi=0 \qquad (4.2)$$

admits eigenvalues

$$\mu_1=1,\ \mu_2=...=\mu_{N+1}=3,\ \mu_{N+2}>3. \qquad (4.3)$$

We then define β_j by:

$$\beta_j = \frac{3-\mu_j}{1+\mu_j}, \quad j = 1, 2, \cdots . \tag{4.4}$$

The following lemma shows the nondegeneracy of $\mathbb{L}$:

Lemma 4.1. *$\mathbb{L}_h$ has no zero eigenvalue if $\beta \neq \beta_j, j = 1, ..., ...$*

Proof. Let $\mathbb{L}_h \binom{\phi}{\psi} = \binom{0}{0}$. Then by an orthonormal transformation it is equivalent to

$$L_{h,1}\widetilde{\phi} = 0, \tag{4.5}$$

$$L_{h,2}\widetilde{\psi} = 0, \tag{4.6}$$

where $L_{h,1} = L_h$, $L_{h,2} = h^2\triangle - (V(x_h + hy) + \lambda) + \frac{3-\beta}{1+\beta}u_h^2$. By Theorem 2.5, we may conclude that $\widetilde{\phi} = 0$. It remains to consider the equation (4.6). As $h \to 0$, the equation (4.6) may tend to the limiting equation given by

$$\triangle\psi - (V(x_0) + \lambda)\psi + \frac{3-\beta}{1+\beta}w_{x_0}^2\psi = 0. \tag{4.7}$$

Since $\beta \neq \beta_j$ i.e. $\frac{3-\beta}{1+\beta} \neq \mu_j$, then by Lemma 4.2 of [30], (4.7) has only trivial solution. Therefore, $\widetilde{\psi} = 0$ and we may complete the proof. $\square$

The next lemma computes the Morse index of $(\frac{1}{\sqrt{1+\beta}}u_h, \frac{1}{\sqrt{1+\beta}}u_h)$. Here the Morse index is defined to be the number of positive eigenvalues of $\mathbb{L}_h$, which is just $n(\mathbb{L}_h)$.

Lemma 4.2. *If $-1 < \beta < 0$ and $\beta \notin \{\beta_2, \cdots, \beta_j, \cdots\}$, then the Morse Index of $(\frac{1}{\sqrt{1+\beta}}u_h, \frac{1}{\sqrt{1+\beta}}u_h)$ is at least $N+2$. If $0 < \beta < 1$, then the Morse Index of $(\frac{1}{\sqrt{1+\beta}}u_h, \frac{1}{\sqrt{1+\beta}}u_h)$ is two. If $\beta > 1$, then the Morse Index of $(\frac{1}{\sqrt{1+\beta}}u_h, \frac{1}{\sqrt{1+\beta}}u_h)$ is one.*

Proof. The eigenvalue problem $\mathbb{L}_h \binom{\phi}{\psi} = \bar{\lambda}\binom{\phi}{\psi}$ can be decomposed to

$$L_{h,1}\widetilde{\phi} = \bar{\lambda}\widetilde{\phi}, \tag{4.8}$$

$$L_{h,2}\widetilde{\psi} = \bar{\lambda}\widetilde{\psi}. \tag{4.9}$$

By Theorem 2.5, $L_{h,1}$ has only one positive eigenvalue. It remains to consider the spectrum of $L_{h,2}$. If $\beta < 0$, then the eigenvalue problem

$$\triangle\psi - (V(x_0) + \lambda)\psi + \frac{3-\beta}{1+\beta}w_{x_0}^2\psi = \bar{\lambda}\psi \tag{4.10}$$

has at least $N+1$ positive eigenvalues. We may define a space of functions by

$$\mathbf{V} = \text{span}\left\{w_{x_0}, \frac{\partial w_{x_0}}{\partial y_j}, \ j = 1, ..., N, \right\}.$$

Since $-1 < \beta < 0$, we have $\frac{3-\beta}{1+\beta} > 3$. Hence

$$\int_{\mathbb{R}^N} \left[|\nabla \phi|^2 + (V(x_0) + \lambda)\phi^2 - \frac{3-\beta}{1+\beta} w_{x_0}^2 \phi^2 \right] < 0, \ \forall \phi \in \mathbf{V}. \tag{4.11}$$

Thus by the variational characterization of the eigenvalues to (4.10),we see that $\lambda_{N+1} > 0$. Moreover, by the perturbation argument, (4.9) has at least $N+1$ positive eigenvalues.

