LAYERED SOLUTIONS FOR A SEMILINEAR ELLIPTIC SYSTEM IN A BALL

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Abstract. We consider the following system of Schrödinger-Poisson equations in the unit ball B_1 of $\mathbb{R}^3$:

$$-\varepsilon^2\Delta v + v + \omega\phi v - \gamma v^p = 0, \quad -\Delta\phi = 4\pi\omega v^2$$

with the Dirichlet boundary conditions $v = \phi = 0$ on ∂B_1. Here $\varepsilon,\ \omega,\ \gamma > 0,\ v,\ \phi : B_1 \to \mathbb{R}$, $1 < p < \frac{7}{3}$. We exhibit a family of positive radially symmetric solutions $(v_\varepsilon, \phi_\varepsilon)$ such that v_ε concentrates on a sphere in the interior of B_1 as $\varepsilon \to 0^+$. Our approach relies upon a finite dimensional reduction by the Liapunov–Schmidt method which allows us to reduce the problem to maximizing a one dimensional functional. The solution of such reduced problem also provides the radius of the concentration sphere.

1. Introduction

According to the mathematical formalism of Quantum Mechanics the dynamical state of a particle constrained to move in a 3-dimensional region Ω is completely defined, at a given instant, by a definite (in general complex) function $\psi(x,t)$: $|\psi(x,t)|^2dx$ gives the probability of finding the particle in the element dx at the instant t. The function ψ is called the *wave function* associated to the particle. Let us consider the motion of a charged particle with mass m and charge ω in an electromagnetic field derived from a vector potential $\mathbf{A}(x,t)$ and a scalar potential $\phi(x,t)$: a wave equation arising in many quantum phenomena is given by the following nonlinear Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla - \frac{\omega}{c}\mathbf{A}\right)^2\psi + \omega\phi\psi - \gamma|\psi|^{p-1}\psi, \quad x\in\Omega,\ t\in\mathbb{R}. \tag{1.1}$$

In (1.1) i is the imaginary unit while $\hbar$ is the *Planck's constant* and c denotes the velocity of light in vacuo. On the right hand side of equation (1.1) the operator $(\frac{\hbar}{i}\nabla - \frac{\omega}{c}\mathbf{A})^2$ designates the *formal* scalar product of the vector operator $\frac{\hbar}{i}\nabla - \frac{\omega}{c}\mathbf{A}$ by itself, i.e.

$$\left(\frac{\hbar}{i}\nabla - \frac{\omega}{c}\mathbf{A}\right)^2\psi = -\hbar^2\Delta\psi + \left(\frac{\omega^2}{c^2}|\mathbf{A}|^2 - \frac{\omega\hbar}{ic}div\mathbf{A}\right)\psi - \frac{\omega\hbar}{ic}(\mathbf{A}\cdot\nabla\psi).$$

Following the same idea of [7] (later developed in [11], [13], [14]), in this paper we consider the case of a charged particle interacting with its own electromagnetic field. Hence, since we do not assume that the electromagnetic field is assigned, we have to solve a system whose unknowns are the wave function ψ associated to the particle together with the potentials $\mathbf{A}$ and ϕ.

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Let B_1 denote the unit ball in $\mathbb{R}^3$:

$$B_1 = \{x \in \mathbb{R}^3 \mid |x| < 1\}.$$

We shall investigate the case $\Omega = B_1$, $\mathbf{A} = 0$ and we assume that ϕ does not depend on the time and the particle is represented by a standing wave of the type

$$\psi(x,t) = v(x) \exp\Big(- i\frac{E}{\hbar}t\Big)$$

where $v : \Omega \to \mathbb{R}$ and $E \in \mathbb{R}$. Furthermore, looking for functions v and ϕ positive in B_1 and vanishing on the boundary ∂B_1, the Schrödinger-Poisson system takes the form of the following Dirichlet problem:

$$\begin{cases} -\varepsilon^2 \Delta v + v + \omega\phi v - \gamma v^p = 0 \text{ in } B_1, \\ \Delta\phi + 4\pi\omega v^2 = 0 \text{ in } B_1, \\ v,\, \phi > 0 \text{ in } B_1, \;\; v = \phi = 0 \text{ on } \partial B_1 \end{cases} \tag{1.2}$$

where, for sake of simplicity, we have set $\varepsilon = \frac{\hbar}{\sqrt{2m}}$ and $E = 1$.

The system (1.2) with $\gamma = 0$ has been studied in [7] in the case in which the charged particle lies in a bounded space region Ω and in [8] in all $\mathbb{R}^3$ where the action of an external nonzero potential is considered. In both cases, for fixed $\varepsilon > 0$, the authors prove the existence of infinitely many solutions $\{v_k, \phi_k, \omega_k\}$. This paper deals with the *semiclassical limit* of the system (1.2), i.e. it is concerned with the problem of finding nontrivial solutions and studying their asymptotic behaviour when $\varepsilon \to 0^+$; hence such solutions are usually referred to as *semiclassical* ones. The analysis of the Schrödinger-Poisson equations in the limit $\varepsilon \to 0^+$ is not only a challenging mathematical task, but also of some relevance for the understanding of a wide class of quantum phenomena. Indeed, according to the *correspondence principle*, letting ε go to zero in the Schrödinger equation formally describes the transition from Quantum Mechanics to Classical Mechanics; it is therefore interesting the problem of analysing which kind of semiclassical phenomena appear for the system (1.2).

While there is a wide literature concerning semiclassical states for the single nonlinear Schrödinger equation in an assigned potential ϕ (we recall, among many others, [1], [2], [3], [4], [9], [15], [16], [17], [18], [20], [21], [22], [25], [26], [27], [33], [35], [36], [38], [39]), there are few papers dealing with the case of an unknown potential. The first time the semiclassical limit for a Schrödinger-Poisson system has been considered seems to be in [12], [13], [14], [37]. In such papers problem (1.2) is studied and it is proved that the solutions exhibit some kind of notable concentration behaviour: their form consists of very sharp peaks which become highly concentrated when ε is small. More precisely, when $\Omega = \mathbb{R}^N$ $(N \geq 3)$ problem (1.2) is known to have *clustered* solutions, i.e. a combination of several interacting peaks concentrating at the same point as $\varepsilon \to 0^+$ ([14]). In [12] *clusters* are proved to exist for (1.2) in a bounded and smooth domain $\Omega \subset \mathbb{R}^3$ and the location of the asymptotic peaks is identified in terms of the Robin's function of Ω. The main goal of this paper is to complete the study begun in [13] and [37] on radial solutions concentrating on spheres for the Schrödinger-Poisson system. While the system in all $\mathbb{R}^3$ was the theme of [13], [37], here we focus on the Dirichlet problem (1.2); the phenomenon is different since the location of the concentration set in our case is determined by the boundary effect, created by the imposed boundary conditions.

Our main purpose here is to establish, for small values of the parameter ε, the existence of a positive radially symmetric wave v_ε and a potential ϕ_ε satisfying (1.2). Furthermore, in the limit when $\varepsilon \to 0^+$, v_ε concentrates around a sphere $\{|x| = s_0\}$. The radius s_0 is determined by the exponent p and the charge ω. More precisely, denote by w the unique solution for the following ODE:

$$\begin{cases} w'' - w + w^p = 0 \text{ in } \mathbb{R}, \\ w > 0, \ \ w(0) = \max\limits_{r\in\mathbb{R}} w(r), \\ w(r) \to 0 \text{ as } |r| \to +\infty. \end{cases}$$

It is well known that w can be written explicitly as

$$w(r) = \Big(\frac{p+1}{2}\Big)^{\frac{1}{p-1}} \Big(\cosh\Big(\frac{p-1}{2} r\Big)\Big)^{-\frac{2}{p-1}}. \tag{1.3}$$

Then, roughly speaking, the limit profiles of the solutions v_ε are obtained as small perturbations of the function w rescaled in such a way to be *very concentrated* near a suitable neighborhood of s_0. Before providing the exact formulation of our result we introduce the following two real functions which play a crucial role in the concentration phenomenon:

$$\rho(t) = (1+t)^{\frac{p-5}{2(p-1)}} t^2 [(3p-7)t + 4(p-1)], \tag{1.4}$$

$$s(t) = \frac{1}{2} + \frac{1}{2}\Big(1 - \frac{1}{\pi\omega^2 \int_{\mathbb{R}} w^2 dr}(1+t)^{\frac{p-5}{2(p-1)}} t\Big)^{1/2}. \tag{1.5}$$

Then the main theorem of this paper is the following.

Theorem 1.1. *Assume that $1 < p < \frac{7}{3}$. Then there exist $\varepsilon_0 > 0$ and $0 < \omega_0 < \omega_1$ such that for every $\varepsilon \in (0, \varepsilon_0)$ and $\omega \in (\omega_0, \omega_1)$ the system (1.2) has a solutions $(v_\varepsilon, \phi_\varepsilon)$ with $\gamma = \gamma_\varepsilon = \varepsilon^{(p-1)/2}$ and:*

(1) v_ε, $\phi_\varepsilon \in H_0^1(B_1)$; v_ε, ϕ_ε *are radially symmetric;*

(2) $v_\varepsilon(r) = \varepsilon^{-\frac{1}{2}} \alpha_\varepsilon^{\frac{1}{p-1}} w\big(\sqrt{\alpha_\varepsilon} \frac{r - s_\varepsilon}{\varepsilon}\big) + o(1)$ *uniformly in B_1,*

(3) $\phi_\varepsilon(r) = 4\pi\omega \alpha_\varepsilon^{\frac{5-p}{2(p-1)}} (\int_{\mathbb{R}} w^2 d\rho) G(r; s_\varepsilon) + o(1)$ *uniformly in B_1, where G is defined as follows*

$$G(r;\rho) = \frac{\rho}{r}\big(\min\{r, \rho\} - r\rho\big). \tag{1.6}$$

Furthermore, as $\varepsilon \to 0^+$,

$$\alpha_\varepsilon \to 1 + t_0, \quad s_\varepsilon \to s_0 := s(t_0),$$

where t_0 *is the unique critical point of the function* $\frac{\rho(t)}{(1-s(t))^2}$.

Remark 1.2. *The existence and the uniqueness of such* t_0 *will be proved in Section 2 provided that* ω *belongs to a suitable interval* (ω_1, ω_2). *Notice that, by* (1.5), *the radius* s_0 *of the concentration sphere belongs to* $(\frac{1}{2}, 1)$. *We will also prove that* $t_0 > \frac{2(p-1)}{7-3p}$, *by which* $t_0 \to +\infty$ *as* $p \to \frac{7}{3}^-$ *and, consequently,* $s_0 \to 1^-$. *This suggests that the range* $1 < p < \frac{7}{3}$ *is also necessary to get concentration around a circle in the interior of* B_1.

Remark 1.3. *By (2) of Theorem 1.1 it immediately follows that*

$$\int_{\mathbb{R}^3} |v_\varepsilon|^2 dx = 4\pi(1+t_0)^{\frac{5-p}{2(p-1)}} s_0^2 \int_{\mathbb{R}} |w|^2 dr + o(1),$$

i.e. the L^2*-norm of the waves* v_ε *becomes bounded from below and from above. Hence* $|v_\varepsilon|^2$ *converges to a homogeneous Dirac measure distributed on the sphere* $\{|x| = s_0\}$.

