Orbital stability of bound states of nonlinear Schrödinger equations with linear and nonlinear lattices

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We study the orbital stability and instability of single-spike bound states of critical semi-classical nonlinear Schrödinger equations (NLS) with linear and nonlinear lattices. These equations may model an inhomogeneous Bose-Einstein condensate and an optical beam in a nonlinear lattice. When the linear lattice is switched off, we derive the asymptotic expansion formulas and obtain necessary conditions for the orbital stability and instability of single-spike bound states, respectively. When the linear lattice is turned on, we consider three different cases and obtain the most general theorem on the orbital stability problem for NLS with linear and nonlinear lattices.

1 Introduction

Nonlinear Schrödinger equations in the presence of the Kerr nonlinearity may describe an optical beam in a nonlinear lattice and an inhomogeneous Bose-Einstein condensate (BEC) given by

$$-i\frac{\partial\psi}{\partial t} = D\Delta\psi - V_{trap}\psi - g|\psi|^2\psi\,, \tag{1.1}$$

for $x \in \mathbb{R}^N$, $N = 2$ and $t > 0$. Here $\psi = \psi(x,t) \in \mathbb{C}$ is the wavefunction, D is the diffraction (or dispersion) coefficient, and V_{trap} is the potential of the linear lattice. Besides, $g = \mu m(x) \sim a$ characterizes the nonlinear lattice, where a denotes the spatially modulated s-wave scattering length, μ is a nonzero constant and $m(x) = m(x_1, \cdots, x_N) > 0$ is a function depending on spatial variables (transverse coordinates) $x_1, \cdots, x_N$ (cf. [1], [6]).

Linear and nonlinear lattices, such as photonic structures for laser beams or optical lattices for atomic BECs, can support stable bright solitons. For instance, V_{trap} the potential of the linear lattice varying along three spatial variables may stabilize bright solitons in BEC experiments (cf. [7]). When the coefficient g varies along two spatial variables, two-dimensional bright solitons can also be observed experimentally in two-dimensional nonlinear lattices (cf. [13]).

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Consequently, under the effect of linear and nonlinear lattices, two-dimensional bright solitons must have suitable stability for experimental observations. However, most theoretical results, e.g., [10] and [11] only focus on orbital (dynamical) stability of one-dimensional bound states, i.e., steady state bright solitons in one-dimensional nonlinear lattices fulfilling specifically asymptotic behaviors. Here we study the orbital stability of two-dimensional single-spike bound states of (1.1) in two-dimensional nonlinear lattices when the coefficient g varies along two spatial variables. Basically, we shall provide rigorous arguments to show the orbital stability and instability of single-spike bound states of (1.1) without using any hypothesis on the asymptotic behavior of nonlinear lattices.

To get single-spike bound states in nonlinear lattices, we may assume the coefficient $D > 0$ and the s-wave scattering length a, i.e., μ is negative and large due to the Feshbach resonance (cf. [9]). Setting $h^2 = D/(-\mu)$, $V(x) = V_{trap}(x)/(-\mu)$ and suitable time scale, the equation (1.1) with negative and large μ can be equivalent to a semi-classical nonlinear Schrödinger equation (NLS) given by

$$-ih\frac{\partial\psi}{\partial t} = h^2\Delta\psi - V\,\psi + m\,|\psi|^2\psi, \quad x \in \mathbb{R}^2\,, t > 0\,, \tag{1.2}$$

where $0 < h \ll 1$ is a small parameter, $V = V(x)$ is a smooth nonnegative function and $m = m(x)$ is a smooth positive function. For the spatial dimension $N \geq 1$, we may generalize the equation (1.2) to a NLS having the following form

$$-ih\frac{\partial\psi}{\partial t} = h^2\Delta\psi - V\,\psi + m\,|\psi|^{p-1}\psi, \quad x \in \mathbb{R}^N\,, t > 0\,, \tag{1.3}$$

with critical exponent

$$p = 1 + \frac{4}{N}\,, \quad N \geq 1\,. \tag{1.4}$$

In particular, when $N = 2$, the equation (1.3) with (1.4) is exactly same as (1.2).

Single-spike bound states of (1.3) are of the form $\psi(x,t) = e^{i\lambda\,t/h}u(x)$, where λ is a positive constant and $u = u(x)$ is a positive solution of the following nonlinear elliptic equation

$$h^2\Delta u - (V + \lambda)\,u + m\,u^p = 0\,, \quad u \in H^1(\mathbb{R}^N)\,, \tag{1.5}$$

with zero Dirichlet boundary condition, i.e., $u(x) \to 0$ as $|x| \to \infty$. When $V \equiv 0$ and $m \equiv 1$, problem (1.5) admits a unique radially symmetric ground state which is stable for any $\lambda > 0$ if $p < 1 + \frac{4}{N}$, and unstable for any $\lambda > 0$ if $p \geq 1 + \frac{4}{N}$ (cf. [4], [8] and [41]). For $V \not\equiv 0$ or $m \not\equiv 1$, there exists u_h a single-spike solution of (1.5), provided both V and m are bounded and satisfy another conditions (cf. [20]). Hereafter, we set $\psi_h(x,t) := e^{i\lambda t/h}u_h(x)$ as a single-spike bound state of (1.3), where u_h is the single-spike solution of (1.5).

In this paper, we want to study the orbital stability of the bound state ψ_h for the equation (1.3) with critical exponent (1.4). One may regard the bound state ψ_h as an orbit of (1.3). From [17], the orbital stability of ψ_h is defined as follows: For all $\epsilon > 0$, there exists $\delta > 0$ such that if $\|\psi_0 - u_h\|_{H^1} < \delta$ and ψ is a solution of (1.3) in some interval $[0, t_0)$ with $\psi|_{t=0} = \psi_0$, then $\psi(\cdot, t)$ can be extended to a solution in $0 \leq t < \infty$ and $\sup_{0<t<\infty}\inf_{s\in\mathbb{R}}\|\psi(\cdot,t) - \psi_h(\cdot,s)\|_{H^1} < \epsilon$. Otherwise, the orbit ψ_h is called orbital unstable.

The functions $V = V(x)$ and $m = m(x)$ may play a crucial role on the orbital stability of ψ_h. When $m \equiv 1$ and V is of class $(V)_a$ and fulfills other conditions in [28]-[29], the orbital stability and instability of ψ_h for the equation (1.3) was established by Lin and Wei [25] if V has non-degenerate critical points. Under different conditions, e.g., $h = 1$ and λ is large, results of the orbital stability problem can be found in [15]. One may also remark that the orbital stability problem of NLS with inhomogeneous nonlinearity has been investigated in [5] but only for the subcritical case, i.e., $1 < p < 1 + \frac{4}{N}$.

To state our main results, we need to introduce some notations. It is well-known that the positive solution of

$$\begin{cases} \Delta w - w + w^p = 0 \quad \text{in } \mathbb{R}^N\,, \\ w(0) = \max\limits_{y \in \mathbb{R}^N} w(y)\,, \quad w(y) \to 0 \text{ as } |y| \to +\infty\,. \end{cases} \tag{1.6}$$

is radial [16] and unique [24]. We denote the solution and its linearized operator as $w = w(r)$ and

$$L_0 := \Delta - 1 + p w^{p-1}\,, \tag{1.7}$$

respectively. For the orbital stability of ψ_h, we set

$$L_h := h^2 \Delta - (V + \lambda) + m\, p u_h^{p-1} \tag{1.8}$$

as the linearized operator of (1.5) with respect to u_h and

$$d(\lambda) = \int_{\mathbb{R}^N} \left[\frac{h^2}{2} |\nabla u_h|^2 + \frac{1}{2} (V + \lambda)\, u_h^2 - \frac{1}{p+1} m\, u_h^{p+1} \right] dx. \tag{1.9}$$

as the energy of u_h. Observe that u_h may depend on the variable λ. Assume that $d(\lambda)$ is non-degenerate, i.e., $d''(\lambda) \neq 0$. Let $p(d'') = 1$ if $d'' > 0$; $p(d'') = 0$ if $d'' < 0$, and $n(L_h)$ be the number of positive eigenvalues of L_h. According to general theory of orbital stability of bound states (cf. [17], [18]), ψ_h is orbital stable if $n(L_h) = p(d'')$, and orbital unstable if $n(L_h) - p(d'')$ is odd (see page 309 of [18]). It is remarkable that if both V and m are constant and $p = 1 + \frac{4}{N}$, then $d''(\lambda) = 0$. Consequently, from now on, we consider the critical exponent $p = 1 + \frac{4}{N}$ and assume the point x_0 as a non-degenerate critical point of the function G defined by (cf. [20], [37])

$$G(x) := \left[V(x) + \lambda \right] m^{-N/2}(x)\,, \quad \forall x \in \mathbb{R}^N\,, \tag{1.10}$$

provided $V \not\equiv 0$ and $m > 0$ in $\mathbb{R}^N$. When $V \equiv 0$ in $\mathbb{R}^N$, x_0 is set as a non-degenerate critical point of the function m.

For simplicity, we firstly switch off the potential V and obtain the following result.

Theorem 1.1. *Let $N \geq 1$ be a positive integer, $p = 1 + \frac{4}{N}$ and the potential $V \equiv 0$. Assume the function $m = m(x)$ satisfies*

$$m \in C^4 \cap L^\infty; \quad |m^{(i)}(x)| \leq C \exp(\gamma |x|),\ \forall\, x \in \mathbb{R}^N\,,\ i = 1, 2, 3, 4, \tag{1.11}$$

where γ and C are positive constants, and $m^{(i)}(x)$ are the i-th derivatives of $m(x)$. Let $\psi_h(x,t) := e^{i\lambda t/h} u_h(x)$ for $x \in \mathbb{R}^N$ and $t > 0$, where u_h is a single-spike solution of (1.5) *concentrating at a non-degenerate critical point x_0 of $m(x)$. Assume*

$$\begin{aligned} m(x_0)\Delta^2 m(x_0) &< C_{N,1}|\Delta m(x_0)|^2 + C_{N,2}\Big[N\|\nabla^2 m(x_0)\|_2^2 - |\Delta m(x_0)|^2\Big] \\ &\quad + C_{N,3} m(x_0)\nabla(\Delta m)(x_0)\cdot\big[\nabla^2 m(x_0)\big]^{-1}\nabla(\Delta m)(x_0)\,, \end{aligned} \tag{1.12}$$

where

$$C_{N,1} = \frac{2(N+2)^2 \int_0^\infty r^{N+1} w^p L_0^{-1}\big(r^2 w^p\big)dr}{N^2 \int_0^\infty r^{N+3} w^{p+1} dr}\,, \tag{1.13}$$

$$C_{N,2} = \frac{4(N+2)\int_0^\infty r^{N+1} w^p \Phi_0 dr}{N^2 \int_0^\infty r^{N+3} w^{p+1} dr}\,, \tag{1.14}$$

$$C_{N,3} = \frac{(N+2)\big(\int_0^\infty r^{N+1} w^{p+1} dr\big)^2}{N\int_0^\infty r^{N-1} w^{p+1} dr \int_0^\infty r^{N+3} w^{p+1} dr}, \tag{1.15}$$

are constants depending only on N. Here $\Phi_0 = \Phi_0(r)$ satisfies

$$\begin{cases} \Phi_0'' + \dfrac{N-1}{r}\Phi_0' - \Phi_0 + p w^{p-1}\Phi_0 - \dfrac{2N}{r^2}\Phi_0 - r^2 w^p = 0,\ r = |x| \in (0,\infty), \\ \Phi_0(0) = \Phi_0'(0) = 0. \end{cases} \tag{1.16}$$

where L_0 is defined in (1.7). Then for any $\lambda > 0$, ψ_h is orbitally stable if h is sufficiently small and x_0 is a non-degenerate local maximum point of $m(x)$. Furthermore, for any $\lambda > 0$, ψ_h is orbitally unstable if h is sufficiently small and the number of positive eigenvalues of the Hessian matrix $\nabla^2 m(x_0)$ is odd.

Remark 1: Theorem 1.1 may include the case that the third order derivatives of the function m at x_0 can be nonzero. When $N = 1$, $x_0 = 0$ and the function m satisfies $m'''(x_0) = 0$, (see (C.2) of [10]), the condition (1.12) of Theorem 1.1 is exactly same as the condition (4.14) of [10]. For $N \geq 2$, G.Fibich and X.-P.Wang considered the function m with radial symmetry, i.e., $m = m(r), r = |x|$ and $m'''(0) = 0$, and studied the orbital stability problem only for radial perturbations (cf. [12]). Here we study the orbital stability problem for general perturbations including the non-radial perturbations and the case that the function m is not radially symmetric.

When the potential V is turned on, we may follow the argument of Theorem 1.1 to obtain

Theorem 1.2. *Let $N \geq 1$ be a positive integer, $p = 1 + \frac{4}{N}$. Assume both the potential $V = V(x)$ and the function $m = m(x)$ satisfy the following conditions: there exist positive constants γ and C such that*

$$V, m \in C^2 \cap L^\infty; \quad |V^{(i)}(x)|, |m^{(i)}(x)| \leq C\exp(\gamma|x|), \quad \forall\, i = 1, 2\,,\ x \in \mathbb{R}^N\,, \tag{1.17}$$

where $V^{(i)}(x), m^{(i)}(x)$ are the i-th derivatives of $V(x), m(x)$, respectively. Let $\psi_h(x,t) := e^{i\lambda t/h}u_h(x)$ for $x \in \mathbb{R}^N$ and $t > 0$, where u_h is a single-spike solution of (1.5) concentrating at a non-degenerate critical point x_0 of the function G defined in (1.10). Then for any $\lambda > 0$, ψ_h is orbitally unstable if h is sufficiently small and x_0 is a non-degenerate local minimum point of G such that $\nabla V(x_0) \neq 0$.

Theorem 1.3. *Under the same hypotheses of Theorem 1.2, assume $\nabla V(x_0) = 0$ and $\Delta V(x_0) \neq 0$. Let n be the number of negative eigenvalues of the matrix $\nabla^2 G(x_0)$. Then for any λ, u_h is orbitally stable if h is sufficiently small and x_0 is a non-degenerate local minimum point of G. Furthermore, for any $\lambda > 0$, u_h is orbitally unstable if h is sufficiently small and $n - \frac{1}{2}(1 + \frac{\Delta V(x_0)}{|\Delta V(x_0)|})$ is even.*

Theorem 1.4. *Under the same hypotheses of Theorem 1.2, assume $\nabla V(x_0) = 0$, $\Delta V(x_0) = 0$ and (1.11) hold for both V and m. Let n be the number of negative eigenvalues of the matrix $\nabla^2 G(x_0)$. Suppose $H(x_0) > 0$, where $H(x_0)$ defined in (4.34) involves the i-th derivatives (for $0 \leq i \leq 4$) of V and m at x_0. Then for any $\lambda > 0$, u_h is orbitally stable if h is sufficiently small and x_0 is a non-degenerate local minimum point of G. Furthermore, for any $\lambda > 0$, u_h is orbitally unstable if n is odd.*

Remark 2: Theorem 1.2-1.4 may include all the cases of values $\nabla V(x_0)$ and $\Delta V(x_0)$ for the orbital stability problem of (1.3) with critical exponent (1.4). Theorem 1.3 may generalize the main result of [25] to the case that the function m is a positive and nonconstant function. As $V \equiv 0$, Theorem 1.4 coincides with Theorem 1.1 since

$$\nabla^2 G(x_0) = m(x_0)^{-\frac{N}{2}-1}\Big[m(x_0)\nabla^2 V(x_0) - \frac{N}{2}\big[V(x_0) + \lambda\big]\nabla^2 m(x_0)\Big],$$

holds in Theorem 1.4.

