SERRIN'S OVERDETERMINED PROBLEM AND CONSTANT MEAN CURVATURE SURFACES

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Abstract. For all $N \geq 9$, we find smooth entire epigraphs in $\mathbb{R}^N$, namely smooth domains of the form $\Omega := \{x \in \mathbb{R}^N \ / \ x_N > F(x_1, \ldots, x_{N-1})\}$, which are not half-spaces and in which a problem of the form $\Delta u + f(u) = 0$ in Ω has a positive, bounded solution with 0 Dirichlet boundary data and constant Neumann boundary data on $\partial\Omega$. This answers negatively for large dimensions a question by Berestycki, Caffarelli and Nirenberg [3]. In 1971, Serrin [25] proved that a bounded domain where such an overdetermined problem is solvable must be a ball, in analogy to a famous result by Alexandrov that states that an embedded compact surface with constant mean curvature (CMC) in Euclidean space must be a sphere. In lower dimensions we succeed in providing examples for domains whose boundary is close to large dilations of a given CMC surface where Serrin's overdetermined problem is solvable.

1. Introduction and statement of the main results

Let Ω be a domain in $\mathbb{R}^N$ with smooth boundary, and ν its inner normal. This paper deals with the *overdetermined* boundary value problem

$$\Delta u + f(u) = 0, \ u > 0 \quad \text{in } \Omega, \quad u \in L^\infty(\Omega), \tag{1.1}$$

$$u = 0, \quad \frac{\partial u}{\partial \nu} = constant \quad \text{on } \partial\Omega \tag{1.2}$$

where f is a sufficiently smooth function. The question we want to analyze in this paper is what type of domains are admissible for this problem to have a solution.

In 1971, Serrin [25] established the following result:

If Ω is bounded, *f is of class C^1 and Problem* (1.1)-(1.2) *has a solution, then Ω must necessarily be a Euclidean ball.*

Serrin's proof was based on the *Alexandrov reflection principle*, introduced in 1956 by Alexandrov [1] to prove the following famous result:

A compact, connected, embedded hypersurface in $\mathbb{R}^N$ whose mean curvature is constant, must necessarily be a Euclidean sphere.

The reflection maximum principle based procedure was used in 1979 by Gidas, Ni and Nirenberg [13] to derive radial symmetry results for positive solutions of semilinear equations. The reflection principle, named after [13] as the *moving plane method*, has become a standard and powerful tool for the analysis of symmetries of solutions of nonlinear elliptic equations.

Serrin had a clever insight into the geometric structure of Problem (1.1)-(1.2) to prove his result as an analog of Alexandrov's. The purpose of this paper is to further explore the parallel between Alexandrov's and Serrin's statements. The underlying question is: how do (noncompact) embedded constant mean curvature (CMC) surfaces relate with (unbounded) domains where Serrin's problem (1.1)-(1.2) is solvable?

A natural class of unbounded domains to be considered are epigraphs, namely domains Ω of the form

$$\Omega = \{x \in \mathbb{R}^N \ / \ x_N > \varphi(x_1, \ldots, x_{N-1})\} \tag{1.3}$$

where $\varphi : \mathbb{R}^{N-1} \to \mathbb{R}$ is a smooth function. In 1997, Berestycki, Caffarelli and Nirenberg [3] proved, under conditions on f that are satisfied for instance by the Fisher-Kolmogorov and Allen-Cahn nonlinearities (1.6) below, the following result: If φ is uniformly Lipschitz and *asymptotically flat* at infinity, and Problem (1.1)-(1.2) is solvable, then φ must be a linear function, in other words Ω must be a *half-space*. This result was improved by Farina and Valdinoci [11], by lifting the asymptotic flatness condition and smoothness on f, under the dimension constraint $N \leq 4$. When the epigraph is coercive they can also consider an arbitrary nonlinearity.

In [3], the following conjecture was made: an unbounded domain Ω with $\mathbb{R}^N \setminus \Omega$ connected, where (1.1)-(1.2) is solvable, must be either

- a half-space, or
- a cylinder $\Omega = B_k \times \mathbb{R}^{N-k}$, where B_k is a k-dimensional Euclidean ball, or
- the complement of a ball or a cylinder.

In the case when Ω is an epigraph (1.3), the conjecture states that if Serrin's problem (1.1)-(1.2) is solvable, then Ω must be a half-space. Our first result, Theorem 1 below, establishes that **this is not the case** if $N \geq 9$.

In all what follows we shall consider a smooth *monostable* nonlinearity f for which (1.1)-(1.2) is indeed solvable in a half-space. We assume that f is a smooth function such that

$$f(0) = 0 = f(1), \quad f(s) > 0 \quad \text{for all} \quad s \in (0,1), \quad f'(1) < 0. \tag{1.4}$$

Under these conditions, there exists a unique positive solution $w(t)$, which is also increasing, to the problem

$$w'' + f(w) = 0 \quad \text{in } (0, \infty), \quad w(0) = 0, \quad w(+\infty) = 1, \tag{1.5}$$

which is implicitly defined by the formula

$$t = \int_0^{w(t)} \frac{ds}{\sqrt{2\int_s^1 f(\tau)d\tau}}.$$

Conditions (1.4) are satisfied by the standard Fisher-Kolmogorov and Allen-Cahn nonlinearities,

$$f(s) = s(1-s), \quad f(s) = s(1-s^2). \tag{1.6}$$

In the latter case, we explicitly have $w(t) = \tanh\left(t/\sqrt{2}\right)$. Let us observe that the function $u(x) = w(x_N)$ solves (1.1)-(1.2) in the half-space $\Omega = \{x \in \mathbb{R}^N \ / \ x_N > 0\}$. Our first main result is the following.

Theorem 1. *Let f satisfy the condition (1.4). If $N \geq 9$, there exists an epigraph domain Ω of the form (1.3), which is not a half-space, such that Problem (1.1)-(1.2) is solvable.*

Let us roughly describe the epigraph of Theorem 1. In 1969, Bombieri, De Giorgi and Giusti [5] found an example of an entire function in $\mathbb{R}^8$ whose graph Γ is a minimal surface in $\mathbb{R}^9$ and it is not a hyperplane (the BDG minimal graph). Let us call Ω_{bdg} its epigraph. Then, for a sufficiently small $\varepsilon > 0$, the epigraph Ω in Theorem 1 lies in a $O(\varepsilon)$-neighborhood of $\varepsilon^{-1}\Omega_{bdg}$. The solution u will be at main order given by $u(x) = w(z) + O(\varepsilon)$ where z designates the normal inner coordinate to $\partial\Omega$.

The result in [5] is a counterexample in large dimensions to *Bernstein's conjecture*, which asserts that all entire minimal graphs in $\mathbb{R}^N$ must be hyperplanes. This statement holds true in dimensions $N \leq 8$, see [26] and its references, so that in analogy, it is natural to think that the question in [3] for epigraphs may have an affirmative answer in low dimensions, but this is not even known in dimension $N = 2$. Another PDE analogue of Bernstein's problem is *De Giorgi's conjecture* (1978) [7], which states that entire solutions, monotone in one direction of the Allen-Cahn equation $\Delta u + u(1-u^2) = 0$ must have level sets which are parallel hyperplanes. This is true in dimensions $N = 2, 3$ [14, 2], and under a certain additional condition for $4 \leq N \leq 8$ [22], see also [10]. This statement is indeed false for $N \geq 9$ as proven in [8] by the construction of an example of a monotone solution whose level sets resemble largely dilated BDG minimal graphs. Serrin's epigraph question in [3] seems to be much harder. We remark, on the other hand, that the example we provide is not uniformly Lipschitz.

The principle behind the proof of Theorem 1 applies, more generally, to domains enclosed by **a large dilation of an embedded CMC surface**, provided that sufficient information about the surface (such as nondegeneracy) is available.

Our second results exhibits two such examples, consisting of non-cylindrical domains of revolution in $\mathbb{R}^3$ where (1.1)-(1.2) is solvable for f satisfying (1.4). Let us consider first the solid region enclosed by the catenoid $r = \cosh z$,

$$\Omega_c = \{(r\cos\theta, r\sin\theta, z) \ / \ 0 \leq r < \cosh z, \ z \in \mathbb{R}\}. \tag{1.7}$$

Theorem 2. *For each $\varepsilon > 0$ sufficiently small there exists a domain of revolution Ω, which lies within an ε-neighborhood of the dilated solid catenoid $\varepsilon^{-1}\Omega_c$, such that Problem (1.1)-(1.2) with f satisfying (1.4) is solvable.*

The boundary of Ω_c is a minimal surface. This result is a part of a more general statement regarding *embedded finite total curvature minimal surfaces in* $\mathbb{R}^3$, a class that includes for instance the Costa and Costa-Hoffmann-Meeks surfaces, which we shall discuss in the next section.

On the other hand, a statement similar to Theorem 2 holds for the classical *Delaunay surfaces*, a one parameter family of constant mean curvature surfaces of revolution in $\mathbb{R}^3$ which are periodic along one axis which, up to a rigid motion, can be taken to be the x_3-axis. These surfaces, which are called Delaunay surfaces and denoted by $\mathcal{D}_\tau$, are the boundary of a smooth domain U_τ and can be parameterized by

$$X_\tau(s,\theta) := (\varphi(s)\cos\theta, \varphi(s)\sin\theta, \psi(s)), \tag{1.8}$$

where the function φ is a nonconstant smooth solution of

$$\dot\varphi^2 + (\varphi^2 + \tau)^2 = \varphi^2, \tag{1.9}$$

and the function ψ is obtained from

$$\dot\psi = \varphi^2 + \tau, \quad \text{with } \psi(0) = 0.$$

Here $\tau \in (0, \frac{1}{2}]$ is a parameter which is usually referred to as the Delaunay parameter.

We have the validity of the following result.

Theorem 3. *For each $\varepsilon > 0$ sufficiently small there exists a domain of revolution Ω, which lies within an ε-neighborhood of the region $\varepsilon^{-1}U_\tau$, such that Problem (1.1)-(1.2) with f satisfying (1.4) is solvable.*

The Delaunay surface is compact when regarded as a submanifold of $\mathbb{R}^3$ with the period of the surface mod out (see Section 2 for explanations). We shall provide in the next section a more general statement, regarding a general manifold and a compact CMC surface in it, from which the above result follows. We will also express in more detail the result of the nontrivial epigraph and state the result regarding a general minimal surface with finite total curvature in $\mathbb{R}^3$. In the later sections we will provide the proof of Theorems 1-3.

Remark 1.1. *The BCN conjecture, in the case of cylindrical domains, was disproved by Sicbaldi in [23], where he provided a counterexample in the case when $N \geq 3$ and $f(t) = \lambda t$, $\lambda > 0$ by constructing a periodic perturbation of the cylinder $B^{N-1} \times R$ which supports a bounded solution to (1.1)-(1.2). In the two-dimensional case the same construction can be done in perioidic perturbations of a strip, see Schlenk and Sicbaldi [24]. A two-dimensional domain where the overdetermined problem in (1.1)-(1.2) is solvable for $f = 0$ was found by Hauswirth, Hèlein and Pacard in [15]. Explicitly, such a domain is given by*

$$\Omega = \{x \in \mathbb{R}^2 \ / \ |x_2| < \frac{\pi}{2} + \cosh(x_1)\},$$

but the solution found is unbounded. A classification of two-dimensional domains where the overdetermined problem in (1.1)-(1.2) is solvable for $f = 0$ is given by Traizet in [27]. Necessary geometric and topological conditions on Ω for solvability in the two-dimensional case have been found by Ros and Sicbaldi in [21]. The overdetermined problem in Riemannian manifolds has been considered by Farina, Mari and Valdinoci in [12].

2. More general statements

In this section we make more precise the statements that lead to Theorems 1-3. Concerning Theorem 1, we will be able to find a positive, bounded solution of (1.1)-(1.2) when Ω is a small perturbation of a large dilation of the epigraph of a nontrivial minimal graph in $\mathbb{R}^9$, found by Bombieri, De Giorgi and Giusti in [5]

$$\Gamma = \{x \in \mathbb{R}^9 \ / \ x_9 = F(x_1, \dots, x_8)\, \}.$$

Let $\nu(y)$ denote the unit normal to Γ with $\nu_9 > 0$. We consider normal perturbations of a large dilation of Γ, namely sets of the form

$$\Gamma_\varepsilon := \varepsilon^{-1}\Gamma, \quad \Gamma_\varepsilon^h = \{x = y + h(\varepsilon y)\nu(\varepsilon y) \ / \ y \in \Gamma_\varepsilon\} \tag{2.1}$$

for a small positive number ε and a smooth function h defined on Γ. We will prove the following result, which makes more precise the statement of Theorem 1.

Theorem 4. *For any sufficiently small $\varepsilon > 0$ there exists a function h defined on Γ, with a uniform C^2 bound independent of ε, such that Γ_ε^h in (2.1) is the graph of a smooth entire function, and letting Ω be its epigraph, then Problem (1.1)-(1.2), with f satisfying (1.4), admits a solution u_ε, with the property that*

$$u_\varepsilon(x) = w(t) + O(\varepsilon), \quad x = y + (t + h(\varepsilon y))\nu(\varepsilon y)$$

uniformly for $0 < t < \delta\varepsilon^{-1}$, some $\delta > 0$. Besides,

$$\partial_\nu u = -w'(0) \quad \text{on } \Gamma_\varepsilon^h.$$

As we have mentioned in the introduction, this result is analogous to that in [8]. The construction in this paper is considerably more delicate and requires new ideas. The linear theory required here deals with a Dirichlet to Neumann map, and it is more subtle than that in [8]. As in that work, an infinite-dimensional Lyapunov-Schmidt procedure reduces the problem to a nonlinear, nonlocal equation involving the Jacobi operator. The lack of symmetry of the seeked surface (unlike the BDG graph itself) induces the presence of large errors, and this is a substantial difficulty in the construction. We succeed in overcoming it, by means of a nontrivial refinement on the invertibility theory for the Jacobi operator.

Next we restrict our attention to the case $N = 3$. The catenoid is the simplest example (besides the plane) of a complete, embedded minimal surface Γ with *finite total curvature*, i.e.

$$\int_\Gamma |K|\, dV \;<\; +\infty$$

where K denotes the Gauss curvature of the manifold Γ. Such surfaces are known to have a finite number of ends, which are either planes or catenoids with a common axis of rotational symmetry. The first nontrivial example of such a manifold, with genus 1, was found in 1982 by Costa [6]. The example was later generalized by Hoffman and Meeks [16] to arbitrary genus $k \geq 1$. These minimal surfaces Γ are known to be *nondegenerate*, after the works by Nayatani and Morabito [18, 19], in the following sense:

The only bounded Jacobi fields, namely functions on Γ that annihilate the Jacobi operator $\mathcal{J}_\Gamma := \Delta_\Gamma - 2K$, are originated in rigid motions: rotations around the axis and translations,
namely they are linear combinations of the vector fields $\nu(x)\cdot e_i$, $i = 1, 2, 3$ and $\nu(x)\cdot x$, where ν is a unit normal vector field (these surfaces are orientable, they split the space into two components).

We fix such a unit normal ν for Γ and define the manifolds Γ_ε and Γ_ε^h as in (2.1).

Given this, we have the validity of the following result, that extends Theorem 2.

Theorem 5. *Let Γ be a complete, embedded minimal surface in $\mathbb{R}^3$ with finite total curvature and nondegenerate. Let f satisfy (1.4). Then for any sufficiently small $\varepsilon > 0$ there exists a function h defined on Γ, with a uniform C^2 bound independent of ε, such that Γ_ε^h is an embedded and orientable surface, and letting Ω be the component of $\mathbb{R}^3$ in the $\nu(\varepsilon y)$-direction, then Problem (1.1)-(1.2) admits a positive bounded solution u_ε, with the property that*

$$u_\varepsilon(x) = w(t) + O(\varepsilon), \quad x = y + (t + h(\varepsilon y))\nu(\varepsilon y)$$

uniformly for $0 < t < \delta\varepsilon^{-1}$, for some $\delta > 0$. Besides,

$$\partial_\nu u = -w'(0) \quad \text{on } \Gamma_\varepsilon^h.$$

In the case of a catenoid, the domain and the solution are axially symmetric.

The corresponding analogue for entire solutions of the Allen-Cahn equation was established in [9].

Theorems 4 and 5 deal with minimal surfaces which have zero mean curvature. It is not surprising that the right analogue of Serrin's overdetermined problem is the CMC surfaces, namely surfaces with constant mean curvature. For a Riemannian manifold M and a nondegenerate CMC compact surface $\Gamma = \partial\Omega_0$, we get a similar statement, which we will make precise next.

Let (M, g) be a Riemannian manifold and Δ_g its Laplace-Beltrami operator. We consider Problem (1.1)-(1.2) in a domain Ω in M, expressed in non-dilated form as

$$\varepsilon^2\Delta_g u + f(u) = 0,\ u > 0 \quad \text{in } \Omega, \quad u \in L^\infty(\Omega), \tag{2.2}$$

$$u = 0, \quad \frac{\partial u}{\partial \nu} = constant \quad \text{on } \partial\Omega. \tag{2.3}$$

Let Γ be a smooth embedded CMC surface in M, the boundary of a smooth domain Ω_0. We consider the problem of finding a domain Ω whose boundary is close to Γ for which Problem (2.2)-(2.3) is solvable.

For some (small) function h defined on Γ, the normal graph of h over Γ is a hypersurface which will be denoted by Γ_h. We have assumed that Γ is the boundary of a domain Ω_0, and we will denote by Ω_h the domain whose boundary is Γ_h. There are two choices since the complement of such a domain is a domain whose boundary is Γ_h and to remove the ambiguity we assume that $h \mapsto \Omega_h$ depends continuously on h (in the Hausdorff topology).

The mean curvature function of Γ_h is denoted by H_{Γ_h} and its differential at $h = 0$ is, by definition, the *Jacobi operator* about Γ. The explicit expression of the Jacobi operator about Γ is given by (see [4])

$$J_\Gamma := \Delta_\Gamma + |A_\Gamma|^2 + \mathrm{Ric}(\mathbf{n}, \mathbf{n}),$$

where Δ_Γ is the Laplace-Beltrami operator on Γ, $|A_\Gamma|^2$ is the square of the norm of A_Γ, the second fundamental form of Γ, namely the sum of the square of the principal curvatures of Γ, Ric is the Ricci tensor on M and $\mathbf{n}$ a choice of normal vector field for Γ. We recall that a compact CMC hypersurface Γ in M is said to be *nondegenerate* if J_Γ is one-to-one.

Theorem 6. *Assume that $\Omega_0 \subset M$ is a smooth bounded domain whose boundary $\Gamma = \partial\Omega_0$ is a nondegenerate hypersurface whose mean curvature is constant. Let f satisfy (1.4). Then, for all small $\varepsilon > 0$ there exists $h_\varepsilon \in C^{2,\alpha}(\Gamma)$ and u_ε, a solution of (2.2)-(2.3) in $\Omega_\varepsilon := \Omega_{h_\varepsilon}$. Moreover, there exists a constant $C > 0$ such that*

$$\|h_\varepsilon\|_{C^{2,\alpha}(\Gamma)} \leq C\,\varepsilon^2.$$

Hypersurfaces whose mean curvature is a constant function are known to exist in abundance and the result of [28] (see also [17]) shows that, for a generic choice of the ambient metric, they are nondegenerate. For example, solutions of the isoperimetric problem, when they are smooth, give rise to hypersurfaces whose mean curvature function is constant. We remark that Theorem 6 corresponds to a parallel of the one by Pacard and Ritoré [20].

Observe that Theorem 6 does not apply to the Delaunay surface, which is noncompact, nor does it apply to the unit ball in Euclidean space since in this case the Jacobi operator about the unit sphere S^m is given by

$$\Delta_{S^m} + m,$$

which is not injective since the coordinate functions $\mathtt{x} \mapsto x_j$, for $j = 1, \dots, m+1$, belong to its kernel. However there is an equivariant version of Theorem 6.

Let $\mathfrak{G} \subset \mathrm{Isom}(M, g)$ be a discrete group of isometries. A compact hypersurface Γ is said to be $\mathfrak{G}$-*nondegenerate* if there is no nontrivial element in the kernel of J_Γ which is invariant by the elements of $\mathfrak{G}$. We have the validity of the following result, from which Theorem 3 follows.

Theorem 7. *Assume that $\Omega_0 \subset M$ is a smooth bounded domain and $\mathfrak{G} \subset Isom(M, g)$ is a discrete group of isometries which leaves Ω_0 globally invariant, namely, $\mathfrak{g}(\Omega_0) = \Omega_0$ for all $\mathfrak{g} \in \mathfrak{G}$. Further assume that $\partial\Omega_0$ is a $\mathfrak{G}$-nondegenerate hypersurface whose mean curvature is constant. Then, the conclusion of Theorem 6 holds for a domain Ω_ε ($O(\varepsilon)$ close to Ω_0) and a solution u_ε which are invariant under the action of the elements of $\mathfrak{G}$.*

For example, in the case of the unit ball in the Euclidean $(m+1)$-dimensional space it is enough to consider the group generated by the symmetry through the origin to apply Theorem 7. A more interesting example is the Delaunay surface given by (1.8)-(1.9) in Section 1. From the definition, $\mathcal{D}_\tau$ is periodic and, if t_τ denotes the fundamental period of the Delaunay surface of parameter $\tau \in (0, \frac{1}{2}]$, we can also understand the Delaunay surfaces as constant mean curvature surfaces in $M_\tau := \mathbb{R}^2 \times (\mathbb{R}/t_\tau \mathbf{Z})$ which is endowed with the Euclidean metric g_{eucl}. With the parameterization (1.8)-(1.9), the Jacobi operator about a Delaunay surface reads

$$J_\tau = \frac{1}{\varphi^2}\left[\partial_s^2 + \partial_\theta^2 + \left(\varphi^2 + \frac{\tau^2}{\varphi^2}\right)\right],$$

it has a nontrivial kernel because of the invariance under the action of translations and in fact, it can be seen from [17] that the functions $(s,\theta) \mapsto \frac{\dot\varphi}{\varphi}$, $(s,\theta) \mapsto \left(\varphi + \frac{\tau}{\varphi}\right)\cos\theta$ and $(s,\theta) \mapsto \left(\varphi + \frac{\tau}{\varphi}\right)\sin\theta$ span the kernel of J_τ. However, if we consider the group of isometries of (M_τ, g_{eucl}) generated by the symmetry with respect to the vertical axis and also with respect to the $x_3 = 0$ plane, no element in the kernel of this operator is invariant with respect to the action of this group. In particular, Theorem 7 applies to $\mathcal{D}_\tau$ in M_τ and, going back to the universal cover $\mathbb{R}^3$, this leads to the:

Corollary 2.1. *Given $\tau \in (0, \frac{1}{2}]$, there exists for all $\varepsilon > 0$ close enough to 0, a cylindrically bounded domain $\Omega_{\tau,\varepsilon}$ which is periodic along the x_3-axis, of period t_τ and in which one can find positive solutions of (1.1)-(1.2). Moreover, the boundary of $\Omega_{\tau,\varepsilon}$ is a normal graph over $\mathcal{D}_\tau$ for some $\mathcal{C}^{2,\alpha}$ function whose norm is bounded by a constant times ε^2.*

The proofs of Theorems 4-7 can be set up into a similar scheme, of which the case of the minimal graph of Theorem 4 is the most complicated, since the surface is noncompact. So in the rest of the paper we concentrate mainly on the proof of Theorem 4. The proofs of the other theorems will be outlined only.

The organization of this paper is as follows. Sections §5-10 contain the proof of Theorem 4: in Section §3 we present in some detail the scheme that leads to the proof, which should help the reader to follow the thread of the subsequent sections. In Sections 10 and 11 we explain the modifications needed to prove Theorem 5 and Theorem 6 respectively. We postpone the proof of solvability of the Jacobi operator of the BDG graph for the Appendix.

3. Scheme of the proof of Theorem 4

We summarize here the proof of Theorem 4, which is carried out in Sections §4-10. Let Γ be a fixed Bombieri-De Giorgi-Giusti minimal graph [5], as in the statement of Theorem 4.