Theorem 1.1 is proved by using an approach relied upon a finite dimensional reduction which is related to the procedure introduced in [23] and [24], and also developed in [13]-[14]. This approach is based on a combination of Lyapunov-Schmidt reduction procedure together with a variational method. The object is to discover the solutions around a small neighborhood of a well chosen first approximation. Then first we introduce the one dimensional surface of approximated solution $\{U_{\varepsilon,t} \,|\, t \in \mathcal{O}\}$, being $\mathcal{O}$ a neighborhood of t_0, such that the profile of each $U_{\varepsilon,t}$ is obtained as a suitable truncation and rescalation of $w(\cdot - s(t))$; then we find a solution of (1.2) in the normal direction of the approximated solution surface as fixed point of a suitable map. Next we study the remaining one dimensional equation. After this reduction process, by using the implicit function theorem, we prove that, in a small neighborhood of the first approximation, solving (1.2) is equivalent to solving the one-dimensional maximization problem

$$\max\Big\{\frac{\rho(t)}{(1-s(t))^2} \,\Big|\, t \in \mathcal{O}\Big\}. \tag{1.7}$$

The solution of such reduced problem also provides the location of peak.

We point out that, in contrast to pointwise concentration, only very recently concentration at higher dimensional sets has been proved. Many progresses have been done for the single nonlinear Schrödinger equation for which different phenomena of concentration on manifolds occur (see [2], [3], [5], [6], [10], [28], [29], [30], [31]). However, regarding the existence of higher dimensional spike-layers for coupled elliptic systems, except for [13] and [37], the only other progress made to date, at our knowledge, was the paper [32], in which solutions concentrating on spheres have been obtained for the stationary Gierer-Meinhardt system in $\mathbb{R}^N$.

The paper consists of four more sections. In Section 2 we study the maximization problem (1.7). Section 3 is devoted to the analysis of the variational structure of the system (1.2). In Sections 4 and 5 we construct the approximate solutions and we carry out the Lyapunov-Schmidt procedure that allows us to reduce the problem to the study of the one dimensional one (1.7). Finally the proof of Theorem 1.1 is given in Section 6.

NOTATION

- If u is a radially symmetric function, we will continue to denote by u the real function $r > 0 \mapsto u(r)$ with $|x| = r$.
- If u is a real valued function, then u_+ is its positive part and u_- its negative part.
- We will often use the symbol C for denoting a positive constant independent on ε. The value of C is allowed to vary from line to line (and also in the same formula).
- Given $\Omega \subset \mathbb{R}^N$ $L^p(\Omega)$ is the usual Lebesgue space endowed with the norm

$$\|u\|^p_{L^p(\Omega)} := \int_\Omega |u|^p\,dx \text{ for } 1 \le p < +\infty, \quad \|u\|_\infty = \sup_{x\in\Omega} |u(x)|.$$

  Furthermore $H^1_0(\Omega)$ is the usual Sobolev space endowed with the norm

$$\|u\|^2_{H^1_0(\Omega)} = \int_\Omega \left(|\nabla u|^2 + |u|^2\right) dx.$$

- $o(1)$ denotes a vanishing quantity.
- Given f_ε, g_ε two families of functions, we write $f_\varepsilon = o(\varepsilon^k) g_\varepsilon$ (resp. $f_\varepsilon = O(\epsilon^k) g_\varepsilon$) to mean that $\frac{f_\varepsilon}{g_\varepsilon} \to 0$ (resp. $|f_\varepsilon| \le C\varepsilon^k |g|$) uniformly as $\varepsilon \to 0^+$.

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2. Preliminaries

In this section we collect some preliminary facts that will be used frequently in the following. First consider the function

$$a(t) = \frac{1}{\pi \int_{\mathbb{R}^3} w^2 dr} (1+t)^{\frac{p-5}{2(p-1)}} t, \quad t > t_1 := \frac{2(p-1)}{7-3p}.$$

We begin by solving the problem (1.7) in the following lemma.

Lemma 2.1. *Assume $1 < p < \frac{7}{3}$ and $a(2t_1) < \omega^2 < a(t_1)$. Then the function $t \in (a^{-1}(\omega^2), +\infty) \mapsto \frac{\rho(t)}{(1-s(t))^2}$ has a unique critical point t_0, where $\rho(t)$ and $s(t)$ have been defined in (1.4)-(1.5). Furthermore t_0 is an absolute maximum point.*

Proof. Since $p < \frac{7}{3}$, it is easy to prove that a is strictly decreasing for $t > t_1$ and $a(t) \to 0$ as $t \to +\infty$. Furthermore by hypothesis we have $\frac{a(t)}{\omega^2} < 1$ for $t > a^{-1}(\omega^2)$, hence s is well defined in such interval. First we split $\left(\frac{\rho}{(1-s)^2}\right)'$ as follows:

$$\left(\frac{\rho}{(1-s)^2}\right)'(t) = \frac{\rho'(t)a(t)}{2\omega^2(1-s(t))^3\sqrt{1-\frac{a(t)}{\omega^2}}}(A(t)-B(t)), \tag{2.8}$$

where

$$A(t) := \omega^2 \frac{\sqrt{1-\frac{a(t)}{\omega^2}} - 1 + \frac{a(t)}{\omega^2}}{a(t)}, \quad B(t) = \frac{a'(t)\rho(t)}{a(t)\rho'(t)}. \tag{2.9}$$

Next we compute

$$\rho'(t) = t(t+1)^{-\frac{p+3}{2(p-1)}}\left(\frac{7p-11}{2(p-1)}t + 4\right)[(3p-7)t + 2(p-1)] \tag{2.10}$$

and

$$A'(t) = \frac{\omega^2 a'(t)}{a^2(t)\sqrt{1-\frac{a(t)}{\omega^2}}}\left(\sqrt{1-\frac{a(t)}{\omega^2}} - 1 + \frac{a(t)}{2\omega^2}\right) \geq 0 \quad \forall t > a^{-1}(\omega^2). \tag{2.11}$$

As regards $B(t)$, it can be easily written in the form

$$B(t) = \frac{((3p-7)t + 4(p-1))\left(\frac{3p-7}{2(p-1)}t + 1\right)}{\left(\frac{7p-11}{2(p-1)}t + 4\right)[(3p-7)t + 2(p-1)]} = \frac{(3p-7)t + 4(p-1)}{(7p-11)t + 8(p-1)}, \tag{2.12}$$

by which we deduce

$$B'(t) = -4\frac{(p-1)(p+3)}{[(7p-11)t + 8(p-1)]^2} \leq 0 \quad \forall t \neq t_2 := \frac{8(p-1)}{11-7p}. \tag{2.13}$$

Hence B is strictly increasing in $(a^{-1}(\omega^2), 2t_1)$, by which, using also (2.9) and (2.12), we get

$$(A-B)(a^{-1}(\omega^2)) = -B(a^{-1}(\omega^2)) < -B(2t_1) = 0. \tag{2.14}$$

On the other hand it is immediate that

$$\lim_{t\to+\infty} A(t) = -\lim_{t\to+\infty} \frac{\sqrt{1-\frac{a(t)}{\omega^2}} - 1}{-\frac{a(t)}{\omega^2}} + 1 = \frac{1}{2}$$

and then

$$\lim_{t\to+\infty} A(t) - B(t) = \frac{p+3}{2(7p-11)} \text{ if } p \neq \frac{11}{7}, \qquad \lim_{t\to+\infty} A(t) - B(t) = +\infty \text{ if } p = \frac{11}{7}, \quad . \tag{2.15}$$

We distinguish the two cases: if $\frac{11}{7} \leq p < \frac{7}{3}$, then, combining (2.11) and (2.13), we obtain that $A-B$ is strictly increasing for $t > a^{-1}(\omega^2)$; then, by using (2.14) and (2.15) we deduce that $A-B$ has a unique zero t_0 in $(a^{-1}(\omega^2), +\infty)$. Since by (2.10) $\rho'(t) < 0$ for $t > t_1$, (2.8) implies that t_0 is the unique critical point of $\frac{\rho}{(1-s)^2}$ in $(a^{-1}(\omega^2), +\infty)$; in particular t_0 is an absolute maximum point. Now assume $1 < p < \frac{11}{7}$. Proceeding as in the previous case, by (2.11) and (2.13) $A - B$ is strictly increasing in $(a^{-1}(\omega^2), t_2) \cup (t_2, +\infty)$ and

$$\lim_{t \to t_2^-} A(t) - B(t) = +\infty, \quad \lim_{t \to t_2^+} A(t) - B(t) = -\infty.$$

Hence, using (2.14) and (2.15), $A - B$ has a unique zero t_0 in $(a^{-1}(\omega^2), t_2)$ and has no zeros in $(t_2, +\infty)$. Furthermore by (2.15) $\rho'(t) < 0$ for $t \in (t_1, t_2)$ and $\rho'(t) > 0$ in $(t_2, +\infty)$; consequently, by (2.8), t_0 is the unique critical point in $(a^{-1}(\omega^2), +\infty)$ for $\frac{\rho}{(1-s)^2}$; again t_0 is an absolute maximum. $\square$

For every $t > 0$ set

$$w_t(r) := (1+t)^{\frac{1}{p-1}} w(r\sqrt{1+t}) \tag{2.16}$$

where w has been defined in (1.3). It is immediate to check that w_t is the unique solution of the problem:

$$\begin{cases} w_t'' - (1+t)w_t + w_t^p = 0 \text{ in } \mathbb{R}, \\ w_t > 0, \quad w_t(0) = \max_{r \in \mathbb{R}} w(r), \quad w_t \to 0 \text{ as } |r| \to +\infty. \end{cases} \tag{2.17}$$

We recall the following non-degeneracy condition of the functions w_t's (see [34]).

Proposition 2.2. *For every $t > 0$ consider the linear operator $A_t : H^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by by*

$$A_t[\phi] = \phi'' - (1+t)\phi + p w_t^{p-1} \phi.$$

Then

$$kernel(A_t) = \{c w_t' \,|\, c \in \mathbb{R}\}.$$

We conclude this section by deriving some further properties on the functions w_t's.

Proposition 2.3. *Assume the hypotheses of Lemma 2.1. Then for every $t > a^{-1}(\omega^2)$ the following hold:*

(i) $\int_{\mathbb{R}} (w_t')^2 dr = \frac{p-1}{2(p+1)} \int_{\mathbb{R}} w_t^{p+1} dr$, $\int_{\mathbb{R}} w_t^2 dr = \frac{p+3}{2(1+t)(p+1)} \int_{\mathbb{R}} w_t^{p+1} dr$;

(ii) *the operator $B_t : H^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by*

$$B_t[\phi] := A_t[\phi] - 8\pi\omega^2 s(t)(1 - s(t)) w_t \int_{\mathbb{R}} w_t \phi dr.$$

satisfies: $kernel(B_t) = \{c w_t' \,|\, c \in \mathbb{R}\}$.