The rest of this paper is organized as follows: In Section 2, we switch off the potential V and study the properties of u_h. Then we state the proof of Theorem 1.1 in Section 3. Finally, the potential V is turned on and we briefly outline the proofs of Theorem 1.2-1.4 in Section 4.

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2 Preliminaries

In this section, we study the properties of u_h a single-spike bound state of (1.5) concentrated at a non-degenerate critical point of $G(x) := \big[V(x) + \lambda\big]\, m^{-N/2}(x)$ (cf. [20], [37]). Let x_h be the unique local maximum point of u_h. So $x_h \to x_0$ as $h \to 0$.

Let $v_h(y) := u_h(hy + x_h)$ for all $y \in \mathbb{R}^N$. Then by (1.5), v_h is a positive solution of

$$\Delta v - \big[V(hy + x_h) + \lambda\big]v + m(hy + x_h)v^p = 0. \tag{2.1}$$

For notation convenience, we still denote

$$L_h := \Delta - \big[V(hy + x_h) + \lambda\big] + m(hy + x_h)pv_h^{p-1} \tag{2.2}$$

as the linearized operator of the equation (2.1) with respect to the solution v_h. As the result of [37], v_h can be written as $v_h = w_{x_h} + \phi_h$, where w_{x_h} is the unique positive solution of

$$\begin{cases} \Delta w - \big[V(x_h)+\lambda\big]w + m(x_h)w^p = 0 \ \text{ in } \ \mathbb{R}^N\,, \\ w(0) = \max\limits_{y\in\mathbb{R}^N} w(y)\,, \quad w(y)\to 0 \text{ as } |y|\to+\infty\,, \end{cases} \tag{2.3}$$

and

$$\|\phi_h\|_\infty \to 0 \quad \text{ as } \ h\to 0\,. \tag{2.4}$$

Moreover,

$$v_h(y) \le C|y|^{\frac{1-N}{2}}\exp\big(-\overline{V}^{1/2}y\big)\,, \quad \forall\, y\in\mathbb{R}^N\,, \tag{2.5}$$

where $\overline{V} := \inf_{\mathbb{R}^N}\big[V(x)+\lambda\big]$. From (2.3), it is easy to check that

$$w_{x_h}(y) = \big[V(x_h)+\lambda\big]^{\frac{1}{p-1}} m(x_h)^{-\frac{1}{p-1}} w\big(\sqrt{V(x_h)+\lambda}\,y\big)\,, \tag{2.6}$$

where w is the positive solution of (1.6).

For the single-spike solution of (1.5), we recall the following result from [36] and [37]:

Lemma 2.1. *Assume that there are positive constants γ and C such that*

$$|\nabla V(x)|, |\nabla m(x)| \le C\exp(\gamma|x|)\,, \quad \forall\, x\in\mathbb{R}^N\,. \tag{2.7}$$

Then

$$\int\limits_{\mathbb{R}^N}\Big[\frac{1}{p+1}\nabla m(hy+x_h)v_h^{p+1} - \frac{1}{2}\nabla V(hy+x_h)v_h^2\Big]dy = 0 \tag{2.8}$$

for $0<h<h_0$, where h_0 is a positive constant depending on γ and λ.

In the rest of this section, for simplicity, we switch off the potential V, i.e., set $V\equiv 0$. Then by Lemma 2.1, we obtain the uniqueness of u_h as follows:

Lemma 2.2. *Suppose (2.7) holds, $V\equiv 0$ and x_0 is a non-degenerate critical point of m. Then u_h is unique.*

Proof. Suppose $u_{h,1}$ and $u_{h,2}$ are different single-spike solutions of (1.5) concentrating at the same point x_0. Let $v_1(y) := u_{h,1}(hy+x_0)$ and $v_2(y) := u_{h,2}(hy+x_0)$. Then both v_1 and v_2 satisfy

$$\Delta v - \lambda v + m(hy+x_0)v^p = 0\,, \quad \text{ for } \ y\in\mathbb{R}^N\,,$$

and $v_1, v_2 \to w_{x_0}$ uniformly on $\mathbb{R}^N$ as $h\to 0$. Due to $v_1 \not\equiv v_2$, we may set

$$\widetilde{v}_h := \frac{v_1 - v_2}{\|v_1-v_2\|_\infty}\,,$$

and then $\widetilde{v}_h$ satisfies

$$\Delta\widetilde{v}_h - \lambda\widetilde{v}_h + m(x_0)pw_{x_0}^{p-1}\widetilde{v}_h + [m(hy+x_0)-m(x_0)]pw_{x_0}^{p-1}\widetilde{v}_h + N(\widetilde{v}_h) = 0, \tag{2.9}$$

where $N(\widetilde{v}_h) = m(hy+x_0)\big[v_1^p - v_2^p - pw_{x_0}^{p-1}(v_1-v_2)\big]/\|v_1-v_2\|_\infty$. Hence by the standard elliptic PDE theorems on the equation (2.9), we may take a subsequence $\widetilde{v}_h \to \widetilde{v}_0$, where $\widetilde{v}_0$ solves

$$\Delta \widetilde{v}_0 - \widetilde{v}_0 + m(x_0)pw_{x_0}^{p-1}\widetilde{v}_0 = 0.$$

Consequently, there exist constants c_j's such that

$$\widetilde{v}_0 = \sum_{j=1}^{N} c_j \partial_j w_{x_0}\,. \tag{2.10}$$

Let y_h be such that $\widetilde{v}_h(y_h) = \|\widetilde{v}_h\|_\infty = 1$ (the same proof applies if $\widetilde{v}_h(y_h) = -1$). Then by the Maximum Principle, we have $|y_h| \le C$. On the other hand, as (2.8), we may obtain

$$\int_{\mathbb{R}^N} \nabla m(hy+x_0) v_1^{p+1} dy = 0 = \int_{\mathbb{R}^N} \nabla m(hy+x_0) v_2^{p+1} dy.$$

Thus

$$\int_{\mathbb{R}^N} \nabla m(hy+x_0)\Big(\frac{v_1^{p+1}-v_2^{p+1}}{v_1-v_2}\Big)\widetilde{v}_h dy = 0. \tag{2.11}$$

Note that for all $i = 1, \cdots, N$, as $h \to 0$,

$$\partial_i m(hy+x_0) = h\sum_{k=1}^{N} \partial_{ik} m(x_0) y_k + \mathrm{o}(h)\,, \text{and } \frac{v_1^{p+1}-v_2^{p+1}}{v_1-v_2} = (p+1)w_{x_0}^p + \mathrm{o}(1)\,.$$

Hence from (2.10) and (2.11), we may obtain

$$\begin{aligned} 0 &= \int_{\mathbb{R}^N} \Big[h\sum_{k=1}^{N} \partial_{ik} m(x_0) y_k\Big](p+1)w_{x_0}^p\Big(\sum_{j=1}^{N} c_j\partial_j w_{x_0}\Big)dy + \mathrm{o}(h) \\ &= -h\sum_{j=1}^{N} \partial_{ij} m(x_0) c_j \int_{\mathbb{R}^N} w_{x_0}^{p+1}\, dy + \mathrm{o}(h)\,. \end{aligned}$$

Hence by the assumption that $\nabla^2 m(x_0)$ is non-degenerate, $c_j = 0$ for $j = 1, \cdots, N$, i.e., $\widetilde{v}_0 \equiv 0$. This may contradict to the fact that $1 = \widetilde{v}_h(y_h) \to \widetilde{v}_0(y_0)$ for some $y_0 \in \mathbb{R}^N$. Therefore, we may complete the proof of Lemma 2.2. □

By Lemma 2.1, we may simplify the proof of [21] and get a shorter proof of the asymptotic behavior of x_h's as follows:

Lemma 2.3. *Under the same hypotheses of Lemma 2.2,*

$$x_h = x_0 + \mathrm{o}(h) \quad \text{as } h \to 0\,. \tag{2.12}$$

Proof. Fix $i \in \{1, \cdots, N\}$ arbitrarily. By Taylor's expansion of $\partial_i m(x)$ and $\nabla m(x_0) = 0$, we obtain

$$\partial_i m(hy + x_h) = \sum_{j=1}^{N} \partial_{ij} m(x_0)(hy_j + x_{h,j} - x_{0,j}) + \mathrm{o}(h) + \mathrm{o}(|x_h - x_0|).$$

Hence by Lemma 2.1 and $v_h = w_{x_0} + \mathrm{o}(1)$, we have

$$\begin{aligned} 0 &= \int_{\mathbb{R}^N} \partial_i m(hy + x_h) v_h^{p+1} dy \\ &= \sum_{j=1}^{N} \partial_{ij} m(x_0)(x_{h,j} - x_{0,j}) \int_{\mathbb{R}^N} w_{x_0}^{p+1} dy + \mathrm{o}(h) + \mathrm{o}(|x_h - x_0|) \end{aligned}$$

Here we have used the fact that $\int_{\mathbb{R}^N} y_j w_{x_0}^{p+1}\, dy = 0$ for $j = 1, \cdots, N$. Using the assumption that $\nabla^2 m(x_0)$ is non-degenerate, we obtain (2.12). □

Following the idea of [25], we may use Lemma 2.3 to show the asymptotic behavior of v_h as follows:

Lemma 2.4. *Under the same hypotheses of Lemma 2.2,*

$$v_h = w_{x_h} + h^2 \phi_2 + \mathrm{o}(h^2), \qquad \text{as } h \to 0, \tag{2.13}$$

where ϕ_2 satisfies

$$\Delta \phi_2 - \lambda \phi_2 + m(x_h) p w_{x_h}^{p-1} \phi_2 + \frac{1}{2} \sum_{i,j=1}^{N} \partial_{ij} m(x_0) y_i y_j w_{x_h}^p = 0\,, \text{and } \nabla \phi_2(0) = 0. \tag{2.14}$$

Proof. Let $\phi_h = v_h - w_{x_h}$. Then it is easy to check that $|\phi_h| \to 0$ uniformly, and ϕ_h satisfies

$$\Delta \phi_h - \lambda \phi_h + m(hy + x_h) p w_{x_h}^{p-1} \phi_h + N(\phi_h) + R(\phi_h) = 0\,, \text{and } \nabla \phi_h(0) = 0, \tag{2.15}$$

where

$$N(\phi_h) = m(hy + x_h) \Big[(w_{x_h} + \phi_h)^p - w_{x_h}^p - p w_{x_h}^{p-1} \phi_h \Big],$$

and

$$R(\phi_h) = \Big[m(hy + x_h) - m(x_h) \Big] w_{x_h}^p.$$

Note that by Lemma 2.3 and $\nabla m(x_0) = 0$,

$$\begin{aligned} m(hy + x_h) - m(x_h) =& hy \cdot \nabla m(x_h) + \frac{h^2}{2} \sum_{i,j=1}^{N} \partial_{ij} m(x_h) y_i y_j + \mathrm{o}(h^2) \\ =& \frac{h^2}{2} \sum_{i,j=1}^{N} \partial_{ij} m(x_0) y_i y_j + \mathrm{o}(h^2). \end{aligned} \tag{2.16}$$

Now we claim that $|\phi_h| \leq c\,h^2$ by contradiction. Suppose that $h^{-2}\|\phi_h\|_\infty \to \infty$. Let $\widetilde{\phi}_h = \phi_h/\|\phi_h\|_\infty$. Then $\widetilde{\phi}_h$ satisfies

$$\Delta\widetilde{\phi}_h - \lambda\widetilde{\phi}_h + m(hy+x_h)pw_{x_h}^{p-1}\widetilde{\phi}_h + \frac{N(\phi_h)}{\|\phi_h\|_\infty} + \frac{R(\phi_h)}{\|\phi_h\|_\infty} = 0\,. \tag{2.17}$$

Note that by (2.16),

$$\frac{R(\phi_h)}{\|\phi_h\|_\infty} \leq C\frac{h^2}{\|\,\phi_h\|_\infty}\,. \tag{2.18}$$

Let y_h be such that $\widetilde{\phi}_h(y_h) = \|\widetilde{\phi}_h\|_\infty = 1$ (the same proof applies if $\widetilde{\phi}_h(y_h) = -1$). Then by (2.17)$-$(2.18) and the Maximum Principle, we have $|y_h| \leq C$. On the other hand, by the usual elliptic regularity theory, we may take a subsequence $\widetilde{\phi}_h \to \widetilde{\phi}_0$, where $\widetilde{\phi}_0$ satisfies

$$\Delta\widetilde{\phi}_0 - \widetilde{\phi}_0 + m(x_0)pw_{x_0}^{p-1}\widetilde{\phi}_0 = 0\,, \text{and } \nabla\widetilde{\phi}_0(0) = 0.$$

Hence $\widetilde{\phi}_0 \equiv 0$. This may contradict to the fact that $1 = \widetilde{\phi}_h(y_h) \to \widetilde{\phi}_0(y_0)$ for some y_0. Therefore, we may complete the claim that $|\,\phi_h| \leq c\,h^2$.