We let $\Gamma_\varepsilon = \varepsilon^{-1}\Gamma$ and describe all points x in a $\frac{\delta}{\varepsilon}$-tubular neighborhood of Γ in the form

$$x = y + z\nu(\varepsilon y), \quad y \in \Gamma_\varepsilon,\ |z| < \frac{\delta}{\varepsilon},$$

where ν is the "upward" choice of unit normal vector field for the graph Γ. We consider a *normal graph* to the surface Γ_ε, namely a surface of the following form: for a given function h defined on Γ we let Γ_ε^h be the set of points $x = y + z\nu(\varepsilon y)$ such that $z = h(\varepsilon y)$. Under suitable smallness for h, Γ_ε^h is also the graph of a function on $\mathbb{R}^8$. In other words, describing all points close to Γ_ε in the form

$$x = y + (t + h(\varepsilon y))\nu(\varepsilon y),$$

then the epigraph corresponds precisely to those with $t > 0$. We let Ω_ε^h be such a domain. We consider problem (1.1)-(1.2) in this domain, and consider $u_0(x) = w(t)$ as a first approximation to a solution, where $w(t)$ is the function defined in (1.5). Using the fact that Γ is a minimal surface, we shall see that the error produced (in the equation, and in the boundary condition) is roughly speaking of size $O(\varepsilon^2)$ for all small ε. In order to compute this error we need to find a convenient expression of the Euclidean Laplacian in terms of the coordinates $(y,t) \in \Gamma_\varepsilon \times \mathbb{R}$. This is precisely what we do in Section §4.

The error of approximation produced by u_0 is computed in Section §5. However, we need to improve the approximation in two iterations (using fairly explicit functions that separate the variables y and t) to obtain an error of size smaller than $O(\varepsilon^3)$. More precisely, we construct functions $\bar h_1 = O(\varepsilon)$, $\bar\phi_1 = O(\varepsilon^2)$ such that the error of approximation for $u = u_1 := u_0 + \bar\phi_1$ and $h = \bar h_1$ produce an error of size $O(\varepsilon^4)$. Quantifying the decay of this error in space variable is also extremely important. The problem is then reduced to finding small functions ϕ and $\mathtt{h}$ such that $u = u_1 + \phi$ and $h = \bar h_1 + \mathtt{h}$ solve the full problem. More precisely, by a *gluing reduction*, in Section §6 we reduce the full problem to one in the space $\Gamma_\varepsilon \times \mathbb{R}$ where the local coordinates (y,t) make sense all the way in t, not just up to $|t| \sim \delta\varepsilon^{-1}$.

In setting up the problem, ϕ is chosen so that it satisfies a suitable set of orthogonality conditions. The resulting system of equations for these functions is solved by perturbation of linear equations in two steps: we define first a projected problem for ϕ for a given small $\mathtt{h}$. Developing a suitable theory of solvability for the linear operators involved is what we do in Sections §7 and §9. This allows us in Section §10 to solve the problem for ϕ as an operator in $\mathtt{h}$. Finally, the problem is reduced to a small perturbation of a linear equation for $\mathtt{h}$ that involves the Jacobi operator of Γ, $J_\Gamma[\mathtt{h}] = \Delta_\Gamma \mathtt{h} + |A_\Gamma|^2\mathtt{h}$. The theory of invertibility of the Jacobi operator in a suitable form for our purposes is used in §8 to conclude the proof of the theorem, and provided in detail in the Appendix. The domain obtained by this procedure is proven to be indeed the epigraph of an entire function in $\mathbb{R}^8$.

4. The Bombieri-De Giorgi-Giusti minimal graph

We describe some general features of Γ in [5] and [8], and we define a useful system of local coordinates in which we compute the Euclidean Laplacian.

Let us consider the minimal surface equation in entire space $\mathbb{R}^8$,

$$H[F] \ := \ \nabla \cdot \left(\frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right) = 0 \quad \text{in } \mathbb{R}^8. \tag{4.1}$$

The quantity $H[F]$ corresponds to the mean curvature of the hypersurface in $\mathbb{R}^9$,

$$\Gamma := \{(x', F(x')) \mid \ x' \in \mathbb{R}^8\}.$$

The Bombieri-De Giorgi-Giusti minimal graph [5] is a nontrivial, entire smooth solution of equation (4.1) that enjoys some simple symmetries which we describe next. Let us write $x' \in \mathbb{R}^8$ as $x' = (\mathbf{u}, \mathbf{v}) \in \mathbb{R}^4 \times \mathbb{R}^4$ and consider the set

$$T \ := \ \{(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^8 \mid \ |\mathbf{v}| > |\mathbf{u}| \ \}. \tag{4.2}$$

The solution found in [5] is radially symmetric in both variables, namely $F = F(|\mathbf{u}|, |\mathbf{v}|)$. In addition, F is positive in T and it vanishes along ∂T. Moreover, it satisfies

$$F(|\mathbf{u}|, |\mathbf{v}|) = -F(|\mathbf{v}|, |\mathbf{u}|) \quad \text{for all} \quad \mathbf{u}, \mathbf{v} \ . \tag{4.3}$$

It is useful to introduce polar coordinates $(|\mathbf{u}|, |\mathbf{v}|) = (r\cos\theta, r\sin\theta)$. In [8] it was found that F is well approximated for large r by a function that separates variables, $F_0(x') = r^3 g(\theta)$, where $g(\theta)$ solves the two-point boundary value problem

$$\frac{21g\,\sin^3 2\theta}{\sqrt{9g^2+g'^2}} + \left(\frac{g'\sin^3 2\theta}{\sqrt{9g^2+g'^2}}\right)' = 0 \quad \text{in } \left(\frac{\pi}{4},\frac{\pi}{2}\right), \quad g\left(\frac{\pi}{4}\right) = 0 = g'\left(\frac{\pi}{2}\right). \tag{4.4}$$

Problem (4.4) has a unique solution $g \in C^2([\frac{\pi}{4},\frac{\pi}{2}])$ such that g and g' are positive in $(\frac{\pi}{4},\frac{\pi}{2})$ and such that $g'(\frac{\pi}{4}) = 1$.

Lemma 4.1. [8] *There exists an entire solution $F = F(|\mathbf{u}|, |\mathbf{v}|)$ to equation (4.1) which satisfies (4.3) and such that*

$$F_0 \le F \le F_0 + \frac{\mathtt{C}}{r^\sigma} \quad \text{in } T, \quad r > R_0, \tag{4.5}$$

where $0 < \sigma < 1$, $\mathtt{C} \ge 1$, and R_0 are positive constants.

In what follows we will denote, for F and F_0 as above,

$$\Gamma = \{(x', F(x')) \mid\ x' \in \mathbb{R}^8\}, \quad \Gamma_0 = \{(x', F_0(x')) \mid\ x' \in \mathbb{R}^8\}.$$

By Γ_ε we will denote the dilated surfaces $\Gamma_\varepsilon = \varepsilon^{-1}\Gamma$. Also, we shall use the notation:

$$\mathtt{r}(x) := \sqrt{1+|x'|^2}, \quad \mathtt{r}_\varepsilon(x) := \mathtt{r}(\varepsilon x), \quad x = (x', x_9) \in \mathbb{R}^8 \times \mathbb{R} = \mathbb{R}^9. \tag{4.6}$$

4.1. **Local coordinates and the Laplacian near Γ_ε.** In [8], the following family of local parametrizations of the surface Γ was considered: given $p \in \Gamma$ with $p = (p', p_9)$, $R = \mathtt{r}(p) >> 1$, we let $\nu(p)$ be its normal vector, and $\Pi_1, \dots, \Pi_8$ an orthonormal basis of its tangent space. Using the known fact that the curvatures of Γ at p are bounded as $O(R^{-1})$ one finds that there exists a $\theta > 0$ independent of p and a smooth function $G_p(\mathtt{y})$ defined on $\mathbb{R}^8$ with $G(0) = G_p'(0) = 0$, such that Γ can be locally parametrized around p by the map

$$\mathtt{y} \in B(0,\theta R) \subset \mathbb{R}^8 \ \longmapsto\ Y_p(\mathtt{y}) := p + \sum_{j=1}^{8} \mathtt{y}_j \Pi_j + G_p(\mathtt{y})\nu(p) \ \in\ \Gamma. \tag{4.7}$$

Besides, for each $m \ge 2$ the following estimate holds:

$$\|D_{\mathtt{y}}^m G_p\|_{L^\infty(B(0,\theta R))} \ \le\ \frac{c_m}{R^{m-1}}$$

where c_m is independent of p.

In what follows, we shall use the following convention: for a function f defined on Γ, we denote indistinctly (when no confusion arises) $f(y)$ or $f(\mathtt{y})$ for the function f evaluated at $y = Y_p(\mathtt{y}) \in \Gamma$.

Let us consider the metric g_{ij} of Γ around p. Then

$$g_{ij}(\mathtt{y}) \ := \ \langle \partial_i Y_p, \partial_j Y_p\rangle = \delta_{ij} + \theta(\mathtt{y}).$$

We will assume in what follows that the metric satisfies the following uniform estimates: There exists a positive number C such that for all $p \in \Gamma$ we have the estimate

$$|\theta(\mathtt{y})| + |D_{\mathtt{y}}\theta(\mathtt{y})| + |D_{\mathtt{y}}^4\theta(\mathtt{y})| \ \le\ C, \quad |\mathtt{y}| < 1. \tag{4.8}$$

4.2. **The Laplace Beltrami operator.** The Laplace-Beltrami operator of Γ is expressed in these local coordinates as

$$\Delta_\Gamma = \frac{1}{\sqrt{\det g(\mathrm{y})}}\,\partial_i\left(\sqrt{\det g(\mathrm{y})}\,g^{ij}(\mathrm{y})\,\partial_j\right).$$

Let us set

$$a^0_{ij}(\mathrm{y}) := g^{ij}(\mathrm{y}),\quad b^0_j(\mathrm{y}) := \frac{1}{\sqrt{\det g(\mathrm{y})}}\,\partial_i\left(\sqrt{\det g(\mathrm{y})}\,g^{ij}(\mathrm{y})\right)$$

so that

$$\Delta_\Gamma \;=\; a^0_{ij}(\mathrm{y})\,\partial_{ij} + b^0_i(\mathrm{y})\,\partial_i,\quad |\mathrm{y}|<\theta R, \tag{4.9}$$

where

$$\begin{aligned} |a^0_{ij}(\mathrm{y})-\delta_{ij}| \;&\le\; c\frac{|\mathrm{y}|^2}{R^2},\quad |D_{\mathrm{y}}a^0_{ij}(\mathrm{y})|\;\le\; c\frac{|\mathrm{y}|}{R^2},\\ |b^0_j(\mathrm{y})|\;&\le\; c\frac{|\mathrm{y}|}{R^2},\quad |D_{\mathrm{y}}b^0_j(\mathrm{y})|\;\le\;\frac{c}{R^2}\quad\text{for all}\quad |\mathrm{y}|<\theta R,\ m\ge 2. \end{aligned} \tag{4.10}$$

4.3. **The Laplacian near Γ.** For a certain $\delta>0$ the map

$$x = X(z,y) := y + z\nu(y),\quad y\in\Gamma,\quad |z|<\delta\mathtt{r}(y) \tag{4.11}$$

defines diffeomorphism onto an expanding tubular neighborhood of Γ. Let us consider the manifold

$$\Gamma^z := \{y+z\nu(y)\ /\ y\in\Gamma\}.$$

The Euclidean Laplacian in $\mathbb{R}^9$ near Γ can be expressed in these coordinates by the well-known formula

$$\Delta_x = \partial_z^2 + \Delta_{\Gamma^z} - H_{\Gamma^z}(y)\partial_z \tag{4.12}$$

where $H_{\Gamma^z}(y)$ denotes mean curvature of Γ^z at the point $y+z\nu(y)$ and the operator Δ_{Γ^z} is understood to act on functions of the variable y.

Using the local coordinates $Y_p(\mathrm{y})$, (4.11) becomes

$$x = X(z,\mathrm{y}) := Y_p(\mathrm{y}) + z\nu(\mathrm{y}),\quad |\mathrm{y}|<\theta R \tag{4.13}$$

and then the metric tensor g^z_{ij} on Γ^z is given by

$$g^z_{ij}(\mathrm{y}) = g_{ij}(\mathrm{y}) \;+\; z\left[\,\langle\partial_i Y_p(\mathrm{y}),\partial_j\nu(\mathrm{y})\rangle + \langle\partial_j Y_p(\mathrm{y}),\partial_i\nu(\mathrm{y})\rangle\,\right] + z^2\,\langle\partial_i\nu(\mathrm{y}),\partial_j\nu(\mathrm{y})\rangle\,.$$

Using that

$$\nu(\mathrm{y}) = \frac{1}{\sqrt{1+|D_{\mathrm{y}}G_p(\mathrm{y})|^2|}}\left[-\sum_{j=1}^{8}\partial_j G_p(\mathrm{y})\Pi_j + \nu(p)\right]$$

for the computation of derivatives we get the expansion

$$g^z_{ij}(\mathrm{y}) = g_{ij}(\mathrm{y}) + z\theta_1(\mathrm{y}) + z^2\theta_2(\mathrm{y})$$

where

$$\begin{aligned} |\theta_1(\mathrm{y})|\;&\le\;\frac{c}{R},\quad |D_{\mathrm{y}}\theta_1(\mathrm{y})|\;\le\;\frac{c}{R^2},\\ |\theta_2(\mathrm{y})|\;&\le\;\frac{c}{R^2},\quad |D_{\mathrm{y}}\theta_2(\mathrm{y})|\;\le\;\frac{c}{R^3}. \end{aligned} \tag{4.14}$$

Therefore if we let

$$a_{ij}(\mathrm{y},z) := g^{zij}(\mathrm{y}),\quad b^0_j(\mathrm{y}) := \frac{1}{\sqrt{\det g^z(\mathrm{y})}}\,\partial_i\left(\sqrt{\det g^z(\mathrm{y})}\,g^{zij}(\mathrm{y})\right)$$

we get

$$\Delta_{\Gamma_z} \;=\; a_{ij}(\mathtt{y},z)\,\partial_{ij} + b_i(\mathtt{y},z)\,\partial_i, \tag{4.15}$$

with a^0_{ij} given in (4.9), where

$$a_{ij}(\mathtt{y},z) = a^0_{ij}(\mathtt{y}) + z a^1_{ij}(\mathtt{y},z), \quad b_j(\mathtt{y},z) = b^0_j(\mathtt{y}) + z b^1_j(\mathtt{y},z) \tag{4.16}$$

and

$$|a^1_{ij}| = O(R^{-1}), \quad |D_{\mathtt{y}} a^1_{ij}| = O(R^{-2})\,,$$
$$|b^1_j| = O(R^{-2}), \quad |D_{\mathtt{y}} b^1_j| = O(R^{-3}). \tag{4.17}$$

On the other hand, it is well-known that if $k_1, \dots, k_8$ denote the principal curvatures of Γ, then

$$H_{\Gamma^z}(y) \;=\; \sum_{i=1}^{8} \frac{k_i(y)}{1 - z k_i(y)}.$$

Since Γ is a minimal surface we have that $\sum_{i=1}^8 k_i = 0$, therefore

$$H_{\Gamma^z}(y) \;=\; z|A_\Gamma|^2 + z^2 \sum_{i=1}^{8} k_i^3 + z^3 \sum_{i=1}^{8} k_i^4 + z^4\theta(y,z) \tag{4.18}$$

where

$$|A_\Gamma|^2 = \sum_{i=1}^{8} k_i^2 := O(\mathtt{r}^{-2}), \tag{4.19}$$

$$|\theta(y,z)| = O(\mathtt{r}(y)^{-5}), \quad |D\theta(y,z)| = O(\mathtt{r}(y)^{-6}). \tag{4.20}$$

4.4. **Coordinates near Γ_ε.** The previous expressions generalize by scaling to Γ_ε in particular the coordinates Y_p induce naturally local coordinates in Γ_ε. If $p_\varepsilon = \varepsilon^{-1}p$, $p \in \Gamma$, we have that the map

$$\mathtt{y} \in B(0, \theta R/\varepsilon) \subset \mathbb{R}^8 \;\longmapsto\; Y_{p_\varepsilon}(\mathtt{y}) := \varepsilon^{-1} Y_p(\varepsilon \mathtt{y}) \;\in\; \Gamma_\varepsilon \tag{4.21}$$

defines a local parametrization. The metric on Γ_ε in these coordinates is simply computed as $g_{ij}(\varepsilon\mathtt{y})$. This yields the expansion

$$\Delta_{\Gamma_\varepsilon} = \Delta_{\mathtt{y}} + (a^0_{ij}(\varepsilon\mathtt{y}) - \delta_{ij})\,\partial^2_{ij} + \varepsilon b^0_i(\varepsilon\mathtt{y})\partial_i, \quad |\mathtt{y}| < \varepsilon^{-1}\theta R. \tag{4.22}$$

For some $\delta > 0$, the following map defines coordinates for a expanding neighborhood of Γ_ε:

$$x = X(y,z) := y + z\nu(\varepsilon y), \quad y \in \Gamma_\varepsilon,\ |z| < \delta\varepsilon^{-1}\,\mathtt{r}(\varepsilon y) \tag{4.23}$$

is computed as

$$\Delta = \partial_z^2 + \Delta_{\Gamma_\varepsilon^{\varepsilon z}} - \varepsilon H_{\Gamma^{\varepsilon z}}(\varepsilon y)\,\partial_z \tag{4.24}$$

where now

$$\Delta_{\Gamma_\varepsilon^{\varepsilon z}} = \Delta_{\Gamma_\varepsilon} + \varepsilon z a^1_{ij}(\varepsilon\mathtt{y}, \varepsilon z)\partial^2_{ij} + \varepsilon^2 z b^1_j(\varepsilon\mathtt{y}, \varepsilon z)\partial_j \tag{4.25}$$

and

$$\varepsilon H_{\Gamma^{\varepsilon z}}(\varepsilon y) = \varepsilon^2 z|A_\Gamma(\varepsilon y)|^2 + \varepsilon^3 z^2 \sum_{i=1}^{8} k_i(\varepsilon y)^3 \,+$$
$$\varepsilon^4 z^3 \sum_{i=1}^{8} k_i^4(\varepsilon y) + \varepsilon^5 z^4 \theta(y,z). \tag{4.26}$$

4.5. **The shifted coordinates.** We consider now a bounded smooth function $h(y)$ defined on Γ and the coordinates near Γ_ε,

$$x = X^h(y,t) := y + (t + h(\varepsilon y))\nu(\varepsilon y), \quad y \in \Gamma_\varepsilon,\ |t| < \delta\varepsilon^{-1}\,\mathtt{r}(\varepsilon y). \tag{4.27}$$

We compute the Laplacian in these coordinates. We obtain now

$$\begin{aligned}\Delta_x \;=&\; \partial_t^2 + a_{ij}(\varepsilon \mathrm{y}, \varepsilon z)\,\partial_{ij} \;+\; \varepsilon b_j(\varepsilon \mathrm{y}, \varepsilon z)\,\partial_j \\ &+\; \varepsilon^2 a_{ij}(\varepsilon \mathrm{y}, \varepsilon z)\,\partial_i h(\varepsilon \mathrm{y})\,\partial_j h(\varepsilon \mathrm{y})\,\partial_t^2 - 2\varepsilon a_{ij}(\varepsilon \mathrm{y}, \varepsilon z)\,\partial_i h(\varepsilon \mathrm{y})\,\partial_{jt} \\ &-\; \left\{\varepsilon^2\left[a_{ij}(\varepsilon \mathrm{y}, \varepsilon z)\,\partial_{ij} h(\varepsilon \mathrm{y}) \;+\; b_j(\varepsilon \mathrm{y}, \varepsilon z)\,\partial_j h(\varepsilon \mathrm{y})\right] \;+\; \varepsilon\, H_{\Gamma^{\varepsilon z}}(\varepsilon \mathrm{y})\right\}\partial_t\end{aligned} \tag{4.28}$$

where $z = \varepsilon(t + h(\varepsilon \mathrm{y}))$.

Since

$$\Delta_{\Gamma_\varepsilon} = a_{ij}(\varepsilon \mathrm{y}, 0)\partial_{ij} + \varepsilon b_i(\varepsilon \mathrm{y}, 0)\partial_i, \tag{4.29}$$

we can also decompose

$$\Delta_x = \partial_{tt} + \Delta_{\Gamma_\varepsilon} + B \tag{4.30}$$

where the small operator B, acting on functions of (y,t) is given in local coordinates by

$$\begin{aligned}B \;=&\; \varepsilon z a^1_{ij}(\varepsilon \mathrm{y}, \varepsilon z)\,\partial_{ij} \;+\; \varepsilon^2 z b^1_j(\varepsilon \mathrm{y}, \varepsilon z)\,\partial_j \\ &+\; \varepsilon^2 a_{ij}(\varepsilon \mathrm{y}, \varepsilon z)\,\partial_i h(\varepsilon \mathrm{y})\,\partial_j h(\varepsilon \mathrm{y})\,\partial_t^2 - 2\varepsilon a_{ij}(\varepsilon \mathrm{y}, \varepsilon z)\,\partial_i h(\varepsilon \mathrm{y})\,\partial_{jt} \\ &-\; \left\{\varepsilon^2\left[a_{ij}(\varepsilon \mathrm{y}, \varepsilon z)\,\partial_{ij} h(\varepsilon \mathrm{y}) \;+\; b_j(\varepsilon \mathrm{y}, \varepsilon z)\,\partial_j h(\varepsilon \mathrm{y})\right] \;+\; \varepsilon\, H_{\Gamma^{\varepsilon z}}(\varepsilon \mathrm{y})\right\}\partial_t.\end{aligned} \tag{4.31}$$

5. A first approximation to the nontrivial epigraph

In this section we shall use the computations of the previous section to compute a good approximation of a dilated BDG surface Γ_ε, around which a perturbation argument will yield the proof of Theorem 1.

5.1. **The perturbed epigraph.** We fix a positive number M and assume for the moment that h is a smooth function such that

$$\|D^2_\Gamma h\|_{L^\infty(\Gamma)} \;+\; \|D_\Gamma h\|_{L^\infty(\Gamma)} + \|h\|_{L^\infty(\Gamma)} \;\le\; M \tag{5.32}$$

uniformly in small ε and set

$$\Gamma^h_\varepsilon = \{y + h(\varepsilon y)\nu(\varepsilon y)\ /\ y \in \Gamma_\varepsilon\}.$$

Γ^h_ε is an embedded manifold provided that ε is sufficiently small, that separates $\mathbb{R}^N$ into two components. We call Ω^h_ε the upper component. Under suitable smallness of h, the implicit function theorem yields that this set is the epigraph of an entire smooth function $F^h_\varepsilon(x')$,

$$\Omega^h_\varepsilon := \{(x', x_9)\ /\ x_9 > F^h_\varepsilon(x')\} \tag{5.33}$$

whose boundary is of course Γ^h_ε.