Proof. By multiplying the equation in (2.17) by w_t' we get $\big((w_t')^2\big)' = (1+t)(w_t^2)' - \frac{2}{p+1}(w_t^{p+1})'$, which implies $(w_t')^2 = (1+t)w_t^2 - \frac{2}{p+1}w_t^{p+1}$. Combining this with the following identity

$$\int_{\mathbb{R}} (w_t')^2 dr + (1+t)\int_{\mathbb{R}} w_t^2 dr = \int_{\mathbb{R}} w_t^{p+1} dr$$

we deduce (i). To prove (ii), straightforward computations gives

$$A_t[w_t] = (p-1)w_t^p, \qquad A_t[rw_t'] = 2(1+t)w_t - 2w_t^p. \tag{2.18}$$

Assume $B_t[\phi] = 0$. Then, using (2.18),

$$A_t\Big[\phi - 4\pi\omega^2 \frac{s(t)(1-s(t))}{1+t}\Big(\frac{2}{p-1}w_t + rw_t'\Big)\int_{\mathbb{R}} w_t\phi d\rho\Big] = B_t[\phi] = 0.$$

By Lemma 2.2 we have

$$\phi - 4\pi\omega^2 \frac{s(t)(1-s(t))}{1+t}\Big(\frac{2}{p-1}w_t + rw_t'\Big)\int_{\mathbb{R}} w_t\phi d\rho = cw_t' \tag{2.19}$$

for some $c \in \mathbb{R}$. Multiplying (2.19) by w_t and integrating over $\mathbb{R}$, we obtain

$$\Big(1 - 8\pi\omega^2 \frac{s(t)(1-s(t))}{(p-1)(1+t)}\int_{\mathbb{R}} w_t^2 dr - 4\pi\omega^2 \frac{s(t)(1-s(t))}{1+t}\int_{\mathbb{R}} rw_t w_t' dr\Big)\int_{\mathbb{R}} w_t\phi dr = 0.$$

After an integration by parts we deduce $\int_{\mathbb{R}} rw_t w_t' = -\frac{1}{2}\int_{\mathbb{R}} w_t^2$, while by (1.5) we have $\int_{\mathbb{R}^3} w_t^2 = (1+t)^{\frac{5-p}{2(p-1)}}\int_{\mathbb{R}^3} w^2 = \frac{t}{4\pi\omega^2 s(t)(1-s(t))}$, by which

$$0 = \Big(1 - \frac{t}{1+t}\frac{5-p}{2(p-1)}\Big)\int_{\mathbb{R}} w_t\phi dr = 0$$

Since according to Lemma 2.1 $t > \frac{2(p-1)}{7-3p}$, then $\frac{t}{1+t}\frac{5-p}{2(p-1)} \neq 1$, and hence $\int_{\mathbb{R}} w_t\phi = 0$ which implies $A_t[\phi] = B_t[\phi] = 0$; again Lemma 2.2 allows us to conclude. □

3. Variational Setting

For every $\varepsilon > 0$ set

$$\Omega_\varepsilon = \varepsilon^{-1}B_1 = \{x \,|\, |x| < \varepsilon^{-1}\}.$$

Then it is convenient to make a change of variables in the system (1.2) to obtain the version

$$\begin{cases} \Delta u - u - \Gamma u\psi + u^p = 0 \text{ in } \Omega_\varepsilon, \\ -\Delta\psi = \varepsilon u^2 \text{ in } \Omega_\varepsilon, \\ u,\, \psi > 0, \;\; u = \psi = 0 \text{ on } \partial\Omega_\varepsilon, \end{cases} \tag{3.1}$$

where

$$\gamma = \varepsilon^{\frac{p-1}{2}}, \;\; \Gamma = 4\pi\omega^2, \;\; u(x) = \sqrt{\varepsilon}v(\varepsilon x), \;\; \psi(x) = \frac{1}{4\pi\omega}\phi(\varepsilon x). \tag{3.2}$$

In order to obtain solutions to the system (3.1) we analyse its variational structure.

For sake of simplicity, in the remaining part of the paper we set $H^1_\varepsilon := H^1_0(\Omega_\varepsilon)$ and $L^2_\varepsilon := L^2(\Omega_\varepsilon)$ and $H^1_{\varepsilon,r}$ and $L^2_{\varepsilon,r}$ denote the subspaces of the Sobolev spaces H^1_ε and L^2_ε respectively formed by the radially symmetric functions. Note that, for every $g \in L^2_\varepsilon$ we have

$$\varepsilon^2 \|g\|^2_{L^2_\varepsilon} = 4\pi\varepsilon^2 \int_0^{1/\varepsilon} r^2 g^2 dr \leq 4\pi \int_0^{1/\varepsilon} g^2 dr. \tag{3.3}$$

We begin with the following proposition .

Proposition 3.1. *For every $g \in L^2_{\varepsilon,r}$ denote by $T_\varepsilon[g]$ the unique solution in $H^1_{\varepsilon,r}$ of*

$$-\Delta\psi = \varepsilon g. \tag{3.4}$$

Then the following representation formula holds:

$$T_\varepsilon[g](r) = \int_0^{1/\varepsilon} G(\varepsilon r, \varepsilon\rho) g(\rho) d\rho, \tag{3.5}$$

where G is the function defined in (1.6)*. Furthermore*

(i) *$T_\varepsilon[g] \geq 0$ for every $g \in L^2_{\varepsilon,r}$ such that $g \geq 0$;*

(ii) *$\int_{\Omega_\varepsilon} T_\varepsilon[g] g dx \geq 0$ for every $g \in L^2_{\varepsilon,r}$;*

(iii) *$\|T_\varepsilon[g]\|_\infty \leq \frac{\sqrt{\varepsilon}}{\sqrt{4\pi}} |g\|_{L^2_\varepsilon}$ for every $g \in L^2_{\varepsilon,r}$;*

(iv) *the functional $J : u \in H^1_{0,r}(\Omega_\varepsilon) \mapsto \int_{\Omega_\varepsilon} u^2 T_\varepsilon[u^2] dx$ is C^1 and*

$$J'(u)[v] = 4 \int_{\Omega_\varepsilon} uv T_\varepsilon[u^2] dx \quad \forall u, v \in H^1_{\varepsilon,r}.$$

Proof. By Lax-Milgram's Lemma for every $g \in L^2_{\varepsilon,r}$ we get the existence of a unique solution in $H^1_{\varepsilon,r}$ of (3.4). It is well known that the Green's function for the ball Ω_ε is given by

$$\frac{1}{4\pi}\Big(\frac{1}{|y-x|} - \frac{|\varepsilon x|}{|y|\varepsilon x|^2 - x|}\Big)$$

(see, for example, [19, pg. 40]). Then, by using the spherical coordinates in the space we can write

$$\begin{aligned} T_\varepsilon[g](r) &= \frac{\varepsilon}{4\pi} \int_{\Omega_\varepsilon} \left(\frac{1}{\sqrt{|y|^2 + r^2 - 2ry_3}} - \frac{\varepsilon r}{\sqrt{|y|^2|\varepsilon r|^4 + r^2 - 2y_3|\varepsilon r|^2 r}} \right) g(y) dy \\ &= \frac{\varepsilon}{2} \int_0^{1/\varepsilon} \rho^2 g(\rho) d\rho \int_0^\pi \left(\frac{\sin\theta}{\sqrt{\rho^2 + r^2 - 2r\rho\cos\theta}} - \frac{\sin\theta}{\sqrt{\rho^2|\varepsilon r|^2 + \varepsilon^{-2} - 2r\rho\cos\theta}} \right) d\theta, \end{aligned}$$

and a direct integration leads to (3.5). Since $G \geq 0$, then (i) follows. Furthermore

$$\int_{\Omega_\varepsilon} T_\varepsilon[g] g dx = \int_{\Omega_\varepsilon} |\nabla T_\varepsilon[g]|^2 dx \geq 0.$$

For every $g \in L^2_{\varepsilon,r}$, by Hölder's inequality we have

$$|T_\varepsilon[g]| \leq \int_0^{1/\varepsilon} \frac{\rho}{r} \min\{\varepsilon r, \varepsilon\rho\} |g(\rho)| d\rho \leq \int_0^{1/\varepsilon} \varepsilon\rho |g(\rho)| d\rho \leq \frac{\sqrt{\varepsilon}}{\sqrt{4\pi}} \|g\|_{L^2_\varepsilon}.$$

The last part of the proposition follows from a direct computation. $\square$

Associated with (3.1) is the following energy functional $E_\varepsilon \in C^1(H^1_{\varepsilon,r},\mathbb{R})$:

$$E_\varepsilon[u] := \frac{1}{2}\int_{\Omega_\varepsilon}\big(|\nabla u|^2+u^2\big)dx - \frac{1}{p+1}\int_{\Omega_\varepsilon} u_+^{p+1}dx + \frac{\Gamma}{4}\int_{\Omega_\varepsilon} u^2 T_\varepsilon[u^2]dx.$$

By using proposition 3.1 the energy functional can be rewritten as

$$\begin{aligned} E_\varepsilon[u] =& 2\pi\int_0^{1/\varepsilon} r^2\big(|u'|^2+|u|^2\big)dr - \frac{4\pi}{p+1}\int_0^{1/\varepsilon} r^2 u_+^{p+1}dr \\ &+ \Gamma\pi\int_0^{1/\varepsilon}\int_0^{1/\varepsilon} r^2 G(\varepsilon r,\varepsilon\rho)u^2(r)u^2(\rho)drd\rho. \end{aligned}$$

In the remaining part of the paper our aim is to capture solutions of (3.1) as critical points of the functional E_ε in $H^1_{\varepsilon,r}$.

4. The Linearized Equation

In the remaining part of the paper we will always assume that the hypotheses of Lemma 2.1 hold. According to Lemma 2.1, let $\eta_\varepsilon \in C^\infty(0,\frac{1}{\varepsilon})$ be a suitable cut-off function satisfying

$$\eta_\varepsilon = 1 \quad \text{for } r\in\Big[\frac{1}{4\varepsilon},\frac{s(3t_0)}{\varepsilon}\Big],\quad \eta_\varepsilon = 0 \ \text{ for } r\in\Big[0,\frac{1}{8\varepsilon}\Big]\cup\Big[\frac{s(4t_0)}{\varepsilon},\frac{1}{\varepsilon}\Big],$$
$$0\le \eta_\varepsilon\le 1,\quad |\eta_\varepsilon'|,\,|\eta_\varepsilon''|,\,|\eta_\varepsilon'''|\le C\varepsilon,$$

and, setting

$$I := [a^{-1}(\omega^2), 2t_0],$$

for every $t\in I$ define approximate solutions $U_{\varepsilon,t}$ as suitable truncation of the function w centered at the point $\frac{s(t)}{\varepsilon}$:

$$U_{\varepsilon,t}(r) := w_t\Big(r-\frac{s(t)}{\varepsilon}\Big)\eta_\varepsilon(r). \tag{4.6}$$

We point out that, by the definition of w in (1.3) and w_t in (2.16), the w_t's satisfy the following decay properties:

$$|w_t(r)|,\ |w_t'(r)|,\ |w_t''(r)|,\ |w_t'''(r)| \le Ce^{-|r|},\quad t\in I,\ r\in\mathbb{R}. \tag{4.7}$$

Since by construction we have $U_{\varepsilon,t}(r+\frac{s(t)}{\varepsilon}) = w_t(r)$ for $t\in I$ and $|r|\le \min\{\frac{1}{4\varepsilon},\frac{s(3t_0)-s(2t_0)}{\varepsilon}\}$; then, using (4.7),

$$\begin{aligned} &\Big|U_{\varepsilon,t}\Big(r+\frac{s(t)}{\varepsilon}\Big)-w_t(r)\Big|,\ \Big|U_{\varepsilon,t}'\Big(r+\frac{s(t)}{\varepsilon}\Big)-w_t'(r)\Big| \le C\varepsilon e^{-\frac{|r|}{2}}, \\ &\Big|U_{\varepsilon,t}''\Big(r+\frac{s(t)}{\varepsilon}\Big)-w_t''(r)\Big|,\ \Big|U_{\varepsilon,t}'''\Big(r+\frac{s(t)}{\varepsilon}\Big)-w_t'''(r)\Big| \le C\varepsilon e^{-\frac{|r|}{2}} \end{aligned} \tag{4.8}$$

uniformly for $r\in\mathbb{R}$ and $t\in I$.