Now we set $\phi_{h,2} = \phi_h - h^2\phi_2$. Then $\phi_{h,2} = \mathrm{O}(h^2)$ and satisfies

$$\Delta\phi_{h,2} - \lambda\phi_{h,2} + m(hy+x_h)pw_{x_h}^{p-1}\phi_{h,2} + N(\phi_{h,2}) + R(\phi_{h,2}) = 0\,, \text{and } \nabla\phi_{h,2}(0) = 0$$

where

$$N(\phi_{h,2}) = m(hy+x_h)\Big[(w_{x_h} + h^2\phi_2 + \phi_{h,2})^p - w_{x_h}^p - pw_{x_h}^{p-1}(h^2\phi_2 + \phi_{h,2})\Big],$$

and

$$R(\phi_{h,2}) = \Big[m(hy+x_h) - m(x_h) - \frac{h^2}{2}\sum_{i,j=1}^N \partial_{ij}m(x_0)y_iy_j\Big]w_{x_h}^p + h^2\Big[m(hy+x_h) - m(x_h)\Big]pw_{x_h}^{p-1}\phi_2.$$

Thus as for previous argument, we may have $\phi_{h,2} = \mathrm{o}(h^2)$ and complete the proof of Lemma 2.4. $\square$

As for Proposition 3.1 of [23], one may get two lemmas as follows:

Lemma 2.5. *For h small enough, the maps*

$$L_{x_h}\phi := \Delta\phi - \big[V(x_h) + \lambda\big]\phi + m(x_h)pw_{x_h}^{p-1}\phi$$

are uniformly invertible from $K_{x_h}^\perp$ to $C_{x_h}^\perp$, where

$$K_{x_h}^\perp = \left\{\phi \in H^2(\mathbb{R}^N)\,\middle|\,\int_{\mathbb{R}^N}\phi\partial_j w_{x_h}dy = 0\,, j = 1,\cdots,N\right\} \subset H^2(\mathbb{R}^N),$$
$$C_{x_h}^\perp = \left\{\phi \in L^2(\mathbb{R}^N)\,\middle|\,\int_{\mathbb{R}^N}\phi\partial_j w_{x_h}dy = 0\,, j = 1,\cdots,N\right\} \subset L^2(\mathbb{R}^N).$$

Lemma 2.6. *The map*

$$L_{x_0}\phi := \Delta\phi - \big[V(x_0)+\lambda\big]\phi + m(x_0)pw_{x_0}^{p-1}\phi$$

has eigenvalues $\mu_j\,, j=1,\cdots,N+2$ *satisfying*

$$\mu_1 > 0 = \mu_2 = \cdots = \mu_{N+1} > \mu_{N+2} \geq \cdots\,,$$

where the kernel of L_{x_0} *is spanned by* $\partial_j w_{x_0}$*,* $j=1,\cdots,N$*.*

In this section, our main result is the small eigenvalue estimates of L_h given by

Theorem 2.7. *Under the same hypotheses of Lemma 2.2, for h small enough, the eigenvalue problem*

$$L_h\varphi_h = \mu_h\varphi_h \tag{2.19}$$

has exactly N *eigenvalues* $\mu_h^j\,, j=1,\cdots,N$ *satisfying*

$$\frac{1}{2}\mu_1 \geq \mu_h^1 \geq \mu_h^2 \geq \cdots \geq \mu_h^N \geq \frac{1}{2}\mu_{N+2}\,,$$

and

$$\frac{\mu_h^j}{h^2} \to c_0\nu_j\,,\ (\textit{up to a subsequence})\quad \textit{as}\ \ h\to 0\,,\quad \textit{for}\ \ j=1,\cdots,N, \tag{2.20}$$

where μ_1 *and* μ_{N+2} *are defined in Lemma 2.6,* ν_j*'s are the eigenvalues of the Hessian matrix* $\nabla^2 m(x_0)$ *and* $c_0=\frac{N}{2m(x_0)}$ *is a positive constant. Furthermore, the corresponding eigenfunctions* φ_h^j*'s satisfy*

$$\varphi_h^j = \sum_{i=1}^N \big[a_{ij}+\mathrm{o}(1)\big]\partial_i w_{x_h} + \mathrm{O}(h^2)\,,\quad j=1,\cdots,N\,, \tag{2.21}$$

where $\boldsymbol{a}_j=(a_{1j},\cdots,a_{Nj})^T$ *is the eigenvector associated with* ν_j*, namely,*

$$\nabla^2 m(x_0)\,\boldsymbol{a}_j = \nu_j\,\boldsymbol{a}_j\,. \tag{2.22}$$

Here $o(1)$ *is a small quantity tending to zero and* $O(1)$ *is a bounded quantity as h goes to zero.*

Proof. We may follow the arguments given in Section 5 of [40]. Assume that $\|\varphi_h\|_{L^2}=1$. It is easy to see that $\mu_h\to 0$ as $h\to 0$, where $\mu_h\in\{\mu_h^1,\cdots,\mu_h^N\}$. Then the corresponding eigenfunctions φ_h's can be written as

$$\varphi_h = \sum_{j=1}^N a_h^j\partial_j w_{x_h} + \varphi_h^\perp, \tag{2.23}$$

where $\varphi_h^\perp\in K_{x_h}^\perp$. Hence by (2.19) and (2.23), $\varphi_h^\perp$ satisfies

$$\Delta\varphi_h^\perp - \lambda\varphi_h^\perp + m(x_h)pw_{x_h}^{p-1}\varphi_h^\perp + R(\varphi_h^\perp) + \sum_{j=1}^N a_h^j L_h\partial_j w_{x_h} = \mu_h\left(\sum_{j=1}^N a_h^j\partial_j w_{x_h} + \varphi_h^\perp\right)\,, \tag{2.24}$$

where

$$R(\varphi_h^\perp) = m(hy+x_h)p(v_h^{p-1} - w_{x_h}^{p-1})\varphi_h^\perp + \Big[m(hy+x_h) - m(x_h)\Big]pw_{x_h}^{p-1}\varphi_h^\perp.$$

Using (2.16) and Lemma 2.4, we have

$$L_h\partial_j w_{x_h} = m(hy+x_h)p(v_h^{p-1} - w_{x_h}^{p-1})\partial_j w_{x_h} + \Big[m(hy+x_h) - m(x_h)\Big]pw_{x_h}^{p-1}\partial_j w_{x_h} = \mathrm{O}(h^2). \tag{2.25}$$

From Lemma 2.5, the map $L_{x_h} = \Delta - \lambda + m(x_h)pw_{x_h}^{p-1}$ is uniformly invertible in the space $K_{x_h}^\perp$. Thus by (2.25) and $\mu_h \to 0$, we have

$$\|\varphi_h^\perp\|_{H^2} \le c(h^2 + |\mu_h|)\sum_{j=1}^N |a_h^j|\,. \tag{2.26}$$

To estimate μ_h and a_h^j's, multiplying (2.24) by $\partial_k w_{x_h}$ and integrating over $\mathbb{R}^N$, we may obtain

$$\int_{\mathbb{R}^N}\left(L_h\varphi_h^\perp\right)\partial_k w_{x_h}dy + \sum_{j=1}^N a_h^j\int_{\mathbb{R}^N}\left(L_h\partial_j w_{x_h}\right)\partial_k w_{x_h}dy = \mu_h\sum_{j=1}^N a_h^j\int_{\mathbb{R}^N}\partial_j w_{x_h}\partial_k w_{x_h}dy\,. \tag{2.27}$$

Here we have used the fact that $\varphi_h^\perp \in K_{x_h}^\perp$. Using (2.25), (2.26), $\mu_h = \mathrm{o}(1)$ and integration by parts, we obtain

$$\int_{\mathbb{R}^N}\left(L_h\varphi_h^\perp\right)\partial_k w_{x_h}dy = \int_{\mathbb{R}^N}\varphi_h^\perp L_h\partial_k w_{x_h}dy = \mathrm{o}(h^2)\,, \tag{2.28}$$

and

$$\int_{\mathbb{R}^N}\left(L_h\partial_j w_{x_h}\right)\partial_k w_{x_h}dy = \frac{h^2}{p+1}\int_{\mathbb{R}^N} w_{x_h}^{p+1}dy\,\partial_{jk}m(x_0) + \mathrm{o}(h^2)\,, \tag{2.29}$$

which we have proved in Appendix A. Substituting (2.28) and (2.29) into (2.27), we may obtain

$$\frac{1}{p+1}\int_{\mathbb{R}^N} w_{x_h}^{p+1}dy\sum_{j=1}^N\partial_{jk}m(x_0)a_h^j = \frac{\mu_h}{h^2}a_h^k\int\limits_{\mathbb{R}^N}\left(\partial_k w_{x_h}\right)^2dy + \mathrm{o}(1).$$

Since $\|\varphi_h\|_{L^2} = 1$, (2.23) implies that $\mathbf{a}_h := (a_h^1, \cdots, a_h^N)^T$ is bound. Moreover, by (2.26), $\mathbf{a}_h$ does not converge to 0. Thus $\frac{\mu_h^j}{h^2} \to c_0\nu_j$ for $j = 1, \cdots, N$ and $\mathbf{a}_h \to \mathbf{a}_j$, where

$$c_0 = \frac{N\int_{\mathbb{R}^N} w_{x_0}^{p+1}dy}{(p+1)\int_{\mathbb{R}^N}|\nabla w_{x_0}|^2dy} = \frac{N}{2m(x_0)},$$

and $\mathbf{a}_j$ is the eigenvector corresponding to ν_j. Here we have use the fact that

$$\int_{\mathbb{R}^N}|\nabla w_{x_0}|^2dy = \frac{N}{N+2}m(x_0)\int_{\mathbb{R}^N} w_{x_0}^{p+1}dy\,,$$

which can be proved by Pohozeve identity. The rest of the proof follows from a perturbation result, similar to page 1473-1474 of [40]. We may omit the details here. □

3 Proof of Theorem 1.1

In this Section, we firstly study the asymptotic expansion of $d''(\lambda)$ as $h \to 0$, and then complete the proof of Theorem 1.1. To drive the $\mathrm{O}(h^4)$ order terms of $d''(\lambda)/h^N$, we need the following lemma:

Lemma 3.1. *Under the same hypotheses of Lemma 2.2,*

$$x_h = x_0 + h^2 \boldsymbol{x}_1 + \mathrm{O}(h^3)\,, \qquad as \ \ h \to 0\,, \tag{3.1}$$

where $\boldsymbol{x}_1 \in \mathbb{R}^N$ *satisfies*

$$\nabla^2 m(x_0)\boldsymbol{x}_1 = -\frac{\int\limits_{\mathbb{R}^N} |y|^2 w^{p+1}\,dy}{2N\lambda \int\limits_{\mathbb{R}^N} w^{p+1}\,dy}\nabla(\Delta m)(x_0). \tag{3.2}$$

Proof. By Lemma 2.3 and $\nabla m(x_0) = 0$, for all $i = 1, \cdots, N$, we have

$$\partial_i m(hy + x_h) = \sum_{j=1}^N \partial_{ij} m(x_0)\big(hy_j + x_{h,j} - x_{0,j}\big) + \mathrm{O}(h^2). \tag{3.3}$$

Then by (2.8), (3.3) and Lemma 2.4, we have

$$\begin{aligned}
0 &= \int\limits_{\mathbb{R}^N} \partial_i m(hy + x_h) v_h^{p+1} dy \\
&= \sum_{j=1}^N \partial_{ij} m(x_0) \int\limits_{\mathbb{R}^N} \big(hy_j + x_{h,j} - x_{0,j}\big)\Big[w_{x_h}^{p+1} + \mathrm{O}(h)\Big] dy + \mathrm{O}(h^2) \\
&= \sum_{j=1}^N \partial_{ij} m(x_0)\big(x_{h,j} - x_{0,j}\big) \int\limits_{\mathbb{R}^N} w_{x_0}^{p+1} dy + \mathrm{O}(h^2)\,.
\end{aligned}$$

Here we have used the fact that $\int\limits_{\mathbb{R}^N} y_j w_{x_h}^{p+1}\,dy = 0$ for $j = 1, \cdots, N$. Thus $x_h = x_0 + \mathrm{O}(h^2)$. Consequently, we may set $x_h = x_0 + h^2 \overline{x}_h$. Then $\overline{x}_h = \mathrm{O}(1)$ and by Taylor's formula of $\partial_i m(x)$, we have

$$\partial_i m(hy + x_h) = \sum_{j=1}^N \partial_{ij} m(x_0)\big(hy_j + h^2 \overline{x}_h\big) + \frac{h^2}{2} \sum_{j,k=1}^N \partial_{ijk} m(x_0) y_j y_k + \mathrm{O}(h^3). \tag{3.4}$$

Hence by(2.8), (3.4) and Lemma 2.4, we may obtain

$$\begin{aligned}
0 =& h^2 \sum_{j=1}^N \partial_{ij} m(x_0)\overline{x}_{h,j} \int\limits_{\mathbb{R}^N} w_{x_h}^{p+1} dy + \frac{h^2}{2} \sum_{j,k=1}^N \partial_{ijk} m(x_0) \int\limits_{\mathbb{R}^N} y_j y_k w_{x_h}^{p+1} dy + \mathrm{O}(h^3) \\
=& h^2 \sum_{j=1}^N \partial_{ij} m(x_0)\overline{x}_{h,j} \int\limits_{\mathbb{R}^N} w_{x_0}^{p+1} dy + \frac{h^2}{2N} \sum_{k=1}^N \partial_{ikk} m(x_0) \int\limits_{\mathbb{R}^N} |y|^2 w_{x_0}^{p+1} dy + \mathrm{O}(h^3).
\end{aligned}$$

Here we have used the fact that

$$\begin{cases} \int\limits_{\mathbb{R}^N} y_j w_{x_0}^{p+1} dy = 0\,, \quad \forall j = 1, \cdots, N, \\ \int\limits_{\mathbb{R}^N} y_j y_k w_{x_0}^{p+1} = \frac{\delta_{jk}}{N} \int\limits_{\mathbb{R}^N} |y|^2 w_{x_0}^{p+1} dy\,, \quad \forall j, k = 1, \cdots, N. \end{cases}$$

Therefore, we may complete the proof because

$$w_{x_0}(y) = \lambda^{N/4} m(x_0)^{-N/4} w(\sqrt{\lambda} y)\,.$$

□

From Lemma 2.4 and 3.1, we may deduce that

Theorem 3.2. *Under the same hypotheses of Lemma 2.2, for h small enough, u_h is smooth on λ. Let $R_h := \frac{\partial u_h}{\partial \lambda}(hy + x_h)$. Then*

$$L_h R_h - v_h = 0. \tag{3.5}$$

and

$$R_h = R_0 + \sum_{j=1}^{N} c_h^j \partial_j w_{x_h} + h^2 R_1 + R_h^{\perp}\,, \tag{3.6}$$

where $R_0 = \lambda^{-1}\big(\frac{1}{p-1} v_h + \frac{1}{2} y \cdot \nabla v_h\big)$, $c_h^j = \mathrm{O}(h)$, $R_h^{\perp} = \mathrm{O}(h^3)$ and R_1 satisfies

$$\Delta R_1 - \lambda R_1 + m(x_h) p w_{x_h}^{p-1} R_1 - \frac{1}{2\lambda} \sum_{i,j=1}^{N} \partial_{ij} m(x_0) y_i y_j w_{x_h}^p = 0\,. \tag{3.7}$$

Furthermore,

$$\nabla^2 m(x_0)\big(h^{-1} \boldsymbol{c}_h\big) \to -\frac{\int\limits_{\mathbb{R}^N} |y|^2 w^{p+1}\, dy}{2N\lambda^2 \int\limits_{\mathbb{R}^N} w^{p+1}\, dy} \nabla(\Delta m)(x_0)\,, \qquad \text{as } h \to 0\,, \tag{3.8}$$

where $\boldsymbol{c}_h := (c_h^1, \cdots, c_h^N)^T$.