5.2. **The problem and a first approximation.** We want to solve the problem

$$S[u] := \Delta u + f(u) = 0 \quad \text{in } \Omega_\varepsilon^h$$

$$u = 0, \quad \partial_\nu u = constant \quad \text{on } \Gamma_\varepsilon^h$$

for a small function h and prove later on that Ω_ε^h has the form (5.33). We observe that in the coordinates

$$x = y + (t + h(\varepsilon y))\,\nu(\varepsilon y), \quad y \in \Gamma_\varepsilon, \quad |t| < \delta\varepsilon^{-1}\mathtt{r}_\varepsilon(y),$$

we have that $x \in \Omega_\varepsilon^h$ if and only if $t > 0$. The problem above then becomes

$$\begin{aligned} S[u] &= \Delta_x u + f(u) = 0 \quad \text{in } \Omega_\varepsilon^h, \\ u(y,0) &= 0 \quad \text{for all} \quad y \in \Gamma_\varepsilon, \\ \partial_t u(y,0) &= constant \quad \text{for all} \quad y \in \Gamma_\varepsilon. \end{aligned} \tag{5.34}$$

We have the existence of a unique solution $w(t)$ to the problem

$$w'' + f(w) = 0 \quad \text{in } (0,\infty),$$

$$w(0) = 0, \quad w(+\infty) = 1.$$

As a first approximation, close to Γ_ε^h we simply take $u_0(x) := w(t)$. Using formula (4.28), we find that the error of approximation is then given by

$$\begin{aligned} S[u_0] &= \varepsilon^2 a_{ij}(\varepsilon \mathtt{y}, \varepsilon z)\,\partial_i h(\varepsilon \mathtt{y})\,\partial_j h(\varepsilon \mathtt{y})\,w''(t) \\ &\quad - \left\{ \varepsilon^2 \left[a_{ij}(\varepsilon \mathtt{y}, \varepsilon z)\,\partial_{ij} h(\varepsilon \mathtt{y}) + b_j(\varepsilon \mathtt{y}, \varepsilon z)\,\partial_j h(\varepsilon \mathtt{y})\right] + \varepsilon\, H_{\Gamma^{\varepsilon z}}(\varepsilon \mathtt{y}) \right\} w'(t) \end{aligned} \tag{5.35}$$

where $z = (t + h(\varepsilon \mathtt{y}))$. Recalling that

$$a_{ij}(\mathtt{y}, z) = a_{ij}^0(\mathtt{y}) + z a_{ij}^1(\mathtt{y}, z), \quad b_j(\mathtt{y}, z) = b_j^0(\mathtt{y}) + z b_j^1(\mathtt{y}, z), \quad \Delta_\Gamma = a_{ij}^0 \partial_{ij} + b_j^0 \partial_j,$$

and using the expansion (4.26) for the mean curvature, we then write

$$\begin{aligned} S[u_0] &= -\varepsilon^2 [\Delta_\Gamma h + |A_\Gamma|^2 h]\, w' + \varepsilon^2 a_{ij}\,\partial_i h\,\partial_j h\, w'' \\ &\quad - \left[\varepsilon^2 t |A_\Gamma|^2 w' + \varepsilon^3 \sum_{i=1}^{8} k_i^3 (t+h)^2 w' + \varepsilon^4 \sum_{i=1}^{8} k_i^4 (t+h)^3 w'\right] \\ &\quad - \left\{ \varepsilon^3 (t+h)\,[a_{ij}^1\,\partial_{ij} h + b_j^1\,\partial_j h]\, w' + \varepsilon^5\,(t+h)^4 \theta \right\} w' \end{aligned} \tag{5.36}$$

where all coefficients are evaluated at $(\varepsilon \mathtt{y}, \varepsilon(t + h(\varepsilon \mathtt{y}))$ or $\varepsilon \mathtt{y}$.

What we will do is to improve this first approximation by choosing h in such a way that at main order the relation

$$\int_{\mathbb{R}} S[u_0]\,(y,t)\,w'(t)\,dt = 0 \quad \text{for all} \quad y \in \Gamma_\varepsilon$$

is satisfied. Under this condition the addition to u_0 of a suitable, explicitly computed small term, reduces the error. We will actually carry out this procedure in successive steps that we describe next. Let us take the function $h(y)$ to have the following form

$$h(y) = h_0 + \varepsilon h_1(y) + \varepsilon^2 h_2(y) + \varepsilon^3 \mathtt{h}(y) \tag{5.37}$$

where the functions h_0, h_1, h_2 will be explicitly chosen later.

5.3. **First improvement of approximation.** We will next add to u_0 a convenient function $\phi_1(y,t)$ of size $O(\varepsilon^2)$ that does not change the boundary conditions and eliminates the quadratic term in ε in the new error $S[u_0]$.

To this end we consider the linear one dimensional problem

$$p'' + f'(w(t))p = q(t), \quad t \in (0,\infty), \quad p(0) = p'(0) = 0. \tag{5.38}$$

The solution to this equation is given by

$$p(t) \;=\; w'(t) \int_0^t \frac{d\tau}{w'(\tau)^2} \int_0^\tau w'(s)\, q(s)\, ds\ . \tag{5.39}$$

If q is a bounded function, p will be bounded if and only if

$$\int_0^\infty q(t)\, w'(t)\, dt = 0.$$

Let c_0 be the number such that

$$\int_0^\infty (t + c_0) w'(t)^2\, dt \;=\; 0,$$

and let $p_0(t)$ be the solution of

$$p_0'' + f'(w(t))p_0 \;=\; (t + c_0) w'(t), \quad t \in (0,\infty), \quad p_0(0) = p_0'(0) = 0, \tag{5.40}$$

given by formula (5.39). We observe that

$$p_0(t) \sim t^3 e^{-t} \quad \text{as } t \to +\infty\ .$$

Let

$$\phi_0(y,t) := \varepsilon^2 |A_\Gamma(\varepsilon y)|^2\, p_0(t).$$

We observe that

$$S[u + \phi] = \Delta\phi + f'(u)\phi + S[u] + N(u,\phi) \tag{5.41}$$

where

$$N(u,\phi) = f(u+\phi) - f(u) - f'(u)\phi.$$

Next we estimate the new error of approximation, $S(u_0 + \phi_0)$. To do this and for later computations, it is useful to have the following lemma, that follows directly from formula (4.28).

Lemma 5.1. *Let $\psi(y)$, $p(t)$ be smooth functions defined respectively on Γ and on $(0,\infty)$. Let us set*

$$\phi(x) \;=\; \psi(\varepsilon y)\, p(t), \quad x = y + (t + h(\varepsilon y))\, \nu(\varepsilon y).$$

Then

$$\begin{aligned} \Delta_x \phi \;=\; & \psi\, p'' \;+\; \varepsilon^2 \big[a_{ij}\, \partial_{ij}\psi \;+\; \varepsilon\, b_j \partial_j \psi \big]\, p' \\ & +\; \varepsilon^2 \psi\, a_{ij}\, \partial_i h\, \partial_j h\, p'' - 2\varepsilon^2 a_{ij}\, \partial_i h\, \partial_j \psi\, p' \\ & -\; \big\{ \varepsilon^2 \left[a_{ij}\, \partial_{ij} h \;+\; b_j\, \partial_j h \right] \;+\; \varepsilon\, H_{\Gamma^{\varepsilon(t+h)}} \big\}\, \psi\, p' \end{aligned} \tag{5.42}$$

where the coefficients are evaluated at $\varepsilon \mathtt{y}$ or $(\varepsilon \mathtt{y}, \varepsilon(t + h(\varepsilon \mathtt{y})))$, and we recall

$$\varepsilon\, H_{\Gamma^{\varepsilon(t+h)}} = \varepsilon^2 (t+h) |A_\Gamma|^2 + \varepsilon^3 (t+h)^2 \sum_{i=1}^8 k_i^3 + \varepsilon^4 (t+h)^3 \sum_{i=1}^8 k_i^4 + \varepsilon^5 (t+h)^3 \theta,$$

$$\theta = O(\mathtt{r}_\varepsilon^{-5}).$$

Using formula (5.42) we then get

$$
\begin{aligned}
\Delta_x\phi_0 + f'(u_0)\phi_0 \;&= \varepsilon^2\,|A_\Gamma|^2(t+c_0)\,w' + \varepsilon^4\big[a_{ij}\,\partial_{ij}|A_\Gamma|^2 \;+\; b_j\partial_j|A_\Gamma|^2\,\big]\,p \\
&\quad +\; \varepsilon^4|A_\Gamma|^2\,a_{ij}\,\partial_i h\,\partial_j h\,p'' - 2\varepsilon^3 a_{ij}\,\partial_i h\,\partial_j|A_\Gamma|^2\,p' \\
&\quad -\; \big\{\,\varepsilon^4\,[a_{ij}\,\partial_{ij}h \;+\; b_j\,\partial_j h\,] \;+\; \varepsilon^3\,H_{\Gamma^{\varepsilon(t+h)}}\,\big\}\,|A_\Gamma|^2\,p'.
\end{aligned}
\tag{5.43}
$$

From formula (5.41) we have

$$
S[u_0+\phi_0] = \Delta\phi_0 + f'(u_0)\phi_0 + S[u_0] + \frac{1}{2}f''(u_0)\phi_0^2 + \frac{\phi_0^3}{6}\int_0^1 f'''(u_0+s\phi_0)\,ds\,. \tag{5.44}
$$

Using (5.36) and (5.42) we then get

$$
\begin{aligned}
S[u_0+\phi_0] \;=\; & -\varepsilon^2\,[\Delta_\Gamma h + |A_\Gamma|^2\,h\,]\,w' \;+\varepsilon^2\,|A_\Gamma|^2(t+c_0)\,w' - \varepsilon^2|A_\Gamma|^2 t w' \\
& +\; \varepsilon^2 a_{ij}\,\partial_i h\,\partial_j h\,w'' \;-\; \varepsilon^3\sum_{i=1}^{8}k_i^3(t+h)^2 w' - \varepsilon^4\sum_{i=1}^{8}k_i^4(t+h)^3 w' \\
& +\; \varepsilon^2\big[a_{ij}\,\partial_{ij}|A_\Gamma|^2 \;+\; b_j\partial_j|A_\Gamma|^2\,\big]\,p \\
& -\; \big\{\,\varepsilon^3(t+h)\,[a^1_{ij}\,\partial_{ij}h \;+\; b^1_j\,\partial_j h\,]\,w' \;+\; \varepsilon^5\,(t+h)^4\theta\,\big\}\,w' \\
& +\; \varepsilon^4|A_\Gamma|^2\,a_{ij}\,\partial_i h\,\partial_j h\,p'' - 2\varepsilon^3 a_{ij}\,\partial_i h\,\partial_j|A_\Gamma|^2\,p' \\
& -\; \varepsilon^4\left\{\,[a_{ij}\,\partial_{ij}h \;+\; b_j\,\partial_j h\,] \;+\; |A_\Gamma|^2(t+h) \;+\; \varepsilon\sum_{i=1}^{8}k_i^3(t+h)^2 + \cdots\right\}|A_\Gamma|^2\,p' \\
& +\; \frac{\varepsilon^4}{2}f''(w)|A_\Gamma|^4p^2 + \frac{\varepsilon^6}{6}p^3\,|A_\Gamma|^6\int_0^1 f'''(w+s\phi_0)\,ds\,.
\end{aligned}
\tag{5.45}
$$

Next we proceed to make a choice at main order of the parameter function h by writing

$$
h = \bar h_1 + \mathtt{h}, \quad \bar h_1 := h_0 + \varepsilon h_1 + \varepsilon^2 h_2.
$$

We choose $h_0 \equiv c_0$ and replace in the above expression. We get

$$
\begin{aligned}
S[u_0+\phi_0] \;=\; & -\varepsilon^2\,[\Delta_\Gamma \mathtt{h} + |A_\Gamma|^2\,\mathtt{h}\,]\,w' \\
& -\; \varepsilon^3\,[\Delta_\Gamma h_1 + |A_\Gamma|^2\,h_1\,]\,w' - \varepsilon^3\sum_{i=1}^{8}k_i^3(t+h_0)^2 w' \\
& -\varepsilon^4\,[\Delta_\Gamma h_2 + |A_\Gamma|^2\,h_2\,]\,w' - \varepsilon^4\sum_{i=1}^{8}k_i^4(t+h_0)^3 w' \\
& +\; \varepsilon^4\Delta_\Gamma\,|A_\Gamma|^2\,p_0 + \frac{\varepsilon^4}{2}f''(w)|A_\Gamma|^4p_0^2 - \varepsilon^4\,|A_\Gamma|^4(t+h_0)\,p' \\
& -\; 2\varepsilon^4\sum_{i=1}^{8}k_i^3 h_1\,(t+h_0)w' \\
& +\; \mathcal{R}_1(h),
\end{aligned}
\tag{5.46}
$$

where

$$
\begin{aligned}
\mathcal{R}_1(h) \;=\; & \varepsilon^2 a_{ij}\,\partial_i h\,\partial_j h\,w'' \\
& -\,\varepsilon^3\sum_{i=1}^{8} k_i^3[\,(t+h)^2-(t+h_0)^2-2(t+h_0)\varepsilon h_1\,]w' \\
& -\,\varepsilon^4\sum_{i=1}^{8} k_i^4[(t+h)^3-(t+h_0)^3]w' \\
& +\,\varepsilon^5(t+h)\big[a^1_{ij}\,\partial_{ij}|A_\Gamma|^2 \;+\; b^1_j\partial_j|A_\Gamma|^2\,\big]\,p_0 \\
& -\,\varepsilon^3(t+h)\,[a^1_{ij}\,\partial_{ij}h \;+\; b^1_j\,\partial_j h\,]\,w' \;+\; \varepsilon^5\,(t+h)^4\theta\,w' \\
& +\;\varepsilon^4|A_\Gamma|^2\,a_{ij}\,\partial_i h\,\partial_j h\,p_0''-2\varepsilon^3 a_{ij}\,\partial_i h\,\partial_j|A_\Gamma|^2\,p_0' \\
& -\;\varepsilon^4\left\{[a_{ij}\,\partial_{ij}h\;+\;b_j\,\partial_j h\,]\;+\;|A_\Gamma|^2(h-h_0)\;+\;\varepsilon\sum_{i=1}^{8}k_i^3(t+h)^2+\cdots\right\}|A_\Gamma|^2\,p' \\
& +\;\frac{\varepsilon^6}{6}p_0^3\,|A_\Gamma|^6\int_0^1 f'''(w+s\phi_0)\,ds\,.
\end{aligned}
\tag{5.47}
$$

5.4. **Second improvement of approximation.** Similar to the introduction of ϕ_0 and h_0, a convenient choice of the functions h_1 and h_2 will allow us to eliminate the largest terms in the error above. To this end, we need to achieve at main order the orthogonality

$$\int_0^\infty S[u_0+\phi_0]\,w'\,dt \;=0.$$

Thus we require first that at main order h_1 and h_2 are such that

$$\int_0^\infty \big(\,[\Delta_\Gamma h_1+|A_\Gamma|^2\,h_1\,]\,w'+\sum_{i=1}^{8}k_i^3(t+h_0)^2w'\,\big)\,w'\,dt \;=\;0,$$

and

$$
\begin{aligned}
\int_0^\infty \big(\,[\Delta_\Gamma h_2+|A_\Gamma|^2\,h_2\,]\,w' &+\sum_{i=1}^{8}k_i^4(t+h_0)^3w'+|\nabla_\Gamma \mathtt{h}_1|^2w'' \\
&+\Delta_\Gamma\,|A_\Gamma|^2\,p_0-\frac{1}{2}f''(w)|A_\Gamma|^4p_0^2+|A_\Gamma|^4(t+h_0)\,p'\,\big)\,w'\,dt \;=\;0.
\end{aligned}
$$

Here, with standard notation, $|\nabla_\Gamma \mathtt{h}|^2 = a^0_{ij}\partial_i\mathtt{h}\partial_j\mathtt{h}$. According to Proposition 12.2 we can find functions $\mathtt{h}_1$, $\mathtt{h}_2$, $\mathtt{h}_3$ such that

$$\Delta_\Gamma \mathtt{h}_1+|A_\Gamma|^2\,\mathtt{h}_1=\sum_{i=1}^{8}k_i^3,$$

$$\Delta_\Gamma \mathtt{h}_2+|A_\Gamma|^2\,\mathtt{h}_2=\sum_{i=1}^{8}k_i^4,$$

$$\Delta_\Gamma \mathtt{h}_3+|A_\Gamma|^2\,\mathtt{h}_3=|A_\Gamma|^4.$$

Besides, thanks to Lemma 12.2 and the same arguments as in Proposition 12.2, we see that up to higher order terms $|\nabla_\Gamma \mathtt{h}_1|^2 \sim \frac{\beta(\theta)}{r^4}$ and that there is a solution $\mathtt{h}_4$ to

$$\Delta_\Gamma \mathtt{h}_4 + |A_\Gamma|^2\, \mathtt{h}_4 = |\nabla_\Gamma \mathtt{h}_1|^2.$$

Now, we let

$$h_1 := c_1 \mathtt{h}_1, \quad h_2 := c_2 \mathtt{h}_2 + c_3 |A_\Gamma|^2 + (c_4 - c_3)\mathtt{h}_3 + c_5 \mathtt{h}_5,$$

where

$$c_1 := -\frac{\int_0^\infty (t+h_0)^2 w'^2\, dt}{\int_0^\infty w'^2\, dt}, \quad c_2 = -\frac{\int_0^\infty (t+h_0)^3\, w'^2\, dt}{\int_0^\infty w'^2\, dt},$$

$$c_3 = \frac{\int_0^\infty p_0\, w'\, dt}{\int_0^\infty w'^2\, dt}, \quad c_4 = \frac{\int_0^\infty \left[\frac{1}{2} f''(w) p_0^2 - (t+h_0) p_0'\right] w'\, dt}{\int_0^\infty w'^2\, dt}, \quad c_5 = -\frac{1}{2} \frac{w'(0)^2}{\int_0^\infty w'^2\, dt}.$$

Then expression (5.46) becomes

$$\begin{aligned} S(u_0 + \phi_0) = &- \varepsilon^2 \left[\Delta_\Gamma \mathtt{h} + |A_\Gamma|^2\, \mathtt{h}\right] w' \\ &+ \varepsilon^3\, \psi_1(\varepsilon y)\, q_1(t) + \varepsilon^4 \sum_{\ell=2}^{5} \psi_\ell(\varepsilon y)\, q_\ell(t) \\ &+ \mathcal{R}_1(h), \end{aligned} \tag{5.48}$$

where

$$\begin{aligned} \psi_1(\varepsilon y)\, q_1(t) &:= -\sum_{i=1}^{8} k_i^3(\varepsilon y)\, [(t+h_0)^2 w'(t) + c_1 w'(t)], \\ \psi_2(\varepsilon y)\, q_2(t) &:= \Delta_\Gamma\, |A_\Gamma(\varepsilon y)|^2\, [p_0(t) - c_3 w'(t)], \\ \psi_3(\varepsilon y)\, q_3(t) &:= -\sum_{i=1}^{8} k_i^4(\varepsilon y)[(t+h_0)^3 w'(t) + c_2 w'(t)], \\ \psi_4(\varepsilon y)\, q_4(t) &:= |A_\Gamma(\varepsilon y)|^4 \left[\frac{1}{2} f''(w(t))\, p_0^2(t) - (t+h_0)\, p_0'(t) - c_4 w'(t)\right], \\ \psi_5(\varepsilon y)\, q_5(t) &:= -2 \sum_{i=1}^{8} k_i^3(\varepsilon y) h_1(\varepsilon y)\, (t+h_0) w'(t), \\ \psi_6(\varepsilon y)\, q_6(t) &:= |\nabla_\Gamma \mathtt{h}_1(\varepsilon y)|^2\, [w''(t) - c_5 w'(t)]. \end{aligned} \tag{5.49}$$

The constants c_ℓ have been chosen so that

$$\int_0^\infty q_\ell(t)\, w'(t)\, dt = 0, \quad \ell = 1, \ldots, 5.$$

Hence the solution $p_\ell(t)$ to the problem

$$p_\ell''(t) + f'(w(t))\, p_\ell(t) = q_\ell(t), \quad t \in (0,\infty), \quad p_\ell(0) = p_\ell'(0) = 0 \tag{5.50}$$

is bounded. In fact, for all ℓ we have

$$p_\ell(t) = O(t^8 e^{-t}) \quad \text{as } t \to +\infty.$$

Let us set

$$\phi_1(y,t) := \varepsilon^3 \psi_1(\varepsilon y)\, p_1(t) + \varepsilon^4 \sum_{\ell=2}^{5} \psi_\ell(\varepsilon y)\, p_\ell(t)$$

and consider as a new approximation the function

$$u_1 = u_0 + \bar\phi_1$$

where

$$\bar\phi_1 := \phi_0 + \phi_1 .$$

Then, according to formula (5.41), we have that

$$\begin{aligned} S[u_0+\phi_0+\phi_1] \;&=\; S[u_0+\phi_0] \;+\; \Delta\phi_1 + f'(u_0)\phi_1 \\ &\quad + [\,f'(u_0+\phi_0) - f'(u_0)\,]\,\phi_1 + N(u_0+\phi_0,\phi_1). \end{aligned}$$

Hence,

$$S[u_0+\phi_0+\phi_1] \;=\; -\,\varepsilon^2\,[\Delta_\Gamma \mathtt{h} + |A_\Gamma|^2\,\mathtt{h}\,]\,w' \;+ \mathcal{R}_2(h), \tag{5.51}$$

where

$$\begin{aligned} \mathcal{R}_2(h) \;=\; & \varepsilon^3(\Delta_x - \partial_t^2)\,[\psi_1 p_1] \;+\; \varepsilon^4 \sum_{\ell=2}^{5} (\Delta_x - \partial_t^2)[\psi_\ell p_\ell] \\ & +\; \mathcal{R}_1(h) \\ & +\; [\,f'(u_0+\phi_0) - f'(u_0)\,]\,\phi_1 \;+\; N(u_0+\phi_0,\phi_1). \end{aligned} \tag{5.52}$$

Now, we can estimate with the aid of Lemma 5.1 the quantities $(\Delta_x - \partial_t^2)[\psi_\ell p_\ell]$, for instance when $\mathtt{h} = 0$. We see that in all these functions the action of the operator is roughly that of adding two powers of ε in smallness and two powers of $\mathtt{r}_\varepsilon$ in decay. Thus we have that

$$(\Delta_x - \partial_t^2)[\psi_\ell p_\ell] = O(\varepsilon^2 \mathtt{r}_\varepsilon^{-4-\mu} e^{-\gamma t}),$$

sizes that do not change with the introduction of $\mathtt{h}$ that decays as $O(\mathtt{r}^{-1})$. The size of $\mathcal{R}_2(h)$ is thus globally estimated as $O(\varepsilon^3 \mathtt{r}_\varepsilon^{-4-\mu} e^{-\gamma t})$. We will make precise these statements in terms of norms that we introduce next.

5.5. **Norms.** We introduce here several norms that will be used in the rest of the paper. Let Λ be an open set of $\mathbb{R}^N$ or of an embedded submanifold. For a function g defined on Λ we denote, as usual,

$$\|g\|^2_{L^2(\Lambda)} := \int_\Lambda |g|^2, \quad \|g\|_{H^m(\Lambda)} := \int_\Lambda |D^m g|^2 + \int_\Lambda |g|^2, \quad m \geq 1.$$

We consider also the following local-uniform norms

$$\|g\|_{L^2_{l.u.}(\Lambda)} := \sup_{x\in\Lambda} \|g\|_{L^2(B(x,1)\cap\Lambda)}, \quad \|g\|_{H^m_{l.u.}(\Lambda)} := \sup_{x\in\Lambda} \|g\|_{L^2(B(x,1)\cap\Lambda)}. \tag{5.53}$$

For a number $0 < \sigma < 1$ we denote, as customary,

$$[\,g\,]_{\sigma,\Lambda} := \sup\left\{ \frac{|g(y_1) - g(y_2)|}{|y_1 - y_2|^\sigma} \;/\; y_1, y_2 \in \Lambda,\; y_1 \neq y_2 \right\}. \tag{5.54}$$

We let

$$\|g\|_{C^{0,\sigma}(\Lambda)} := \|g\|_{L^\infty(\Lambda)} + [\,g\,]_{\sigma,\Lambda} \tag{5.55}$$

and for $k \geq 1$,

$$\|g\|_{C^{k,\sigma}_0(\Lambda)} := \|g\|_{C^{0,\sigma}(\Lambda)} + \|D^k g\|_{C^{0,\sigma}(\Lambda)}. \tag{5.56}$$

Now, we consider a submanifold Γ in $\mathbb{R}^{m+1}$ and its dilation Γ_ε. We denote as usual,

$$\mathtt{r}(y', y_{m+1}) := \sqrt{1+|y'|^2}, \quad y \in \Gamma, \tag{5.57}$$

and

$$\mathtt{r}_\varepsilon(y) := \mathtt{r}(\varepsilon y), \quad y \in \Gamma_\varepsilon. \tag{5.58}$$

We consider local-uniform weighted Hölder norms involving powers of $\mathtt{r}_\varepsilon$. Let g be a function defined on Γ_ε. For a number $\nu \geq 0$ we define

$$\|g\|_{C^{0,\sigma}_\nu(\Gamma_\varepsilon)} := \|\mathtt{r}_\varepsilon^\nu\, g\|_{L^\infty(\Gamma_\varepsilon)} + [\mathtt{r}_\varepsilon^\nu\, g\,]_{\sigma,\Gamma_\varepsilon}, \tag{5.59}$$

and for $m \geq 1$,

$$\|g\|_{C^{m,\sigma}_\nu(\Gamma_\varepsilon)} := \|D^m_{\Gamma_\varepsilon} g\|_{C^{0,\sigma}_\nu(\Gamma_\varepsilon)} + \|g\|_{C^{0,\sigma}_\nu(\Gamma_\varepsilon)}. \tag{5.60}$$

We will use the same notation to refer to the corresponding norm in $\mathbb{R}^9$ rather than on Γ_ε.