Next lemma provides an estimate of the error committing when w_t is replaced by $U_{\varepsilon,t}$.

Lemma 4.1. *The following estimates hold:*

$$\varepsilon^2 E_\varepsilon[U_{\varepsilon,t}] = 2\pi s^2(t)\int_{\mathbb{R}}\Big((w_t')^2+\Big(1+\frac{t}{2}\Big)w_t^2\Big)dr - \frac{4\pi s^2(t)}{p+1}\int_{\mathbb{R}} w_t^{p+1}dr + O(\varepsilon) \ \ as\ \varepsilon\to 0^+$$

$$\left|\Gamma T_\varepsilon[U^2_{\varepsilon,t}]\Big(r+\frac{s(t)}{\varepsilon}\Big)-t\right| = O(\varepsilon)(1+r)\ \ as\ \varepsilon\to 0^+ \tag{4.9}$$

uniformly for $t\in I$ and $r\in[0,\frac{1}{\varepsilon}]$.

Proof. By (4.7) and (4.8) we easily get

$$\begin{aligned}\frac{\varepsilon^2}{2}\int_{\Omega_\varepsilon}\big(|\nabla U_{\varepsilon,t}|^2+U^2_{\varepsilon,t}\big)dx - \frac{\varepsilon^2}{p+1}\int_{\Omega_\varepsilon}U^{p+1}_{\varepsilon,t}dx&\\ = 2\pi\int_{\mathbb{R}}(\varepsilon r+s(t))^2\big((w_t')^2+w_t^2\big)dr - \frac{4\pi}{p+1}\int_{\mathbb{R}}(\varepsilon r+s(t))^2 w_t^{p+1}dr+O(\varepsilon)&\\ = 2\pi s^2(t)\int_{\mathbb{R}}\big((w_t')^2+w_t^2\big)dr - \frac{4\pi s^2(t)}{p+1}\int_{\mathbb{R}}w_t^{p+1}dr+O(\varepsilon)&\end{aligned}$$

uniformly for $t\in I$. It remains to estimate the potential term. First it is easy to show that

$$|G(\varepsilon r,\varepsilon\rho)-s(1-s)|\le C\big(|\varepsilon r-s|+|\varepsilon\rho-s|\big)\quad \forall r,\rho\in\Big[0,\frac{1}{\varepsilon}\Big],\ \forall s\in[0,1]. \tag{4.10}$$

On the other hand, by using again (4.7), we have

$$|(\varepsilon\rho-s(t))U^2_{\varepsilon,t}(\rho)|\le C|\varepsilon\rho-s(t)|e^{-|\rho-\frac{s(t)}{\varepsilon}|}\le C\varepsilon e^{-\frac{1}{2}|\rho-\frac{s(t)}{\varepsilon}|}\quad \forall\rho\in\Big[0,\frac{1}{\varepsilon}\Big],\ \forall t\in I, \tag{4.11}$$

by which

$$\begin{aligned}T_\varepsilon[U^2_{\varepsilon,t}](r) &= s(t)(1-s(t))\int_0^{1/\varepsilon}U^2_{\varepsilon,t}d\rho + O(\varepsilon r-s(t))+O(\varepsilon)\\ &= s(t)(1-s(t))\int_{\mathbb{R}}w_t^2 d\rho + O(\varepsilon r-s(t))+O(\varepsilon)\\ &= s(t)(1-s(t))(1+t)^{\frac{5-p}{2(p-1)}}\int_{\mathbb{R}}w^2d\rho+O(\varepsilon r-s(t))+O(\varepsilon) = \frac{t}{\Gamma}+O(\varepsilon r-s(t))+O(\varepsilon)\end{aligned}$$

uniformly for $r\in[0,\frac{1}{\varepsilon}]$ and $t\in I$, where in the last equality we have used (1.5); then (4.9) follows. By multiplying the above identity by $U^2_{\varepsilon,t}$ and using again (4.11),

$$\Gamma U^2_{\varepsilon,t}T_\varepsilon[U^2_{\varepsilon,t}] = tU^2_{\varepsilon,t}+O(\varepsilon)e^{-\frac{1}{2}|r-\frac{s(t)}{\varepsilon}|},$$

by which, after integration, we obtain

$$\begin{aligned}\Gamma\varepsilon^2\int_{\Omega_\varepsilon}T_\varepsilon[U^2_{\varepsilon,t}]U^2_{\varepsilon,t}dx &= 4\pi t\varepsilon^2\int_0^{1/\varepsilon}r^2U^2_{\varepsilon,t}dr+O(\varepsilon^3)\int_0^{1/\varepsilon}r^2e^{-\frac{1}{2}|r-\frac{s(t)}{\varepsilon}|}dr\\ &= 4\pi t\int_{\mathbb{R}}(\varepsilon r+s(t))^2w_t^2dr+O(\varepsilon) = 4\pi t s^2(t)\int_{\mathbb{R}}w_t^2dr+O(\varepsilon)\end{aligned}$$

uniformly for $t\in I$. □

Let us equip H^1_ε and L^2_ε with the following scalar product respectively:

$$(u,v)_\varepsilon=\int_{\Omega_\varepsilon}\big(\nabla u\nabla v+(1+\Gamma T_\varepsilon[U^2_{\varepsilon,t}])uv\big)\,dx=4\pi\int_0^{1/\varepsilon}\rho^2\big(u'v'+(1+\Gamma T_\varepsilon[U^2_{\varepsilon,t}])uv\big)\,dr,$$

$$\langle u,v\rangle_\varepsilon=\int_{\Omega_\varepsilon}uv\,dx=4\pi\int_0^{1/\varepsilon}\rho^2 uv\,dr.$$

For every $t\in I$ we introduce the following functions:

$$Z_{\varepsilon,t}(r)=U'''_{\varepsilon,t}(r)+\frac{2}{r}U''_{\varepsilon,t}(r)-(1+\Gamma T_\varepsilon[U^2_{\varepsilon,t}])U'_{\varepsilon,t}(r),\quad r\in\Big[0,\frac{1}{\varepsilon}\Big].$$

Taking into account that by construction $|\frac{1}{r}|\le 8\varepsilon$ on $supp\,U_{\varepsilon,t}$ by using (4.8) and (4.9) we deduce

$$Z_{\varepsilon,t}\Big(r+\frac{s(t)}{\varepsilon}\Big)=w'''_t(r)-(1+t)w'_t(r)+o\big(e^{-\frac{|r|}{2}}\big)=-pw_t^{p-1}(r)w'_t(r)+o\big(e^{-\frac{|r|}{2}}\big) \tag{4.12}$$

uniformly for $r\in[0,\frac{1}{\varepsilon}]$ and $t\in I$. After integration by parts it is immediate to prove that

$$(\phi,U'_{\varepsilon,t})_\varepsilon=-\langle\phi,Z_{\varepsilon,t}\rangle_\varepsilon\quad\forall\phi\in H^1_{\varepsilon,r}, \tag{4.13}$$

then orthogonality to the functions $U'_{\varepsilon,t}$ in $H^1_{\varepsilon,r}$ is equivalent to orthogonality to $Z_{\varepsilon,t}$ in $L^2_{\varepsilon,r}$.

We study a linear problem: given $h\in C(\overline{\Omega}_\varepsilon)$, h radial, find a function ϕ and a constant $\beta\in\mathbb{R}$ satisfying

$$\begin{cases}L_{\varepsilon,t}[\phi]=h+\beta Z_{\varepsilon,t},\\ \phi\in H^2(\Omega_\varepsilon)\cap H^1_{\varepsilon,r},\quad\langle\phi,Z_{\varepsilon,t}\rangle_\varepsilon=0,\end{cases} \tag{4.14}$$

where

$$L_{\varepsilon,t}[\phi]:=\phi''+\frac{2}{r}\phi'-(1+\Gamma T_\varepsilon[U^2_{\varepsilon,t}])\phi+pU^{p-1}_{\varepsilon,t}\phi-2\Gamma T_\varepsilon[U_{\varepsilon,t}\phi]U_{\varepsilon,t},\quad r\in\Big[0,\frac{1}{\varepsilon}\Big].$$

First we prove the following a priori estimate for (4.14).

Lemma 4.2. *There exists a constant $C>0$ such that, provided that ε is sufficiently small, if $t\in I$ and (ϕ,β,h) satisfy (4.14), the following holds:*

$$\|\phi\|_\infty\le C(\varepsilon\|h\|_{L^2_\varepsilon}+\|h\|_\infty).$$

Proof. We argue by contradiction. Assume the existence of a sequence $\varepsilon_k\to 0$, $(t_k)\subset I$ and $(\tilde\phi_k,\tilde\beta_k)\in\big(H^2(\Omega_{\varepsilon_k})\cap H^1_{\varepsilon_k,r}\big)\times\mathbb{R}$, $\tilde h_k\in C(\overline{\Omega}_{\varepsilon_k})$, $\tilde h_k$ radial, satisfying (4.14) such that

$$\|\tilde\phi_k\|_\infty>k(\varepsilon_k\|\tilde h_k\|_{L^2_{\varepsilon_k}}+\|\tilde h_k\|_\infty).$$

Since by Sobolev's embeddings we have $\tilde\phi_k\in L^\infty(\Omega_{\varepsilon_k})$, it makes sense to set $\phi_k=\frac{\tilde\phi_k}{\|\tilde\phi_k\|_\infty}$, $\beta_k=\frac{\tilde\beta_k}{\|\tilde\phi_k\|_\infty}$, $h_k=\frac{\tilde h_k}{\|\tilde\phi_k\|_\infty}$. We obtain that (ϕ_k,β_k,h_k) satisfies (4.14) and

$$\|\phi_k\|_\infty=1,\quad\varepsilon_k\|h_k\|_{L^2_{\varepsilon_k}}+\|h_k\|_\infty=o(1).$$

For the sake of simplicity, in the remaining part of the proof we replace the subscripts (ε_k, t_k) and ε_k by k. By multiplying the equation in (4.14) by $r^2\phi_k$ and integrating on $(0, \frac{1}{\varepsilon_k})$, using (i) and (ii) of Lemma 3.1 we immediately get

$$\begin{aligned}\varepsilon_k^2 \int_{\Omega_k} (|\nabla\phi_k|^2 + |\phi_k|^2)dx &\leq 4\pi p \varepsilon_k^2 \int_0^{1/\varepsilon_k} r^2 U_k^{p-1}\phi_k^2 dr + 4\pi\varepsilon_k^2 \int_0^{1/\varepsilon_k} r^2|h_k\phi_k|dr \\ &\leq 4\pi p \int_{\mathbb{R}} U_k^{p-1} + \varepsilon_k^2 \|h_k\|_{L_k^2}\|\phi_k\|_{L_k^2} \leq C + o(\varepsilon_k\|\phi_k\|_{L_k^2}).\end{aligned}$$