Proof. By Lemma 2.2 and Theorem 2.7, u_h is unique and non-degenerate. Consequently, u_h is smooth on λ and R_h satisfies (3.5). Now we decompose R_h as

$$R_h = R_0 + \sum_{j=1}^{N} c_h^j \partial_j w_{x_h} + h^2 R_1 + R_h^{\perp}\,,$$

where $R_h^{\perp} \in K_{x_h}^{\perp}$. Then $R_h^{\perp}$ satisfies

$$L_h R_h^{\perp} + \big[L_h R_0 + h^2 L_h R_1 - v_h\big] + \sum_{j=1}^{N} c_h^j L_h \partial_j w_{x_h} = 0. \tag{3.9}$$

As for the proof of Theorem 2.7, we have

$$\|R_h^{\perp}\|_{H^2} \le c\Big(\|L_hR_0 + h^2L_hR_1 - v_h\|_{L^2} + \sum_{j=1}^{N}|c_h^j|h^2\Big)\,. \tag{3.10}$$

It is easy to check

$$L_hR_0 = v_h - \frac{h}{2\lambda}y\cdot\nabla m(hy+x_h)v_h^p. \tag{3.11}$$

Hence by Lemma 2.4, 3.1, (3.7) and (3.11), we obtain

$$\begin{aligned} & L_hR_0 + h^2L_hR_1 - v_h \\ =& -\frac{h^3}{2\lambda}\Big[\sum_{i,j=1}^{N}\partial_{ij}m(x_0)x_{1,i}y_j + \frac{1}{2}\sum_{i,j,k=1}\partial_{ijk}m(x_0)y_iy_jy_k\Big]w_{x_h}^p + \mathrm{O}(h^4)\,. \end{aligned} \tag{3.12}$$

Consequently, by (3.10),

$$\|R_h^{\perp}\|_{H^2} \le c\Big(h^3 + \sum_{j=1}^{N}|c_h^j|h^2\Big)\,. \tag{3.13}$$

To estimate c_h^j's, we may multiply (3.9) by $\partial_k w_{x_h}$ and integrate over $\mathbb{R}^N$. Then

$$\begin{aligned} &\int_{\mathbb{R}^N}(L_hR_h^{\perp})\partial_k w_{x_h}dy + \int_{\mathbb{R}^N}\Big[L_hR_0 + h^2L_hR_1 - v_h\Big]\partial_k w_{x_h}dy \\ &+\sum_{j=1}^{N}c_h^j\int_{\mathbb{R}^N}(L_h\partial_j w_{x_h})\partial_k w_{x_h}dy = 0. \end{aligned} \tag{3.14}$$

Hence by (2.29), (3.14) may imply

$$|c_h^j| \le \frac{C}{h^2}\Bigg[\Big|\int_{\mathbb{R}^N}(L_hR_h^{\perp})\partial_k w_{x_h}dy\Big| + \Big|\int_{\mathbb{R}^N}\big[L_hR_0 + h^2L_hR_1 - v_h\big]\partial_k w_{x_h}dy\Big|\Bigg]\,. \tag{3.15}$$

Using integration by parts and (2.25), we have

$$\int_{\mathbb{R}^N}(L_hR_h^{\perp})\partial_k w_{x_h}dy = \int_{\mathbb{R}^N}R_h^{\perp}L_h\partial_k w_{x_h}dy = \|R_h^{\perp}\|_{L^2}\mathrm{O}(h^2)\,. \tag{3.16}$$

Therefore, by (3.12), (3.13), (3.15) and (3.16), we may obtain $|c_h^j| = \mathrm{O}(h)$. Consequently, by (3.13), $R_h^{\perp} = \mathrm{O}(h^3)$. Thus by (3.16),

$$\int_{\mathbb{R}^N}(L_hR_h^{\perp})\partial_k w_{x_h}dy = \mathrm{O}(h^5). \tag{3.17}$$

Hence by (2.29), (3.12) and (3.17), (3.14) gives

$$\begin{aligned} &\frac{1}{p+1}\int_{\mathbb{R}^N}w_{x_0}^{p+1}dy\sum_{j=1}^{N}\partial_{jk}m(x_0)\big(h^{-1}c_h^j\big) \\ =&\frac{1}{2\lambda}\int_{\mathbb{R}^N}\Big[\sum_{i,j=1}^{N}\partial_{ij}m(x_0)x_{1,i}y_j + \frac{1}{2}\sum_{i,j,l=1}^{N}\partial_{ijl}m(x_0)y_iy_jy_l\Big]w_{x_h}^p\partial_k w_{x_h}dy + \mathrm{o}(1). \end{aligned} \tag{3.18}$$

Using integration by parts, we obtain

$$\begin{cases} \int\limits_{\mathbb{R}^N} y_j w_{x_h}^p \partial_k w_{x_h} dy = -\frac{\delta_{jk}}{p+1} \int\limits_{\mathbb{R}^N} w_{x_h}^{p+1} dy\,, \\ \int\limits_{\mathbb{R}^N} y_i y_j y_l w_{x_h}^p \partial_k w_{x_h} dy = -\frac{\delta_{ik}\delta_{jl}+\delta_{jk}\delta_{il}+\delta_{lk}\delta_{ij}}{N(p+1)} \int\limits_{\mathbb{R}^N} |y|^2 w_{x_h}^{p+1} dy\,, \end{cases}$$

where δ is the Kronecker symbol. Hence by (3.18), $|c_h^j| = \mathrm{O}(h)$ for $j = 1, \cdots, N$. Moreover, by (3.2), we obtain (3.8) and complete the proof. □

Let us now compute $d''(\lambda)$. From (1.9), it is easy to get

$$d'(\lambda) = \frac{1}{2} \int\limits_{\mathbb{R}^N} u_h^2 dx$$

and hence

$$d''(\lambda) = \int\limits_{\mathbb{R}^N} u_h \frac{\partial u_h}{\partial \lambda} dx = h^N \int\limits_{\mathbb{R}^N} v_h R_h dy\,. \tag{3.19}$$

Using integration by parts and (3.5), we have

$$\int\limits_{\mathbb{R}^N} v_h R_0 dy = \int\limits_{\mathbb{R}^N} v_h \lambda^{-1} \big(\frac{1}{p-1} v_h + \frac{1}{2} y \cdot \nabla v_h\big) dy = \lambda^{-1} \big(\frac{1}{p-1} - \frac{N}{4}\big) \int\limits_{\mathbb{R}^N} v_h^2 dy = 0\,, \tag{3.20}$$

since $p = 1 + \frac{4}{N}$. Hence, by (3.19) and Theorem 3.2, we have

$$\begin{aligned} \frac{d''(\lambda)}{h^N} &= \int\limits_{\mathbb{R}^N} v_h \Big[R_0 + \sum_{j=1}^N c_h^j \partial_j w_{x_h} + h^2 R_1 + R_h^\perp\Big] dy \\ &= \int\limits_{\mathbb{R}^N} v_h \Big[\sum_{j=1}^N c_h^j \partial_j w_{x_h} + h^2 R_1 + R_h^\perp\Big] dy \qquad \Big(\text{because} \int\limits_{\mathbb{R}^N} v_h R_0 dy = 0\Big) \\ &= \int\limits_{\mathbb{R}^N} R_h \Big[\sum_{j=1}^N c_h^j L_h \partial_j w_{x_h} + h^2 L_h R_1 + L_h R_h^\perp\Big] dy \qquad \Big(\text{because} \quad L_h R_h = v_h\Big) \\ &= \int\limits_{\mathbb{R}^N} \Big[R_0 + \sum_{j=1}^N c_h^j \partial_j w_{x_h} + h^2 R_1 + R_h^\perp\Big] \Big[\sum_{j=1}^N c_h^j L_h \partial_j w_{x_h} + h^2 L_h R_1 + L_h R_h^\perp\Big] dy\,. \end{aligned}$$

Therefore, by (2.25), (3.9) and $c_h^j = \mathrm{O}(h)$,

$$\begin{aligned} \frac{d''(\lambda)}{h^N} = &\int\limits_{\mathbb{R}^N} R_0 \Big[v_h - L_h R_0\Big] dy + \sum_{j,k=1}^N c_h^j c_h^k \int\limits_{\mathbb{R}^N} \partial_k w_{x_h} \big(L_h \partial_j w_{x_h}\big) dy \\ &+ h^4 \int\limits_{\mathbb{R}^N} R_1 \big(L_h R_1\big) dy + \mathrm{O}(h^5)\,. \end{aligned} \tag{3.21}$$

For the integral $\int\limits_{\mathbb{R}^N} R_0\Big[v_h - L_h R_0\Big]dy$, by (3.11)and using integration by parts, we have

$$\int\limits_{\mathbb{R}^N} R_0\Big[v_h - L_h R_0\Big]dy = \int\limits_{\mathbb{R}^N} \lambda^{-1}\big(\frac{1}{p-1}v_h + \frac{1}{2}y\cdot\nabla v_h\big)\Big[\frac{h}{2\lambda}y\cdot\nabla m(hy+x_h)v_h^p\Big]dy$$
$$=\frac{1}{2\lambda^2}\int\limits_{\mathbb{R}^N}\frac{N}{4(N+2)}\Big[hy\cdot\nabla m(hy+x_h) - h^2\sum_{i,j=1}^N \partial_{ij}m(hy+x_h)y_iy_j\Big]v_h^{p+1}dy.$$

Note that by Lemma 2.4, 3.1 and Theorem 3.2, we have

$$hy\cdot\nabla m(hy+x_h) - h^2\sum_{i,j=1}^N \partial_{ij}m(hy+x_h)y_iy_j$$
$$=hy\cdot\nabla m(x_h) - \frac{h^3}{2}\sum_{i,j,k=1}^N \partial_{ijk}m(x_h)y_iy_jy_k - \frac{h^4}{3}\sum_{i,j,k,l=1}^N \partial_{ijkl}m(x_h)y_iy_jy_ky_l + \mathrm{o}(h^4)\,,$$

and

$$v_h^p = w_{x_h}^p + h^2pw_{x_h}^{p-1}\phi_2 + \mathrm{O}(h^3)\,. \tag{3.22}$$

Hence

$$\int\limits_{\mathbb{R}^N} R_0\Big[v_h - L_hR_0\Big]dy = \frac{N}{8(N+2)}\lambda^{-2}\int\limits_{\mathbb{R}^N}\Big[-\frac{h^4}{3}\sum_{i,j,k,l=1}^N \partial_{ijkl}m(x_h)y_iy_jy_ky_l\Big]w_{x_h}^{p+1}dy + \mathrm{o}(h^4)$$
$$= -\frac{h^4}{8(N+2)^2}\lambda^{-2}\int\limits_{\mathbb{R}^N}|y|^4w_{x_h}^{p+1}dy\Delta^2m(x_0) + \mathrm{o}(h^4)$$
$$= -\frac{h^4}{8(N+2)^2}\lambda^{-3}m(x_0)^{-\frac{N}{2}-1}\int\limits_{\mathbb{R}^N}|y|^4w^{p+1}dy\Delta^2m(x_0) + \mathrm{o}(h^4)\,. \tag{3.23}$$

Here we have used the following identities:

$$\begin{cases} \int\limits_{\mathbb{R}^N} y_iw_{x_h}^{p+1}dy = \int\limits_{\mathbb{R}^N} y_iy_jy_kw_{x_h}^{p+1}dy = 0\,, \quad \text{for all } i,j,k = 1,\cdots,N\,; \\ \int\limits_{\mathbb{R}^N} y_iy_jy_ky_lw_{x_h}^{p+1}dy = 0\,, \quad \text{if } y_iy_jy_ky_l \text{ is an odd function on one of its variate}\,; \\ \int\limits_{\mathbb{R}^N} y_i^4w_{x_h}^{p+1}dy = \frac{3}{N(N+2)}\int\limits_{\mathbb{R}^N}|y|^4w_{x_h}^{p+1}dy\,, \quad \text{for all } i = 1,\cdots,N\,; \\ \int\limits_{\mathbb{R}^N} y_i^2y_j^2w_{x_h}^{p+1}dy = \frac{1}{N(N+2)}\int\limits_{\mathbb{R}^N}|y|^4w_{x_h}^{p+1}dy\,, \quad \text{for all } i \neq j\,, \end{cases}$$

which can be proved by polar coordinates.

For the sum $\sum\limits_{j,k=1}^{N} c_h^j c_h^k \int\limits_{\mathbb{R}^N} \partial_k w_{x_h}\big(L_h \partial_j w_{x_h}\big) dy$, we may use (2.29) and (3.8) to get

$$\begin{aligned}
&\sum_{j,k=1}^{N} c_h^j c_h^k \int_{\mathbb{R}^N} \partial_k w_{x_h}\left(L_h \partial_j w_{x_h}\right) dy \\
=&\frac{h^4}{p+1}\int_{\mathbb{R}^N} w_{x_h}^{p+1} dy \sum_{j,k=1}^{N} (h^{-1}c_h^j)(h^{-1}c_h^k)\partial_{jk} m(x_0) + \mathrm{o}(h^4) \\
=&\frac{h^4}{8N(N+2)}\lambda^{-3} m(x_0)^{-\frac{N}{2}-1} \frac{\Big(\int\limits_{\mathbb{R}^N} |y|^2 w^{p+1} dy\Big)^2}{\int\limits_{\mathbb{R}^N} w^{p+1} dy} \nabla(\Delta m)(x_0)\cdot \big[\nabla^2 m(x_0)\big]^{-1}\nabla(\Delta m)(x_0) + \mathrm{o}(h^4).
\end{aligned} \tag{3.24}$$

For the integral $h^4\int\limits_{\mathbb{R}^N} R_1\big(L_h R_1\big) dy$, by (3.7), it is obvious that $R_1(\lambda^{-\frac{1}{2}}y)$ satisfies

$$\Delta R - R + p w^{p-1} R - \frac{1}{2}\lambda^{\frac{N}{4}-2} m(x_h)^{-\frac{N}{4}-1}\sum_{i,j=1}^{N} \partial_{ij} m(x_0) y_i y_j w^p = 0\,. \tag{3.25}$$

Hence

$$\begin{aligned}
&h^4 \int\limits_{\mathbb{R}^N} R_1\big(L_h R_1\big) dy = h^4 \int\limits_{\mathbb{R}^N} R_1\big(L_{x_h} R_1\big) dy + \mathrm{O}(h^6) \\
=&\frac{h^4}{4}\lambda^{-3} m(x_0)^{-\frac{N}{2}-2}\sum_{i,j,k,l=1}^{N} \partial_{ij} m(x_0)\partial_{kl} m(x_0)\int\limits_{\mathbb{R}^N} y_i y_i w^p L_0^{-1}\big(y_k y_l w^p\big) dy + \mathrm{O}(h^6) \\
=&\frac{h^4}{4N^2}\lambda^{-3} m(x_0)^{-\frac{N}{2}-2}|\Delta m(x_0)|^2 \int\limits_{\mathbb{R}^N} r^2 w^p L_0^{-1}(r^2 w^p) dy \\
&+\frac{h^4}{2N(N+2)}\lambda^{-3} m(x_0)^{-\frac{N}{2}-2}\|\nabla^2 m(x_0)\|_2^2 \int\limits_{\mathbb{R}^N} r^2 w^p \Phi_0(r) dy \\
&-\frac{h^4}{2N^2(N+2)}\lambda^{-3} m(x_0)^{-\frac{N}{2}-2}|\Delta m(x_0)|^2 \int\limits_{\mathbb{R}^N} r^2 w^p \Phi_0(r) dy + \mathrm{O}(h^6).
\end{aligned} \tag{3.26}$$

Here $\|\nabla^2 m(x_0)\|_2^2 = \sum_{i,j=1}^{N} m_{ij}^2(x_0)$ and we have used the following identities:

$$\int_{\mathbb{R}^N} y_N^2 w^p L_0^{-1}(y_N^2 w^p)dy = \frac{1}{N^2}\int_{\mathbb{R}^N} r^2 w^p L_0^{-1}(r^2 w^p)dy + \frac{2(N-1)}{N^2(N+2)}\int_{\mathbb{R}^N} r^2 w^p \Phi_0(r)dy\,, \tag{3.27}$$

$$\int_{\mathbb{R}^N} y_{N-1}^2 w^p L_0^{-1}(y_N^2 w^p)dy = \frac{1}{N^2}\int_{\mathbb{R}^N} r^2 w^p L_0^{-1}(r^2 w^p)dy - \frac{2}{N^2(N+2)}\int_{\mathbb{R}^N} r^2 w^p \Phi_0(r)dy\,, \tag{3.28}$$

$$\int_{\mathbb{R}^N} y_{N-1}y_N w^p L_0^{-1}(y_{N-1}y_N w^p)dy = \frac{1}{N(N+2)}\int_{\mathbb{R}^N} r^2 w^p \Phi_0(r)dy\,, \tag{3.29}$$

where Φ_0 satisfies (1.16), which we have proved in Appendix B.