So when $\beta < 0$, $\mathbb{L}_h$ has at least $N+2$ positive eigenvalues.

When $0 < \beta < 1$, $1 < \frac{3-\beta}{1+\beta} < 3$, (4.10) has only one positive eigenvalue. So the Morse index is two.

When $\beta > 1$, (4.10) has no positive eigenvalue. So the Morse index is one. $\square$

Since $\mathbb{L}_h$ is invertible, $\left(\frac{1}{\sqrt{1+\beta}} u_h, \frac{1}{\sqrt{1+\beta}} u_h\right)$ is nondegenerate. Thus the equation

$$\begin{cases} h^2 \triangle u - (V(x) + \lambda + \lambda_1)u + u^3 + \beta u v^2 = 0, & \text{in } \mathbb{R}^N, \\ h^2 \triangle v - (V(x) + \lambda + \lambda_2)v + v^3 + \beta u^2 v = 0, & \text{in } \mathbb{R}^N, \end{cases} \tag{4.12}$$

has a solution $(u_{h,\lambda_1,\lambda_2}, v_{h,\lambda_1,\lambda_2})$ satisfying

$$(u_{h,\lambda_1,\lambda_2}, v_{h,\lambda_1,\lambda_2}) = \left(\frac{1}{\sqrt{1+\beta}} u_h, \frac{1}{\sqrt{1+\beta}} u_h\right) + \mathrm{O}\left((|\lambda_1| + |\lambda_2|)h^{-2}\right) \tag{4.13}$$

as $|\lambda_1| + |\lambda_2| << 1$.

Let us define

$$\begin{aligned} d(\lambda_1, \lambda_2) = & \int_{\mathbb{R}^N} \frac{h^2}{2} |\nabla u_{h,\lambda_1,\lambda_2}|^2 + \frac{V(x) + \lambda + \lambda_1}{2} u_{h,\lambda_1,\lambda_2}^2 - \frac{1}{4} \int_{\mathbb{R}^N} u_{h,\lambda_1,\lambda_2}^4 \\ & + \int_{\mathbb{R}^N} \frac{h^2}{2} |\nabla v_{h,\lambda_1,\lambda_2}|^2 + \frac{V(x) + \lambda + \lambda_2}{2} v_{h,\lambda_1,\lambda_2}^2 - \frac{1}{4} \int_{\mathbb{R}^N} v_{h,\lambda_1,\lambda_2}^4 \\ & - \frac{\beta}{2} \int_{\mathbb{R}^N} u_{h,\lambda_1,\lambda_2}^2 v_{h,\lambda_1,\lambda_2}^2. \end{aligned} \tag{4.14}$$

It is easy to see that

$$\begin{aligned} \frac{\partial d}{\partial \lambda_1} &= \frac{1}{2} \int_{\mathbb{R}^N} u_{h,\lambda_1,\lambda_2}^2, \quad \frac{\partial^2 d}{\partial \lambda_1^2} = \int_{\mathbb{R}^N} u_{h,\lambda_1,\lambda_2} \frac{\partial u_{h,\lambda_1,\lambda_2}}{\partial \lambda_1}, \\ \frac{\partial d}{\partial \lambda_2} &= \frac{1}{2} \int_{\mathbb{R}^N} v_{h,\lambda_1,\lambda_2}^2, \quad \frac{\partial^2 d}{\partial \lambda_2^2} = \int_{\mathbb{R}^N} v_{h,\lambda_1,\lambda_2} \frac{\partial v_{h,\lambda_1,\lambda_2}}{\partial \lambda_2}, \\ \frac{\partial^2 d}{\partial \lambda_1 \partial \lambda_2} &= \int_{\mathbb{R}^N} u_{h,\lambda_1,\lambda_2} \frac{\partial u_{h,\lambda_1,\lambda_2}}{\partial \lambda_2}. \end{aligned}$$