Then the sequence $\varepsilon_k\|\phi_k\|_{H_k^1}$ is bounded. Now, setting $\bar\phi_k := \phi_k(\cdot + \frac{s(t_k)}{\varepsilon_k})$ and recalling that by (1.5) $s(t) > \frac{1}{2}$, we compute

$$\int_{-1/(4\varepsilon_k)}^{+\infty} \left(|\bar\phi_k'|^2 + |\bar\phi_k|^2\right)dr \leq \int_{1/(4\varepsilon_k)}^{+\infty} \left(|\phi_k'|^2 + |\phi_k|^2\right)dr \leq 16\varepsilon_k^2 \int_0^{+\infty} r^2\left(|\phi_k'|^2 + |\phi_k|^2\right)dr = \frac{\varepsilon_k^2}{4\pi}\|\phi_k\|_{H_k^1}^2,$$

hence the sequence $\|\bar\phi_k\|_{H^1(-\frac{1}{4\varepsilon_k}, +\infty)}$ is bounded too; up to a subsequence, we have $\bar\phi_k \rightharpoonup \bar\phi$ in $H^1_{loc}(\mathbb{R})$ and $\bar\phi_k \to \bar\phi$ uniformly on compact sets for some $\bar\phi \in H^1(\mathbb{R})$. Assume, without loss of generality, $t_k \to t \in I$. By multiplying the equation in (4.14) by U_k' and integrating over $[0, \frac{1}{\varepsilon_k}]$, we get

$$\beta_k \int_0^{1/\varepsilon_k} Z_k U_k' dr = -\int_0^{1/\varepsilon_k} h_k U_k' dr + \int_0^{1/\varepsilon_k} L_k[\phi_k]U_k' dr. \tag{4.15}$$

First examine the left hand side of (4.15): by using (4.8) and (4.12) we deduce

$$\int_0^{1/\varepsilon_k} Z_k U_k' dr \to -\int_{\mathbb{R}} p w_t^{p-1}(w_t')^2 dr. \tag{4.16}$$

The first term on the right hand side of (4.15) can be estimated as follows

$$\int_0^{1/\varepsilon_k} |h_k U_k'|dr \leq \|h_k\|_\infty \int_{\mathbb{R}} |U_k'|dr = o(1). \tag{4.17}$$

The last term in (4.15) equals

$$\begin{aligned}&\int_0^{1/\varepsilon_k} L_k[\phi_k]U_k' dr \\ &= \int_0^{1/\varepsilon_k} \phi_k\Big[U_k''' + \frac{2}{r^2}U_k' - \frac{2}{r}U_k'' - (1 + \Gamma T_k[U_k^2])U_k' + pU_k^{p-1}U_k'\Big]dr - 2\Gamma\int_0^{1/\varepsilon_k} T_k[U_k\phi_k]U_kU_k' dr \\ &= \int_0^{1/\varepsilon_k} \phi_k\Big[Z_k - \frac{4}{r}U_k'' + \frac{2}{r^2}U_k' + pU_k^{p-1}U_k'\Big]dr - 2\Gamma\int_0^{1/\varepsilon_k} T_k[U_k\phi_k]U_kU_k' dr;\end{aligned}$$

since, by construction, $\frac{1}{\rho} \leq 8\varepsilon_k$ on $supp\, U_k$, by (4.8) and (4.12) we obtain

$$\int_0^{1/\varepsilon_k} \phi_k\Big[Z_k - \frac{4}{r}U_k'' + \frac{2}{r}U_k' + pU_k^{p-1}U_k'\Big]dr \to 0. \tag{4.18}$$

On the other hand, since proceeding as in (4.11) $|\varepsilon_k\rho - s_k||U_k| \le C\varepsilon_k e^{-\frac{1}{2}|\rho - \frac{s(t_k)}{\varepsilon_k}|}$, by using (4.10) we deduce

$$\begin{aligned} T_k[U_k\phi_k](r)U_k(r) &= s(t_k)(1-s(t_k))U_k(r)\int_0^{1/\varepsilon} U_k\phi_k d\rho + O(\varepsilon_k r - s(t_k))U_k(r) + O(\varepsilon_k) \\ &= s(t)(1-s(t))U_k(r)\int_{\mathbb{R}} w_t\bar{\phi}d\rho + o(1) \end{aligned} \tag{4.19}$$

uniformly for $r \in (0, \frac{1}{\varepsilon_k})$, and, consequently,

$$\int_0^{1/\varepsilon_k} T_k[U_k\phi_k]U_kU_k^{'}dr \to s(t)(1-s(t))\int_{\mathbb{R}} w_t\bar{\phi}d\rho\int_{\mathbb{R}} w_tw_t'dr = 0.$$

Hence we have proved that

$$\int_0^{1/\varepsilon_k} L_k[\phi_k]U_k'dr = o(1). \tag{4.20}$$

Combining (4.15), (4.16), (4.17) and (4.20) we achieve $|\beta_k| = o(1)$, by which $\|h_k + \beta_k Z_k\|_\infty = o(1)$.

Now observe that, by (4.8), (4.9) and (4.19), $\bar{\phi}$ satisfies

$$\bar{\phi}'' - (1+t)\bar{\phi} + pw_t^{p-1}\bar{\phi} - 2s(t)(1-s(t))\Gamma\Big(\int_{\mathbb{R}} w_t\bar{\phi}dr\Big)w_t = 0 \text{ in } \mathbb{R}, \quad \|\bar{\phi}\|_\infty \le 1.$$

Lemma 2.3 implies $\bar{\phi} = cw_t'$. On the other hand by (4.12) $0 = \varepsilon_k^2\langle\phi_k, Z_k\rangle_{\varepsilon_k} = 4\pi\int_{\mathbb{R}}\bar{\phi}_kZ_k(r + \frac{s(t_k)}{\varepsilon_k})(\varepsilon_k r + s(t_k))^2 \to -4\pi cps^2(t)\int_{\mathbb{R}} w_t^{p-1}(w_t')^2$, which implies $c = 0$ and, consequently, $\bar{\phi} = 0$. In particular we have that $\bar{\phi}_k \to 0$ in any compact interval of $\mathbb{R}$. Taking into account of (4.7), this shows that

$$\|U_k^{p-1}\phi_k\|_\infty \le C \sup_{r \ge -\frac{s(t_k)}{\varepsilon_k}} |e^{-(p-1)|r|}\bar{\phi}_k| = o(1),$$

while, replacing $\bar{\phi}$ with 0 in (4.19),

$$\big\|T_k[U_k\phi_k]U_k\big\|_\infty = o(1)$$

Thus we have arrived at the following situation: ϕ_k satisfies

$$\|\Delta\phi_k - (1 + \Gamma T_k[U_k^2])\phi_k\|_\infty = o(1).$$

By standard regularity results $\phi_k \in C^2(\Omega_k) \cap C(\overline{\Omega}_k)$. Let $\overline{x}_k$ be the maximum point for $|\phi_k|$ in Ω_k. Then we get $|\phi_k|(\overline{x}_k) = 1$ and $\Delta|\phi_k|(\overline{x}_k) \le 0$, by which $|\Delta\phi_k(\overline{x}_k) - (1+\Gamma T_k[U_k^2])\phi_k(\overline{x}_k)| = |\Delta|\phi_k|(\overline{x}_k) - (1+\Gamma T_k[U_k^2])|\phi_k|(\overline{x}_k)| > 1$, and the contradiction follows. □

Now we are in position to provide the existence of a solution for the system (4.14).

Lemma 4.3. *For $\varepsilon > 0$ sufficiently small, for every $t \in I$ and $h \in C(\overline{\Omega}_\varepsilon)$, h radial, there exists a unique pair (ϕ, β) solving (4.14). Moreover*

$$\varepsilon\|\phi\|_{H_\varepsilon^1} + \|\phi\|_\infty \le C(\varepsilon\|h\|_{L_\varepsilon^2} + \|h\|_\infty). \tag{4.21}$$

Proof. The existence follows from Fredholm's alternative. To this aim, let us consider $\mathcal{H}_\varepsilon$ the closed subset of $H^1_{\varepsilon,r}$ defined by

$$\mathcal{H}_\varepsilon = \{u \in H^1_{\varepsilon,r} \mid (u, U'_{\varepsilon,t})_\varepsilon = 0\}.$$

Notice that, by (4.13), ϕ solves the system (4.14) if and only if $\phi \in \mathcal{H}_\varepsilon$ and

$$(\phi,\psi)_\varepsilon - p\langle U^{p-1}_{\varepsilon,t}\phi,\psi\rangle_\varepsilon + 2\Gamma\langle T_\varepsilon[U_{\varepsilon,t}\phi]U_{\varepsilon,t},\psi\rangle_\varepsilon = -\langle h,\psi\rangle_\varepsilon, \ \ \forall\psi \in \mathcal{H}_\varepsilon. \tag{4.22}$$

Indeed, once we know ϕ we can determine the unique β from the equation

$$-p\langle U^{p-1}_{\varepsilon,t}\phi, U'_{\varepsilon,t}\rangle_\varepsilon + 2\Gamma\langle T_\varepsilon[U_{\varepsilon,t}\phi]U_{\varepsilon,t}, U'_{\varepsilon,t}\rangle_\varepsilon = -\langle h, U'_{\varepsilon,t}\rangle_\varepsilon + \beta(U'_{\varepsilon,t}, U'_{\varepsilon,t})_\varepsilon.$$

Thus it remains to solve (4.22). According to Riesz's representation theorem, take $\mathcal{K}_\varepsilon(\phi)$, $\bar{h} \in \mathcal{H}_\varepsilon$ such that

$$(\mathcal{K}_{\varepsilon,t}(\phi),\psi)_\varepsilon = -p\langle U^{p-1}_{\varepsilon,t}\phi,\psi\rangle_\varepsilon + 2\Gamma\langle T_\varepsilon[U_{\varepsilon,t}\phi]U_{\varepsilon,t},\psi\rangle_\varepsilon, \ \ \forall\psi \in \mathcal{H}_\varepsilon,$$

$$(\bar{h},\psi)_\varepsilon = -\langle h,\psi\rangle_\varepsilon \ \ \forall\psi \in \mathcal{H}_\varepsilon.$$

Then problem (4.22) consists in finding $\phi \in \mathcal{H}_\varepsilon$ such that

$$\phi + \mathcal{K}_{\varepsilon,t}(\phi) = \bar{h}. \tag{4.23}$$

It is easy to prove that $\mathcal{K}_{\varepsilon,t}$ is a linear compact operator form $\mathcal{H}_\varepsilon$ to $\mathcal{H}_\varepsilon$.