Let us consider now the case of a function $g(y,t)$ defined on the space $\Gamma_\varepsilon \times (0,\infty)$. We consider a norm similar to that above, but that measures also exponential decay in the t-direction. For numbers $\nu, \gamma \geq 0$ we denote

$$\|g\|_{C^{0,\sigma}_{\nu,\gamma}(\Gamma_\varepsilon\times(0,\infty))} := \|\mathtt{r}_\varepsilon^\nu\, e^{\gamma t}\, g\|_{L^\infty(\Gamma_\varepsilon\times(0,\infty))} + [\,\mathtt{r}_\varepsilon^\nu\, e^{\gamma t}\, g\,]_{\sigma,\Gamma_\varepsilon\times(0,\infty)} \tag{5.61}$$

which we observe, is also equivalent to the norm

$$\|\mathtt{r}_\varepsilon^\nu\, e^{\gamma t}\, g\|_{L^\infty(\Gamma_\varepsilon\times(0,\infty))} + \sup_{(y,t)\in\Gamma_\varepsilon\times(0,\infty)} e^{\gamma t}\mathtt{r}_\varepsilon^\nu(y)\,[\,g\,]_{\sigma,\{B((x,t),1)\cap\Gamma_\varepsilon\times(0,\infty)\}}\,. \tag{5.62}$$

We define, correspondingly,

$$\|g\|_{C^{m,\sigma}_{\nu,\gamma}(\Gamma_\varepsilon\times(0,\infty))} := \|g\|_{C^{0,\sigma}_{\nu,\gamma}(\Gamma_\varepsilon\times(0,\infty))} + \|D^m_{\Gamma_\varepsilon} g\|_{C^{0,\sigma}_{\nu,\gamma}(\Gamma_\varepsilon\times(0,\infty))}. \tag{5.63}$$

We shall denote, also, by simplicity

$$\|p\|_{C^{0,\sigma}_0(\Gamma_\varepsilon)} =: \|p\|_{C^{0,\sigma}(\Gamma_\varepsilon)}, \quad \|g\|_{C^{0,\sigma}_{0,0}(\Gamma_\varepsilon\times(0,\infty))} =: \|g\|_{C^{0,\sigma}(\Gamma_\varepsilon)}.$$

The weighted norms introduced above are appropriate for functions that share the same decay properties as its derivatives. It is important for our purposes to consider a different set of norms that are suitably adapted to a function $\mathtt{g}$ defined on a subset Λ of Γ such that when differentiated it gains decay in successive negative powers of $\mathtt{r}(y)$ Let us assume that

$$\|\,\mathtt{r}^\nu\,\mathtt{g}\|_{L^\infty(\Lambda)} < +\infty. \tag{5.64}$$

Roughly speaking, we expect that when differentiated m times, also

$$\|\,\mathtt{r}^{\nu+m}\,D^m\mathtt{g}\|_{L^\infty(\Lambda)} < +\infty.$$

In this context, since the Hölder seminorm (5.54) corresponds roughly to differentiate σ times, then it is natural to require that, besides (5.64), the following quantity be finite:

$$[\mathtt{g}]_{\sigma,\nu,\Gamma} := \sup_{y\in\Gamma} \mathtt{r}(y)^{\nu+\sigma}[\,\mathtt{g}\,]_{\sigma,\{B(y,1)\cap\Gamma\}}.$$

We define

$$\|\mathtt{g}\|_{\sigma,\nu,\Lambda} := \|\mathtt{r}^\nu\mathtt{g}\|_{L^\infty(\Lambda)} + [\mathtt{g}]_{\sigma,\nu,\Gamma}, \tag{5.65}$$

which is actually equivalent to the norm

$$\|\mathtt{r}^\nu\mathtt{g}\|_{L^\infty(\Lambda)} + [\mathtt{r}^{\nu+\sigma}\mathtt{g}]_{\sigma,\Gamma}.$$

Let us observe that if $\mathtt{g}$ is of class $C^1(\Lambda)$ then we have

$$\|\mathtt{g}\|_{\sigma,\nu,\Lambda} \ \leq \ C[\,\|\mathtt{r}^\nu \mathtt{g}\|_{L^\infty(\Lambda)} + \|\mathtt{r}^{\nu+1} D_\Gamma \mathtt{g}\|_{L^\infty(\Lambda)}\,].$$

We define correspondingly

$$\|\mathtt{g}\|_{k,\sigma,\nu,\Lambda} := \|\, D^k_\Gamma \mathtt{g}\|_{\sigma,\nu+k,\Lambda} + \|\mathtt{g}\|_{k,\sigma,\nu,\Lambda}\ . \tag{5.66}$$

5.6. **Connection between norms.** A simple but very important fact is the connection existing between the norms $\|\ \|_{C^{0,\sigma}_\nu(\Gamma_\varepsilon)}$ in (5.59) and $\|\ \|_{\sigma,\nu,\Gamma}$ in (5.65) as described in the following result.

Let p be a function defined on Γ, so that $q(y) := p(\varepsilon y)$ is defined on Γ_ε.

Lemma 5.2. *Let p and q be functions related as above, $\nu > 1$. Then there exists a $C > 0$ such that the following inequalities hold*

$$\|q\|_{C^{0,\sigma}_\nu(\Gamma_\varepsilon)} \ \leq \ C\|p\|_{\sigma,\nu,\Gamma} \tag{5.67}$$

and

$$\|p\|_{\sigma,\nu-\sigma,\Gamma} \ \leq \ C\varepsilon^{-\sigma}\|q\|_{C^{0,\sigma}_\nu(\Gamma_\varepsilon)}. \tag{5.68}$$

Proof. On the one hand, we have that $\|\mathtt{r}_\varepsilon^\nu\, q\|_{L^\infty(\Gamma_\varepsilon)} \ = \ \|\mathtt{r}^\nu\, p\|_{L^\infty(\Gamma)}$. On the other hand, for $y_1, y_2 \in \Gamma_\varepsilon$ we have that for $\tilde y_l := \varepsilon y_l$,

$$\frac{(\mathtt{r}^\nu_\varepsilon q)(y_1) - (\mathtt{r}^\nu_\varepsilon q)(y_2)}{|y_1 - y_2|^\sigma} \ = \ \varepsilon^\sigma \frac{(\mathtt{r}^\nu p)(\tilde y_1) - (\mathtt{r}^\nu q)(\tilde y_2)}{|\tilde y_1 - \tilde y_2|^\sigma}.$$

Assuming that $|\tilde y_1 - \tilde y_2| < 1$ we get

$$\frac{(\mathtt{r}^\nu p)(\tilde y_1) - (\mathtt{r}^\nu p)(\tilde y_2)}{|\tilde y_1 - \tilde y_2|^\sigma} = \mathtt{r}^\nu(\tilde y_1)\frac{p(\tilde y_1) - p(\tilde y_2)}{|\tilde y_1 - \tilde y_2|^\sigma} + p(\tilde y_2)\frac{\mathtt{r}^\nu(\tilde y_1) - \mathtt{r}^\nu(\tilde y_2)}{|\tilde y_1 - \tilde y_2|^\sigma}.$$

Hence

$$\frac{|(\mathtt{r}^\nu_\varepsilon q)(y_1) - (\mathtt{r}^\nu_\varepsilon q)(y_2)|}{|y_1 - y_2|^\sigma} \ \leq \ C\varepsilon^\sigma[\,\|\mathtt{r}^\nu p\|_{L^\infty(B(\tilde y_1,1))} + \mathtt{r}^\nu(\tilde y_1)[p]_{\sigma,B(\tilde y_1,1)}] \leq C\varepsilon^\sigma\|p\|_{\sigma,\nu,\Gamma}.$$

If $|\tilde y_1 - \tilde y_2| \geq 1$ we get that

$$\frac{|(\mathtt{r}^\nu_\varepsilon q)(y_1) - (\mathtt{r}^\nu_\varepsilon q)(y_2)|}{|y_1 - y_2|^\sigma} \ \leq \ C\varepsilon^\sigma\|\mathtt{r}^\nu p\|_{L^\infty(\Gamma)}.$$

As a conclusion, we get the validity of inequality (5.67).

Now, let us consider the inequality in the opposite direction. Let $\tilde y_0 \in \Gamma$ and consider $\tilde y_1, \tilde y_2 \in B(\tilde y_0, 1) \cap \Gamma$, and set correspondingly $y_l = \varepsilon^{-1}\tilde y_l$. Let us assume first that $|\tilde y_1 - \tilde y_2| \leq \varepsilon$. We have that

$$\mathtt{r}^\nu(\tilde y_0)\frac{|p(\tilde y_1) - p(\tilde y_2)|}{|\tilde y_1 - \tilde y_2|^\sigma} \ \leq \ C\mathtt{r}^\nu_\varepsilon(y_0)\varepsilon^{-\sigma}\frac{|q(y_1) - q(y_2)|}{|y_1 - y_2|^\sigma} \ \leq C\mathtt{r}^\nu_\varepsilon(y_0)\varepsilon^{-\sigma}[\,q\,]_{\sigma,B(y_0,1)}.$$

On the other hand if $|\tilde y_1 - \tilde y_2| > \varepsilon$ we have

$$\mathtt{r}^\nu(\tilde y_0)\frac{|p(\tilde y_1) - p(\tilde y_2)|}{|\tilde y_1 - \tilde y_2|^\sigma} \ \leq \ C\varepsilon^{-\sigma}\|\mathtt{r}^\nu_\varepsilon\, q\|_{L^\infty(\Gamma_\varepsilon)}.$$

Combining these two inequalities yields

$$[\,p\,]_{\sigma,\nu-\sigma,\Gamma} \ \leq \ C\varepsilon^{-\sigma}\|q\|_{C^{0,\sigma}_\nu}.$$

Hence, we have obtained the inequality (5.68) and the proof is concluded. ∎

Remark 5.1. *A typical term to which we want to measure its size in $\Gamma_\varepsilon \times (0,\infty)$ has the form*

$$g(y,t) \;=\; a(y,t)\, p(\varepsilon y)\, \zeta(t)$$

where ζ is such that $\zeta(t) = O(e^{-\gamma t})$ as $t \to +\infty$, as well as its derivatives. Arguing as in the proof of Lemma 5.2 we find the estimate:

$$\|g\|_{C^{0,\sigma}_{\nu,\gamma}(\Gamma_\varepsilon \times (0,\infty))} \;\le\; C\, \|a\|_{C^{0,\sigma}(\Gamma_\varepsilon\times(0,\infty))}\, \|p\|_{\sigma,\nu,\Gamma}.$$

5.7. **Conclusion of the construction of the first approximation and computation of error size.** We consider then h of the form

$$h(y) \;=\; h_0 + \varepsilon h_1(y) + \varepsilon^2 h_2(y) + \mathtt{h}(y), \quad y \in \Gamma. \tag{5.69}$$

On the ε-dependent parameter function $\mathtt{h}$, we assume in what follows that for some fixed $\mu > 0$,

$$\|\mathtt{h}\|_{2,2+\mu,\sigma,\Gamma} := \|D^2_\Gamma \mathtt{h}\|_{4+\mu,\sigma,\Gamma} + \|D_\Gamma \mathtt{h}\|_{3+\mu,\sigma,\Gamma} + \|\mathtt{h}\|_{2+\mu,\sigma,\Gamma} \;\le\; \varepsilon. \tag{5.70}$$

We observe that from Corollary 12.1, we have that the functions h_1 and h_2 satisfy

$$\|h_1\|_{2,1,\sigma,\Gamma} \;<\; +\infty, \quad \|h_2\|_{2,2-\tau,\sigma,\Gamma} \;<\; +\infty, \tag{5.71}$$

for any small $\tau > 0$. Since h_0 is a constant, we point out that we have in particular

$$|\nabla_\Gamma h(y)| \le C\varepsilon \mathtt{r}(y)^{-2}. \tag{5.72}$$

The approximation already built, $u_0 + \phi_0 + \phi_1$ is sufficient for our purposes, except that it is only defined near Γ^h_ε. Since we have that

$$w(t) = 1 - O(e^{-\gamma t}) \quad \text{as } t \to +\infty$$

we consider a simple interpolation with the function 1. We let $\eta(s)$ be a smooth function with $\eta(s) = 1$ is $s < 1$ and $= 0$ if $s > 2$. For a sufficiently small $\delta > 0$ we let

$$\eta_m(x) = \eta(s), \quad s = \varepsilon t - m\delta, \quad x = y + (t + h(\varepsilon y))\, \nu(\varepsilon y)$$

understood this function as identically zero at any point outside its support. Thus we set

$$u_1(x) = \eta_{10}(x)(u_0 + \phi_0 + \phi_1) + (1 - \eta_{10}(x))\,(+1), \quad s = \varepsilon t - 10\delta\, \mathtt{r}_\varepsilon(y) \tag{5.73}$$

where the function is understood to be identically equal to $+1$, all over the space, outside the support of η_{10}.

5.8. **Error size and Lipschitz property.** We shall investigate the size of the error in terms of the Hölder type norms introduced in the previous section, as well as its Lipschitz dependence on $\mathtt{h}$. The main part of the error of approximation is of course in the region close to Γ_ε. In reality, we can consider the error as a function defined in the entire space $(y,t) \in \Gamma_\varepsilon \times (0,\infty)$, by setting

$$E(y,s) \;:=\; \eta_3 S[u_1] \;=\; \eta_3\, S[u_0 + \phi_0 + \phi_1]\ , \tag{5.74}$$

where this function understood to be identically zero outside its support.

Let us consider the expansion (5.46) of the first approximation error and the operator $\mathcal{R}_1(h)$, there appearing, defined in (5.47) as

$$\begin{aligned}
\mathcal{R}_1(h) \;=\; & \varepsilon^2 a_{ij}\,\partial_i h\,\partial_j h\, w'' \\
& -\;\varepsilon^3 \sum_{i=1}^{8} k_i^3 [\,(t+h)^2 - (t+h_0)^2 - 2(t+h_0)\varepsilon h_1\,] w' \\
& -\;\varepsilon^4 \sum_{i=1}^{8} k_i^4 [(t+h)^3 - (t+h_0)^3] w' \\
& +\;\varepsilon^5 (t+h)\big[a_{ij}^1\,\partial_{ij}|A_\Gamma|^2 \,+\, b_j^1 \partial_j |A_\Gamma|^2\,\big]\, p_0 \\
& -\;\varepsilon^3 (t+h)\,[a_{ij}^1\,\partial_{ij} h \,+\, b_j^1\,\partial_j h\,]\, w' \,+\, \varepsilon^5\,(t+h)^4 \theta\, w' \\
& +\;\varepsilon^4 |A_\Gamma|^2\, a_{ij}\,\partial_i h\,\partial_j h\, p_0'' - 2\varepsilon^3 a_{ij}\,\partial_i h\,\partial_j |A_\Gamma|^2\, p_0' \\
& -\;\varepsilon^4 \left\{ [a_{ij}\,\partial_{ij} h \,+\, b_j\,\partial_j h\,] \,+\, |A_\Gamma|^2 (h-h_0) \,+\, \varepsilon \sum_{i=1}^{8} k_i^3 (t+h)^2 + \cdots \right\} |A_\Gamma|^2 p' \\
& +\;\frac{\varepsilon^6}{6} p_0^3\, |A_\Gamma|^6 \int_0^1 f'''(w + s\phi_0)\, ds\,.
\end{aligned} \tag{5.75}$$

Let us consider for instance the term,

$$\mathcal{R}_{11}(h) = \varepsilon^2\, a_{ij}(\varepsilon y, \varepsilon(t+h))\,\partial_i h\,\partial_j h\, w''$$

where we recall

$$h = h_0 + \varepsilon h_1 + \varepsilon^2 h_2 + \mathtt{h}$$

with $\mathtt{h}$ satisfying (5.70). Then we have

$$|\eta_3 \mathcal{R}_{11}(h)| \;\le\; C\,\varepsilon^2\, |D_\Gamma h(\varepsilon y)|^2\, e^{-\gamma t} \le C\,\varepsilon^4\, \mathtt{r}_\varepsilon^{-4-\mu}\, e^{-\gamma t},$$

so that

$$\| e^{\gamma t} \mathtt{r}_\varepsilon^{4+\mu} \eta_3 \mathcal{R}_1(h) \|_{L^\infty(\Gamma_\varepsilon \times (0,\infty))} \;\le\; C\,\varepsilon^4\,.$$

Using Remark 5.1, we find moreover that

$$\|\eta_3 \mathcal{R}_{11}(h)\|_{C^{0,\sigma}_{4+\mu,\gamma}(\Gamma_\varepsilon \times (0,\infty))} \;\le\; C\,\varepsilon^4.$$

Similar estimates are obtained for the remaining terms in $\mathcal{R}_1(h)$. We then find

$$\|\eta_3 \mathcal{R}_1(h)\|_{C^{0,\sigma}_{4+\mu,\gamma}(\Gamma_\varepsilon \times (0,\infty))} \;\le\; C\,\varepsilon^4.$$

We want to investigate next the Lipschitz character of the operator $\mathcal{R}_1(h)$. Let us consider again our model operator $\mathcal{R}_{11}(h)$. We have that

$$\begin{aligned}
\varepsilon^{-2}[\eta_3 \mathcal{R}_{11}(h_1) - \mathcal{R}_{11}(h_2)] \;=\; & [\,\eta_3 a_{ij}(\varepsilon y, \varepsilon(t+h_1)) \,-\, \eta_3 a_{ij}(\varepsilon y, \varepsilon(t+h_2))\,]\,\partial h_i^1 \partial h_j^1\, w'' \\
& + \eta_3 a_{ij}(\varepsilon y, \varepsilon(t+h_2))\,[\,\partial h_i^1 \partial h_j^1 - \partial h_i^2 \partial h_j^2\,]\, w'',
\end{aligned}$$

and hence from Remark 5.1

$$\varepsilon^{-2} \|\eta_3 \mathcal{R}_{11}(h_1) - \eta_3 \mathcal{R}_{11}(h_2)\|_{C^{0,\sigma}_{4+\mu,\gamma}(\Gamma_\varepsilon \times (0,\infty))} \;\le$$

$$\begin{aligned}
C \|\eta_3 a_{ij}(\varepsilon y, \varepsilon(t+h_1)) \,-\, \eta_3 a_{ij}(\varepsilon y, \varepsilon(t+h_2))\,\|_{C^{0,\sigma}(\Gamma_\varepsilon \times (0,\infty))}\; \|\partial h_i^1 \partial h_j^1\,\|_{\sigma, 4+\mu, \Gamma} \\
+\, C \|\eta_3 a_{ij}(\varepsilon y, \varepsilon(t+h_2))\,\|_{C^{0,\sigma}(\Gamma_\varepsilon \times (0,\infty))}\; \|\partial h_i^1 \partial h_j^1 - \partial h_i^2 \partial h_j^2\,\|_{\sigma, 4+\mu, \Gamma}.
\end{aligned}$$

Thus

$$\|\eta_3 \mathcal{R}_{11}(h_1) - \eta_3 \mathcal{R}_{11}(h_2)\|_{C^{0,\sigma}_{4+\mu,\gamma}(\Gamma_\varepsilon\times(0,\infty))} \le C\varepsilon^3 \|D_\Gamma h_1 - D_\Gamma h_1\|_{\sigma,2+\mu,\Gamma} + C\varepsilon^5 \|h_1 - h_1\|_{\sigma,0,\Gamma}.$$

Similar estimates are obtained for the remaining terms. In all we have,

$$\|\eta_3 \mathcal{R}_1(h_1) - \eta_3 \mathcal{R}_1(h_2)\|_{C^{0,\sigma}_{4+\mu,\gamma}(\Gamma_\varepsilon\times(0,\infty))} \le C\varepsilon^3 \|h_1 - h_1\|_{2,\sigma,2+\mu,\Gamma} \tag{5.76}$$

uniformly on h_1, h_2 satisfying (5.70).

Now, let us consider expression (5.51) for the error at $u_0 + \phi_0 + \phi_1$

$$S[u_0 + \phi_0 + \phi_1] = -\varepsilon^2 \left[\Delta_\Gamma \mathtt{h} + |A_\Gamma|^2 \,\mathtt{h}\right] w' + \mathcal{R}_2(h), \tag{5.77}$$

where

$$\begin{aligned} \mathcal{R}_2(h) &= \varepsilon^3 (\Delta_x - \partial_t^2)\,[\psi_1 p_1] + \varepsilon^4 \sum_{\ell=2}^{5} (\Delta_x - \partial_t^2)[\psi_\ell p_\ell] + \mathcal{R}_1(h) \\ &\quad + [f'(u_0 + \phi_0) - f'(u_0)]\,\phi_1 + N(u_0 + \phi_0, \phi_1). \end{aligned} \tag{5.78}$$

The terms $(\Delta_x - \partial_t^2)[\psi_\ell p_\ell]$ are of the same nature as those in the operator $\mathcal{R}_1(h)$, which was computed from $(\Delta_x - \partial_t^2)[|A_\Gamma|^2 p_0]$. We get in fact extra smallness and decay. Therefore we get

$$\|\eta_3 \mathcal{R}_2(h)\|_{C^{0,\sigma}_{4+\mu,\gamma}(\Gamma_\varepsilon\times(0,\infty))} \le C\,\varepsilon^4$$

and

$$\|\eta_3 \mathcal{R}_2(h_1) - \eta_3 \mathcal{R}_2(h_2)\|_{C^{0,\sigma}_{4+\mu,\gamma}(\Gamma_\varepsilon\times(0,\infty))} \le C\varepsilon^3 \|h_1 - h_1\|_{2,\sigma,2+\mu,\Gamma} \tag{5.79}$$

uniformly on h, h_1, h_2 satisfying (5.70).

As a conclusion of the above considerations it follows that we can write the error (5.74) as

$$E = -\varepsilon^2 \left[\Delta_\Gamma \mathtt{h} + |A_\Gamma|^2 \mathtt{h}\right] w' + \mathcal{R}_3(\mathtt{h}) \tag{5.80}$$

where the operator $\mathcal{R}_3(\mathtt{h})$ satisfies

$$\|\mathcal{R}_3(h)\|_{C^{0,\sigma}_{4+\mu,\gamma}(\Gamma_\varepsilon\times(0,\infty))} \le C\,\varepsilon^4 \tag{5.81}$$

and

$$\|\mathcal{R}_3(h_1) - \mathcal{R}_3(h_2)\|_{C^{0,\sigma}_{4+\mu,\gamma}(\Gamma_\varepsilon\times(0,\infty))} \le C\varepsilon^3 \|h_1 - h_1\|_{2,\sigma,2+\mu,\Gamma}\,. \tag{5.82}$$

6. The gluing reduction

We want to solve the problem

$$S(u) := \Delta u + f(u) = 0 \quad \text{in } \Omega^h_\varepsilon$$

$$u = 0, \quad \partial_\nu u = \mathit{constant} \quad \text{on } \Gamma^h_\varepsilon$$

where we have our first approximation u_1, built in §5.7. We write $u = u_1 + \tilde\phi$. We recall that

$$\partial_\nu u_0 = w'(0) = \mathit{constant} \quad \text{on } \Gamma^h_\varepsilon.$$

Thus the problem gets rewritten as

$$\begin{aligned} \Delta\tilde\phi + f'(u_1)\tilde\phi + N(\tilde\phi) + E = 0 \quad &\text{in } \Omega^h_\varepsilon \\ \tilde\phi = 0, \quad \partial_\nu\tilde\phi = 0 \quad &\text{on } \Gamma^h_\varepsilon \\ E = S[u_1], \quad N(\tilde\phi) = f(u_1 + \tilde\phi) - f(u_1) - f'(u_1)\tilde\phi. & \end{aligned} \tag{6.1}$$

We recall that we set in §5.7,

$$\eta_m(y,t) = \eta(m^{-1}(\varepsilon t - \delta \mathbf{r}_\varepsilon(y))) \ .$$

We look for a solution of Problem (6.1) with the form

$$\tilde\phi = \eta_2 \phi + \psi,$$

with $\phi(y,t)$ is defined in entire $\Gamma_\varepsilon \times (0,\infty)$. We obtain a solution to Problem (6.1) if we solve the following system

$$\begin{aligned} \partial_t^2\phi + \Delta_{\Gamma_\varepsilon}\phi + f'(w(t))\phi &= -\eta_3[S[u_1] + B\phi + N(\eta_1\phi+\psi) + (f'(u_0)+1)\psi] \quad \text{in } \Gamma_\varepsilon \times (0,\infty),\\ \phi(y,0) &= 0 \quad \text{for all} \quad y \in \Gamma_\varepsilon,\\ \partial_t\phi(y,0) &= -\partial_t\psi(y,0) \quad \text{for all} \quad y \in \Gamma_\varepsilon, \end{aligned} \tag{6.2}$$

$$\begin{aligned} \Delta\psi - \psi &= -[2\nabla\phi\cdot\nabla\eta_1 + \phi\Delta\eta_1\,] - (1-\eta_1)\,[\,(f'(u_1)+1)\,\psi + S[u_1] + N(\eta_1\phi+\psi)\,] \quad \text{in } \Omega_\varepsilon\\ \psi &= 0 \quad \text{on } \Gamma_\varepsilon. \end{aligned} \tag{6.3}$$

We have the following result.