Now we may define functions as $\Phi_1 = \left.\frac{\partial u_{h,\lambda_1,\lambda_2}}{\partial \lambda_1}\right|_{(\lambda_1,\lambda_2)=(0,0)}$ and $\Psi_1 = \left.\frac{\partial u_{h,\lambda_1,\lambda_2}}{\partial \lambda_2}\right|_{(\lambda_1,\lambda_2)=(0,0)}$. Then by (4.12), $\binom{\Phi_1}{\Psi_1}$ satisfies

$$\mathbb{L}_h \begin{pmatrix} \Phi_1 \\ \Psi_1 \end{pmatrix} = \begin{pmatrix} u_{h,0,0} \\ 0 \end{pmatrix}. \tag{4.15}$$

Similarly, if we set $\Phi_2 = \frac{\partial v_{h,\lambda_1,\lambda_2}}{\partial \lambda_1}\Big|_{(\lambda_1,\lambda_2)=(0,0)}$ and $\Psi_2 = \frac{\partial v_{h,\lambda_1,\lambda_2}}{\partial \lambda_2}\Big|_{(\lambda_1,\lambda_2)=(0,0)}$, then by (4.12), we have

$$\Psi_2 = \Phi_1, \quad \Phi_2 = \Psi_1. \tag{4.16}$$

Let $\mathbb{B} = \left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$. Then (4.15) is equivalent to

$$\mathbb{B}\mathbb{L}_h \begin{pmatrix} \Phi_1 \\ \Psi_1 \end{pmatrix} = \begin{pmatrix} L_{h,1}(\Phi_1 + \Psi_1) \\ L_{h,2}(\Phi_1 - \Psi_1) \end{pmatrix} = \begin{pmatrix} u_{h,0,0} \\ u_{h,0,0} \end{pmatrix}. \tag{4.17}$$

So

$$\Phi_1 + \Psi_1 = R_{h,1} \quad \text{and} \quad \Phi_1 - \Psi_1 = R_{h,2}, \tag{4.18}$$

where

$$R_{h,1} = \frac{1}{\sqrt{1+\beta}} R_h \quad \text{and} \quad R_{h,2} = L_{h,2}^{-1}\left(\frac{1}{\sqrt{1+\beta}} u_h\right). \tag{4.19}$$

Now we compute the Hessian matrix

$$(\nabla^2 d)\big|_{(\lambda_1,\lambda_2)=(0,0)} = \begin{pmatrix} \frac{1}{\sqrt{1+\beta}} \int_{\mathbb{R}^N} u_h \Phi_1 & \frac{1}{\sqrt{1+\beta}} \int_{\mathbb{R}^N} u_h \Psi_1 \\ \frac{1}{\sqrt{1+\beta}} \int_{\mathbb{R}^N} u_h \Psi_1 & \frac{1}{\sqrt{1+\beta}} \int_{\mathbb{R}^N} u_h \Phi_1 \end{pmatrix},$$

$$\begin{aligned} \mathbb{B}\,(\nabla^2 d)\big|_{(\lambda_1,\lambda_2)=(0,0)}\,\mathbb{B}^T &= \begin{pmatrix} \frac{1}{\sqrt{1+\beta}} \int_{\mathbb{R}^N} u_h R_{h_1} & \frac{1}{\sqrt{1+\beta}} \int_{\mathbb{R}^N} u_h R_{h_1} \\ \frac{1}{\sqrt{1+\beta}} \int_{\mathbb{R}^N} u_h R_{h_2} & -\frac{1}{\sqrt{1+\beta}} \int_{\mathbb{R}^N} u_h R_{h_2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{2}{\sqrt{1+\beta}} \int_{\mathbb{R}^N} u_h R_{h_1} & 0 \\ 0 & \frac{2}{\sqrt{1+\beta}} \int_{\mathbb{R}^N} u_h R_{h_2} \end{pmatrix}. \end{aligned}$$

By the results in Section 3, $\frac{2}{\sqrt{1+\beta}} \int_{\mathbb{R}^N} u_h R_{h_1} = \frac{2}{1+\beta} \int_{\mathbb{R}^N} u_h R_h > 0$. It is enough to compute $\sqrt{1+\beta} \int_{\mathbb{R}^N} u_h R_{h_2} = \int_{\mathbb{R}^N} u_h L_{h,2}^{-1}(u_h)$. Note that as $h \to 0+$,

$$\int_{\mathbb{R}^N} u_h L_{h,2}^{-1}(u_h) \to \int_{\mathbb{R}^N} w_{x_0} L_\mu^{-1}(w_{x_0}),$$

where

$$L_\mu \phi = \triangle \phi - (V(x_0) + \lambda)\phi + \mu w_{x_0}^2 \phi, \tag{4.20}$$

with $\mu = \frac{3-\beta}{1+\beta}$.