Using Fredholm's alternatives, (4.23) has a unique solution for each $\bar{h}$, if and only if (4.23) has a unique solution for $\bar{h} = 0$. Let $\phi \in \mathcal{H}_\varepsilon$ be a solution of $\phi + \mathcal{K}_{\varepsilon,t}(\phi) = 0$; then ϕ solves the system (4.14) with $h = 0$ for some $\beta \in \mathbb{R}$. Lemma 4.2 implies $\phi \equiv 0$. Finally, by multiplying the equation in (4.14) by $r^2\phi$ and integrating by parts, using (i) and (ii) of Lemma 3.1 and applying Hölder's inequality we immediately get

$$\begin{aligned}
\varepsilon^2 \int_{\Omega_\varepsilon} \big(|\nabla\phi|^2 + \phi^2\big)dx \leq & 4\pi\varepsilon^2 \int_0^{1/\varepsilon} r^2 U^{p-1}_{\varepsilon,t}\phi^2 dr + 4\pi\varepsilon^2 \int_0^{1/\varepsilon} r^2|h\phi| dr \\
& \leq \varepsilon^2\Big(4\pi\int_0^{1/\varepsilon} r^2 U^{2(p-1)}_{\varepsilon,t} dr\Big)^{1/2} \|\phi\|_\infty \|\phi\|_{L^2_\varepsilon} + \varepsilon^2\|h\|_{L^2_\varepsilon}\|\phi\|_{L^2_\varepsilon} \\
& \leq \varepsilon\Big(4\pi\int_0^{1/\varepsilon} U^{2(p-1)}_{\varepsilon,t} dr\Big)^{1/2} \|\phi\|_\infty \|\phi\|_{L^2_\varepsilon} + \varepsilon^2\|h\|_{L^2_\varepsilon}\|\phi\|_{L^2_\varepsilon} \\
& \leq C(\|\phi\|_\infty + \varepsilon\|h\|_{L^2_\varepsilon})\varepsilon\|\phi\|_{H^1_\varepsilon},
\end{aligned}$$

by which

$$\varepsilon\|\phi\|_{H^1_\varepsilon} \leq C(\|\phi\|_\infty + \varepsilon\|h\|_{L^2_\varepsilon})$$

and we conclude by using Lemma 4.2 □

5. Finite Dimensional Reduction

This section is devoted to solve the following nonlinear system with the unknowns (ϕ, β):

$$\begin{cases} S_\varepsilon[U_{\varepsilon,t} + \phi] = \beta Z_{\varepsilon,t} \text{ in } \Omega_\varepsilon, \\ \phi \in H^2(\Omega_\varepsilon) \cap H^1_{\varepsilon,r}, \quad \langle \phi, Z_{\varepsilon,t} \rangle_\varepsilon = 0, \end{cases} \tag{5.24}$$

where

$$S_\varepsilon[\psi] = \psi'' + \frac{2}{r}\psi' - \psi + \psi_+^p - \Gamma \psi T_\varepsilon[\psi^2].$$

Lemma 5.1. *Fix $\sigma \in (\frac{1}{2}, 1)$. Provided that $\varepsilon > 0$ is sufficiently small, for every $s \in I$ there is a unique pair $(\phi_{\varepsilon,t}, \beta_{\varepsilon,t}) \in \left(H^2(\Omega_\varepsilon) \cap H^1_{\varepsilon,r}\right) \times \mathbb{R}$ satisfying (5.24) and*

$$\|\phi_{\varepsilon,t}\|_\infty \le \varepsilon^\sigma, \quad \|\phi_{\varepsilon,t}\|_{H^1_\varepsilon} \le \varepsilon^{\sigma-1}. \tag{5.25}$$

Proof. We write the equation in (5.24) in the following form:

$$L_{\varepsilon,t}[\phi] = -S_\varepsilon[U_{\varepsilon,t}] + N_{\varepsilon,t}[\phi] + \beta Z_{\varepsilon,t} \tag{5.26}$$

and use contraction mapping theorem. Here

$$N_{\varepsilon,t}[\phi] = -(U_{\varepsilon,t} + \phi)_+^p + U_{\varepsilon,t}^p + pU_{\varepsilon,t}^{p-1}\phi + \Gamma\left[(U_{\varepsilon,t} + \phi)T_\varepsilon[\phi^2] + 2\phi T_\varepsilon[U_{\varepsilon,t}\phi]\right].$$

Consider the matric space $\mathcal{B}_\varepsilon = \{\phi \in C(\overline{\Omega}_\varepsilon) \,|\, \|\phi\|_\infty \le \varepsilon^\sigma, \|\phi\|_{L^2_\varepsilon} \le \varepsilon^{\sigma-1}\}$ endowed with the norm $\|\cdot\|_{\varepsilon,*} = \|\cdot\|_\infty + \varepsilon\|\cdot\|_{L^2_\varepsilon}$. Taking into account that $\frac{1}{r} \le 8\varepsilon$ on *supp* $U_{\varepsilon,t}$, then, by (4.8) and (4.9) we immediately obtain

$$S_\varepsilon[U_{\varepsilon,t}] = U''_{\varepsilon,t} - (1+t)U_{\varepsilon,t} + U^p_{\varepsilon,t} + O(\varepsilon)e^{-\frac{1}{2}|r - \frac{s(t)}{\varepsilon}|} = O(\varepsilon)e^{-\frac{1}{2}|r - \frac{s(t)}{\varepsilon}|}$$

uniformly for $t \in I$ and $r \in [0, \frac{1}{\varepsilon}]$. By using (3.3) we deduce

$$\|S_\varepsilon[U_{\varepsilon,t}]\|_{\varepsilon,*} \le C\varepsilon \quad \forall t \in I.$$

Now we have to estimate the term $N_{\varepsilon,t}[\phi]$. Taken ϕ_1, $\phi_2 \in \mathcal{B}_\varepsilon$, we compute

$$\begin{aligned} &\| - (U_{\varepsilon,t} + \phi_1)_+^p + pU_{\varepsilon,t}^{p-1}\phi_1 + (U_{\varepsilon,t} + \phi_2)_+^p - pU_{\varepsilon,t}^{p-1}\phi_2\|_{\varepsilon,*} \\ &\qquad \le \|\phi_1 - \phi_2\|_{\varepsilon,*} \sup_{\xi \in \mathcal{B}_\varepsilon} \|p(U_{\varepsilon,t} + \xi)_+^{p-1} - pU_{\varepsilon,t}^{p-1}\|_\infty \le C\varepsilon^{\sigma(p-1)}\|\phi_1 - \phi_2\|_{\varepsilon,*}. \end{aligned}$$

(4.7) and (3.3) lead to $\|U_{\varepsilon,t}\|_{\varepsilon,*} \le C$; then, by (iii) of Proposition 3.1 we get

$$\begin{aligned} &\|(U_{\varepsilon,t} + \phi_1)T_\varepsilon[\phi_1^2] - (U_{\varepsilon,t} + \phi_2)T_\varepsilon[\phi_2^2]\|_{\varepsilon,*} \\ &\qquad \le \|U_{\varepsilon,t} + \phi_1\|_{\varepsilon,*}\|T_\varepsilon[\phi_1^2 - \phi_2^2]\|_\infty + \|T_\varepsilon[\phi_2^2]\|_\infty\|\phi_1 - \phi_2\|_{\varepsilon,*} \\ &\qquad \le C\sqrt{\varepsilon}\|\phi_1^2 - \phi_2^2\|_{L^2_\varepsilon} + C\sqrt{\varepsilon}\|\phi_2^2\|_{L^2_\varepsilon}\|\phi_1 - \phi_2\|_{\varepsilon,*} \\ &\qquad \le C\varepsilon^{\frac{1}{2}+\sigma}\|\phi_1 - \phi_2\|_{L^2_\varepsilon} + \varepsilon^{\frac{1}{2}+\sigma}\|\phi_2\|_{L^2_\varepsilon}\|\phi_1 - \phi_2\|_{\varepsilon,*} \le C\varepsilon^{\sigma - \frac{1}{2}}\|\phi_1 - \phi_2\|_{\varepsilon,*}. \end{aligned}$$

In a similar way

$$\|\phi_1 T_{\varepsilon,t}[U_{\varepsilon,t}\phi_1] - \phi_2 T_{\varepsilon,t}[U_{\varepsilon,t}\phi_2]\|_{\varepsilon,*} \leq C\varepsilon^{\sigma-\frac{1}{2}}\|\phi_1 - \phi_2\|_{\varepsilon,*},$$

by which

$$\|N_{\varepsilon,t}(\phi_1) - N_{\varepsilon,t}(\phi_2)\|_{\varepsilon,*} \leq C(\varepsilon^{\sigma-\frac{1}{2}} + \varepsilon^{\sigma(p-1)})\|\phi_1 - \phi_2\|_{\varepsilon,*} \ \ \forall \phi_1,\, \phi_2 \in \mathcal{B}_\varepsilon, \ \ \forall t \in I. \tag{5.27}$$

According to Lemma 4.3, for every $\phi \in \mathcal{B}_\varepsilon$ we define $\mathcal{A}_{\varepsilon,t}[\phi] \in H^2(\Omega_\varepsilon) \cap H^1_{\varepsilon,r}$ to be the unique solution to the system (4.14) with $h = -S_\varepsilon[U_{\varepsilon,t}] + N_{\varepsilon,t}[\phi]$. Then by (4.21), provided that $\varepsilon > 0$ is sufficiently small, we have

$$\varepsilon\|\mathcal{A}_{\varepsilon,t}[\phi]\|_{H^1_\varepsilon} + \|\mathcal{A}_{\varepsilon,t}[\phi]\|_\infty \leq C\| - S_\varepsilon[U_{\varepsilon,t}] + N_{\varepsilon,t}[\phi]\|_{\varepsilon,*} \leq C(\varepsilon + \varepsilon^{2\sigma-\frac{1}{2}} + \varepsilon^{\sigma p}) \leq \varepsilon^\sigma$$

and hence $\mathcal{A}_{\varepsilon,t}[\phi] \in \mathcal{B}_\varepsilon$. Moreover, since $\mathcal{A}_{\varepsilon,t}[\phi_1] - \mathcal{A}_{\varepsilon,t}[\phi_2]$ solves the system (4.14) with $h = N_{\varepsilon,t}[\phi_1] - N_{\varepsilon,t}[\phi_2]$, by (5.27) and again (4.21) we also have that

$$\|\mathcal{A}_{\varepsilon,t}[\phi_1] - \mathcal{A}_{\varepsilon,t}[\phi_2]\|_\infty \leq C\|N_{\varepsilon,t}[\phi_1] - N_{\varepsilon,t}[\phi_2]\|_{\varepsilon,*} < \|\phi_1 - \phi_2\|_{\varepsilon,*} \ \ \forall s \in [s_1, s_2], \tag{5.28}$$

for $\varepsilon > 0$ small enough, i.e. the map $\mathcal{A}_{\varepsilon,t}$ is a contraction map from $\mathcal{B}_\varepsilon$ to $\mathcal{B}_\varepsilon$. By the contraction mapping theorem, (5.24) has a unique solution $(\phi_{\varepsilon,t}, \beta_{\varepsilon,t}) \in \mathcal{B}_\varepsilon \times \mathbb{R}$.