Lemma 6.1. *Let ϕ be given with $\|\phi\|_{C^{2,\sigma}_{\nu,\gamma}(\Gamma_\varepsilon\times(0,\infty))} < 1$ and $\nu \geq 2$. Then equation (6.3) can be solved as $\psi = \Psi(\phi)$ where for some $a>0$,*

$$\|\Psi(\phi_1) - \Psi(\phi_2)\|_{C^{2,\sigma}_{\nu}(\Omega_\varepsilon)} \leq e^{-\frac{a}{\varepsilon}}\|\phi_1-\phi_2\|_{C^{2,\sigma}_{\nu,\gamma}(\Gamma_\varepsilon\times(0,\infty))},$$

$$\|\Psi(0)\|_{C^{2,\sigma}_{\nu}(\Omega_\varepsilon)} \ \leq \ e^{-\frac{a}{\varepsilon}}.$$

Proof. Let us consider first the linear problem

$$\Delta\psi - \psi = g \quad \text{in } \Omega_\varepsilon, \quad \psi = 0 \quad \text{on } \Gamma_\varepsilon^h. \tag{6.4}$$

By a standard barrier argument, this problem has a unique bounded solution $\psi = T(g)$ if g is bounded. In fact

$$\|\psi\|_{L^\infty(\Omega_\varepsilon)} \leq \|g\|_{L^\infty(\Omega_\varepsilon)}.$$

Let us further assume that $\|g\|_{C^{0,\sigma}(\Omega_\varepsilon)} < +\infty$. Applying local boundary and interior Schauder estimates we also get

$$\|\psi\|_{C^{2,\sigma}(\Omega_\varepsilon)} \leq C\|g\|_{C^{0,\sigma}(\Omega_\varepsilon)} \tag{6.5}$$

for a constant C uniform in all small ε. Finally, by writing $\psi = r_\varepsilon^{-\nu}\tilde\psi$, examining the equation for $\tilde\psi$ and using estimate (6.5) we obtain that

$$\|T(g)\|_{C^{2,\sigma}_{\nu}(\Omega_\varepsilon)} \ \leq \ C\|g\|_{C^{0,\sigma}_{\nu}(\Omega_\varepsilon)}. \tag{6.6}$$

Now, we can solve Problem (6.3) as the fixed point problem

$$\psi = -T(\,[2\nabla\phi\cdot\nabla\eta_1 + \phi\Delta\eta_1\,] + (1-\eta_1)\,[\,(f'(u_0)+1)\,\psi + E + N(\eta_1\phi+\psi)\,]\,).$$

It is readily checked that the right hand side of this equation defines a contraction mapping in a region of the form $\|\psi\|_{C^{2,\sigma}_{\nu}(\Omega_\varepsilon)} \ \leq \ e^{-\frac{a}{\varepsilon}}$. Banach fixed point then gives a solution of this problem with the desired properties. □

Through this lemma we have reduced our original equation to solving the nonlinear, nonlocal problem for a ϕ with $\|\phi\|_{C^{2,\sigma}_{\nu,\gamma}(\Gamma_\varepsilon\times(0,\infty))} < 1$,

$$\begin{aligned} \partial_t^2\phi + \Delta_{\Gamma_\varepsilon}\phi + f'(w(t))\phi \;&= -E - \mathcal{N}(\phi) \quad \text{in } \Gamma_\varepsilon \times (0,\infty),\\ \phi(y,0) = 0 \quad &\text{for all} \quad y \in \Gamma_\varepsilon,\\ \partial_t\phi(y,0) = -\partial_t\Psi(\phi)(y,0) \quad &\text{for all} \quad y \in \Gamma_\varepsilon, \end{aligned} \tag{6.7}$$

where

$$\mathcal{N}(\phi) := \eta_3[\,[f'(u_1) - f'(u_0)]\,\phi + B\phi + N(\eta_1\phi + \Psi(\phi)) + (f'(u_1)+1)\Psi(\phi)], \tag{6.8}$$

and as in (5.74),

$$E \;:=\; \eta_3\, S[u_1].$$

Let us recall that we decomposed in (5.80)

$$E \;=\; -\varepsilon^2(\Delta_\Gamma \mathtt{h} + |A_\Gamma|^2\mathtt{h})\, w'(t) + \mathcal{R}_3(\mathtt{h}),$$

where $\mathcal{R}_3(\mathtt{h})$ is a small operator, satisfying (5.81), (5.82).

6.1. **The projected problem.** Rather than solving Problem 6.7 directly, we consider a projected version of it, namely the problem of finding ϕ and α such that

$$\begin{aligned} \partial_t^2\phi + \Delta_{\Gamma_\varepsilon}\phi + f'(w(t))\phi \;&=\; \alpha(y)\, w'(t) \;-\mathcal{R}_3(\mathtt{h}) - \mathcal{N}(\phi) \quad \text{in } \Gamma_\varepsilon \times (0,\infty),\\ \phi(y,0) = 0 \quad &\text{for all} \quad y \in \Gamma_\varepsilon,\\ \partial_t\phi(y,0) = -\partial_t\Psi(\phi)(y,0) \quad &\text{for all} \quad y \in \Gamma_\varepsilon. \end{aligned} \tag{6.9}$$

7. Linearized problem in a half-space

In order to solve Problem (6.9) we shall develop a uniform invertibility theory for the associated linear problem, so that we later proceed just by contraction mapping principle to solve the nonlinear equation. This is also the procedure to prove the results of Theorems 5 and 6 so that we consider a more general surface Γ in Euclidean space $\mathbb{R}^{m+1}$, $m \geq 1$. We consider in this section the (linear) problem in $\mathbb{R}^{m+1}$ of finding, for given functions $g(y,t)$, $\beta(y)$, a solution (α,ϕ) to the problem

$$\begin{aligned} \Delta\phi \;+\; f'(w(t))\phi \;&= \alpha(y)\, w'(t) + g(y,t) \quad \text{in } \mathbb{R}^{m+1}_+,\\ \phi(y,0) \;=\; 0 \qquad &\text{for all} \quad y \in \mathbb{R}^m,\\ \partial_t\phi(y,0) \;=\beta(y) \quad &\text{for all} \quad y \in \mathbb{R}^m. \end{aligned} \tag{7.1}$$

Here $\mathbb{R}^{m+1}_+ := \mathbb{R}^m \times (0,\infty)$. We want to solve Problem (7.1) using Hölder norms. The main result of this section is the following

Proposition 7.1. *Given β and g such that*

$$\|\beta\|_{C^{1,\sigma}(\mathbb{R}^m)} \;+\; \|g\|_{C^{0,\sigma}(\mathbb{R}^{m+1}_+)} \;<\; +\infty$$

there exists a solution $(\phi,\alpha) \in C^{2,\sigma}(\mathbb{R}^{m+1}_+) \times C^{0,\sigma}(\mathbb{R}^m)$ of Problem (7.1) that defines a linear operator of the pair (β,g), satisfying the estimate

$$\|\phi\|_{C^{2,\sigma}(\mathbb{R}^{m+1}_+)} + \|\alpha\|_{C^{0,\sigma}(\mathbb{R}^m)} \;\leq\; C\,[\,\|\beta\|_{C^{1,\sigma}(\mathbb{R}^m)} \;+\; \|g\|_{C^{0,\sigma}(\mathbb{R}^{m+1}_+)}\,]\,. \tag{7.2}$$

We will first solve this problem in an L^2 setting by means of Fourier transform and then use classical elliptic regularity theory to solve it in Hölder spaces.

We consider first the special case $g = 0$, namely the problem of finding, for a given function $\beta(y)$ defined in $\mathbb{R}^m$, functions $\alpha(y)$ and $\phi(y,t)$ that solve the problem

$$\begin{aligned} \Delta\phi + f'(w(t))\phi &= \alpha(y)\, w'(t) \quad \text{in } \mathbb{R}^{m+1}_+, \\ \phi(y,0) &= 0 \qquad \text{for all} \quad y \in \mathbb{R}^m, \\ \partial_t\phi(y,0) &= \beta(y) \quad \text{for all} \quad y \in \mathbb{R}^m. \end{aligned} \tag{7.3}$$

We will solve first this problem in L^2 by means of Fourier transform. We have the following result.

Lemma 7.1. *Given $\beta \in H^1(\mathbb{R}^m)$, there exists a unique solution*

$$(\phi, \alpha) \in H^2(\mathbb{R}^{m+1}_+) \times L^2(\mathbb{R}^m)$$

of Problem (7.3) that defines a linear operator of β. Besides, we have the estimate

$$\|\phi\|_{H^2(\mathbb{R}^{m+1}_+)} + \|\alpha\|_{L^2(\mathbb{R}^m)} \le C\, \|\beta\|_{H^1(\mathbb{R}^m)}\,. \tag{7.4}$$

Proof. Let us assume for the moment that b is a smooth, rapidly decaying function. Then we can write Problem (7.3) in terms of Fourier transforms for its unknowns, $\hat\phi(\xi,t)$, $\hat\alpha(\xi)$ as

$$\begin{aligned} \partial_t^2\hat\phi - |\xi|^2\hat\phi + f'(w(t))\hat\phi &= \hat\alpha(\xi)\, w'(t) \quad \text{in } \mathbb{R}^m \times (0,\infty), \\ \hat\phi(\xi,0) &= 0 \qquad \text{for all} \quad \xi \in \mathbb{R}^m, \\ \partial_t\hat\phi(\xi,0) &= \hat\beta(\xi) \quad \text{for all} \quad \xi \in \mathbb{R}^m. \end{aligned} \tag{7.5}$$

Let us consider first the ODE problem

$$p_0''(t) + f'(w(t))\, p_0(t) \;=\; -w'(t) \quad \text{in } (0,\infty), \quad p_0(0) \;=\; 0\,. \tag{7.6}$$

This equation has a unique bounded solution, given by

$$p_0(t) \;=\; w'(t) \int_0^t \frac{d\tau}{w'(\tau)^2} \int_\tau^\infty w'(s)^2\, ds\ .$$

In particular, $p_0'(0) = w'(0)^{-1} \int_0^\infty w'(t)^2\, dt > 0$. Similarly, for $\xi \neq 0$ the equation

$$\begin{aligned} p_\xi''(t) - |\xi|^2 p_\xi + f'(w(t))\, p_\xi(t) &= -\, w'(t) \quad \text{in } (0,\infty), \\ p_\xi(0) &= 0 \end{aligned}$$

has a unique bounded solution, which by maximum principle it is positive. Since $p_\xi''(0) = -w'(0) < 0$, we must have $p_\xi'(0) > 0$. This last value defines a smooth function of ξ. On the other hand for large values of ξ we have that $p_\xi'(0) \approx q_\xi'(0)$, where q_ξ solves

$$\begin{aligned} q_\xi''(t) - |\xi|^2 q_\xi(t) &= -\, w'(t) \quad \text{in } (0,\infty), \\ q_\xi(0) &= 0\,. \end{aligned}$$

Thus we have, for large values of $|\xi|$,

$$p_\xi'(0) \approx q_\xi'(0) = \int_0^\infty w'(t)\, e^{-|\xi| t}\, dt \approx \frac{w'(0)}{|\xi|}.$$

The solution of problem (7.5) is then given by

$$\hat\alpha(\xi) = -\frac{1}{p_\xi'(0)}\hat\beta(\xi), \quad \hat\phi(\xi,t) = \hat\alpha(\xi) q_\xi(t).$$

Observe in particular that

$$\int_{\mathbb{R}^m} |\hat\alpha(\xi)|^2\, d\xi \;\le\; c\int_{\mathbb{R}^m} (1+|\xi|^2)|\hat\beta(\xi)|^2\, d\xi.$$

Now, from the fact that for some $a>0$ and any function $q\in H_0^1(0,\infty)$ we have that

$$a\int_0^\infty |q|^2\, dt \le \int_0^\infty (\,|q'|^2 - f'(w)q^2)\, dt. \tag{7.7}$$

We deduce that for some $C>0$, and any β,

$$\int_0^\infty \int_{\mathbb{R}^m} [\,|\partial_t\hat\phi|^2 + (1+|\xi|^2)\,|\hat\phi|^2\, d\xi\, dt \;\le\; C\int_{\mathbb{R}^m} (1+|\xi|^2)\,|\hat\beta(\xi)|^2\, d\xi\ .$$

Taking inverse Fourier transform of $\hat\phi(\xi,t)$ and $\hat\alpha(\xi)$ we obtain a solution (ϕ,α) to the original problem (7.3). The above inequalities translate into

$$\int_{\mathbb{R}^m} |\alpha|^2\, dy + \int_{\mathbb{R}^{m+1}_+} [\,|\nabla\phi|^2 + |\phi|^2\,]\, dy\, dt \;\le\; C\int_{\mathbb{R}^m} [\,|\nabla\beta|^2 + |\beta|^2\,]\, dy. \tag{7.8}$$

By density, and the standard L^2 regularity theory, we get, given $\beta\in H^1(\mathbb{R}^m)$ the existence of a solution $(\phi,\alpha)\in H^2(\mathbb{R}^{m+1}_+)\times L^2(\mathbb{R}^m)$. This solution satisfies the desired estimate. Finally, for uniqueness we need to prove that if $\beta=0$ in Problem (7.3), then α and ϕ vanish identically. Taking Fourier transform we arrive at the following family of ODEs in $H^2(\mathbb{R})$

$$\begin{aligned}
\partial_t^2\hat\phi - |\xi|^2\hat\phi + f'(w(t))\hat\phi &= \hat\alpha(\xi)\, w'(t) \quad \text{in } \mathbb{R}^m\times(0,\infty),\\
\hat\phi(\xi,0) &= 0 \qquad \text{for all} \quad \xi\in\mathbb{R}^m,\\
\partial_t\hat\phi(\xi,0) &= 0 \quad \text{for all} \quad \xi\in\mathbb{R}^m.
\end{aligned} \tag{7.9}$$

For $\xi\neq 0$ we see that setting

$$\phi = p - \frac{\hat\alpha(\xi)}{|\xi|^2} w'(t),$$

then p satisfies as a function of t,

$$p'' - |\xi|^2 p + f'(w(t))p = 0 \quad \text{in } (0,\infty), p'(0) = 0 \quad \text{for all} \quad \xi\in\mathbb{R}^m. \tag{7.10}$$

The function p and its equation can be evenly extended to all $\mathbb{R}$. Then

$$\int_{\mathbb{R}} (|p'|^2 - f'(w)|p|^2)\, dt + \int_{\mathbb{R}} |\xi|^2|p|^2 = 0.$$

Since the first integral above is always nonnegative, we get that $p\equiv 0$. In particular this tells us that

$$0 = \phi(\xi,0) = -\frac{\hat\alpha(\xi)}{|\xi|^2} w'(0),$$

and hence $\hat\alpha(\xi)=0$ almost everywhere in ξ. We conclude that $\alpha\equiv 0$. The fact that $\hat\phi\equiv 0$ comes directly testing its equation against ϕ itself. The proof is concluded. □

In order to solve Problem (7.1) and for later purposes, we consider the problem

$$\begin{aligned} \Delta\phi \,+\, f'(w(t))\phi \,=\, g(y,t) \quad \text{in } \mathbb{R}^{m+1}_+, \\ \phi(y,0) \,=\, 0 \qquad \text{for all} \quad y\in\mathbb{R}^m, \end{aligned} \tag{7.11}$$

and establish the following result.

Lemma 7.2. *Assume that $g \in L^2(\mathbb{R}^{m+1}_+)$. Then Problem (7.11) has a unique solution $\phi \in H^2(\mathbb{R}^{m+1}_+)$. This solution satisfies in addition*

$$\|\phi\|_{H^2(\mathbb{R}^{m+1}_+)} \,\le\, C\,\|g\|_{L^2(\mathbb{R}^{m+1}_+)} \tag{7.12}$$

and if $\phi \in H^1(\mathbb{R}^{m+1}_+)$,

$$\|\partial_t\phi(y,0)\|_{H^1(\mathbb{R}^m)} \,\le\, C\,\|g\|_{H^1(\mathbb{R}^{m+1}_+)}. \tag{7.13}$$

Proof. As in the previous arguments, we consider the version of (7.11) after Fourier transform,

$$\begin{aligned} \partial_t^2\hat\phi - |\xi|^2\hat\phi \,+\, f'(w(t))\hat\phi \,=\hat g(\xi,t) \quad \text{in } \mathbb{R}^m\times(0,\infty), \\ \hat\phi(\xi,0) \,=\, 0 \qquad \text{for all} \quad \xi\in\mathbb{R}^m. \end{aligned}$$

Using inequality (7.7), we see that for each ξ this problem can be solved uniquely in such a way that

$$\int_0^\infty [\,|\partial_t\hat\phi|^2 + (1+|\xi|^2)\,|\hat\phi|^2\,]\,dt \;\le\; C\int_0^\infty |\hat g|^2\,dt \tag{7.14}$$

so that

$$\int_{\mathbb{R}^{m+1}_+} [\,|\partial_t\hat\phi|^2 + (1+|\xi|^2)\,|\hat\phi|^2\,]\,d\xi\,dt \;\le\; C\int_{\mathbb{R}^{m+1}_+} |\hat g|^2\,d\xi\,dt\,.$$

By taking Fourier transform back, and then using L^2 regularity we get a solution ϕ satisfying (7.12). Now, testing equation (7.11) against $e^{-|\xi|t}$ and setting $\hat\beta(\xi) := \partial_t\hat\phi(\xi,0)$ we see that

$$\hat\beta(\xi) = -\int_0^\infty f'(w(t))e^{-|\xi|t}\,\hat\phi(\xi,t)\,dt + \int_0^\infty f'(w(t))e^{-|\xi|t}\,\hat g(\xi,t)\,dt.$$

Hence

$$|\hat\beta(\xi)| \le -C\left(\int_0^\infty e^{-2|\xi|t}\,dt\right)^{\frac12}\left(\int_0^\infty [\,|\hat\phi(\xi,t)|^2 + \hat g(\xi,t)|^2\,dt\right)^{\frac12}.$$

Using inequality (7.14) we then get that

$$(1+|\xi|^2)|\hat\beta(\xi)|^2 \;\le\; C\int_0^\infty (1+|\xi|)|\hat g(\xi,t)|^2\,dt.$$

From here, estimate (7.13) immediately follows. Observe that the control is in reality stronger. In terms of fractional Sobolev spaces we have

$$\|\beta\|_{H^1(\mathbb{R}^m)} \;\le\; C\|g\|_{H^{\frac12}(\mathbb{R}^{m+1}_+)}$$

but we will not need this. The proof is concluded. ∎

Using Lemmas 7.1, 7.2 and simple superposition we conclude the following result.

Lemma 7.3. *Given $\beta \in H^1(\mathbb{R}^m)$, $g \in H^1(\mathbb{R}^{m+1}_+)$, there exists a unique solution*

$$(\phi, \alpha) \in H^2(\mathbb{R}^{m+1}_+) \times L^2(\mathbb{R}^m)$$

of Problem (7.1). This solution defines a linear operator of the pair (β, g), that satisfies the estimate

$$\|\phi\|_{H^2(\mathbb{R}^{m+1}_+)} + \|\alpha\|_{L^2(\mathbb{R}^m)} \le C\,[\,\|\beta\|_{H^1(\mathbb{R}^m)} + \|g\|_{H^1(\mathbb{R}^{m+1}_+)}\,]\,. \tag{7.15}$$

We are interested in solving Problem (7.3) for functions β that are only locally in $H^1(\mathbb{R}^m)$ however in a uniform way, in the sense of the local uniform norms introduced in (5.53).

We would like to solve Problem (7.3) for a β with $\|\beta\|_{H^1_{l.u.}(\mathbb{R}^m)} < +\infty$, obtaining a linear operator with an estimate similar to (7.4) in its "local uniform version". We have the following result.

Lemma 7.4. *Given $\beta \in H^1(\mathbb{R}^m)$, there exists a solution*

$$(\phi, \alpha) \in H^2_{loc}(\mathbb{R}^{m+1}_+) \times L^2_{loc}(\mathbb{R}^m)$$

of Problem (7.3) that defines a linear operator of β. Besides, we have the estimate

$$\|\phi\|_{H^2_{l.u.}(\mathbb{R}^{m+1}_+)} + \|\alpha\|_{L^2_{l.u.}(\mathbb{R}^m)} \le C\,\|\beta\|_{H^1_{l.u.}(\mathbb{R}^m)}\,. \tag{7.16}$$

Proof. For the moment, let us further assume that $\beta \in H^1(\mathbb{R}^m)$ and consider the solution (α, ϕ) to Problem (7.3) predicted by Lemma 7.1. We will prove that the a priori estimate (7.4) holds.

Let $p \in \mathbb{R}^m$ and for small values δ consider the function

$$\rho(y) := \sqrt{1 + \delta^2 |y-p|^2}.$$

Let us write ϕ in the form

$$\phi = \rho^\nu \tilde\phi.$$

Then Problem (7.3) becomes in terms of $\tilde\phi$

$$\begin{aligned} \Delta\tilde\phi + f'(w(t))\tilde\phi &= \tilde\alpha(y)\, w'(t) + B_\delta\tilde\phi \quad \text{in } \mathbb{R}^{m+1}_+, \\ \tilde\phi(y,0) &= 0 \qquad \text{for all} \quad y \in \mathbb{R}^m, \\ \partial_t\tilde\phi(y,0) &= \tilde\beta(y) \quad \text{for all} \quad y \in \mathbb{R}^m \end{aligned} \tag{7.17}$$

where $\tilde\beta = \rho^{-\nu}\beta$, $\tilde\alpha = \rho^{-\nu}\alpha$. Here B_δ is a small linear operator of the form

$$B_\delta\tilde\phi = O(\delta)\cdot\nabla_y\tilde\phi + O(\delta^2)\tilde\phi.$$

We observe that for all small δ,

$$\|B_\delta\tilde\phi\|_{H^1(\mathbb{R}^{m+1}_+)} \le C\delta\|\tilde\phi\|_{H^2(\mathbb{R}^{m+1}_+)}$$

where C is independent of the point p defining ρ. By uniqueness of the H^2 solution in Lemma 7.3, and using the estimate (7.15) we get, after fixing δ sufficiently small,

$$\|\tilde\alpha\|_{L^2(\mathbb{R}^m)} + \|\tilde\phi\|_{H^1(\mathbb{R}^{m+1}_+)} \le C\|\tilde\beta\|_{H^1(\mathbb{R}^m)}.$$

Now, if ν was chosen large, then $\|\tilde\beta\|_{H^1(\mathbb{R}^m)} \le C\|\beta\|_{H^1_{l.u.}(\mathbb{R}^m)}$, while

$$\|\alpha\|_{L^2(B(p,1)} + \|\phi\|_{H^2(B(p,1)} \le C[\,\|\tilde\alpha\|_{L^2(\mathbb{R}^m)} + \|\tilde\phi\|_{H^2(\mathbb{R}^{m+1}_+)}]$$

where the constants C are uniform on the location of the origin p. Then we get

$$\|\alpha\|_{L^2_{l.u.}(\mathbb{R}^m)} + \|\phi\|_{H^2_{l.u.}(\mathbb{R}^{m+1}_+)} \le C \|\beta\|_{H^1_{l.u.}(\mathbb{R}^m)}, \tag{7.18}$$

as desired. Using this estimate on the solution of Lemma 7.1, we extend it by density to a solution of Problem (7.3) satisfying (7.18) whenever $\|\beta\|_{H^1_{l.u.}(\mathbb{R}^m)} < +\infty$. The proof is concluded. □

Our next task is to estimate the solution thus found using local uniform Hölder norms.