Let $\rho(\mu) = \int_{\mathbb{R}^N} w_{x_0} L_\mu^{-1}(w_{x_0})$ and let ϕ_μ be the unique solution of $\triangle\phi_\mu - (V(x_0) + \lambda)\phi_\mu + \mu w_{x_0}^2 \phi_\mu = w_{x_0}$ i.e. $L_\mu \phi_\mu = w_{x_0}$ for $\mu \neq \mu_j, j = 1, 2, \cdots$. Then $\frac{\partial \phi_\mu}{\partial \mu}$ satisfies

$$L_\mu\left(\frac{\partial \phi_\mu}{\partial \mu}\right) = -w_{x_0}^2 \phi_\mu, \quad \text{i.e.} \quad \frac{\partial \phi_\mu}{\partial \mu} = -L_\mu^{-1}(w_{x_0}^2 \phi_\mu).$$

Hence

$$\rho'(\mu) = \int_{\mathbb{R}^N} w_{x_0} \frac{\partial \phi_\mu}{\partial \mu} = -\int_{\mathbb{R}^N} w_{x_0} L_\mu^{-1}(w_{x_0}^2 \phi_\mu)$$
$$= -\int_{\mathbb{R}^N} (L_\mu^{-1} w_{x_0})(w_{x_0}^2 \phi_\mu)$$
$$= -\int_{\mathbb{R}^N} w_{x_0}^2 \phi_\mu^2 < 0 \quad \text{for} \quad \mu \neq \mu_j \, , j = 1, 2, \cdots .$$

i.e.

$$\rho'(\mu) < 0 \quad \text{for} \quad \mu \neq \mu_j \, , j = 1, 2, \cdots . \tag{4.21}$$

Due to (4.3), ρ is smooth on $(-\infty, 1) \cup (1, 3) \cup (3, \infty) \setminus \{\mu_j : j = N+2, N+3, \cdots\}$. On the other hand, as $\mu \to 3$,

$$\phi_\mu \to L_0^{-1} w_{x_0} = \frac{1}{2} w_{x_0} + \frac{1}{2} y \cdot \nabla w_{x_0},$$
$$\rho(\mu) \to \int_{\mathbb{R}^2} w_{x_0} \left(\frac{1}{2} w_{x_0} + \frac{1}{2} y \cdot \nabla w_{x_0} \right) = 0.$$

Here we have used the fact that $N = 2$ and $p = 3$. Thus for $1 < \mu < 3$, $\rho(\mu) > 0$. This implies that for $0 < \beta < 1$, $\int_{\mathbb{R}^2} u_h R_{h,2} > 0$ and thus $(\nabla^2 d(0,0))$ has **two** positive eigenvalues.

Now we consider $\mu \in (-\infty, 1)$ i.e. $\beta > 1$. By the standard maximal principle, $\phi_\mu < 0$ in $\mathbb{R}^2$ for $\mu < 0$. Consequently, $\rho(\mu) < 0$ for $\mu < 0$. Hence by (4.21), $\rho(\mu) < 0$ for $\mu \in (-\infty, 1)$ i.e. $\beta > 1$. This implies that $\int_{\mathbb{R}^2} u_h R_{h,2} < 0$ and thus $(\nabla^2 d(0,0))$ has only **one** positive eigenvalue.

In conclusion, we see that the matrix $(\nabla^2 d(0,0))$ has two positive eigenvalues when $0 < \beta < 1$ and one positive eigenvalue when $\beta > 1$. That is $p = 2$ when $0 < \beta < 1$ and $p = 1$ when $\beta > 1$. By Lemma 4.1 and 4.2, we also deduce that $n(\mathbb{L}_h) = 2$ when $0 < \beta < 1$ and $n(\mathbb{L}_h) = 1$ when $\beta > 1$. Hence for $\beta > 0, \beta \neq 1$, we have $n(\mathbb{L}_h) = p$. Therefore, we conclude that $(u_{h,0,0}, v_{h,0,0}) = \left(\frac{1}{\sqrt{1+\beta}} u_h, \frac{1}{\sqrt{1+\beta}} v_h \right)$ is orbitally stable if $0 < \beta, \ \beta \neq 1$. This completes the proof of Theorem 1.2.

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