Lemma 5.2. *For $\varepsilon > 0$ sufficiently small, the map $t \in I \mapsto \phi_{\varepsilon,t} \in H^1_{\varepsilon,r}$ constructed in Lemma 5.1 is C^1.*

Proof. Consider the following map $A_\varepsilon : I \times H^1_{\varepsilon,r} \times \mathbb{R} \to H^1_{\varepsilon,r} \times \mathbb{R}$ of class C^1:

$$A_\varepsilon(t,\phi,\beta) = \begin{pmatrix} \left(1 + \Gamma T_\varepsilon[U^2_{\varepsilon,t}] - \Delta\right)^{-1}\left(S_\varepsilon[U_{\varepsilon,t} + \phi] - \beta Z_{\varepsilon,t}\right) \\ (\phi, U'_{\varepsilon,t})_\varepsilon \end{pmatrix},$$

where $v = \left(1 + \Gamma T_\varepsilon[U^2_{\varepsilon,t}] - \Delta\right)^{-1}(h)$ is defined as the unique solution in $H^1_{\varepsilon,r}$ of

$$\left(1 + \Gamma T_\varepsilon[U^2_{\varepsilon,t}]\right)v - \Delta v = h. \tag{5.29}$$

It is immediate that (ϕ, β) solves the system (5.24) if and only if $A_\varepsilon(t,\phi,\beta) = 0$. We are going to prove that, provided that ε is sufficiently small, for every $t \in I$ the linear operator

$$\frac{\partial A_\varepsilon(t,\phi,\beta)}{\partial(\phi,\beta)}\Big|_{(t,\phi_{\varepsilon,t},\beta_{\varepsilon,t})} : H^1_{\varepsilon,r} \times \mathbb{R} \to H^1_{\varepsilon,r} \times \mathbb{R}$$

is invertible. But first notice how, assuming this, the thesis easily follows: indeed the uniqueness of the local solution $(\phi_{\varepsilon,t}, \beta_{\varepsilon,t})$ proved in Lemma 5.1 implies that the map $t \in I \mapsto \phi_{\varepsilon,t} \in H^1_{\varepsilon,r}$ actually coincides with the implicit function associated to A_ε, hence the C^1-regularity will follows from the Implicit Function Theorem.

Now we compute

$$\frac{\partial A_\varepsilon(t,\phi,\beta)}{\partial(\phi,\beta)}|_{(t,\phi_{\varepsilon,t},\beta_{\varepsilon,t})}[\hat\phi,\hat\beta]=\begin{pmatrix}\big(1+\Gamma T_\varepsilon[U_{\varepsilon,t}^2]-\Delta\big)^{-1}\big(S'_\varepsilon[U_{\varepsilon,t}+\phi_{\varepsilon,t}](\hat\phi)-\hat\beta Z_{\varepsilon,t}\big)\\ (\hat\phi,U'_{\varepsilon,t})_\varepsilon\end{pmatrix}.$$

Proceeding as in the proof of Lemma 4.3, $(\hat\phi,\hat\beta)$ solves the system

$$\frac{\partial A_\varepsilon(t,\phi,\beta)}{\partial(\phi,\beta)}|_{(s,\phi_{\varepsilon,t},\beta_{\varepsilon,t})}[\hat\phi,\hat\beta]=(\theta,\gamma)$$

if and only if (since by (5.29) $\big(\big(1+\Gamma T_\varepsilon[U_{\varepsilon,t}^2]-\Delta\big)^{-1}(h),\psi\big)_\varepsilon=\langle h,\psi\rangle_\varepsilon$ and since (4.13) holds) $\hat\phi$ satisfies

$$\langle S'_\varepsilon[U_{\varepsilon,t}+\phi_{\varepsilon,t}](\hat\phi),\psi\rangle_\varepsilon=(\theta,\psi)_\varepsilon\quad\forall\psi\in\mathcal{H}_\varepsilon,\tag{5.30}$$

being $\mathcal{H}_\varepsilon$ is the closed subset of $H^1_{\varepsilon,r}$ defined in Lemma 4.3, and

$$(\hat\phi,U'_{\varepsilon,t})_\varepsilon=\gamma.\tag{5.31}$$

Indeed, once we know $\hat\phi$, the related $\hat\beta$ is given by the identity

$$\langle S'_\varepsilon[U_{\varepsilon,t}+\phi_{\varepsilon,t}](\hat\phi),U'_{\varepsilon,t}\rangle_\varepsilon+\hat\beta(U'_{\varepsilon,t},U'_{\varepsilon,t})_\varepsilon=(\theta,U'_{\varepsilon,t})_\varepsilon.$$

Thus it remains to solve (5.30)-(5.31). We can decompose $\hat\phi=\bar\phi+dU'_{\varepsilon,t}$, where $\bar\phi\in\mathcal{H}_\varepsilon$ and $d=\frac{\gamma}{(U'_{\varepsilon,t},U'_{\varepsilon,t})_\varepsilon}U'_{\varepsilon,t}$, so that (5.31) is satisfied. (5.30) may be rewritten as

$$\begin{aligned}(\bar\phi,\psi)_\varepsilon-p\langle(U_{\varepsilon,t}+\phi_{\varepsilon,t})_+^{p-1}\hat\phi,\psi\rangle_\varepsilon+\Gamma\langle T_\varepsilon[\phi_{\varepsilon,t}^2+2U_{\varepsilon,t}\phi_{\varepsilon,t}]\hat\phi,\psi\rangle_\varepsilon&\\ +2\Gamma\langle(U_{\varepsilon,t}+\phi_{\varepsilon,t})T_\varepsilon[(U_{\varepsilon,t}+\phi_{\varepsilon,t})\hat\phi],\psi\rangle_\varepsilon=-(\theta,\psi)_\varepsilon\quad\forall\psi\in\mathcal{H}_\varepsilon.&\end{aligned}$$

According to Riesz's representation theorem, take $\mathcal{W}_{\varepsilon,t}(\bar\phi)$, $\overline{\theta}\in\mathcal{H}_\varepsilon$ such that

$$\begin{aligned}(\mathcal{W}_{\varepsilon,t}(\bar\phi),\psi)_\varepsilon=&-p\langle(U_{\varepsilon,t}+\phi_{\varepsilon,t})_+^{p-1}\bar\phi,\psi\rangle_\varepsilon+\Gamma\langle T_\varepsilon[\phi_{\varepsilon,t}^2+2U_{\varepsilon,t}\phi_{\varepsilon,t}]\bar\phi,\psi\rangle_\varepsilon\\ &+2\Gamma\langle(U_{\varepsilon,t}+\phi_{\varepsilon,t})T_\varepsilon[(U_{\varepsilon,t}+\phi_{\varepsilon,t})\bar\phi],\psi\rangle_\varepsilon,\quad\forall\psi\in\mathcal{H}_\varepsilon,\end{aligned}$$

$$\begin{aligned}(\bar\theta,\psi)_\varepsilon=&-pd\langle(U_{\varepsilon,t}+\phi_{\varepsilon,t})_+^{p-1}U'_{\varepsilon,t},\psi\rangle+\Gamma d\langle T_\varepsilon[\phi_{\varepsilon,t}^2+2U_{\varepsilon,t}\phi_{\varepsilon,t}]U'_{\varepsilon,t},\psi\rangle_\varepsilon\\ &+2\Gamma d\langle(U_{\varepsilon,t}+\phi_{\varepsilon,t})T_\varepsilon[(U_{\varepsilon,t}+\phi_{\varepsilon,t})U'_{\varepsilon,t}],\psi\rangle_\varepsilon,\quad\forall\psi\in\mathcal{H}_\varepsilon,\end{aligned}$$

Then problem (5.30) consists in finding $\bar\phi\in\mathcal{H}_\varepsilon$ such that

$$\bar\phi+\mathcal{W}_{\varepsilon,t}(\bar\phi)=-\theta-\overline{\theta}.\tag{5.32}$$

By (5.25) and (iii) of Lemma 3.1

$$\|T_\varepsilon[\phi_{\varepsilon,t}^2+2U_{\varepsilon,t}\phi_{\varepsilon,t}]\bar\phi\|_{L^2_\varepsilon}\le\|T_\varepsilon[\phi_{\varepsilon,t}^2+2U_{\varepsilon,t}\phi_{\varepsilon,t}]\|_\infty\|\bar\phi\|_{L^2_\varepsilon}\le C\varepsilon^{\sigma-\frac12}\|\bar\phi\|_{L^2_\varepsilon},\tag{5.33}$$

$$\|(U_{\varepsilon,t}+\phi_{\varepsilon,t})T_\varepsilon[(U_{\varepsilon,t}+\phi_{\varepsilon,t})\bar\phi]-U_{\varepsilon,t}T_\varepsilon[U_{\varepsilon,t}\bar\phi]\|_{L^2_\varepsilon}\le C\varepsilon^{\sigma-\frac12}\|\bar\phi\|_{L^2_\varepsilon};\tag{5.34}$$

furthermore

$$\|(U_{\varepsilon,t}+\phi_{\varepsilon,t})_+^{p-1}\bar\phi-U_{\varepsilon,t}^{p-1}\bar\phi\|_{L^2_\varepsilon}\le C\varepsilon^{\sigma(p-1)}\|\bar\phi\|_{L^2_\varepsilon}\tag{5.35}$$

then, combining (5.33)-(5.35) with the definitions of $\mathcal{W}_{\varepsilon,t}$ and $\mathcal{K}_{\varepsilon,t}$ (see Lemma 4.3), when $\varepsilon \to 0^+$ we have

$$\mathcal{W}_{\varepsilon,t} - \mathcal{K}_{\varepsilon,t} \to 0 \text{ uniformly for } t \in I.$$

Since $I + \mathcal{K}_{\varepsilon,t}$ is invertible, then the theory of the linear bounded operator assures the invertibility of $I + \mathcal{W}_{\varepsilon,t}$ for small ε. This concludes the proof of the lemma. □

6. The Reduced Energy Functional: Proof of Theorem 1.1

For $\varepsilon > 0$ sufficiently small consider the following reduced functional

$$M_\varepsilon : t \in I \mapsto \frac{\varepsilon^2}{4\pi} E_\varepsilon[U_{\varepsilon,t} + \phi_{\varepsilon,t}] \in \mathbb{R},$$

where $\phi_{\varepsilon,t}$ has been constructed in Lemma 5.1. First we provide the following estimate.

Lemma 6.1. *For $\varepsilon > 0$ sufficiently small the following holds:*

$$M_\varepsilon(t) = \frac{1}{8(p+1)} \frac{\int_{\mathbb{R}} w^{p+1} dy}{(\int_{\mathbb{R}} w^2 dy)^2} \frac{\rho(t)}{(1-s(t))^2} + o(1) \ \text{ as } \varepsilon \to 0^+$$

uniformly for $t \in I$, where $\rho(t)$ has been defined in 1.4.