Therefore, combining (3.21), (3.23), (3.24) and (3.26), we obtain

$$\begin{aligned}
&\frac{d''(\lambda)}{h^N} + \mathrm{o}(h^4)\\
=&-\frac{h^4}{8(N+2)^2}\lambda^{-3}m(x_0)^{-\frac{N}{2}-1}\int_{\mathbb{R}^N}|y|^4 w^{p+1}dy\Delta^2 m(x_0)\\
&+\frac{h^4}{8N(N+2)}\lambda^{-3}m(x_0)^{-\frac{N}{2}-1}\frac{\left(\int_{\mathbb{R}^N}|y|^2 w^{p+1}dy\right)^2}{\int_{\mathbb{R}^N} w^{p+1}dy}\nabla(\Delta m)(x_0)\cdot\left[\nabla^2 m(x_0)\right]^{-1}\nabla(\Delta m)(x_0)\\
&+\frac{h^4}{4N^2}\lambda^{-3}m(x_0)^{-\frac{N}{2}-2}|\Delta m(x_0)|^2\int_{\mathbb{R}^N}|y|^2 w^p L_0^{-1}\left(|y|^2 w^p\right)dy\\
&+\frac{h^4}{2N^2(N+2)}\lambda^{-3}m(x_0)^{-\frac{N}{2}-2}\left[N\|\nabla^2 m(x_0)\|_2^2 - |\Delta m(x_0)|^2\right]\int_{\mathbb{R}^N}|y|^2 w^p\Phi_0(|y|)dy.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\frac{8(N+2)^2 m(x_0)^{\frac{N}{2}+2}\lambda^3}{h^{N+4}\int_{\mathbb{R}^N}|y|^4 w^{p+1}dy}d''(\lambda) =& C_{N,1}|\Delta m(x_0)|^2 + C_{N,2}\left(N\|\nabla^2 m(x_0)\|_2^2 - |\Delta m(x_0)|^2\right)\\
&+ C_{N,3}m(x_0)\Big[\nabla(\Delta m)(x_0)\cdot\left[\nabla^2 m(x_0)\right]^{-1}\nabla(\Delta m)(x_0)\Big]\\
&- m(x_0)\Delta^2 m(x_0) + \mathrm{o}(h)\,,
\end{aligned}$$

where $C_{N,1}, C_{N,2}, C_{N,3}$ are constants given by (1.13), (1.14), (1.15), respectively.

Now we may prove Theorem 1.1 as follows: Suppose that x_0 is a non-degenerate local maximum point of the function $m(x)$, then the Hessian matrix $\nabla^2 m(x_0)$ of m at x_0 is negative definite. By Theorem 2.7, we have $n(L_h) = 1$. On the other hand, we have $p(d'') = 1$. Thus ψ_h is orbital stable by the orbital stability criteria of [17]-[18]. For orbital instability, we denote the number of positive eigenvalues of the Hessian matrix $\nabla^2 m(x_0)$ by n. Then by Theorem 2.7, we obtain $n(L_h) = n+1$. On the other hand, we have $p(d'') = 1$. Thus by the instability criteria of [18], we conclude that ψ_h is orbital unstable if n is odd. This may complete the proof of Theorem 1.1.

4 Proof of Theorem 1.2-1.4

In this section, we may follow the argument in Section 2 and 3 to prove Theorem 1.2-1.4. Let $v_h(y) := u_h(hy + x_h)$, where u_h is a single-spike bound state of (1.5) with a unique local maximum point at x_h. Then v_h satisfies

$$\Delta v_h - \Big[V(hy + x_h) + \lambda\Big] v_h + m(hy + x_h) v_h^p = 0 \quad \text{in } \mathbb{R}^N . \tag{4.1}$$

Suppose (2.7) hold. By (2.8) and [37], we have

$$m(x_0)\nabla V(x_0) = \frac{N}{2}\left[V(x_0) + \lambda\right]\nabla m(x_0) , \tag{4.2}$$

so x_0 may depend on λ. Note that by (4.2), $\nabla m(x_0) = 0$ if and only if $\nabla V(x_0) = 0$. By direct computation on the function G,

$$\begin{aligned}\partial_{ij}G(x_0) = m(x_0)^{-\frac{N}{2}-1}\Big[&m(x_0)\partial_{ij}V(x_0) + (1 - \frac{N}{2})\partial_i V(x_0)\partial_j m(x_0) \\ &- \frac{N}{2}\left[V(x_0) + \lambda\right]\partial_{ij} m(x_0)\Big] .\end{aligned}$$

In particular, if $\nabla m(x_0) = 0$, then

$$\nabla^2 G(x_0) = m(x_0)^{-\frac{N}{2}-1}\Big[m(x_0)\nabla^2 V(x_0) - \frac{N}{2}\Big[V(x_0) + \lambda\Big]\nabla^2 m(x_0)\Big] .$$

Using the identity (2.8), one may follow the arguments of Lemma 2.2-2.4 to get the uniqueness of u_h and

$$x_h = x_0 + \mathrm{o}(h) ; \tag{4.3}$$

$$v_h = w_{x_h} + h\phi_1 + h^2\phi_2 + \mathrm{o}(h^2) , \tag{4.4}$$

where ϕ_1 and ϕ_2 satisfy $\nabla\phi_1(0) = \nabla\phi_2(0) = 0$,

$$\Delta\phi_1 - \left[V(x_0) + \lambda\right]\phi_1 + m(x_0)pw_{x_0}^{p-1}\phi_1 - y\cdot\nabla V(x_0)w_{x_0} + y\cdot\nabla m(x_0)w_{x_0}^p = 0 , \tag{4.5}$$

and

$$\begin{aligned}&\Delta\phi_2 - \Big[V(x_h) + \lambda\Big]\phi_2 + m(x_h)pw_{x_h}^{p-1}\phi_2 - y\cdot\nabla V(x_0)\phi_1 - \frac{1}{2}\sum_{i,j=1}^N \partial_{ij}V(x_0)y_iy_jw_{x_h} \\ &+ y\cdot\nabla m(x_0)pw_{x_h}^{p-1}\phi_1 + \frac{1}{2}\sum_{i,j=1}^N \partial_{ij}m(x_0)y_iy_jw_{x_h}^p + \frac{1}{2}m(x_0)p(p-1)w_{x_h}^{p-2}\phi_1^2 = 0 .\end{aligned} \tag{4.6}$$

Here we have used the hypothesis that x_0 is a non-degenerate point of the function G. And the only difference in the proof is that we need to estimate the term

$$\frac{1}{p+1}\nabla m(x_0)\int_{\mathbb{R}^N} v_h^{p+1}dy - \frac{1}{2}\nabla V(x_0)\int_{\mathbb{R}^N} v_h^2 dy ,$$

to estimate which one may use the following Pohozaev identity (cf. [33])

$$\int_{\mathbb{R}^N} \Big[\frac{2}{N+2} m(hy+x_h) + \frac{h}{p+1} y\cdot\nabla m(hy+x_h)\Big] v_h^{p+1} dy$$
$$= \int_{\mathbb{R}^N} \Big[V(hy+x_h) + \lambda + \frac{h}{2} y\cdot\nabla V(hy+x_h)\Big] v_h^2 dy\,.$$

For the small eigenvalue estimates of L_h, one may generalize the idea of Theorem 2.7 to get

Theorem 4.1. *For h small enough, the eigenvalue problem*

$$L_h\varphi_h = \mu_h\varphi_h \tag{4.7}$$

has exactly N eigenvalues $\mu_h^j, j=1,\cdots N$ satisfying

$$\frac{1}{2}\mu_1 \geq \mu_h^1 \geq \mu_h^2 \geq \cdots \geq \mu_h^N \geq \frac{1}{2}\mu_{N+2}\,, \tag{4.8}$$

and

$$\frac{\mu_h^j}{h^2} \to c_0\nu_j\,, \qquad as\ \ h\to 0\,, \qquad for\ \ j=1,\cdots N\,, \tag{4.9}$$

where μ_1 and μ_{N+2} are defined Lemma 2.6, ν_j's are the eigenvalues of the Hessian matrix $\nabla^2 G(x_0)$, and $c_0 = -\frac{m(x_0)^{\frac{N}{2}}}{V(x_0)+\lambda}$ is a negative constant. Furthermore, the corresponding eigenfunctions φ_h^j's satisfy

$$\varphi_h^j = \sum_{i=1}^N \big[a_{ij} + \mathrm{o}(1)\big]\big(\partial_i w_{x_h} + h\psi_i\big) + \mathrm{O}(h^2)\,, \quad j=1,\cdots,N\,, \tag{4.10}$$

where each ψ_i is the solution of

$$\begin{aligned}&\Delta\psi_i - \Big[V(x_h)+\lambda\Big]\psi_i + m(x_h)pw_{x_h}^{p-1}\psi_i\\ &+\Big[-y\cdot\nabla V(x_h) + y\cdot\nabla m(x_h)pw_{x_h}^{p-1} + m(x_h)p(p-1)w_{x_h}^{p-2}\phi_1\Big]\partial_i w_{x_h} = 0\,,\end{aligned} \tag{4.11}$$

and $\boldsymbol{a}_j = (a_{1j},\cdots,a_{Nj})^T$ is the eigenvector corresponding to ν_j, namely,

$$\nabla^2 G(x_0)\boldsymbol{a}_j = \nu_j\boldsymbol{a}_j\,. \tag{4.12}$$

Remark: To prove it, one may follow the arguments in the proof of Theorem 2.7 and use the following identity

$$\int_{\mathbb{R}^N} \partial_k w_{x_h} L_h\big(\partial_j w_{x_h} + h\psi_j\big)dy = -\frac{h^2}{N+2}\int_{\mathbb{R}^N} w^{p+1}dy\,\partial_{jk}G(x_0) + \mathrm{o}(h^2), \tag{4.13}$$

which replaces (2.29) (see Appendix C). The main difference between Theorem 2.7 and 4.1 is the solution ψ_i of (4.11) which comes from

$$\begin{aligned} L_h\partial_i w_{x_h} = h\Big[& -y\cdot\nabla V(x_0)+y\cdot\nabla m(x_0)pw_{x_h}^{p-1} \\ & +m(x_0)p(p-1)w_{x_h}^{p-2}\phi_1\Big]\partial_i w_{x_h}+\mathrm{O}(h^2). \end{aligned} \tag{4.14}$$

Since the potential function V is nonzero, then x_0 may depend on λ and the asymptotic expansion of $d''(\lambda)$ becomes more complicated. Indeed, when $m\equiv 1$ and $\Delta V(x_0)\neq 0$, the result in [25] shows that the effect of potential function V on $d''(\lambda)$ is $\mathrm{O}(h^2)$. On the other hand, when $V\equiv 0$ and condition (1.12) holds, the effect of m on $d''(\lambda)$ is $\mathrm{O}(h^4)$ (see Section 3). Generally, when both m and V are not constant, we may show

(I) The effect of V and m on $d''(\lambda)$ is $\mathrm{O}(1)$ if $\nabla V(x_0)\neq 0$ (see Theorem 1.2);

(II) The effect of V and m on $d''(\lambda)$ is $\mathrm{O}(h^2)$ if $\nabla V(x_0)=0$ and $\Delta V(x_0)\neq 0$ (see Theorem 1.3);

(III)The effect of V and m on $d''(\lambda)$ is $\mathrm{O}(h^4)$ if $\nabla V(x_0)=0\,,\Delta V(x_0)=0$ and some local condition hold (see Theorem 1.4).

Now we divide three cases to prove these results.

Case I: $\nabla V(x_0)\neq 0$.

Let $R_h:=\frac{\partial u_h}{\partial\lambda}(hy+x_h)$. Then (3.5) and (3.20) hold. Hence one may apply the idea of Theorem 3.2 to get

$$R_h=\sum_{i=1}^N c_h^i\big(\partial_i w_{x_h}+h\psi_i\big)+R_0+R_h^\perp\,, \tag{4.15}$$

where as $h\to 0$, $\mathbf{c}_h=(c_h^1,\cdots,c_h^N)$ satisfies

$$\nabla^2 G(x_0)(h\mathbf{c}_h)\to -\frac{N}{2}m(x_0)^{-\frac{N}{2}-1}\nabla m(x_0)\,, \tag{4.16}$$

and

$$R_0=\big[V(x_h)+\lambda\big]^{-1}\Big(\frac{1}{p-1}v_h+\frac{1}{2}y\cdot\nabla v_h\Big)\,,\quad R_h^\perp=\mathrm{O}(h)\,. \tag{4.17}$$

Thus

$$\begin{aligned} \frac{d''(\lambda)}{h^N} &= \int_{\mathbb{R}^N} v_hR_h dy=\int_{\mathbb{R}^N} v_h\Big[\sum_{i=1}^N c_h^i\big(\partial_i w_{x_h}+h\psi_i\big)+R_0+R_h^\perp\Big]dy \\ &= \int_{\mathbb{R}^N} v_h\sum_{i=1}^N c_h^i\big(\partial_i w_{x_h}+h\psi_i\big)dy+\mathrm{O}(h) \qquad \Big(\text{because} \int_{\mathbb{R}^N} v_hR_0dy=0\Big) \\ &= \int_{\mathbb{R}^N} R_h\sum_{i=1}^N c_h^iL_h\big(\partial_i w_{x_h}+h\psi_i\big)dy+\mathrm{O}(h) \qquad \Big(\text{because } L_hR_h=v_h\Big) \\ &= \int_{\mathbb{R}^N} \Big[\sum_{k=1}^N c_h^k\big(\partial_k w_{x_h}+h\psi_k\big)+R_0+R_h^\perp\Big]\sum_{i=1}^N c_h^iL_h\big(\partial_i w_{x_h}+h\psi_i\big)dy+\mathrm{O}(h)\,. \end{aligned}$$

Therefore, by (4.11), (4.14), (4.13), (4.16) and (4.17), we obtain

$$\frac{d''(\lambda)}{h^N} = -\frac{N^2}{4(N+2)} m(x_0)^{-N-2} \int_{\mathbb{R}^N} w^{p+1} dy \nabla m(x_0) \cdot \left[\nabla^2 G(x_0)\right]^{-1} \nabla m(x_0) + \mathrm{O}(h) . \tag{4.18}$$

Consequently, if x_0 is a non-degenerate local minimum point of G, then the Hessian matrix $\nabla^2 G(x_0)$ is positive definite. By Theorem 4.1, we have $n(L_h) = 1$. On the other hand, by (4.18), we have $p(d'') = 0$. Thus we complete the proof of Theorem 1.2 by the orbital instability criteria of [17]-[18].

Case II: $\nabla V(x_0) = 0$ and $\Delta V(x_0) \neq 0$.