7.1. **Proof of Proposition 7.1.** We argue first that it suffices to establish the above statement replacing $\|g\|_{C^{0,\sigma}(\mathbb{R}^{m+1}_+)}$ by the stronger norm $\|g\|_{C^{1,\sigma}_0(\mathbb{R}^{m+1}_+)}$, namely finding a solution such that

$$\|\phi\|_{C^{2,\sigma}(\mathbb{R}^{m+1}_+)} + \|\alpha\|_{C^{0,\sigma}(\mathbb{R}^m)} \le C\,[\,\|\beta\|_{C^{1,\sigma}(\mathbb{R}^m)} + \|g\|_{C^{1,\sigma}(\mathbb{R}^{m+1}_+)}\,]. \tag{7.19}$$

To see this, we let (α, ϕ) solve Problem (7.1) and write

$$\phi = \tilde{\phi} + \bar{\phi}$$

where $\bar{\phi}$ is the unique bounded solution to the problem

$$\Delta\bar{\phi} - \bar{\phi} = g \quad \text{in } \mathbb{R}^{m+1}_+,$$
$$\bar{\phi}(y,0) = 0 \qquad \text{for all} \quad y \in \mathbb{R}^m.$$

It is standard that $\bar{\phi}$ satisfies the estimate

$$\|\bar{\phi}\|_{C^{2,\sigma}(\mathbb{R}^{m+1}_+)} \le \|g\|_{C^{0,\sigma}(\mathbb{R}^{m+1}_+)}.$$

Problem (7.1) can be written in the equivalent form

$$\Delta\tilde{\phi} + f'(w(t))\tilde{\phi} = \alpha(y)\,w'(t) + \tilde{g} \quad \text{in } \mathbb{R}^{m+1}_+,$$
$$\tilde{\phi}(y,0) = 0 \qquad \text{for all} \quad y \in \mathbb{R}^m,$$
$$\partial_t\phi(y,0) = \tilde{\beta}(y) \quad \text{for all} \quad y \in \mathbb{R}^m$$

where

$$\tilde{g} = -(1 + f'(w))\bar{\phi}, \quad \tilde{\beta} = \beta - \partial_t\bar{\phi}(y,0),$$

so that

$$\|\tilde{g}\|_{C^{1,\sigma}_0(\mathbb{R}^{m+1}_+)} + \|\tilde{\beta}\|_{C^{1,\sigma}_0(\mathbb{R}^m)} \le C\,[\,\|g\|_{C^{0,\sigma}(\mathbb{R}^{m+1}_+)} + \|\tilde{\beta}\|_{C^{1,\sigma}_0(\mathbb{R}^m)}\,],$$

and the claim follows.

The proof of the proposition consists of establishing the a priori estimate (7.2) for the solution built in Lemma 7.4. Let us fix a point $p \in \mathbb{R}^m$. We consider the unique solution of the equation

$$\Delta_y\gamma = \alpha \quad \text{in } B(p,3)$$
$$\gamma = 0 \quad \text{on } \partial B(p,3).$$

Let us write, for $|y| < 3$

$$\phi(y,t) = w'(t)\,\gamma(y) + \psi(y,t)$$

so that ψ satisfies

$$\Delta\psi + f'(w(t))\psi = g \quad \text{in } B(p,3) \times (0,\infty),$$
$$\partial_t\psi(y,0) = \beta(y) \quad \text{for all} \quad y \in B(p,3). \tag{7.20}$$

Standard boundary regularity estimates for the Laplacian yield the estimate

$$\|\psi\|_{C_0^{2,\sigma}(B(p,1)\times(0,1))} \le C\,[\,\|\beta\|_{C^{1,\sigma}(B(p,2))} + \|g\|_{C^{0,\sigma}(B(p,2)\times(0,2))} + \|\psi\|_{H^2(B(p,2)\times(0,2))}\,]\,. \tag{7.21}$$

On the other hand, using Lemma 7.4,

$$\begin{aligned}\|\gamma\|_{H^2(B(p,2))} &\le C\,\|\alpha\|_{L^2(B(p,3)}\\ &\le C\,[\|\beta\|_{H^1_{l.u.}(\mathbb{R}^m)} + \|g\|_{H^1_{l.u.}(\mathbb{R}^{m+1}_+)}\\ &\le C\,[\,\|\beta\|_{C^{1,\sigma}(\mathbb{R}^m)} + \|g\|_{C_0^{1,\sigma}(\mathbb{R}^{m+1}_+)}\,],\end{aligned} \tag{7.22}$$

while

$$\begin{aligned}\|\phi\|_{H^2(B(p,2)\times(0,2))} &\le C\,\|\phi\|_{H^2_{l.u.}(\mathbb{R}^{m+1}_+)}\\ &\le C\,[\,\|\beta\|_{H^1_{l.u.}(\mathbb{R}^m)} + \|g\|_{H^1_{l.u.}(\mathbb{R}^{m+1}_+)}\,]\\ &\le C\,[\|\beta\|_{C_0^{1,\sigma}(\mathbb{R}^m)} + \|g\|_{C_0^{1,\sigma}(\mathbb{R}^{m+1}_+)}\,]\,.\end{aligned} \tag{7.23}$$

Since

$$\|\psi\|_{H^2(B(p,2)\times(0,2))} \le C\,[\,\|\gamma\|_{H^2(B(p,2))} + \|\phi\|_{H^2(B(p,2)\times(0,2))}\,],$$

it follows that

$$\|\psi\|_{H^2(B(p,2)\times(0,2))} \le C\,[\,\|\beta\|_{C_0^{1,\sigma}(\mathbb{R}^m)} + \|g\|_{C_0^{1,\sigma}(\mathbb{R}^{m+1}_+)}\,]\,.$$

Hence, from (7.21) we conclude

$$\|\psi\|_{C_0^{2,\sigma}(B(p,1)\times(0,1))} \le C\,[\,\|\beta\|_{C_0^{1,\sigma}(\mathbb{R}^m)} + \|g\|_{C_0^{1,\sigma}(\mathbb{R}^{m+1}_+)}\,]\,. \tag{7.24}$$

Since $\psi(y,0) = -w'(0)\gamma(y)$ for $y \in B(p,3)$ we find then that

$$\|\alpha\|_{C^{0,\sigma}(B(p,1))} = \|\Delta_y \gamma\|_{C^{0,\sigma}(B(p,1))} \le C\,[\,\|\beta\|_{C_0^{1,\sigma}(\mathbb{R}^m)} + \|g\|_{C_0^{1,\sigma}(\mathbb{R}^{m+1}_+)}\,]\,.$$

The constants C accumulated above are all independent of the point $p \in \mathbb{R}^m$ chosen, therefore

$$\|\alpha\|_{C^{0,\sigma}(\mathbb{R}^m)} \le C\,[\,\|\beta\|_{C_0^{1,\sigma}(\mathbb{R}^m)} + \|g\|_{C_0^{1,\sigma}(\mathbb{R}^{m+1}_+)}\,]\,. \tag{7.25}$$

Besides, the definition of ψ yields

$$\|\phi\|_{C_0^{2,\sigma}(\mathbb{R}^m\times(0,1))} \le C\,[\,\|\beta\|_{C_0^{1,\sigma}(\mathbb{R}^m)} + \|g\|_{C_0^{1,\sigma}(\mathbb{R}^{m+1}_+)}\,]\,. \tag{7.26}$$

Finally, estimate (7.25) and interior elliptic estimates for the equation satisfied by ϕ yield that for any $\tau > 0$

$$\begin{aligned}\|\phi\|_{C_0^{2,\sigma}(B(p,1)\times(\tau+1,\tau+2))} &\le C\,[\,\|\phi\|_{H^2(B(p,3)\times(\tau,\tau+3))} +\\ &\qquad \|g\|_{C^{0,\sigma}(B(p,3)\times(\tau,\tau+3)))} + \|\alpha\|_{C^{0,\sigma}(B(p,3))}\,]\\ &\le C\,[\,\|\phi\|_{H^2_{l.u.}(\mathbb{R}^m)} + \|g\|_{C_0^{1,\sigma}(\mathbb{R}^{m+1}_+)} + \|\beta\|_{C^{1,\sigma}(\mathbb{R}^m)}\,]\\ &\le C\,[\,\|\beta\|_{C^{1,\sigma}(\mathbb{R}^m)} + \|g\|_{C_0^{1,\sigma}(\mathbb{R}^{m+1}_+)}\,]\end{aligned} \tag{7.27}$$

where, again, C is uniform on p and τ. Combining (7.26) and (7.27) we obtain

$$\|\phi\|_{C_0^{2,\sigma}(\mathbb{R}^{m+1}_+)} \le C\,[\,\|\beta\|_{C^{1,\sigma}(\mathbb{R}^m)} + \|g\|_{C_0^{1,\sigma}(\mathbb{R}^{m+1}_+)}\,]$$

and estimate (7.2) has been established. The proof is concluded. □

We have the validity of a similar result for Problem (7.11).

Lemma 7.5. *Given g such that*

$$\|g\|_{C^{0,\sigma}(\mathbb{R}^{m+1}_+)} < +\infty$$

there exists a solution $\phi \in C^{2,\sigma}(\mathbb{R}^{m+1}_+)$ of Problem (7.11) that defines a linear operator of g, satisfying the estimate

$$\|\phi\|_{C^{2,\sigma}(\mathbb{R}^{m+1}_+)} \leq C\,\|g\|_{C^{0,\sigma}(\mathbb{R}^{m+1}_+)}\,. \tag{7.28}$$

Proof. The proof goes along the same lines as that of Lemma 7.1, being actually easier. A solution satisfying the estimate

$$\|\phi\|_{H^2_{l.u.}(\mathbb{R}^{m+1}_+)} \leq C\,\|g\|_{L^2_{l.u.}(\mathbb{R}^{m+1}_+)} \tag{7.29}$$

is found with the argument in Lemma 7.4, using the result of Lemma 7.2. Estimate (7.28) for this solution then follows right away from local boundary and interior elliptic estimates. □

8. Linearized problem in the half-cylinder

Let Γ be a smooth, complete, embedded manifold in $\mathbb{R}^{m+1}$ that separates the space into two components. For each point $p \in \Gamma$ we assume that we can find a parametrization

$$Y_p : B(0,1) \subset \mathbb{R}^m \mapsto \Gamma \subset \mathbb{R}^{m+1}$$

onto a neighborhood $\mathcal{U}_p$ of p in Γ, so that if we write

$$g_{ij}(\mathtt{y}) \;:=\; \langle \partial_i Y_p, \partial_j Y_p\rangle = \delta_{ij} + \theta_p(\mathtt{y}),$$

we may assume that θ_p is smooth with $\theta_p(0) = 0$ and with second order derivatives bounded in $B(0,1)$, uniformly in p.

In this section we want to extend the linear theory developed so far to the same problem considered in the region $\Gamma_\varepsilon \times (0,\infty)$. Thus we consider the problem of finding, for given functions $g(y,t)$, $\beta(y)$, a solution (α,ϕ) to the problem

$$\begin{aligned}
\partial_t^2\phi + \Delta_{\Gamma_\varepsilon}\phi \,+\, f'(w(t))\phi \,&= \alpha(y)\,w'(t) + g(y,t) \quad \text{for all} \quad (y,t) \in \Gamma_\varepsilon \times (0,\infty),\\
\phi(y,0) \,&=\, 0 \qquad \text{for all} \quad y \in \Gamma_\varepsilon,\\
\partial_t\phi(y,0) \,&= \beta(y) \quad \text{for all} \quad y \in \Gamma_\varepsilon.
\end{aligned} \tag{8.1}$$

We also consider the problem

$$\begin{aligned}
\partial_t^2\phi + \Delta_{\Gamma_\varepsilon}\phi \,+\, f'(w(t))\phi \,&= g(y,t) \quad \text{for all} \quad (y,t) \in \Gamma_\varepsilon \times (0,\infty),\\
\phi(y,0) \,&=\, 0 \qquad \text{for all} \quad y \in \Gamma_\varepsilon.
\end{aligned} \tag{8.2}$$

For an open subset Λ of a manifold embedded in $\mathbb{R}^N$, we call $C^{k,\sigma}(\Lambda)$ the Banach space of functions $h \in C^{k,\sigma}_0(\Lambda)$ for which

$$\|h\|_{C^{k,\sigma}(\Lambda)} < +\infty.$$

We prove first the following result.

Lemma 8.1. *For all sufficiently small ε the following statement holds: given*

$$(\beta, g) \in C^{1,\sigma}(\Gamma_\varepsilon) \times C^{0,\sigma}(\Gamma_\varepsilon \times (0,\infty))$$

such that

$$\|\beta\|_{C^{1,\sigma}(\Gamma_\varepsilon)} \,+\, \|g\|_{C^{0,\sigma}(\Gamma_\varepsilon\times(0,\infty))} < +\infty$$

there exists a solution $(\phi, \alpha) = \mathcal{T}(\beta, g)$ *of Problem* (8.1)*, linear in its argument, that satisfies the estimate*

$$\|\phi\|_{C^{2,\sigma}(\Gamma_\varepsilon\times(0,\infty))} + \|\alpha\|_{C^{0,\sigma}(\Gamma_\varepsilon)} \le C\,[\,\|\beta\|_{C^{1,\sigma}(\Gamma_\varepsilon)} + \|g\|_{C^{0,\sigma}}(\Gamma_\varepsilon \times (0,\infty))\,] \tag{8.3}$$

where C *is independent of* ε.

Proof. We shall construct a solution by gluing solutions built through Lemma 7.1 to problems of the form (7.1), associated to local Euclidean parametrizations of Γ_ε.

The local coordinates of the surface Γ induce corresponding ones for the neighborhood $\varepsilon^{-1}\mathcal{U}_p$ of the point $p_\varepsilon = \varepsilon^{-1}p$ in Γ_ε, by means of the map

$$\mathtt{y} \in B(0, \varepsilon^{-1}) \longmapsto \varepsilon^{-1}Y_p(\varepsilon\mathtt{y}) \in \varepsilon^{-1}\mathcal{U}_p.$$

The Laplace-Beltrami operator is represented in these coordinates by

$$\Delta_{\Gamma_\varepsilon} = \frac{1}{\sqrt{\det g(\varepsilon\mathtt{y})}}\,\partial_i\left(\sqrt{\det g(\varepsilon\mathtt{y})}\,g^{ij}(\varepsilon\mathtt{y})\,\partial_j\right)$$

where the g^{ij} represent the coefficients of the inverse matrix of g_{ij}. Then we can expand

$$\Delta_{\Gamma_\varepsilon} = \Delta_y + B_{p_\varepsilon}, \quad B_{p_\varepsilon} := b_{ij}(\varepsilon\mathtt{y})\,\partial_{ij} + \varepsilon b_i(\varepsilon\mathtt{y})\,\partial_i, \quad |\mathtt{y}| < \varepsilon^{-1}, \tag{8.4}$$

where the coefficients b_{ij}, b_i have derivatives bounded, uniformly in p, and $b_{ij}(0) = 0$. Observe in particular that $|b_{ij}(\varepsilon\mathtt{y})| \le C\varepsilon|\mathtt{y}|$ with C uniform in p, so that for $\delta > 0$ small but fixed we have

$$|b_{ij}| \le C\delta, \quad |D_{\mathtt{y}}b_{ij}| + |D_{\mathtt{y}}b_j| \le C\varepsilon, \quad \mathtt{y} \in B(0, \delta\varepsilon^{-1}),$$

so that the coefficients are uniformly small with δ as $\varepsilon \to 0$, in other words $\Delta_{\Gamma_\varepsilon}$ differs from Euclidean Laplacian by an operator δ-small, uniformly in p.

We fix a small number δ and choose a sequence of points p_j such that Γ is covered by the union of the open sets

$$\mathcal{U}_k := Y_{p_k}(B(0, \delta/2)) \tag{8.5}$$

and so that each $\mathcal{U}_j$ does not intersect more than a finite, uniform number of $\mathcal{U}_\ell$ with $\ell \ne j$. Let us consider a smooth cut-off function η, with $\eta(s) = 1$ for $s < 1$, $= 0$ for $s > 2$. We define on Γ_ε the smooth cut-off functions,

$$\eta_{km}(y) := \eta(\varepsilon\,|\mathtt{y}|/m\delta), \quad y = \varepsilon^{-1}Y_{p_k}(\varepsilon\mathtt{y}),$$

extended as zero to all Γ_ε outside their supports.

We look for a solution to Problem (8.1) with the following form.

$$\phi = \sum_{k=1}^{\infty} \eta_{k1}\phi_k + \psi, \quad \alpha = \sum_{j=1}^{\infty} \eta_{k1}\alpha_k \tag{8.6}$$

where ϕ_k is defined, for instance, on $\mathcal{U}_k \times (0,\infty)$ with

$$\mathcal{U}_k := \varepsilon^{-1}Y_{p_k}(B(0, 2\delta)),$$

and the function $\eta_{k1}\phi_k$ is extended by zero outside its support. Using Einstein's summation convention, equation (8.1) can be written as

$$\begin{aligned} \eta_{k1}[\,(\partial_t^2+\Delta_{\Gamma_\varepsilon})\phi_k + f'(w)\phi_k] + 2\nabla_{\Gamma_\varepsilon}\phi_k\cdot\nabla_{\Gamma_\varepsilon}\eta_{k1} \ + \sum_{k=1}^{\infty}\eta_{k1}\alpha_k w' \\ +\phi_k\Delta_{\Gamma_\varepsilon}\eta_{k1} - h + (\partial_t^2+\Delta_{\Gamma_\varepsilon})\psi + f'(w)\psi \ &= 0. \\ \eta_{k1}\phi_k(y,0)+\psi(y,0) \ &= 0, \\ \eta_{k1}\partial_t\phi_k(y,0)+\partial_t\psi(y,0) \ &= \ \beta(y) \\ \text{for all} \quad y\in\Gamma_\varepsilon. \end{aligned} \tag{8.7}$$

We separate further the above equation as

$$\begin{aligned} \eta_{k1}[\,(\partial_t^2+\Delta_{\Gamma_\varepsilon})\phi_k + f'(w)\phi_k\,] \ + \ (\sum_{k=1}^{\infty}\eta_{k1})(f'(w)+1)\psi \ + \ \sum_{k=1}^{\infty}\eta_{k1}\alpha_k w' \ + \\ (\partial_t^2+\Delta_{\Gamma_\varepsilon})\psi - (\sum_{k=1}^{\infty}\eta_{k1})\psi + 2\nabla_{\Gamma_\varepsilon}\phi_k\cdot\nabla_{\Gamma_\varepsilon}\eta_{k1} + \phi_k\Delta_{\Gamma_\varepsilon}\eta_{k1} - h\,. \end{aligned} \tag{8.8}$$

Since the sets $\mathcal{U}_k$ cover Γ and for each j, there is at most a uniformly bounded number of $\ell\neq j$ is such that $\mathcal{U}_k\cap\mathcal{U}_\ell\neq\emptyset$, it follows that for some constant C uniform in ε.

$$1\le V := \sum_{k=1}^{\infty}\eta_{k1}\le C. \tag{8.9}$$

Now, defining

$$\beta_k \ := \ \frac{\eta_{k2}\beta}{\sum_{\ell=1}^{\infty}\eta_{\ell 1}}, \tag{8.10}$$

and using that $\eta_{k1}\eta_{k2}=\eta_{k1}$ we get that

$$\beta = \sum_{k=1}^{\infty}\eta_{k1}\beta_k.$$

Then Equation (5.46) will hold if we have the following infinite system of equations satisfied

$$\begin{aligned} \partial_t^2\phi_k + \Delta_{\Gamma_\varepsilon}\phi_k + f'(w)\phi_k = \alpha_k w' - (f'(w)+1)\psi \quad \text{in } \mathcal{U}_k\times(0,\infty) \\ \phi_k(y,0)=0, \\ \partial_t\phi_k(y,0) = -\partial_t\psi(y,0)+\beta_k(y) \\ \text{for all} \quad y\in\mathcal{U}_k, \quad k=1,2,\dots, \end{aligned} \tag{8.11}$$

$$\begin{aligned} \partial_t^2\psi + \Delta_{\Gamma_\varepsilon}\psi - V(y)\,\psi \ = \ -\sum_{k=1}^{\infty}[\,2\nabla_{\Gamma_\varepsilon}\phi_k\cdot\nabla_{\Gamma_\varepsilon}\eta_{k1} + \phi_k\Delta_{\Gamma_\varepsilon}\eta_{k1}\,] + g \\ \text{in } \mathcal{U}_k\times(0,\infty) \\ \psi(y,0)=0 \quad \text{for all} \quad y\in\Gamma_\varepsilon. \end{aligned} \tag{8.12}$$

Using local coordinates $y=\varepsilon^{-1}Y_{p_k}(\varepsilon \mathrm{y})$ in $\mathcal{U}_k$, we get equation (8.11) expressed as

$$\begin{aligned} \partial_t^2 \phi_k + \Delta_{\mathtt{y}} \phi_k + B_{p_k} \phi_k + f'(w) \phi_k &= \alpha_k w' - (f'(w)+1)\psi, \\ &\text{for all} \quad (\mathtt{y}, t) \in B(0, 2\delta\varepsilon^{-1}) \times (0,\infty), \\ \phi_k(\mathtt{y}, 0) = 0, &\quad \mathtt{y} \in B(0, 2\delta\varepsilon^{-1}), \\ \partial_t \phi_k(\mathtt{y}, 0) = -\partial_t \psi(\mathtt{y}, 0) + \beta_k(\mathtt{y}), &\quad \mathtt{y} \in B(0, 2\delta\varepsilon^{-1}) \end{aligned} \tag{8.13}$$

where B_{p_k} is the small operator in (8.4), and by slight abuse of notation we denote in the same way $h(\mathtt{y})$ and $h(y)$, a function h defined on $\varepsilon^{-1}\tilde{\mathcal{U}}_k$ evaluated at the point $y = \varepsilon^{-1} Y_k(\varepsilon \mathtt{y})$. Equation (8.13) can be extended to all of $\mathbb{R}^9_+$, with all its coefficients well-defined. Now, $\eta_{k3}(\mathtt{y}) = \eta(\varepsilon|\mathtt{y}|/3\delta)$, and we will have a solution of (8.11)-(8.12) if we solve the system

$$\begin{aligned} \partial_t^2 \phi_k + \Delta_{\mathtt{y}} \phi_k + \eta_{k3} B_{p_k} \phi_k + f'(w) \phi_k = \alpha_k w' - \eta_{k3}(f'(w)+1)\psi, &\quad \text{in } \mathbb{R}^9_+, \\ \phi_k(\mathtt{y}, 0) = 0 \quad \text{for all} \quad \mathtt{y} \in \mathbb{R}^m, & \\ \partial_t \phi_k(\mathtt{y}, 0) = -\eta_{k3} \partial_t \psi(\mathtt{y}, 0) + \beta_k(\mathtt{y}) \quad \text{for all} \quad \mathtt{y} \in \mathbb{R}^m, & \end{aligned} \tag{8.14}$$

$$\begin{aligned} \partial_t^2 \psi + \Delta_{\Gamma_\varepsilon} \psi - V(y)\,\psi \ = \ - \sum_{j=1}^{\infty} [\, 2\nabla_{\Gamma_\varepsilon} \phi_k \cdot \nabla_{\Gamma_\varepsilon} \eta_{k1} + \phi_k \Delta_{\Gamma_\varepsilon} \eta_{k1} \,] + g & \\ \text{in } \Gamma_\varepsilon \times (0,\infty) & \\ \psi(y, 0) = 0 \quad \text{for all} \quad y \in \Gamma_\varepsilon. & \end{aligned} \tag{8.15}$$