Proof. First note that, by (4.7) and (3.3),

$$\varepsilon \|U_{\varepsilon,t}\|_{L^2_\varepsilon} \le C, \ \ \varepsilon \|U'_{\varepsilon,t}\|_{L^2_\varepsilon} \le C. \tag{6.1}$$

Hence, by using (5.25) and (iii) of Lemma 3.1,

$$\varepsilon^2 \int_{\Omega_\varepsilon} \big((U_{\varepsilon,t} + \phi_{\varepsilon,t})^2 T_\varepsilon[(U_{\varepsilon,t} + \phi_{\varepsilon,t})^2] - U^2_{\varepsilon,t} T_\varepsilon[U^2_{\varepsilon,t}]\big) dx = O(\varepsilon^{\sigma - \frac{1}{2}}),$$

by which compute

$$\begin{aligned} 4\pi M_\varepsilon(t) =& \varepsilon^2 E_\varepsilon[U_{\varepsilon,t}] + \frac{\varepsilon^2}{2} \|\phi_{\varepsilon,t}\|^2_{H^1_\varepsilon} + \varepsilon^2 \int_{\Omega_\varepsilon} (\nabla U_{\varepsilon,t} \nabla \phi_{\varepsilon,t} + U_{\varepsilon,t}\phi_{\varepsilon,t}) dx \\ &- \frac{\varepsilon^2}{p+1} \int_{\Omega_\varepsilon} ((U_{\varepsilon,t} + \phi_{\varepsilon,s})^{p+1}_+ - U^{p+1}_{\varepsilon,t}) dx + O(\varepsilon^{\sigma - \frac{1}{2}}). \end{aligned}$$

Since $|(U_{\varepsilon,t} + \phi_{\varepsilon,s})^{p+1}_+ - U^{p+1}_{\varepsilon,t}| \le C(U^p_{\varepsilon,t}|\phi_{\varepsilon,t}| + |\phi_{\varepsilon,t}|^{p+1})$, then, by (5.25) and (6.1) we obtain

$$4\pi M_\varepsilon(t) = \varepsilon^2 E_\varepsilon[U_{\varepsilon,t}] + o(1).$$

Finally combining (i) of Lemma 2.3 and Lemma 4.1 we conclude

$$\begin{aligned} M_\varepsilon(t) &= s^2(t) \frac{(3p-7)t + 4(p-1)}{8(p+1)(1+t)} \int_{\mathbb{R}} w_t^{p+1} dr + o(1) \\ &= s^2(t) \frac{(3p-7)t + 4(p-1)}{8(p+1)} (1+t)^{\frac{5-p}{2(p-1)}} \int_{\mathbb{R}} w^{p+1} dr + o(1) \\ &= \frac{s^2(t)\rho(t)}{8t^2(p+1)} (1+t)^{\frac{5-p}{p-1}} \int_{\mathbb{R}} w^{p+1} dr + o(1) = \frac{1}{8(p+1)} \frac{\int_{\mathbb{R}} w^{p+1} dr}{\left(\Gamma \int_{\mathbb{R}} w^2 dr\right)^2} \frac{\rho(t)}{(1-s(t))^2} + o(1). \end{aligned}$$

□

Corollary 6.2. *For ε sufficiently small, the function $t \in I \mapsto M_\varepsilon(t)$ has a maximum point $t_\varepsilon = t_0 + o(1)$.*

Proof. Assume first $1 < p < \frac{11}{7}$. Then, letting $t_\varepsilon \in I$ be a maximum point for M_ε in I, then combining Lemma 2.1 and Lemma 6.1 it follows immediately that t_ε tends to t_0 as $\varepsilon \to 0$. The same holds for $\frac{11}{7} \le p < \frac{7}{3}$ be taking t_ε the minimum of M_ε in I. □

Proof of Theorem 1.1. Fix $\varepsilon > 0$ sufficiently small such that Lemma 5.1 and Lemma 5.2 hold for $\varepsilon \in (0, \varepsilon_0)$. According to Lemma 5.1 for every $\varepsilon \in (0, \varepsilon_0)$ and $t \in I$ the function $u_{\varepsilon,t} := U_{\varepsilon,t} + \phi_{\varepsilon,t}$ solves the equation

$$\Delta u_{\varepsilon,t}(x) - u_{\varepsilon,t}(x) + (u_{\varepsilon,t})_+^p(x) - \Gamma u_{\varepsilon,t} T_\varepsilon[u_{\varepsilon,t}^2](x) = \beta_{\varepsilon,t} Z_{\varepsilon,t}(|x|) \text{ in } \Omega_\varepsilon. \tag{6.2}$$

Set $u_\varepsilon = u_{\varepsilon,t_\varepsilon}$, where $t_\varepsilon \in I$ is the maximum point of M_ε, according to Corollary 6.2. Then we have

$$\frac{d}{dt} M_\varepsilon(t)\Big|_{t=t_\varepsilon} = 0; \tag{6.3}$$

by using the C^1 regularity of the map $t \in I \mapsto \phi_{\varepsilon,t} \in H^1_{\varepsilon,r}$, (6.3) can be rewritten as

$$\int_{\Omega_\varepsilon} \Big(\nabla u_\varepsilon \nabla \frac{du_{\varepsilon,t}}{dt} + \big(u_\varepsilon - (u_\varepsilon)_+^p + \Gamma T_\varepsilon[u_\varepsilon^2]\big) \frac{du_{\varepsilon,t}}{dt} \Big) dx \Big|_{t=t_\varepsilon} = 0.$$

which is equivalent, by (6.2), to

$$\varepsilon^2 \beta_{\varepsilon,t_\varepsilon} \int_{\Omega_\varepsilon} Z_{\varepsilon,t_\varepsilon} \frac{du_{\varepsilon,t}}{dt} dx \Big|_{t=t_\varepsilon} = 0. \tag{6.4}$$

A direct computation leads to

$$\begin{aligned} \frac{d}{dt} U_{\varepsilon,t}(r) &= \frac{U_{\varepsilon,t}(r)}{(p-1)(1+t)} + \Big(\frac{r - \frac{s(t)}{\varepsilon}}{2(1+t)} - \frac{s'(t)}{\varepsilon} \Big) w_t'\Big(r - \frac{s(t)}{\varepsilon}\Big) \eta_\varepsilon(r) \\ &= -\frac{s'(t)}{\varepsilon} w_t'\Big(r - \frac{s(t)}{\varepsilon}\Big) + O(1) e^{-\frac{1}{2}|r - \frac{s(t)}{\varepsilon}|} \end{aligned} \tag{6.5}$$

uniformly for $r \in \mathbb{R}$ and $t \in I$, where we have used the estimate (4.7). Hence by (4.12) and (6.5) we get

$$\begin{aligned} \varepsilon^2 \int_{\Omega_\varepsilon} Z_{\varepsilon,t_\varepsilon} \frac{d}{dt} U_{\varepsilon,t} dx \Big|_{t=t_\varepsilon} &= p \frac{s'(t_\varepsilon)}{\varepsilon} \int_{\mathbb{R}} (\varepsilon r + s(t_\varepsilon))^2 w_{t_\varepsilon}^{p-1} (w_{t_\varepsilon}')^2 dr + o(\varepsilon^{-1}) \\ &= p \frac{s'(t_0) s^2(t_0)}{\varepsilon} \int_{\mathbb{R}} w_{t_0}^{p-1} (w_{t_0}')^2 dr + o(\varepsilon^{-1}). \end{aligned} \tag{6.6}$$

Similar computations as in (6.5) lead to

$$\frac{d}{dt} Z_{\varepsilon,t}(r) = O(\varepsilon^{-1}) e^{-\frac{1}{2}|r - \frac{s(t)}{\varepsilon}|} \tag{6.7}$$

uniformly for $r \in [0, \frac{1}{\varepsilon}]$ and $t \in I$. Differentiating the equation $\langle Z_{\varepsilon,t}, \phi_{\varepsilon,t}\rangle_\varepsilon = 0$ with respect to t, and using (5.25) and (6.7), we get

$$\varepsilon^2 \int_{\Omega_\varepsilon} Z_{\varepsilon,t} \frac{d\phi_{\varepsilon,t}}{dt} dx = -\varepsilon^2 \int_{\Omega_\varepsilon} \phi_{\varepsilon,t} \frac{dZ_{\varepsilon,t}}{dt} dx = O(\varepsilon^{\sigma-1}) \int_{\mathbb{R}} (\varepsilon r + s(t))^2 e^{-\frac{1}{2}|r|} dr = O(\varepsilon^{\sigma-1}). \tag{6.8}$$

Combining (6.4) with (6.6) and (6.8) we arrive at

$$0 = \beta_{\varepsilon,t_\varepsilon} \Big(p \frac{s'(t_0) s^2(t_0)}{\varepsilon} \int_{\mathbb{R}} w_{t_0}^p (w'_{t_0})^2 dr + o(\varepsilon^{-1}) \Big);$$

since by Lemma 2.1 $t_0 > \frac{2(p-1)}{7-3p}$, then $s'(t_0) \neq 0$; hence the term in the brackets is nonzero, which implies $\beta_{\varepsilon,t_\varepsilon} = 0$. Hence u_ε solves the equation

$$\Delta u_\varepsilon - u_\varepsilon + (u_\varepsilon)_+^p - \Gamma u_\varepsilon T_\varepsilon[u_\varepsilon^2] = 0 \text{ in } \Omega_\varepsilon. \tag{6.9}$$

By multiplying both members of (6.9) by $(u_\varepsilon)_-$ and integrating by parts we get

$$\int_{\Omega_\varepsilon} |\nabla (u_\varepsilon)_-|^2 dx + \int_{\Omega_\varepsilon} |(u_\varepsilon)_-|^2 + \Gamma \int_{\Omega_\varepsilon} (u_\varepsilon)_-^2 T_\varepsilon[u_\varepsilon^2] dx = 0,$$

by which $(u_\varepsilon)_- = 0$. From the strong Maximum Principle it follows that $u_\varepsilon > 0$ in Ω_ε. Hence u_ε and $\psi_\varepsilon = T_\varepsilon[u_\varepsilon^2]$ actually solve the system (3.1). In order to obtain (2) of Theorem 1.1, using (4.8) and (5.25) we obtain

$$\begin{aligned} \frac{1}{\sqrt{\varepsilon}} u_\varepsilon\Big(\frac{r}{\varepsilon}\Big) &= \frac{1}{\sqrt{\varepsilon}} U_{\varepsilon,t_\varepsilon}\Big(\frac{r}{\varepsilon}\Big) + \frac{1}{\sqrt{\varepsilon}} \phi_{\varepsilon,t_\varepsilon}\Big(\frac{r}{\varepsilon}\Big) \\ &= \frac{1}{\sqrt{\varepsilon}} (1+t_\varepsilon)^{\frac{1}{p-1}} w\Big(\sqrt{1+t_\varepsilon}\frac{r - s(t_\varepsilon)}{\varepsilon}\Big) + O(\varepsilon^{\sigma - \frac{1}{2}}). \end{aligned} \tag{6.10}$$

Finally, since G is a Lipschitz function in $[0,1] \times [0,1]$, by (6.10) we deduce

$$\begin{aligned} \psi_\varepsilon\Big(\frac{r}{\varepsilon}\Big) &= \frac{1}{\varepsilon} \int_0^1 G(r,\rho) u_\varepsilon^2\Big(\frac{\rho}{\varepsilon}\Big) d\rho \\ &= \frac{1}{\varepsilon} G(r, s(t_\varepsilon)) \int_0^1 u_\varepsilon^2\Big(\frac{\rho}{\varepsilon}\Big) d\rho + \frac{O(1)}{\varepsilon} \int_0^1 |\rho - s(t_\varepsilon)| u_\varepsilon^2\Big(\frac{\rho}{\varepsilon}\Big) d\rho \\ &= G(r, s(t_\varepsilon))(1+t_\varepsilon)^{\frac{5-p}{2(p-1)}} \int_{\mathbb{R}} w^2 d\rho + O(1). \end{aligned}$$

Taking into account of (3.2), Theorem 1.1 is proved. □

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