Firstly, note that in this case, $\phi_1 \equiv 0$ and $\psi_i \equiv 0$. Then one may apply the idea of Lemma 3.1 and Theorem 3.2 to obtain

$$x_h = x_0 + h^2 \mathbf{x}_1 + \mathrm{O}(h^3) ; \tag{4.19}$$

$$R_h = R_0 + \sum_{j=1}^{N} c_h^j \partial_j w_{x_h} + h^2 R_1 + R_h^{\perp} , \tag{4.20}$$

where $\mathbf{x}_1 \in \mathbb{R}^N$ satisfies

$$\begin{aligned} \nabla^2 G(x_0) \mathbf{x}_1 = & -\frac{N+2}{4N} \left[V(x_0) + \lambda\right]^{-1} m(x_0)^{-\frac{N}{2}} \left(\frac{\int_{\mathbb{R}^N} |y|^2 w^2 dy}{\int_{\mathbb{R}^N} w^{p+1} dy} \right) \nabla(\Delta V)(x_0) \\ & + \frac{1}{4} m(x_0)^{-\frac{N}{2}-1} \left(\frac{\int_{\mathbb{R}^N} |y|^2 w^{p+1} dy}{\int_{\mathbb{R}^N} w^{p+1} dy} \right) \nabla(\Delta m)(x_0) , \end{aligned} \tag{4.21}$$

R_1 satisfies

$$\begin{aligned} & \Delta R_1 - \left[V(x_h) + \lambda\right] R_1 + m(x_h) p w_{x_h}^{p-1} R_1 \\ & + \left[V(x_h) + \lambda\right]^{-1} \Big[\sum_{i,j=1}^{N} \partial_{ij} V(x_0) y_i y_j w_{x_h} - \frac{1}{2} \sum_{i,j=1}^{N} \partial_{ij} m(x_0) y_i y_j w_{x_h}^p \Big] = 0 , \end{aligned} \tag{4.22}$$

$R_h^{\perp} = \mathrm{O}(h^3)$ and $c_h^j = \mathrm{O}(h)$ for $j = 1, \cdots, N$. Moreover, $\mathbf{c}_h := (c_h^1, \cdots, c_h^N)$ satisfies

$$\nabla^2 G(x_0) \big(h^{-1} \mathbf{c}_h\big) = \mathbf{c}_0 + \mathrm{o}(1) , \tag{4.23}$$

where

$$\begin{aligned} \mathbf{c}_0 = & - \left[V(x_0) + \lambda\right]^{-1} m(x_0)^{-\frac{N}{2}} \nabla^2 V(x_0) \mathbf{x}_1 \\ & - \frac{N+2}{2N} \left[V(x_0) + \lambda\right]^{-2} m(x_0)^{-\frac{N}{2}} \left(\frac{\int_{\mathbb{R}^N} |y|^2 w^2 dy}{\int_{\mathbb{R}^N} w^{p+1} dy} \right) \nabla(\Delta V)(x_0) \\ & + \frac{1}{4} \left[V(x_0) + \lambda\right]^{-1} m(x_0)^{-\frac{N}{2}-1} \left(\frac{\int_{\mathbb{R}^N} |y|^2 w^{p+1} dy}{\int_{\mathbb{R}^N} w^{p+1} dy} \right) \nabla(\Delta m)(x_0). \end{aligned} \tag{4.24}$$

Hence

$$
\begin{aligned}
\frac{d''(\lambda)}{h^N} &= \int_{\mathbb{R}^N} v_h R_h dy = \int_{\mathbb{R}^N} v_h \Big[R_0 + \sum_{j=1}^N c_h^j \partial_j w_{x_h} + h^2 R_1 + R_h^{\perp} \Big] dy \\
&= \int_{\mathbb{R}^N} v_h \Big[\sum_{j=1}^N c_h^j \partial_j w_{x_h} + h^2 R_1 + R_h^{\perp} \Big] dy \quad \Big(\text{because} \int_{\mathbb{R}^N} v_h R_0 dy = 0 \Big) \\
&= \int_{\mathbb{R}^N} R_h \Big[\sum_{j=1}^N c_h^j L_h \partial_j w_{x_h} + h^2 L_h R_1 + L_h R_h^{\perp} \Big] dy \quad \Big(\text{because}\, L_h R_h = v_h \Big) \\
&= \int_{\mathbb{R}^N} \Big[R_0 + \sum_{k=1}^N c_h^k \partial_k w_{x_h} + h^2 R_1 + R_h^{\perp} \Big] \Big[\sum_{j=1}^N c_h^j L_h \partial_j w_{x_h} + h^2 L_h R_1 + L_h R_h^{\perp} \Big] dy \,.
\end{aligned}
$$

Therefore, by (4.11), (4.14) and (4.20), we obtain

$$
\begin{aligned}
\frac{d''(\lambda)}{h^N} = & \int_{\mathbb{R}^N} R_0 \big[v_h - L_h R_0 \big] dy + \sum_{j,k=1}^N c_h^j c_h^k \int_{\mathbb{R}^N} \partial_k w_{x_h} L_h \big(\partial_j w_{x_h} \big) dy \\
& + h^4 \int_{\mathbb{R}^N} R_1 \big(L_h R_1 \big) dy + \mathrm{O}(h^5) \,.
\end{aligned} \tag{4.25}
$$

For the integral $\int_{\mathbb{R}^N} R_0 \big[v_h - L_h R_0 \big] dy$, by direct computation, we have

$$
\begin{aligned}
v_h - L_h R_0 = & - \big[V(x_h) + \lambda \big]^{-1} \Big[V(hy + x_h) - V(x_h) + \frac{h}{2} y \cdot \nabla V(hy + x_h) \Big] \\
& + \frac{h}{2} \big[V(x_h) + \lambda \big]^{-1} y \cdot \nabla m(hy + x_h).
\end{aligned} \tag{4.26}
$$

Thus by (4.4), (4.19) and (2.6), we obtain

$$
\int_{\mathbb{R}^N} R_0 \big[v_h - L_h R_0 \big] dy = \frac{h^2}{2N} \big[V(x_0) + \lambda \big]^{-3} m(x_0)^{-\frac{N}{2}} \int_{\mathbb{R}^N} |y|^2 w^2 dy \Delta V(x_0) + \mathrm{O}(h^4) \,. \tag{4.27}
$$

For the sum $\sum_{j,k=1}^N c_h^j c_h^k \int_{\mathbb{R}^N} \partial_k w_{x_h} \big(L_h \partial_j w_{x_h} \big) dy$, by (4.11), (4.14) and $c_h^j = \mathrm{O}(h)$ for $j = 1, \cdots, N$, we have

$$
\sum_{j,k=1}^N c_h^j c_h^k \int_{\mathbb{R}^N} \partial_k w_{x_h} \big(L_h \partial_j w_{x_h} \big) dy = \mathrm{O}(h^4) \,. \tag{4.28}
$$

Combining (4.27), (4.28) and (4.25), we obtain

$$
\frac{d''(\lambda)}{h^N} = \frac{h^2}{2N} \big[V(x_0) + \lambda \big]^{-3} m(x_0)^{-\frac{N}{2}} \int_{\mathbb{R}^N} |y|^2 w^2 dy \Delta V(x_0) + \mathrm{O}(h^4) \,. \tag{4.29}
$$

Consequently, by (4.29), we have $p(d'') = \frac{1}{2}(1 + \frac{\Delta V(x_0)}{|\Delta V(x_0)|})$. On the other hand, by Theorem 4.1, we have $n(L_h) = n + 1$. Thus we complete the proof of Theorem 1.3 by the orbital stability and instability criteria of [17]-[18].

Case III: $\nabla V(x_0) = 0\,, \Delta V(x_0) = 0$.

In this case, we shall use (4.24), (4.21) and (4.25) to compute the $\mathrm{O}(h^4)$ term of $d''(\lambda)/h^N$. For the integral $\int\limits_{\mathbb{R}^N} R_0\Big[v_h - L_h R_0\Big]dy$, by (4.26) and integration by parts, we obtain

$$
\begin{aligned}
&\int\limits_{\mathbb{R}^N} R_0\Big[v_h - L_h R_0\Big]dy\\
=&-\big[V(x_h)+\lambda\big]^{-2}\int\limits_{\mathbb{R}^N}\big(\frac{1}{p-1}v_h+\frac{1}{2}y\cdot\nabla v_h\big)\big[V(hy+x_h)-V(x_h)+\frac{h}{2}y\cdot\nabla V(hy+x_h)\big]v_h dy\\
&+\big[V(x_h)+\lambda\big]^{-2}\int\limits_{\mathbb{R}^N}\big(\frac{1}{p-1}v_h+\frac{1}{2}y\cdot\nabla v_h\big)\frac{h}{2}y\cdot\nabla m(hy+x_h)v_h^p dy\\
=&\frac{1}{8}\big[V(x_h)+\lambda\big]^{-2}\int\limits_{\mathbb{R}^N}\Big[3hy\cdot\nabla V(hy+x_h)+h^2\sum_{i,j=1}^N\partial_{ij}V(hy+x_h)y_iy_j\Big]v_h^2dy\\
&+\frac{N}{8(N+2)}\big[V(x_h)+\lambda\big]^{-2}\int\limits_{\mathbb{R}^N}\Big[hy\cdot\nabla m(hy+x_h)-h^2\sum_{i,j=1}^N\partial_{ij}m(hy+x_h)y_iy_j\Big]v_h^{p+1}dy\,.
\end{aligned}
$$

Hence by (4.19), (4.20) and Taylor's formulas of V and m, we have

$$
\begin{aligned}
&\int\limits_{\mathbb{R}^N} R_0\Big[v_h - L_h R_0\Big]dy\\
=&\frac{1}{8}\big[V(x_h)+\lambda\big]^{-2}\int\limits_{\mathbb{R}^N}\Big[4h^2\sum_{i,j=1}^N\partial_{ij}V(x_0)y_iy_jw_{x_h}^2+8h^4\sum_{i,j=1}^N\partial_{ij}V(x_0)y_iy_jw_{x_h}\phi_2\\
&+4h^4\sum_{i,j,k=1}^N V_{ijk}(x_0)x_{1,i}y_jy_kw_{x_h}^2+h^4\sum_{i,j,k,l=1}^N V_{ijkl}(x_0)y_iy_jy_ky_lw_{x_h}^2\Big]dy\\
&+\frac{N}{8(N+2)}\big[V(x_h)+\lambda\big]^{-2}\int\limits_{\mathbb{R}^N}\Big[-\frac{h^4}{3}\sum_{i,j,k,l=1}^N\partial_{ijkl}m(x_0)y_iy_jy_ky_l\Big]w_{x_h}^{p+1}dy+\mathrm{o}(h^4).
\end{aligned}\tag{4.30}
$$

For the sum $\sum\limits_{j,k=1}^N c_h^jc_h^k\int\limits_{\mathbb{R}^N}\partial_kw_{x_h}\big(L_h\partial_jw_{x_h}\big)dy$, by (4.13) and (4.23), we obtain

$$
\sum_{j,k=1}^N c_h^jc_h^k\int\limits_{\mathbb{R}^N}\partial_kw_{x_h}\big(L_h\partial_jw_{x_h}\big)dy=-\frac{h^4}{N+2}\int\limits_{\mathbb{R}^N}w^{p+1}dy\nabla^2G(x_0)\mathbf{c}_0\cdot\mathbf{c}_0+\mathrm{o}(h^4)\,.\tag{4.31}
$$

For the integral $\int\limits_{\mathbb{R}^N} R_1\big(L_h R_1\big)dy$, by (4.22), $R_1\big(\frac{y}{\sqrt{V(x_h)+\lambda}}\big)$ satisfies

$$\Delta R - R + pw^{p-1}R + \big[V(x_h)+\lambda\big]^{\frac{N}{4}-3} m(x_h)^{-\frac{N}{4}} \sum_{i,j=1}^{N} \partial_{ij}V(x_0)y_iy_jw$$
$$-\frac{1}{2}\big[V(x_h)+\lambda\big]^{\frac{N}{4}-2} m(x_h)^{-\frac{N}{4}-1} \sum_{i,j=1}^{N} \partial_{ij}m(x_0)y_iy_jw^p = 0\,. \tag{4.32}$$

Hence

$$\int\limits_{\mathbb{R}^N} R_1\big(L_hR_1\big)dy = \int\limits_{\mathbb{R}^N} R_1(L_{x_h}R_1)dy + \mathrm{O}(h^2)$$
$$= \big[V(x_h)+\lambda\big]^{-5} m(x_h)^{-\frac{N}{2}} \sum_{i,j,k,l=1}^{N} \partial_{ij}V(x_0)V_{kl}(x_0) \int\limits_{\mathbb{R}^N} y_iy_jwL_0^{-1}(y_ky_lw)dy$$
$$-\big[V(x_h)+\lambda\big]^{-4} m(x_h)^{-\frac{N}{2}-1} \sum_{i,j,k,l=1}^{N} \partial_{ij}V(x_0)m_{kl}(x_0) \int\limits_{\mathbb{R}^N} y_iy_jwL_0^{-1}(y_ky_lw^p)dy \tag{4.33}$$
$$+\frac{1}{4}\big[V(x_h)+\lambda\big]^{-3} m(x_h)^{-\frac{N}{2}-2} \sum_{i,j,k,l=1}^{N} \partial_{ij}m(x_0)m_{kl}(x_0) \int\limits_{\mathbb{R}^N} y_iy_jw^pL_0^{-1}(y_ky_lw^p)dy + \mathrm{O}(h^2).$$