We will solve first equation (8.15) for given ϕ_k's and g. Let us consider the Banach space

$$\ell^\infty(C^{m,\sigma}(\Lambda))$$

of bounded sequences in $C^{m,\sigma}(\Lambda)$, endowed with the norm

$$\|\mathtt{h}\|_{\ell^\infty(C^{m,\sigma}(\Lambda))} := \sup_{k\geq 1} \|h_k\|_{C^{m,\sigma}(\Lambda)}.$$

Let

$$\Phi = (\phi_k)_{k\geq 1} \in \ell^\infty(C^{2,\sigma}(\mathbb{R}^9_+)), \quad g \in C^{0,\sigma}(\Gamma_\varepsilon \times (0,\infty))$$

be given. We write equation (8.15) as

$$\begin{aligned} \partial_t^2 \psi + \Delta_{\Gamma_\varepsilon} \psi - V(y)\,\psi \ = \ \tilde{g} \quad \text{in } \Gamma_\varepsilon \times (0,\infty) & \\ \psi(y, 0) = 0 \quad \text{for all} \quad y \in \Gamma_\varepsilon. & \end{aligned} \tag{8.16}$$

We see that $\tilde{g}$ defines a linear operator of the pair (ϕ, g) and

$$\|\tilde{g}\|_{C^{2,\sigma}(\Gamma_\varepsilon \times (0,\infty))} \ \leq \ C\, [\, \|g\|_{C^{0,\sigma}(\Gamma_\varepsilon \times (0,\infty))} + \varepsilon \|\phi\|_{\ell^\infty(C^{2,\sigma}(\mathbb{R}^9_+))} \,]$$

where the constant C is uniform in all small ε. Here we have used the fact that

$$|\nabla_{\Gamma_\varepsilon} \eta_{k1}| + |\Delta_{\Gamma_\varepsilon} \eta_{k1}| \leq C\varepsilon.$$

Now, since $1 \leq V \leq C$, the use of barriers yields the existence of a unique bounded solution ψ to (8.16), which satisfies

$$\|\psi\|_{L^\infty(\Gamma_\varepsilon \times (0,\infty))} \ \leq \ C \|\tilde{g}\|_{L^\infty(\Gamma_\varepsilon \times (0,\infty))}.$$

Then the use of local interior and boundary Schauder estimates, invoking the representation of the equation in local coordinates and the uniform Hölder character of the coefficients, yields that $\psi = \Psi(\Phi, g)$ satisfies

$$\|\Psi(\Phi,g)\|_{C^{2,\sigma}(\Gamma_\varepsilon\times(0,\infty))} \ \le\ C\,[\,\|g\|_{C^{0,\sigma}(\Gamma_\varepsilon\times(0,\infty))} + \varepsilon\|\Phi\|_{\ell^\infty(C^{2,\sigma}(\mathbb{R}^9_+))}\,]\ . \tag{8.17}$$

Next we consider a given $\beta \in C^{1,\sigma}(\Gamma_\varepsilon)$ and define β_k as in (8.10). Then $\mathfrak{b} := (\beta_k)_{k\ge 1}$ satisfies

$$\|\mathfrak{b}\|_{\ell^\infty(C^{1,\sigma}(\mathbb{R}^m))} \ \le C\ \|\beta\|_{C^{1,\sigma}(\Gamma_\varepsilon)}.$$

Let

$$\begin{aligned} T : \ell^\infty(C^{1,\sigma}(\mathbb{R}^m)) \times \ell^\infty(C^{0,\sigma}(\mathbb{R}^9_+)) \ &\to\ \ell^\infty(C^{2,\sigma}(\mathbb{R}^9_+)) \\ (\beta, g)\ &\longmapsto\ T(\beta, g) := \phi \end{aligned}$$

be the linear operator built in Lemma 7.1 as a solution of Problem (7.1), so that

$$\|T(\beta,g)\|_{C^{2,\sigma}(\mathbb{R}^9_+)} \ \le\ [\,\|g\|_{C^{0,\sigma}(\mathbb{R}^9_+)} + \|\beta\|_{C^{1,\sigma}(\mathbb{R}^m)}\,]\,. \tag{8.18}$$

Then the equation (8.14) after substituting ψ by $\Psi(\phi, g)$ becomes

$$\begin{aligned} \partial_t^2\phi_k + \Delta_{\mathtt{y}}\phi_k + f'(w)\phi_k &= \alpha_k w' + \tilde g_k, \quad \text{in } \mathbb{R}^9_+, \\ \phi_k(\mathtt{y},0) &= 0 \quad \text{for all} \quad \mathtt{y}\in\mathbb{R}^m, \\ \partial_t\phi_k(\mathtt{y},0) &= \tilde\beta_k(\mathtt{y}) \quad \text{for all} \quad \mathtt{y}\in\mathbb{R}^m, \end{aligned} \tag{8.19}$$

where

$$\tilde g_k := -\eta_{k3}B_{p_k}\phi_k - \eta_{k3}(f'(w)+1)\Psi(\phi,g), \quad \tilde\beta_k(\mathtt{y}) := -\eta_{k3}\partial_t\Psi(\phi,g)\,(\mathtt{y},0)\ +\ \beta_k(\mathtt{y})$$

so that we find a solution if we solve the linear fixed point problem

$$\Phi \ =\ \mathfrak{A}(\Phi) + \mathfrak{g}, \quad \Phi \in \ell^\infty(C^{2,\sigma}(\mathbb{R}^9_+)) \tag{8.20}$$

where

$$\begin{aligned} \mathfrak{A}(\Phi)_k \ &:=\ T(-\chi\partial_t\Psi(\Phi,0)\,(\cdot,0)\,,\, -\eta_{k3}B_{p_k}\phi_k - \eta_{k3}(f'(w)+1)\Psi(\Phi,0)\,), \\ \mathfrak{g}_k &:= T(-\eta_{k3}\partial_t\Psi(0,g)\,(\cdot,0)\ +\ \beta_k\,,\, -\eta_{k3}(f'(w)+1)\Psi(0,g)\,). \end{aligned}$$

From estimates (8.17), (8.18) and the δ-smallness of the operator B_{p_k} we readily get that for all small ε

$$\|\mathfrak{A}(\Phi)\|_{\ell^\infty(C^{2,\sigma}(\mathbb{R}^9_+))} \le C\delta\|\Phi\|_{\ell^\infty(C^{2,\sigma}(\mathbb{R}^9_+))},$$

and also

$$\|\mathfrak{g}\|_{\ell^\infty(C^{2,\sigma}(\mathbb{R}^9_+))} \le C\ [\,\|g\|_{C^{0,\sigma}(\Gamma_\varepsilon\times(0,\infty))} + \|\beta\|_{C^{1,\sigma}(\Gamma_\varepsilon)}\,]\,.$$

It follows that if δ is fixed sufficiently small, then for all small ε, Problem (8.20) has a unique solution $\Phi = \Phi(\beta, g)$ which is linear in its argument and satisfies the estimate

$$\|\Phi(\beta,g)\|_{\ell^\infty(C^{2,\sigma}(\mathbb{R}^9_+))} \le C\ [\,\|g\|_{C^{0,\sigma}(\Gamma_\varepsilon\times(0,\infty))} + \|\beta\|_{C^{1,\sigma}(\Gamma_\varepsilon)}\,]\,. \tag{8.21}$$

It is straightforward to check that the solution (ϕ, α) thus obtained by formula (8.6), satisfies the desired properties. The proof is concluded. □

We consider next the case in which g and β have a uniformly weighted Hölder control. Let us consider a positive function $\rho(y,t)$ defined on $\Gamma_\varepsilon \times (0,\infty)$ which we assume of class $C^{2,\sigma}$. Let us write $\phi = \rho\tilde\phi$, and consider Problem (8.1) written in terms of $\tilde\phi$. We have

$$\begin{aligned} \partial_t^2 \tilde\phi + \Delta_{\Gamma_\varepsilon} \tilde\phi + B\tilde\phi + f'(w)\tilde\phi &= \tilde\alpha(y)\, w'(t) + \tilde g(y,t) \quad \text{for all} \quad (y,t) \in \Gamma_\varepsilon \times (0,\infty), \\ \tilde\phi(y,0) &= 0 \qquad \text{for all} \quad y \in \Gamma_\varepsilon, \\ \partial_t \tilde\phi(y,0) &= \tilde\beta(y) \quad \text{for all} \quad y \in \Gamma_\varepsilon \end{aligned} \tag{8.22}$$

where

$$B\tilde\phi = \rho^{-1}[\partial_t^2 \rho + \Delta_{\Gamma_\varepsilon}\rho]\, \tilde\phi + 2\rho^{-1}\left[\nabla_{\Gamma_\varepsilon}\rho \cdot \nabla_{\Gamma_\varepsilon}\tilde\phi + \partial_t \rho\, \partial_t \tilde\phi\right],$$

$$\tilde\alpha = \rho^{-1}\alpha, \quad \tilde g = \rho^{-1} g, \quad \tilde\beta = \rho^{-1}\beta .$$

If we have that

$$\|B\tilde\phi\|_{C^{0,\sigma}(\Gamma_\varepsilon\times(0,\infty)} \le c\|\tilde\phi\|_{C^{2,\sigma}(\Gamma_\varepsilon\times(0,\infty))} \tag{8.23}$$

for a c sufficiently small, then there is a small linear perturbation of the operator built in Lemma 8.1 which solves problem (8.1) whenever

$$\|\tilde\beta\|_{C^{1,\sigma}(\Gamma_\varepsilon)} + \|\tilde g\|_{C^{0,\sigma}(\Gamma_\varepsilon\times(0,\infty)} < +\infty$$

and a corresponding estimate is obtained for $\tilde\phi = \rho^{-1}\phi$. Condition (8.23) will be achieved provided that

$$\|\rho^{-1}D^2\rho\|_{C^{0,\sigma}(\Gamma_\varepsilon\times(0,\infty))} + \|\rho^{-1}D\rho\|_{C^{0,\sigma}(\Gamma_\varepsilon\times(0,\infty))} \ \le\ c, \tag{8.24}$$

with c sufficiently small. Under this condition we have obtained a solution ϕ to Problem (8.1) such that

$$\begin{aligned} \|\rho^{-1}D^2\phi\|_{C^{0,\sigma}(\Gamma_\varepsilon\times(0,\infty))} + \|\rho^{-1}D\phi\|_{C^{0,\sigma}(\Gamma_\varepsilon\times(0,\infty))} + \|\rho^{-1}\phi\|_{C^{0,\sigma}(\Gamma_\varepsilon\times(0,\infty))} \\ \le\ C\,\|\tilde\phi\|_{C^{2,\sigma}(\Gamma_\varepsilon\times(0,\infty))} \ \le \\ C\,[\,\|\rho^{-1}D\beta\|_{C^{0,\sigma}(\Gamma_\varepsilon)} + \|\rho^{-1}\beta\|_{C^{0,\sigma}(\Gamma_\varepsilon)} + \|\rho^{-1}g\|_{C^{0,\sigma}(\Gamma_\varepsilon\times(0,\infty))}\,]\,. \end{aligned} \tag{8.25}$$

We recall that for $y\in\Gamma$ we denote $\mathtt{r}(y',y_9) = \sqrt{1+|y'|^2}$ and

$$\mathtt{r}_\varepsilon(y) := \mathtt{r}(\varepsilon y), \quad y \in \Gamma_\varepsilon.$$

For positive numbers ν, γ let us consider

$$\rho(y,t) \ =\ \mathtt{r}_\varepsilon(y)^{-\nu} e^{-\gamma t}, \quad (y,t) \in \Gamma_\varepsilon \times (0,\infty).$$

Then we observe that if γ is fixed sufficiently small, then for any fixed $\nu \ge 0$ and all small ε we have the validity of condition (8.24).

Let us consider the weighted Hölder norms defined in (5.60)-(5.61). Then the following result has been obtained.

Proposition 8.1. *If $\gamma \ge 0$ is fixed sufficiently small and $\nu \ge 0$ is arbitrary, then for all sufficiently small ε the following statement holds: given (β, g) such that*

$$\|\beta\|_{C^{1,\sigma}_{\nu}(\Gamma_\varepsilon)} + \|g\|_{C^{0,\sigma}_{\nu,\gamma}(\Gamma_\varepsilon\times(0,\infty))} \ <\ +\infty$$

there exists a solution $\phi = \mathcal{T}(\beta,g)$ of Problem (8.1), linear in its argument, that satisfies the estimate

$$\|\phi\|_{C^{2,\sigma}_{\nu,\gamma}(\Gamma_\varepsilon\times(0,\infty))} \ \le\ C\,[\,\|\beta\|_{C^{1,\sigma}_{\nu}(\Gamma_\varepsilon)} + \|g\|_{C^{0,\sigma}_{\nu,\gamma}(\Gamma_\varepsilon\times(0,\infty))}\,] \tag{8.26}$$

where C is independent of ε.

9. Solving the projected problems

In terms of the operator $\mathcal{T}(\beta, g)$ defined in Proposition 8.1, we will have a solution to (6.9) if we solve the fixed point problem

$$\phi \;=\; -\mathcal{T}(\,\partial_t \Psi(\phi)(y,0)\,,\, \mathcal{R}_3(\mathtt{h}) + \mathcal{N}(\phi)\,), \quad \phi \in C^{2,\sigma}_{\nu,\gamma}(\Gamma_\varepsilon \times (0,\infty)). \tag{9.1}$$

Taking into account that for $\nu = 4 + \mu$ we have

$$\|\partial_t \Psi(0)(y,0)\|_{C^{1,\sigma}_{\nu,\gamma}(\Gamma_\varepsilon)} + \|\mathcal{R}_3(\mathtt{h}) + \mathcal{N}(0)\|_{C^{0,\sigma}_{\nu,\gamma}(\Gamma\times(0,\infty))} \;\le\; C\varepsilon^4$$

and the fact, straightforward to check, that the operator on the right hand side of equation (9.1), defines a contraction mapping on the set of functions ϕ with

$$\|\phi\|_{C^{2,\sigma}_{\nu,\gamma}(\Gamma\times(0,\infty))} \le M\varepsilon^3,$$

then fixing M sufficiently large, contraction mapping principle provides a unique solution of (9.1) in that region. Using the Lipschitz property (5.82) for $\mathcal{R}_3(\mathtt{h})$, corresponding properties are inherited for the solution. The following result holds.

Lemma 9.1. *For all small ε sufficiently small the following holds. There exists a solution $\phi = \Phi(\mathtt{h})$ of problem (6.2) that defines an operator on functions $\mathtt{h}$ satisfying constraints (5.70). For such functions and some $\mu > 0$ we have*

$$\|\Phi(\mathtt{h}^1) - \Phi(\mathtt{h}^2)\|_{C^{2,\sigma}_{4+\mu,\gamma}(\Gamma_\varepsilon\times(0,\infty))} \;\le\; C\,\varepsilon^3 \|\mathtt{h}_1 - \mathtt{h}_2\|_{2,4+\mu,\Gamma},$$

$$\|\Phi(\mathtt{h})\|_{C^{2,\sigma}_{4+\mu,\gamma}(\Gamma_\varepsilon\times(0,\infty))} \le C\varepsilon^4$$

and

$$\|\mathcal{N}(\Phi(\mathtt{h}^1)) - \mathcal{N}(\Phi(\mathtt{h}^2))\|_{C^{0,\sigma}_{4+\mu,\gamma}(\Gamma_\varepsilon\times(0,\infty))} \;\le\; C\,\varepsilon^4\,\|\mathtt{h}^1 - \mathtt{h}^2\|_{2,2+\mu,\Gamma}$$

for all $\mathtt{h}$, $\mathtt{h}^1$, $\mathtt{h}^2$ satisfying (5.70).

9.1. **The reduced problem: proof of Theorem 4.** We are ready to solve the full problem. Let us consider the solution $\Phi(\mathtt{h})$ to (6.2) predicted by Lemma 9.1, and call $\alpha[\mathtt{h}]$ the corresponding α. Then

$$\begin{aligned}
\partial_t^2\Phi(\mathtt{h}) + \Delta_{\Gamma_\varepsilon}\Phi(\mathtt{h}) + f'(w(t))\Phi(\mathtt{h}) \;=\; \alpha[\mathtt{h}]\,w'(t) \,-\mathcal{R}_3(\mathtt{h}) - \mathcal{N}(\Phi(\mathtt{h})) \quad &\text{in } \Gamma_\varepsilon\times(0,\infty),\\
\phi(y,0) = 0 \quad &\text{for all} \quad y\in\Gamma_\varepsilon,\\
\partial_t\Phi(\mathtt{h})(y,0) = -\partial_t\Psi(\Phi(\mathtt{h}))(y,0) \quad &\text{for all} \quad y\in\Gamma_\varepsilon.
\end{aligned} \tag{9.2}$$

We can express the parameter function $\alpha[\mathtt{h}]$ for the solution predicted of Problem (6.9), as an operator in $\mathtt{h}$, by integration against w', for $y \in \Gamma_\varepsilon$,

$$\alpha[\mathtt{h}](y)\int_0^\infty w'^2\,dt \;=\; \int_0^\infty [\Delta_{\Gamma_\varepsilon}\Phi(\mathtt{h}) \,+\, \mathcal{R}_3(\mathtt{h}) + \mathcal{N}(\Phi(\mathtt{h}))]\,w'\,. \tag{9.3}$$

Then we will have solved our original problem if we find a solution $\mathtt{h}$ within constraints (5.70) of the equation

$$\Delta_\Gamma \mathtt{h} + |A_\Gamma(y)|^2\mathtt{h} = \mathcal{B}[\mathtt{h}]\,(y) \quad \text{in } \Gamma \tag{9.4}$$

where

$$\mathcal{B}[h]\,(y) \;:=\; \varepsilon^{-2}\alpha[\mathtt{h}]\,(\varepsilon^{-1}y), \quad y\in\Gamma$$

and α is the function in (9.3). We see then that

$$\|\alpha[\mathtt{h}]\|_{C^{0,\sigma}_{4+\mu}(\Gamma_\varepsilon)} \ \leq\ C\,\varepsilon^4\ ,$$

$$\|\alpha[\mathtt{h}_1]-\alpha[\mathtt{h}_2]\|_{C^{0,\sigma}_{4+\mu}(\Gamma_\varepsilon)} \ \leq\ C\,\varepsilon^3\|\mathtt{h}_1-\mathtt{h}_2\|_{C^{2,\sigma}_{4+\mu}(\Gamma_\varepsilon)}\ .$$

Using Lemma 5.2 we then get

$$\begin{aligned}\|\mathcal{B}[\mathtt{h}]\|_{\sigma,4+\mu-\sigma,\Gamma} &\leq\ C\varepsilon^{-\sigma-2}\|\alpha[\mathtt{h}]\|_{C^{0,\sigma}_{4+\mu}(\Gamma_\varepsilon)}\\ &\leq\ C\,\varepsilon^{2-\sigma}, \qquad (9.5)\\ \|\mathcal{B}[\mathtt{h}^1]-\mathcal{B}[\mathtt{h}^2]\|_{\sigma,4+\mu-\sigma,\Gamma} &\leq\ C\varepsilon^{-\sigma-2}\|\alpha[\mathtt{h}^1]-\alpha[\mathtt{h}^2]\|_{C^{0,\sigma}_{4+\mu}(\Gamma_\varepsilon)}\\ &\leq\ C\,\varepsilon^{1-\sigma}\|\mathtt{h}_1-\mathtt{h}_2\|_{C^{2,\sigma}_{4+\mu}(\Gamma_\varepsilon)}\,. \qquad (9.6)\end{aligned}$$

9.2. **The proof of Theorem 4.** At this point, we make use of a linear result. for the problem

$$\mathcal{J}_\Gamma[\mathtt{h}] := \Delta_\Gamma\mathtt{h} + |A_\Gamma|^2\mathtt{h} = \mathtt{g} \quad \text{in } \Gamma. \qquad (9.7)$$

Lemma 9.2. *Let $\mu' > 0$. Then there exists a positive constant $C > 0$ such that if $\mathtt{g}$ satisfies*

$$\|\mathtt{g}\|_{\sigma,4+\mu',\Gamma}\ <\ +\infty$$

then there is a solution $\mathtt{h} = \mathcal{T}(\mathtt{g})$ of equation (9.7) that defines a linear operator of $\mathtt{g}$ that satisfies

$$\|\mathtt{h}\|_{2,\sigma,4+\mu',\Gamma} := \|D^2_\Gamma\mathtt{h}\|_{\sigma,4+\mu',\Gamma} + \|\mathtt{h}\|_{\sigma,2+\mu',\Gamma}\ \leq\ C\,\|\mathtt{g}\|_{\sigma,4+\mu',\Gamma}.$$

This result follows from Proposition 12.1 and Corollary 12.1 in the appendix. We will use it here to conclude the result. We find a solution to Equation (9.4) if we solve the fixed point problem

$$\mathtt{h}\ =\ \mathcal{T}(\tilde{B}[\mathtt{h}]), \qquad (9.8)$$

where we choose $\mu' = \frac{\mu}{2}$, $(0 < \mu < 1)$, and $\sigma \leq \frac{\mu}{2}$. From the lemma, and estimates (9.5), (9.6) we find that the operator on the right hand side of (9.8) is a contraction mapping of the region where

$$\|\mathtt{h}\|_{2,\sigma,4+\mu',\Gamma}\ \leq\ C\varepsilon^{2-\sigma} < \varepsilon^{\frac{3}{2}}.$$

Hence there is a solution of (9.8) and hence of (9.4) in this region. Obviously constraint (5.70) is satisfied by this solution for all small ε.

We claim that Γ^h_ε is the graph of an entire function. This is equivalent to showing that $\Gamma^{\varepsilon h} = \{y + \varepsilon h(y)\ /\ y \in \Gamma\}$ is a graph, provided that ε is sufficiently small. The map

$$(y, z) \in \Gamma \times (-\delta, \delta) \mapsto y + z\nu(y)$$

defines a diffeomorphism onto a tubular neighborhood of Γ, for sufficiently small δ. Moreover, since curvatures of Γ are actually decaying at infinity, we have moreover that its inverse $x \mapsto (y(x), z(x))$ has uniformly bounded derivatives. Now, we have that $\Gamma^{\varepsilon h}$ is described by the equation $V(x) := z(x) - \varepsilon h(y(x)) = 0$. We observe then that

$$\partial_{x_9}V(x) = \partial_{x_9}z - \varepsilon\nabla_\Gamma h(y)\cdot\partial_{x_9}y = \nu_9 + O(\varepsilon\mathtt{r}^{-2})$$

thanks to estimate (5.72). Since

$$\nu_9 = \frac{1}{\sqrt{1+|\nabla F|^2}} \geq \frac{c}{\mathtt{r}^2}$$

for some $c > 0$ it follows that $\partial_{x_9}V(x) > 0$ at every point of $\Gamma^{\varepsilon h}$, and hence this manifold can be described locally about each of its points as a graph of a function of $(x_1, \dots, x_8)$, hence this is also globally the case. This function is clearly entire. The proof of the Theorem is thus concluded. □

In the next section we shall conclude the proofs of Theorems 5-6 by solving the reduced problem (9.4) in the situations there considered. In order not to lose the main thread of the presentation of the results, we postpone the rather delicate analysis to Section 12.