As in Section 3, we have used the following identities:

$$\sum_{i,j=1}^{N} \partial_{ij}V(x_0) \int\limits_{\mathbb{R}^N} y_iy_jw_{x_h}^2 dy = \frac{1}{N}\int\limits_{\mathbb{R}^N} |y|^2 w_{x_h}^2 dy \Delta V(x_0) = 0\,,$$

$$\sum_{i,j=1}^{N} \partial_{ij}V(x_0) \int\limits_{\mathbb{R}^N} y_iy_jw_{x_h}\phi_2 dy$$
$$= \frac{1}{2}\big[V(x_h)+\lambda\big]^{-3} m(x_h)^{-\frac{N}{2}} \sum_{i,j,k,l=1}^{N} \partial_{ij}V(x_0)V_{kl}(x_0) \int\limits_{\mathbb{R}^N} y_iy_jwL_0^{-1}(y_ky_lw)dy$$
$$-\frac{1}{2}\big[V(x_h)+\lambda\big]^{-2} m(x_h)^{-\frac{N}{2}-1} \sum_{i,j,k,l=1}^{N} \partial_{ij}V(x_0)m_{kl}(x_0) \int\limits_{\mathbb{R}^N} y_iy_jwL_0^{-1}(y_ky_lw^p)dy\,,$$

$$\begin{cases} \sum\limits_{i,j,k=1}^{N} V_{ijk}(x_0)x_{1,i} \int\limits_{\mathbb{R}^N} y_jy_kw_{x_h}^2 dy = \frac{1}{N}\int\limits_{\mathbb{R}^N} |y|^2 w_{x_h}^2 dy \nabla(\Delta V)(x_0)\cdot \mathbf{x}_1\,, \\ \sum\limits_{i,j,k,l=1}^{N} V_{ijkl}(x_0) \int\limits_{\mathbb{R}^N} y_iy_jy_ky_lw_{x_h}^2 = \frac{3}{N(N+2)}\int\limits_{\mathbb{R}^N} |y|^4 w_{x_h}^2 dy \Delta^2 V(x_0)\,, \\ \sum\limits_{i,j,k,l=1}^{N} \partial_{ijkl}m(x_0) \int\limits_{\mathbb{R}^N} y_iy_jy_ky_lw_{x_h}^{p+1} = \frac{3}{N(N+2)}\int\limits_{\mathbb{R}^N} |y|^4 w_{x_h}^{p+1} dy \Delta^2 m(x_0)\,, \end{cases}$$

$$\begin{cases} \int\limits_{\mathbb{R}^N} y_N^2 w L_0^{-1}(y_N^2 w)dy = \frac{1}{N^2}\int\limits_{\mathbb{R}^N} r^2 w L_0^{-1}(r^2 w)dy + \frac{2(N-1)}{N^2(N+2)}\int\limits_{\mathbb{R}^N} r^2 w\Phi_1(r)dy\,, \\ \int\limits_{\mathbb{R}^N} y_{N-1}^2 w L_0^{-1}(y_N^2 w)dy = \frac{1}{N^2}\int\limits_{\mathbb{R}^N} r^2 w L_0^{-1}(r^2 w)dy - \frac{2}{N^2(N+2)}\int\limits_{\mathbb{R}^N} r^2 w\Phi_1(r)dy\,, \\ \int\limits_{\mathbb{R}^N} y_{N-1}y_N w L_0^{-1}(y_{N-1}y_N w)dy = \frac{1}{N(N+2)}\int\limits_{\mathbb{R}^N} r^2 w\Phi_1(r)dy\,, \end{cases}$$

$$\begin{cases} \int\limits_{\mathbb{R}^N} y_N^2 w L_0^{-1}(y_N^2 w^p)dy = \frac{1}{N^2}\int\limits_{\mathbb{R}^N} r^2 w L_0^{-1}(r^2 w^p)dy + \frac{2(N-1)}{N^2(N+2)}\int\limits_{\mathbb{R}^N} r^2 w\Phi_0(r)dy\,, \\ \int\limits_{\mathbb{R}^N} y_{N-1}^2 w L_0^{-1}(y_N^2 w^p)dy = \frac{1}{N^2}\int\limits_{\mathbb{R}^N} r^2 w L_0^{-1}(r^2 w^p)dy - \frac{2}{N^2(N+2)}\int\limits_{\mathbb{R}^N} r^2 w\Phi_0(r)dy\,, \\ \int\limits_{\mathbb{R}^N} y_{N-1}y_N w L_0^{-1}(y_{N-1}y_N w^p)dy = \frac{1}{N(N+2)}\int\limits_{\mathbb{R}^N} r^2 w\Phi_0(r)dy\,, \end{cases}$$

where Φ_0, Φ_1 satisfy

$$\begin{cases} \Phi_0'' + \frac{N-1}{r}\Phi_0' - \Phi_0 + pw^{p-1}\Phi_0 - \frac{2N}{r^2}\Phi_0 - r^2 w^p = 0,\ r\in(0,\infty)\,, \\ \Phi_0(0) = \Phi_0'(0) = 0\,, \end{cases}$$

and

$$\begin{cases} \Phi_1'' + \frac{N-1}{r}\Phi_1' - \Phi_0 + pw^{p-1}\Phi_1 - \frac{2N}{r^2}\Phi_1 - r^2 w = 0,\ r\in(0,\infty)\,, \\ \Phi_1(0) = \Phi_1'(0) = 0\,, \end{cases}$$

which can be proved as in Appendix B.

Therefore, combining (4.25), (4.30), (4.31) and (4.33), we obtain

$$\frac{d''(\lambda)}{h^{N+4}} + \mathrm{o}(h^4) = H_2(x_0) + H_3(x_0) + H_4(x_0) \equiv H(x_0)\,, \tag{4.34}$$

where

$$\begin{aligned} H_2(x_0) =& \frac{3}{N(N+2)}\big[V(x_0)+\lambda\big]^{-5} m(x_0)^{-\frac{N}{2}} \int\limits_{\mathbb{R}^N} |y|^2 w\Phi_1(|y|)dy \|\nabla^2 V(x_0)\|_2^2 \\ & - \frac{3}{N(N+2)}\big[V(x_0)+\lambda\big]^{-4} m(x_0)^{-\frac{N}{2}-1} \int\limits_{\mathbb{R}^N} |y|^2 w\Phi_0(|y|)dy \nabla^2 V(x_0)\cdot\nabla^2 m(x_0) \\ & + \frac{1}{4N^2}\big[V(x_0)+\lambda\big]^{-3} m(x_0)^{-\frac{N}{2}-2} \int\limits_{\mathbb{R}^N} |y|^2 w^p L_0^{-1}(|y|^2 w^p)dy |\Delta m(x_0)|^2 \\ & + \frac{1}{2N(N+2)}\big[V(x_0)+\lambda\big]^{-3} m(x_0)^{-\frac{N}{2}-2} \int\limits_{\mathbb{R}^N} |y|^2 w^p\Phi_0(|y|)dy \|\nabla^2 m(x_0)\|_2^2 \\ & - \frac{1}{2N^2(N+2)}\big[V(x_0)+\lambda\big]^{-3} m(x_0)^{-\frac{N}{2}-2} \int\limits_{\mathbb{R}^N} |y|^2 w^p\Phi_0(|y|)dy |\Delta m(x_0)|^2\,, \end{aligned} \tag{4.35}$$

$$
\begin{aligned}
H_3(x_0) =& \frac{1}{2N}\big[V(x_0)+\lambda\big]^{-3} m(x_0)^{-\frac{N}{2}} \int\limits_{\mathbb{R}^N} |y|^2 w^2 dy \nabla(\Delta m)(x_0)\cdot \mathbf{x}_1 \\
& - \frac{1}{N+2}\int\limits_{\mathbb{R}^N} w^{p+1} dy \mathbf{c}_0 \cdot \big[\nabla^2 G(x_0)\big]^{-1}\mathbf{c}_0 \,,
\end{aligned} \tag{4.36}
$$

$$
\begin{aligned}
H_4(x_0) =& \frac{3}{8N(N+2)}\big[V(x_0)+\lambda\big]^{-4} m(x_0)^{-\frac{N}{2}} \int\limits_{\mathbb{R}^N} |y|^4 w^2 dy \Delta^2 V(x_0) \\
& - \frac{1}{8(N+2)^2}\big[V(x_0)+\lambda\big]^{-3} m(x_0)^{-\frac{N}{2}-1} \int\limits_{\mathbb{R}^N} |y|^4 w^{p+1} dy \Delta^2 m(x_0)\,.
\end{aligned} \tag{4.37}
$$

Consequently, $p(d'') = 1$ if $H(x_0) > 0$, where $H(x_0)$ defined in (4.34) involves the i-th derivatives (for $0 \le i \le 4$) of V and m at x_0. On the other hand, by Theorem 4.1, we have $n(L_h) = n+1$. Thus we complete the proof of Theorem 1.4 by the orbital stability and instability criteria of [17]-[18].

5 Appendix A

In this Appendix we will prove (2.29) of Section 2, i.e., we shall prove

$$
\int_{\mathbb{R}^N} (L_h \partial_j w_{x_h})\, \partial_k w_{x_h} dy = \frac{h^2}{p+1}\int_{\mathbb{R}^N} w_{x_h}^{p+1} dy \partial_{jk} m(x_0) + \mathrm{o}(h^2)\,. \tag{5.1}
$$

Proof. Note that by Lemma 2.3 and 2.4, we obtain

$$
\begin{aligned}
L_h \partial_j w_{x_h} =& \Big[m(hy+x_h) - m(x_h)\Big] p w_{x_h}^{p-1} \partial_j w_{x_h} + m(hy+x_h) p (v_h^{p-1} - w_{x_h}^{p-1}) \partial_j w_{x_h} \\
=& \frac{h^2}{2}\sum_{i,l}^{N} \partial_{il} m(x_0) y_i y_l p w_{x_h}^{p-1} \partial_j w_{x_h} + h^2 m(x_h) p(p-1) w_{x_h}^{p-2} \phi_2 \partial_j w_{x_h} + \mathrm{o}(h^2)\,.
\end{aligned}
$$

Hence we may write the integral $\int_{\mathbb{R}^N} (L_h \partial_j w_{x_h})\, \partial_k w_{x_h} dy$ as follows:

$$
\int_{\mathbb{R}^N} (L_h \partial_j w_{x_h})\, \partial_k w_{x_h}\, dy = I_1 + I_2 + \mathrm{o}(h^2), \tag{5.2}
$$

where

$$
I_1 = \frac{h^2}{2}\sum_{i,l=1}^{N} \partial_{il} m(x_0) \int_{\mathbb{R}^N} y_i y_l p w_{x_h}^{p-1} \partial_j w_{x_h} \partial_k w_{x_h} dy\,, \tag{5.3}
$$

$$
I_2 = h^2 \int_{\mathbb{R}^N} m(x_h) p(p-1) w_{x_h}^{p-2} \phi_2 \partial_j w_{x_h} \partial_k w_{x_h} dy\,. \tag{5.4}
$$

Note that from (2.3), we have

$$
\Big[\Delta - \lambda + m(x_h) p w_{x_h}^{p-1}\Big] \partial_{jk} w_{x_h} + m(x_h) p(p-1) w_{x_h}^{p-2} \partial_j w_{x_h} \partial_k w_{x_h} = 0\,. \tag{5.5}
$$

Hence by (2.14), (5.4) and (5.5), we may use integration by parts to get

$$
\begin{aligned}
I_2 =& - h^2 \int_{\mathbb{R}^N} \phi_2 \Big[\Delta - \lambda + m(x_h) p w_{x_h}^{p-1} \Big] \partial_{jk} w_{x_h} dy \\
=& - h^2 \int_{\mathbb{R}^N} \partial_{jk} w_{x_h} \Big[\Delta - \lambda + m(x_h) p w_{x_h}^{p-1} \Big] \phi_2 dy \\
=& \frac{h^2}{2} \sum_{i,l=1}^{N} \partial_{il} m(x_0) \int_{\mathbb{R}^N} y_i y_l w_{x_h}^p \partial_{jk} w_{x_h} dy \\
=& - \frac{h^2}{2} \sum_{i,l=1}^{N} \partial_{il} m(x_0) \int_{\mathbb{R}^N} \frac{\partial (y_i y_l w_{x_h}^p)}{\partial y_j} \partial_k w_{x_h} dy \\
=& - \frac{h^2}{2} \sum_{i,l=1}^{N} \partial_{il} m(x_0) \int_{\mathbb{R}^N} y_i y_l p w_{x_h}^{p-1} \partial_j w_{x_h} \partial_k w_{x_h} dy - h^2 \partial_{jk} m(x_0) \int_{\mathbb{R}^N} y_k w_{x_h}^p \partial_k w_{x_h} dy \\
=& - \frac{h^2}{2} \sum_{i,l=1}^{N} \partial_{il} m(x_0) \int_{\mathbb{R}^N} y_i y_l p w_{x_h}^{p-1} \partial_j w_{x_h} \partial_k w_{x_h} dy + \frac{h^2}{p+1} \partial_{jk} m(x_0) \int_{\mathbb{R}^N} w_{x_h}^{p+1} dy \,. \qquad (5.6)
\end{aligned}
$$

Combining (5.2), (5.3) and (5.6), we obtain (5.1). □

6 Appendix B

In this Appendix, we shall prove (3.27), (3.28) and (3.29) of Section 3, i.e., we will prove

$$
\int_{\mathbb{R}^N} y_N^2 w^p L_0^{-1}(y_N^2 w^p) dy = \frac{1}{N^2} \int_{\mathbb{R}^N} r^2 w^p L_0^{-1}(r^2 w^p) dy + \frac{2(N-1)}{N^2(N+2)} \int_{\mathbb{R}^N} r^2 w^p \Phi_0(r) dy \,, \qquad (6.1)
$$

$$
\int_{\mathbb{R}^N} y_{N-1}^2 w^p L_0^{-1}(y_N^2 w^p) dy = \frac{1}{N^2} \int_{\mathbb{R}^N} r^2 w^p L_0^{-1}(r^2 w^p) dy - \frac{2}{N^2(N+2)} \int_{\mathbb{R}^N} r^2 w^p \Phi_0(r) dy \,, \qquad (6.2)
$$

$$
\int_{\mathbb{R}^N} y_{N-1} y_N w^p L_0^{-1}(y_{N-1} y_N w^p) dy = \frac{1}{N(N+2)} \int_{\mathbb{R}^N} r^2 w^p \Phi_0(r) dy \,, \qquad (6.3)
$$

where $r := |y|$ and Φ_0 satisfies

$$
\begin{cases}
\Phi_0'' + \dfrac{N-1}{r} \Phi_0' - \Phi_0 + p w^{p-1} \Phi_0 - \dfrac{2N}{r^2} \Phi_0 - r^2 w^p = 0, \; r \in (0, \infty), \\
\Phi_0(0) = \Phi_0'(0) = 0.
\end{cases} \qquad (6.4)
$$

Proof. From (6.4), it is easy to check that

$$
L_0 \left[\Phi_0 \frac{y_N^2}{r^2} + \frac{1}{N} L_0^{-1}(r^2 w^p) - \frac{1}{N} \Phi_0 \right] = y_N^2 w^p \,, \text{and}\; L_0 \left[\Phi_0 \frac{y_{N-1} y_N}{r^2} \right] = y_{N-1} y_N w^p. \qquad (6.5)
$$

Then using the polar coordinate, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^N} y_N^2 w^p L_0^{-1}(y_N^2 w^p)dy \\
=&\int_{\mathbb{R}^N} y_N^2 w_{x_0}^p \left[\Phi_0(r)\frac{y_N^2}{r^2} - \frac{1}{N}\Phi_0(r) + \frac{1}{N}L_0^{-1}(r^2 w^p)\right] dy \\
=&\int_{\mathbb{R}^N} r^2\cos^2\theta_{N-1} w^p \left[\Phi_0(r)\frac{r^2\cos^2\theta_{N-1}}{r^2} - \frac{1}{N}\Phi_0(r) + \frac{1}{N}L_0^{-1}(r^2 w^p)\right] dy \\
=&\frac{\int_0^\pi \cos^4\theta_{N-1}\sin^{N-2}\theta_{N-1}d\theta_{N-1}}{\int_0^\pi \sin^{N-2}\theta_{N-1}d\theta_{N-1}}\int_{\mathbb{R}^N} r^2 w^p\Phi_0(r)dy \\
&+\frac{\int_0^\pi \cos^2\theta_{N-1}\sin^{N-2}\theta_{N-1}d\theta_{N-1}}{\int_0^\pi \sin^{N-2}\theta_{N-1}d\theta_{N-1}}\int_{\mathbb{R}^N} r^2 w_{x_0}^p\left[-\frac{1}{N}\Phi_0(r) + \frac{1}{N}L_0^{-1}(r^2 w_0^p)\right] dy \\
=&\frac{3}{N(N+2)}\int_{\mathbb{R}^N} r^2 w^p\Phi_0(r)dy + \frac{1}{N}\int_{\mathbb{R}^N} r^2 w^p\left[-\frac{1}{N}\Phi_0(r) + \frac{1}{N}L_0^{-1}(r^2 w^p)\right] dy \\
=&\frac{1}{N^2}\int_{\mathbb{R}^N} r^2 w^p L_0^{-1}(r^2 w^p)dy + \frac{2(N-1)}{N^2(N+2)}\int_{\mathbb{R}^N} r^2 w^p\Phi_0(r)dy\,.
\end{aligned}
$$

This completes the proof of (6.1). Similarly, one may obtain (6.2) and (6.3), respectively. □

7 Appendix C

In this Appendix we will prove (4.13) of Section 4, i.e., we shall prove

$$\int_{\mathbb{R}^N} \partial_k w_{x_h} L_h\big(\partial_j w_{x_h} + h\psi_j\big)dy = -\frac{h^2}{N+2}\int_{\mathbb{R}^N} w^{p+1}dy\partial_{jk}G(x_0) + \mathrm{o}(h^2). \tag{7.1}$$

Proof. Note that by (4.3), (4.4) and (4.11), we obtain

$$
\begin{aligned}
L_h\partial_j w_{x_h} =& L_{x_h}\partial_j w_{x_h} + \Big[m(hy+x_h)-m(x_h)\Big]pw_{x_h}^{p-1}\partial_j w_{x_h} \\
&+ m(hy+x_h)p(v_h^{p-1}-w_{x_h}^{p-1})\partial_j w_{x_h} - \Big[V(hy+x_h)-V(x_h)\Big]\partial_j w_{x_h} \\
=& h\Big[y\cdot\nabla m(x_h)pw_{x_h}^{p-1} + m(x_h)p(p-1)w_{x_h}^{p-2}\phi_1 - y\cdot\nabla V(x_h)\Big]\partial_j w_{x_h} \\
&+ h^2\Big[\frac{1}{2}\sum_{i,l}^{N}\partial_{il}m(x_h)y_iy_l p w_{x_h}^{p-1} + y\cdot\nabla m(x_h)p(p-1)w_{x_h}^{p-2}\phi_1 + m(x_h)p(p-1)w_{x_h}^{p-2}\phi_2 \\
&+ \frac{1}{2}m(x_h)p(p-1)(p-2)w_{x_h}^{p-3}\phi_1^2 - \frac{1}{2}\sum_{i,l}^{N}\partial_{il}V(x_h)y_iy_l\Big]\partial_j w_{x_h} + \mathrm{o}(h^2)\,,
\end{aligned}
$$

and

$$
\begin{aligned}
L_h\psi_j =& L_{x_h}\psi_j + \Big[m(hy+x_h)-m(x_h)\Big]pw_{x_h}^{p-1}\psi_j \\
&+ m(hy+x_h)p(v_h^{p-1}-w_{x_h}^{p-1})\psi_j - \Big[V(hy+x_h)-V(x_h)\Big]\psi_j \\
=& -\Big[y\cdot\nabla m(x_h)pw_{x_h}^{p-1} + m(x_h)p(p-1)w_{x_h}^{p-2}\phi_1 - y\cdot\nabla V(x_h)\Big]\partial_j w_{x_h} \\
&+ h\Big[y\cdot\nabla m(x_h)pw_{x_h}^{p-1} + m(x_h)p(p-1)w_{x_h}^{p-2}\phi_1 - y\cdot\nabla V(x_h)\Big]\psi_j + \mathrm{O}(h^2).
\end{aligned}
$$

Hence we may write the integral $\int\limits_{\mathbb{R}^N}\partial_k w_{x_h} L_h\big(\partial_j w_{x_h} + h\psi_j\big)dy$ as follows:

$$
\int\limits_{\mathbb{R}^N}\partial_k w_{x_h} L_h\big(\partial_j w_{x_h} + h\psi_j\big)dy = I_0 + I_1 + I_2 + \mathrm{o}(h^2), \tag{7.2}
$$

where

$$
I_0 = h^2\int_{\mathbb{R}^N}\Big[y\cdot\nabla m(x_h)pw_{x_h}^{p-1} + m(x_h)p(p-1)w_{x_h}^{p-2}\phi_1 - y\cdot\nabla V(x_h)\Big]\psi_j\partial_k w_{x_h}dy\,, \tag{7.3}
$$

$$
\begin{aligned}
I_1 =& h^2\int_{\mathbb{R}^N}\Big[\frac{1}{2}\sum_{i,l}^{N}\partial_{il}m(x_h)y_iy_l p w_{x_h}^{p-1} + y\cdot\nabla m(x_h)p(p-1)w_{x_h}^{p-2}\phi_1 \\
&+ \frac{1}{2}m(x_h)p(p-1)(p-2)w_{x_h}^{p-3}\phi_1^2 - \frac{1}{2}\sum_{i,l}^{N}\partial_{il}V(x_h)y_iy_l\Big]\partial_j w_{x_h}\partial_k w_{x_h}dy\,,
\end{aligned} \tag{7.4}
$$

$$
I_2 = h^2\int_{\mathbb{R}^N} m(x_h)p(p-1)w_{x_h}^{p-2}\phi_2\partial_j w_{x_h}\partial_k w_{x_h}dy\,. \tag{7.5}
$$

Note that from (2.3), we have

$$
\Big[\Delta - \big(V(x_h)+\lambda\big) + m(x_h)pw_{x_h}^{p-1}\Big]\partial_{jk}w_{x_h} + m(x_h)p(p-1)w_{x_h}^{p-2}\partial_j w_{x_h}\partial_k w_{x_h} = 0\,. \tag{7.6}
$$

Hence by (4.6), (7.4) and (7.5), we may use integration by parts to get

$$
\begin{aligned}
I_2 =& - h^2 \int_{\mathbb{R}^N} \phi_2 \Big[\Delta - \big(V(x_h)+\lambda\big) + m(x_h) p w_{x_h}^{p-1}\Big] \partial_{jk} w_{x_h} dy \\
=& - h^2 \int_{\mathbb{R}^N} \partial_{jk} w_{x_h} \Big[\Delta - \big(V(x_h)+\lambda\big) + m(x_h) p w_{x_h}^{p-1}\Big] \phi_2 dy \\
=& h^2 \int_{\mathbb{R}^N} \Big[- y\cdot\nabla V(x_h)\phi_1 - \frac{1}{2}\sum_{i,l=1}^N \partial_{il} V(x_h) y_i y_l w_{x_h} + y\cdot\nabla m(x_h) p w_{x_h}^{p-1}\phi_1 \\
& \qquad + \frac{1}{2}\sum_{i,l=1}^N \partial_{il} m(x_h) y_i y_l w_{x_h}^p + \frac{1}{2} m(x_h) p(p-1) w_{x_h}^{p-2}\phi_1^2\Big] \partial_{jk} w_{x_h} dy \\
=& h^2 \int_{\mathbb{R}^N} \Big[\partial_j V(x_h)\phi_1 + y\cdot\nabla V(x_h)\partial_j\phi_1 + \frac{1}{2}\sum_{i,l=1}^N \partial_{il} V(x_h) y_i y_l \partial_j w_{x_h} + \partial_{jk} V(x_h) y_k w_{x_h} \\
& \qquad - \partial_j m(x_h) p w_{x_h}^{p-1}\phi_1 - y\cdot\nabla m(x_h) p(p-1) w_{x_h}^{p-2}\phi_1\partial_j w_{x_h} - y\cdot\nabla m(x_h) p w_{x_h}^{p-1}\partial_j\phi_1 \\
& \qquad - \frac{1}{2}\sum_{i,l=1}^N \partial_{il} m(x_h) y_i y_l p w_{x_h}^{p-1}\partial_j w_{x_h} - \partial_{jk} m(x_h) y_k w_{x_h}^p \\
& \qquad - \frac{1}{2} m(x_h) p(p-1)(p-2) w_{x_h}^{p-3}\phi_1^2 \partial_j w_{x_h} - m(x_h) p(p-1) w_{x_h}^{p-2}\phi_1\partial_j\phi_1\Big] \partial_k w_{x_h} dy \\
=& - I_1 - h^2 \int_{\mathbb{R}^N} \Big[y\cdot\nabla m(x_h) p w_{x_h}^{p-1} + m(x_h) p(p-1) w_{x_h}^{p-2}\phi_1 - y\cdot\nabla V(x_h)\Big] \partial_j\phi_1 \partial_k w_{x_h} dy \\
& + h^2 \int_{\mathbb{R}^N} \Big[\partial_j V(x_h)\phi_1 + \partial_{jk} V(x_h) y_k w_{x_h} - \partial_j m(x_h) p w_{x_h}^{p-1}\phi_1 - \partial_{jk} m(x_h) y_k w_{x_h}^p\Big] \partial_k w_{x_h} dy.
\end{aligned} \tag{7.7}
$$

Note that from (4.5), we have

$$
\begin{aligned}
& \Delta(\partial_j\phi_1) - [V(x_0)+\lambda]\,\partial_j\phi_1 + m(x_0) p w_{x_0}^{p-1}\partial_j\phi_1 + m(x_0) p(p-1) w_{x_0}^{p-2}\phi_1\partial_j w_{x_0} \\
& - y\cdot\nabla V(x_0)\partial_j w_{x_0} - \partial_j V(x_0) w_{x_0} + y\cdot\nabla m(x_0) p w_{x_0}^{p-1}\partial_j w_{x_0} + \partial_j m(x_0) w_{x_0}^p = 0\,,
\end{aligned} \tag{7.8}
$$

and by direct computation,

$$
\begin{cases}
L_{x_0} w_{x_0} = (p-1) m(x_0) w_{x_0}^p\,, \\
L_{x_0}(\frac{1}{p-1} w_{x_0} + \frac{1}{2} y\cdot\nabla w_{x_0}) = [V(x_0)+\lambda]\, w_{x_0}\,.
\end{cases} \tag{7.9}
$$

Thus we may use (7.3)-(7.9) and integration by parts to get

$$
\begin{aligned}
& I_0+I_1+I_2\\
=& h^2\int_{\mathbb{R}^N}\Big[y\cdot\nabla m(x_h)pw_{x_h}^{p-1}+m(x_h)p(p-1)w_{x_h}^{p-2}\phi_1-y\cdot\nabla V(x_h)\Big]\Big(\psi_j-\partial_j\phi_1\Big)\partial_k w_{x_h}dy\\
&+h^2\int_{\mathbb{R}^N}\Big[\partial_jV(x_h)\phi_1+\partial_{jk}V(x_h)y_kw_{x_h}-\partial_jm(x_h)pw_{x_h}^{p-1}\phi_1-\partial_{jk}m(x_h)y_kw_{x_h}^p\Big]\partial_kw_{x_h}dy\\
=&-h^2\int_{\mathbb{R}^N}\Big(\psi_j-\partial_j\phi_1\Big)L_{x_h}\psi_kdy+h^2\int_{\mathbb{R}^N}\Big[\partial_jV(x_h)-\partial_jm(x_h)pw_{x_h}^{p-1}\Big]\phi_1\partial_kw_{x_h}dy\\
&+h^2\int_{\mathbb{R}^N}\Big[\partial_{jk}V(x_h)y_kw_{x_h}-\partial_{jk}m(x_h)y_kw_{x_h}^p\Big]\partial_kw_{x_h}dy\\
=&h^2\int_{\mathbb{R}^N}\Big[\partial_jV(x_0)w_{x_0}-\partial_jm(x_0)w_{x_0}^p\Big]\psi_kdy-h^2\int_{\mathbb{R}^N}\Big[\partial_jV(x_h)w_{x_h}-\partial_jm(x_h)w_{x_h}^p\Big]\partial_k\phi_1dy\\
&-h^2\int_{\mathbb{R}^N}\Big[\frac12\partial_{jk}V(x_h)w_{x_h}^2-\frac{1}{p+1}\partial_{jk}m(x_h)w_{x_h}^{p+1}\Big]dy+\mathrm{o}(h^2)\\
=&h^2\int_{\mathbb{R}^N}\Big[\partial_jV(x_0)\big(V(x_0)+\lambda\big)^{-1}\big(\frac{1}{p-1}w_{x_0}+\frac12y\cdot\nabla w_{x_0}\big)\\
&\qquad-\frac{1}{p-1}m(x_0)^{-1}\partial_jm(x_0)w_{x_0}\Big]L_{x_0}\Big(\psi_k-\partial_k\phi_1\Big)dy\\
&-h^2\int_{\mathbb{R}^N}\Big[\frac12\partial_{jk}V(x_0)w_{x_0}^2-\frac{1}{p+1}\partial_{jk}m(x_0)w_{x_0}^{p+1}\Big]dy+\mathrm{o}(h^2)\\
=&-h^2\int_{\mathbb{R}^N}\Big[\partial_jV(x_0)\big(V(x_0)+\lambda\big)^{-1}\big(\frac{1}{p-1}w_{x_0}+\frac12y\cdot\nabla w_{x_0}\big)\\
&\qquad-\frac{1}{p-1}m(x_0)^{-1}\partial_jm(x_0)w_{x_0}\Big]\Big[\partial_kV(x_0)w_{x_0}-\partial_km(x_0)w_{x_0}^p\Big]dy\\
&-h^2\int_{\mathbb{R}^N}\Big[\frac12\partial_{jk}V(x_0)w_{x_0}^2-\frac{1}{p+1}\partial_{jk}m(x_0)w_{x_0}^{p+1}\Big]dy+\mathrm{o}(h^2)\\
=&-h^2\Big[\big(\frac{1}{p-1}-\frac N4\big)\left[V(x_0)+\lambda\right]^{-1}\partial_jV(x_0)\partial_kV(x_0)\\
&\qquad-\frac{1}{p-1}m(x_0)^{-1}\partial_jm(x_0)\partial_kV(x_0)\Big]\int_{\mathbb{R}^N}w_{x_0}^2dy\\
&+h^2\Big[\big(\frac{1}{p-1}-\frac12\frac{N}{p+1}\big)\left[V(x_0)+\lambda\right]^{-1}\partial_jV(x_0)\partial_km(x_0)\\
&\qquad-\frac{1}{p-1}m(x_0)^{-1}\partial_jm(x_0)\partial_km(x_0)\Big]\int_{\mathbb{R}^N}w_{x_0}^{p+1}dy\\
&-h^2\int_{\mathbb{R}^N}\Big[\frac12\partial_{jk}V(x_0)w_{x_0}^2-\frac{1}{p+1}\partial_{jk}m(x_0)w_{x_0}^{p+1}\Big]dy+\mathrm{o}(h^2)\,.
\end{aligned}\tag{7.10}
$$

Recall that

$$
\begin{cases}
w_{x_0}(y)=\big[V(x_0)+\lambda\big]^{\frac{1}{p-1}}m(x_0)^{-\frac{1}{p-1}}w(\sqrt{V(x_0)+\lambda}y)\,,\\
m(x_0)\nabla V(x_0)=\frac N2\left[V(x_0)+\lambda\right]\nabla m(x_0)\,,\\
\partial_{ij}G(x_0)=m(x_0)^{-\frac N2-1}\Big[m(x_0)\partial_{ij}V(x_0)+(1-\frac N2)\partial_iV(x_0)\partial_jm(x_0)-\frac N2\left[V(x_0)+\lambda\right]\partial_{ij}m(x_0)\Big]\,,
\end{cases}
$$

and the integral identity

$$[V(x_0)+\lambda]\int_{\mathbb{R}^N} w_{x_0}^2 dy = \frac{2}{N+2} m(x_0)\int_{\mathbb{R}^N} w_{x_0}^{p+1} dy\,.$$

Combining (7.2) and (7.10), we obtain (7.1). $\square$

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