10. Proof of Theorem 5

The proof follows the same scheme as that for the epigraph case, so that we will only describe the differences. The step of the improvement of the approximation is actually identical. The coefficients of the metric, as well as the curvatures, decay faster in r than those of the minimal graph. In fact the Jacobi operator is at main order Laplacian along each of its leaves. The slight difference is that now we will need to find sufficiently far away a solution of

$$J_\Gamma[h] = p \quad \text{in } \Gamma$$

where $p = O(r^{-2-\sigma})$ as $r \to \infty$. We do not solve this problem directly but rather its projected version

$$J_\Gamma[h] = p - \sum_{j=1}^{4} c_i \frac{z_i}{1+r^3} \quad \text{in } \Gamma$$

,

$$\int_\Gamma \frac{z_i}{1+r^3} h = 0 \quad \text{for all} \quad i = 1, \dots, 4$$

where z_i's are the Jacobi fields associated to rigid motions. At this point we refer to the theory developed in [9] that allow to solve this problem for bounded h in which in addition one has fast decay of first and second derivatives. We can adapt the theory to the use of weights with Holder norms like in this paper in a straightforward way. The final problem that is to solve is actually a projected version of (9.4),

$$\Delta_\Gamma \mathtt{h} + |A_\Gamma(y)|^2 \mathtt{h} = \mathcal{B}[\mathtt{h}]\,(y) - \sum_{j=1}^{4} c_i \frac{z_i}{1+r^3} \quad \text{in } \Gamma. \tag{10.1}$$

$$\int_\Gamma \frac{z_i}{1+r^3} \mathtt{h} = 0 \quad \text{for all} \quad i = 1, \dots, 4$$

We conclude, as in [9], the existence of a solution u and coefficients c_i such that

$$\Delta u + f(u) = \sum_{i=1}^{4} c_i \frac{z_i}{1+r^3} w'(t), \ u > 0 \quad \text{in } \Omega_\varepsilon, \quad u \in L^\infty(\Omega_\varepsilon), \tag{10.2}$$

$$u = 0, \quad \frac{\partial u}{\partial \nu} = constant \quad \text{on } \partial\Omega_\varepsilon \tag{10.3}$$

An interesting, but important fact is that for Serrin's overdetermined problem Pohozaev's identity also holds and the boundary term vanishes. A slight variation of the argument given in [9] pp. 99-102, that exploits the invariance under rotation and translations of the problem, yields that $c_i = 0$ for all i, and the construction is concluded. In the case of the catenoid we can further restrict ourselves to the space of axially symmetric functions to conclude the existence of an axially symmetric solution. □

11. Proofs of Theorems 6 and 7

In this section, we sketch the proofs of Theorems 6 and 7 and Corollary 2.1 which follow from the general scheme of proof of Theorem 4. The notable difference here is that the first error is $\mathcal{O}(\varepsilon)$ only but thanks to the CMC condition the first approximation can be made to depend on the signed distance to the surface only. As before we need to improve the error up to order $\mathcal{O}(\varepsilon^4)$. The reduced problem–the Jacobi operator-can be solved easily in this case, thanks to the compactness and nondegeneracy condition. So we shall concentrate only on the part of improving the errors.

11.1. **Fermi coordinates and the expression of the Laplace-Beltrami operator.** In this section, we assume that Γ is an oriented smooth hypersurface embedded in M. We first define the Fermi coordinates about Γ and then, we provide some asymptotic expansion of the Laplace-Beltrami operator in Fermi coordinates about Γ.

We denote by $\mathbf{n}$ a unit normal vector field on Γ and we define

$$Z(\mathtt{x}, z) := \mathrm{Exp}_{\mathtt{x}}(z\,\mathbf{n}(\mathtt{x})), \tag{11.1}$$

where $\mathtt{x} \in \Gamma$, $z \in \mathbb{R}$ and Exp is the exponential map. The implicit function theorem implies that Z is a local diffeomorphism from a neighborhood of a point $(\mathtt{x}, 0) \in \Gamma \times \mathbf{R}$ onto a neighborhood of $\mathtt{x} \in M$.

Remark 11.1. *In the special case where (M, g) is the Euclidean space, we simply have*

$$Z(\mathtt{x}, z) := \mathtt{x} + z\,\mathbf{n}(\mathtt{x}).$$

Given $z \in \mathbb{R}$, we define Γ_z by

$$\Gamma_z := \{Z(\mathbf{x}, z) \in M \,:\, \mathbf{x} \in \Gamma\}.$$

Observe that for z small enough (depending on the point $y \in \Gamma$ where one is working), Γ_z restricted to a neighborhood of y is a smooth hypersurface which will be referred to as the *hypersurface parallel to* Γ *at height* z. The induced metric on Γ_z will be denoted by g_z.

The following result is a consequence of Gauss' Lemma. It gives the expression of the metric g on the domain of M which is parameterized by Z.

Lemma 11.1. *We have*

$$Z^*\, g = g_z + dz^2,$$

where g_z is considered as a family of metrics on $T\Gamma$, smoothly depending on z, which belongs to a neighborhood of $0 \in \mathbb{R}$.

Proof. It is easier to work in local coordinates. Given $y \in \Gamma$, we fix local coordinates $x := (x_1, \ldots, x_n)$ in a neighborhood of $0 \in \mathbb{R}^n$ to parameterize a neighborhood of y in Γ by Φ, with $\Phi(0) = y$. We consider the mapping

$$\tilde{F}\,(x, z) = \mathrm{Exp}_{\Phi(x)}(z\,N(\Phi(x))),$$

which is a local diffeomorphism from a neighborhood of $0 \in \mathbb{R}^{n+1}$ into a neighborhood of y in M. The corresponding coordinate vector fields are denoted by

$$X_0 := \tilde{F}_*(\partial_z) \qquad \text{and} \qquad X_j := \tilde{F}_*(\partial_{x_j}),$$

for $j = 1, \ldots, n$. The curve $x_0 \longmapsto \tilde{F}(x_0, x)$ being a geodesic we have $g(X_0, X_0) \equiv 1$. This also implies that $\nabla^g_{X_0} X_0 \equiv 0$ and hence we get

$$\partial_z g(X_0, X_j) = g(\nabla^g_{X_0} X_0, X_j) + g(\nabla^g_{X_0} X_j, X_0) = g(\nabla^g_{X_0} X_j, X_0).$$

The vector fields X_0 and X_j being coordinate vector fields we have $\nabla^g_{X_0} X_j = \nabla^g_{X_j} X_0$ and we conclude that

$$2\,\partial_z g(X_0, X_j) = 2\, g(\nabla^g_{X_j} X_0, X_0) = \partial_{x_j} g(X_0, X_0) = 0.$$

Therefore, $g(X_0, X_j)$ does not depend on z and since on Γ this quantity is 0 for $j = 1, \ldots, n$, we conclude that the metric g can be written as

$$g = g_z + dz^2,$$

where g_z is a family of metrics on Γ smoothly depending on z (this is nothing but Gauss' Lemma). □

The next result expresses, for z small, the expansion of g_z in terms of geometric objects defined on Γ. In particular, in terms of $\mathring{g}$ the induced metric on Γ, $\mathring{h}$ the second fundamental form on Γ, which is defined by

$$\mathring{h}(t_1, t_2) := -\mathring{g}(\nabla^g_{t_1} N, t_2\,),$$

and in terms of the square of the second fundamental form which is the tensor defined by

$$\mathring{h} \otimes \mathring{h}(t_1, t_2) := \mathring{g}(\nabla^g_{t_1} N, \nabla^g_{t_2} N),$$

for all $t_1, t_2 \in T\Gamma$. Observe that, in local coordinates, we have

$$(\mathring{h} \otimes \mathring{h})_{ij} = \sum_{a,b} \mathring{h}_{ia}\, \mathring{g}^{ab}\, \mathring{h}_{bj}.$$

With these notations at hand, we have the :

Lemma 11.2. *The induced metric g_z on Γ_z can be expanded in powers of z as*

$$g_z = \mathring{g} - 2\, z\, \mathring{h} + z^2\, \left(\mathring{h} \otimes \mathring{h} + g(R_g(\,\cdot\,, N),\,\cdot\,, N)\right) + \mathcal{O}(z^3),$$

where R_g denotes the Riemannian tensor on (M, g).

Proof. We keep the notations introduced in the previous proof. By definition of $\mathring{g}$, we have

$$g_z = \mathring{g} + \mathcal{O}(z).$$

We now derive the next term the expansion of g_z in powers of z. To this aim, we compute

$$\partial_z\, g(X_i, X_j) = g(\nabla^g_{X_i} X_0, X_j) + g(\nabla^g_{X_j} X_0, X_i)\,,$$

for all $i, j = 1, \ldots, n$. Since $X_0 = N$ on Γ, we get

$$\partial_z\, \bar{g}_z\,_{|z=0} = -2\, \mathring{h}, \tag{11.2}$$

by definition of the second fundamental form. This already implies that

$$g_z = \mathring{g} - 2\, \mathring{h}\, z + \mathcal{O}(z^2)\,.$$

Using the fact that the X_0 and X_j are coordinate vector fields, we can compute

$$\partial^2_z\, g(X_i, X_j) = g(\nabla^g_{X_0}\, \nabla^g_{X_i} X_0, X_j) + g(\nabla^g_{X_0}\, \nabla^g_{X_j} X_0, X_i) + 2\, g(\nabla^g_{X_i} X_0, \nabla^g_{X_j} X_0). \tag{11.3}$$

By definition of the curvature tensor, we can write

$$\nabla^g_{X_0}\, \nabla^g_{X_j} = R_g(X_0, X_j) + \nabla^g_{X_j}\, \nabla^g_{X_0} + \nabla^g_{[X_0, X_j]}\,,$$

which, using the fact that X_0 and X_j are coordinate vector fields, simplifies into

$$\nabla^g_{X_0}\, \nabla^g_{X_j} = R_g(X_0, X_j) + \nabla^g_{X_j}\, \nabla^g_{X_0}\,.$$

Since $\nabla^g_{X_0} X_0 \equiv 0$, we get

$$\nabla^g_{X_0} \nabla^g_{X_j} X_0 = R_g(X_0, X_j)\, X_0 .$$

Inserting this into (11.3) yields

$$\partial^2_z\, g(X_i, X_j) = 2\, g(R_g(X_0, X_i)\, X_0, X_j) + 2\, g(\nabla^g_{X_i} X_0, \nabla^g_{X_j} X_0) .$$

Evaluation at $x_0 = 0$ gives

$$\partial^2_z\, g_z\,_{|z=0} = 2\, g(R(N, \cdot)\, N, \cdot) + 2\, g(\nabla^g_{\cdot} N, \nabla^g_{\cdot} N).$$

The formula then follows at once from Taylor's expansion. □

Similarly, the mean curvature H_z of Γ_z can be expressed in term of $\mathring{g}$ and $\mathring{h}$. We have the :

Lemma 11.3. *The following expansion holds*

$$H_z = \mathrm{Tr}_{\mathring{g}} \mathring{h} + z \left(\mathrm{Tr}_{\mathring{g}} \mathring{h} \otimes \mathring{h} + \mathrm{Ric}_g(N, N) \right) + \mathcal{O}(z^2),$$

for z close to 0.

Proof. The mean curvature appears in the first variation of the volume form of parallel hypersurfaces, namely

$$H_z = -\frac{1}{\sqrt{\det g_z}} \frac{d}{dz} \sqrt{\det g_z}.$$

The result then follows at once from the expansion of the metric g_z in powers of z together with the well known formula

$$\det(I + A) = 1 + \mathrm{Tr}A + \frac{1}{2} \left((\mathrm{Tr}A)^2 - \mathrm{Tr}A^2 \right) + \mathcal{O}(\|A\|^3),$$

where $A \in M_n(\mathbb{R})$. □

Recall that, in local coordinates, the Laplace Beltrami operator is given by

$$\Delta_g = \frac{1}{\sqrt{|g|}}\, \partial_{x_i} \left(g^{ij} \sqrt{|g|}\, \partial_{x_j} \right) .$$

Therefore, in a fixed tubular neighborhood of Γ, the Euclidean Laplacian in $\mathbb{R}^{n+1}$ can be expressed in Fermi coordinates by the (well-known) formula

$$\Delta_{g_e} = \partial^2_z - H_z\, \partial_z + \Delta_{g_z} . \tag{11.4}$$

In the case where the ambient manifold is the Euclidean space, we get

Lemma 11.4. *The induced metric g_z on Γ_z is given by*

$$g_z = g_0 - 2\, z\, k_0 + z^2\, k_0 \otimes k_0,$$

in other words

$$g_z := g_0((I - zA)\,\cdot\,, (I - zA)\,\cdot\,)$$

where A is the shape operator defined by

$$A = (\mathring{h}_{ij}). \tag{11.5}$$

Proof. We just need to compute

$$\partial_{x_i} X_z \cdot \partial_{x_j} X_z = \partial_{x_i} X \cdot \partial_{x_j} X + z \, \left(\partial_{x_i} X \cdot \partial_{x_j} \tilde N + \partial_{x_i} \tilde N \cdot \partial_{x_j} X \right) + z^2 \, \partial_{x_i} \tilde N \cdot \partial_{x_j} \tilde N,$$

where $\tilde N := X^* N$. We can use (11.2) to write

$$\partial_{x_i} \tilde N \cdot \partial_{x_j} \tilde N = A \, \partial_{x_i} X \cdot A \, \partial_{x_j} X$$

And, using the definition of the first and second fundamental forms on Γ, we conclude that

$$\partial_{x_i} X_z \cdot \partial_{x_j} X_z = \partial_{x_i} X \cdot \partial_{x_j} X - 2 \, z \, (A \, \partial_{x_i} X) \cdot \partial_{x_j} X + z^2 \, (A \, \partial_{x_i} X) \cdot (A \, \partial_{x_j} X) \, .$$

This completes the proof of the result. □

Similarly, the mean curvature H_z of Γ_z can be expressed in term of z and A, the shape operator about Γ which has been defined in (11.5). We have the :

Lemma 11.5. *The following expansion holds*

$$H_z = \sum_{k=0}^{\infty} \mathrm{Tr} \, (A^{k+1}) \, z^k \, .$$

Proof. The mean curvature appears in the first variation of the volume form of parallel hypersurfaces, namely

$$H_z = - \frac{1}{\sqrt{\det g_z}} \, \frac{d}{dz} \sqrt{\det g_z}.$$

Hence we find that

$$H_z = - \frac{1}{\det \, (I - z \, A)} \, \frac{d}{dz} \det (I - z \, A) = \mathrm{Tr} \, (A \, (I - z \, A)^{-1})$$

and the result follows. □

11.2. **Construction of an approximate solution.** Given any (sufficiently small) smooth function h defined on Γ, we define Γ_h to be the normal graph over Γ for the function h. Namely

$$\Gamma_h := \{ y + h(y) \, N(y) \in \mathbb{R}^{n+1} \ : \ y \in \Gamma \} \, .$$

We also define the epigraph

$$\Omega_h := \{ y + t \, N(y) \in \mathbb{R}^{n+1} \ : \ y \in \Gamma, \quad t \geq h(y) \} \, .$$

We would like to solve the equation

$$\Delta u + f(u) = 0 \, ,$$

in Ω_h, with $u = 0$ and $\partial_\nu u = \text{constant}$ on $\partial \Omega_h$. In this section, we explain how to build a function h and a function u which solve this overdetermined problem to high order of accuracy. The construction makes use of an iteration scheme which can be used to determine all the orders successively.

We keep the notations of the previous section and, in a tubular neighborhood of Γ we write

$$u(z, y) = v \left(\frac{z - h(y)}{\varepsilon}, y \right) \, ,$$

where h is a (sufficiently small) smooth function defined on Γ. It will be convenient to denote by t the variable

$$t := \frac{z - h(y)}{\varepsilon} \, .$$

Using the expression of the Laplacian in Fermi coordinates which has been derived in (11.4), we find with little work that the equation we would like to solve can be rewritten as

$$\Big[(1+\|dh\|^2_{g_\zeta})\,\partial_t^2 v+\varepsilon^2\Delta_{g_\zeta}v-\varepsilon\,\big(H_\zeta+\Delta_{g_\zeta}h\big)\,\partial_t v \\ -\varepsilon\,(dh,d\,\partial_t v)_{g_\zeta}\Big]_{|\zeta=\varepsilon\,t+h}\quad +f(v)=0 \tag{11.6}$$

for $t>0$ close to 0 and $y\in\Gamma$. Some comments are due about the notations. In this equation and below all computations of the quantities between the square brackets [ ] are performed using the metric g_ζ defined in Lemma 11.4 and considering that ζ is a parameter, and once this is done, we set $\zeta=t+h(y)$.

The fact that we ask that u has 0 boundary data translates into

$$v(0,y)=0$$

on Γ. Finally, the Neumann data of u reads

$$\mathfrak{N}(v,h):=\Big(1+\|dh\|^2_{g_\zeta}\Big)^{1/2}_{\zeta=h(y)}\;\partial_t v$$

where this time the expression between the square brackets is evaluated at $t=0$.

We set

$$v(t,y)=w(t)+\phi(t,y)$$

where w is the solution of (1.5). In this case, the equation (11.6) becomes

$$\mathfrak{M}(v,h)=0\,,$$

where we have defined

$$\begin{aligned}\mathfrak{M}(v,h)\;&:=\;\Big[\big(\partial_t^2+\varepsilon^2\Delta_{g_\zeta}+f'(w)\big)\,\phi-\varepsilon\,(\Delta_{g_\zeta}h+H_\zeta)\,(w'+\partial_t\phi)\\ &-\;\;\|dh\|^2_{g_\zeta}\,(w''+\partial_t^2\phi)-\varepsilon\,(dh,d\,\partial_t\phi)_{g_\zeta}\Big]_{|\zeta=\varepsilon t+h}\\ &+\;\;(f(w+\phi)-f(w)-f'(w)\,\phi)\,.\end{aligned} \tag{11.7}$$

Now we perform some formal computations to determine the solution ϕ and h so that $\mathfrak{N}(w+\phi,h)$ is constant. We assume that ϕ and h can be expanded in power of ε as

$$\phi=\varepsilon\phi_0+\varepsilon^2\,\phi_1+\varepsilon^3\,\phi_2+\varepsilon^4\,\phi_3+\dots$$

and

$$h=\varepsilon\,h_0+\varepsilon^2\,h_1+\varepsilon^3\,h_2+\dots$$

where all functions ϕ_j depend on t and y while the functions h_j only depend on y. We naturally assume that ϕ_j nor h_j depend on ε and, since this will turn out to be the case and since this simplifies the computations, we also assume that h_0 is constant and $\phi_0=\phi_0(t)$.

Lemma 11.6. *The following expansion holds*

$$
\begin{array}{rcl}
\mathfrak{M}(w+\phi,h) & = & \varepsilon\left((\partial_t^2+f'(w))\phi_0 - \mathrm{Tr}(A)\,w'\right) \\
& + & \varepsilon^2\left((\partial_t^2+f'(w))\phi_1 - \mathrm{Tr}(A^2)\,(t+h_0)\,w' - \mathrm{Tr}(A)\partial_t\phi_0 + \frac{1}{2}f''(w)\phi_0^2\right) \\
& + & \varepsilon^3\Big((\partial_t^2+f'(w))\phi_2 - (\mathrm{Tr}(A^3)\,(t+h_0)^2 + J_\Gamma\,h_1)\,w' \\
& - & \mathrm{Tr}(A^2)(t+h_0)\partial_t\phi_0 + \frac{1}{2}f''(w)\phi_0\phi_1 + \frac{1}{6}f'''(w)\phi_0^3\Big) \\
& + & \varepsilon^4\Big((\partial_t^2+f'(w))\phi_3 - (\mathrm{Tr}(A^4)\,(t+h_0)^3 + J_\Gamma\,h_2)\,w' \\
& & \quad -\mathrm{Tr}(A^2)\,(t+h_0)\,\partial_t\phi_1 + \Delta_{g_0}\,\phi_1 + \frac{1}{2}\,f''(w)\phi_1^2 - \|dh_1\|^2_{g_0}\,w'' \\
& & \quad -\left(2\,\mathrm{Tr}(A^3)\,h_1 + [\partial_\zeta\Delta_{g_\zeta}h_1]_{|\zeta=0}\right)\,(t+h_0)\,w' \\
& & -\mathrm{Tr}(A^2)h_1\partial_t\phi_0 - \mathrm{Tr}(A^3)(t+h_0)^2\partial_t\phi_0 \\
& & +f^{(2)}(w)\phi_0\phi_2 + \frac{f^{(3)}(w)}{3}\phi_0\phi_1 + \frac{f^{(4)}(w)}{4!}\phi_0^4\Big) \\
& + & \mathcal{O}(\varepsilon^5)
\end{array}
\tag{11.8}
$$

where

$$J_\Gamma := (\Delta_{g_0} + \mathrm{Tr}(A^2))$$

is the Jacobi operator about Γ *and* $f^{(j)}$ *denotes the* $j-th$ *order derivative of* f*.*

Proof. Under the above assumptions, the following expansion is easy to derive

$$\left[\varepsilon^2\Delta_{g_\zeta}\phi\right]_{|\zeta=\varepsilon t+h} = \varepsilon^4\,\Delta_{g_0}\phi_1 + \mathcal{O}(\varepsilon^5)$$

since $\phi_0 = \phi_0(t)$ which does not depend on y. Using the fact that g_ζ depends smoothly on ζ together with the facts that h_0 is constant and $\phi_0 = \phi_0(t)$, we get

$$\left[\Delta_{g_\zeta}h\right]_{|\zeta=\varepsilon t+h} = \varepsilon\,\Delta_{g_0}(h_1+\varepsilon\,h_2+\varepsilon^2\,h_3) + \varepsilon^3\,\left[\partial_\zeta\Delta_{g_\zeta}h_1\right]_{|\zeta=0}\,(t+h_0) + \mathcal{O}(\varepsilon^4)\,.$$

Using the result of Lemma 11.3, we obtain the expansion

$$
\begin{array}{rcl}
[H_\zeta]_{|\zeta=\varepsilon t+h} & = & \mathrm{Tr}(A) + \varepsilon\,\mathrm{Tr}(A^2)\,(t+h_0+\varepsilon\,h_1+\varepsilon^2\,h_2) \\
& + & \varepsilon^2\,\mathrm{Tr}(A^3)(t+h_0)(t+h_0+2\,\varepsilon\,h_1) \\
& + & \varepsilon^3\,\mathrm{Tr}(A^4)(t+h_0)^3 + \mathcal{O}(\varepsilon^4).
\end{array}
$$

Next, we have

$$\left[\|dh\|^2_{g_\zeta}\right]_{|\zeta=\varepsilon t+h} = \varepsilon^4\,\|dh_1\|^2_{g_0} + \mathcal{O}(\varepsilon^5)$$

since h_0 is assumed to be constant. Similarly, we get

$$\left[(dh, d\,\partial_t\phi)_{g_\zeta}\right]_{|\zeta=\varepsilon t+h} = \mathcal{O}(\varepsilon^4)$$

since h_0 is constant and $\phi_0 = \phi_0(t)$. Finally, Taylor's expansion yields

$$f(w+\phi) - f(w) - f'(w)\,\phi \;=\; \tfrac{1}{2}f''(w)\phi^2 + \tfrac{1}{3!}f^{(3)}(w)\phi^3 + \tfrac{1}{4!}f^{(4)}(w)\phi^4 + \mathcal{O}(\varepsilon^5).$$

To derive the expansion, it is enough to insert these expression in (11.7) and rearrange the result in powers of ε. □

Using similar arguments, we also get the expansion of the normal derivative of $v + \phi$ in powers of ε.

Lemma 11.7. *The following expansion holds*

$$\mathfrak{N}(w+\phi,h) = w'(0) + \varepsilon\phi_0'(0) + \left(\varepsilon^2\,\partial_t\phi_1 + \varepsilon^3\,\partial_t\phi_2 + \varepsilon^4\,\partial_t\phi_3\right)_{|t=0} + \|dh_1\|^2_{g_0} + \mathcal{O}(\varepsilon^5).$$

In order to construct the approximate solution the idea is first to find the functions $\phi_0, \phi_1, \dots, \phi_3$ so that $\mathfrak{M}(w+\phi,h) = \mathcal{O}(\varepsilon^5)$. Thanks to Lemma 11.6, we obtain the following system of equations

$$\begin{cases} (\partial_t^2 + f'(w))\phi_0 &= \mathrm{Tr}(A)\,w' \\ (\partial_t^2 + f'(w))\phi_1 &= \mathrm{Tr}(A^2)\,(t+h_0)\,w' \\ & + \ \mathrm{Tr}(A)\partial_t\phi_0 + \frac{1}{2}f''(w)\phi_0^2 \\ (\partial_t^2 + f'(w))\phi_2 &= (\mathrm{Tr}(A^3)\,(t+h_0)^2 - J_\Gamma\,h_1)\,w' \\ & + \ \mathrm{Tr}(A^2)(t+h_0)\partial_t\phi_0 + \frac{1}{2}f''(w)\phi_0\phi_1 + \frac{1}{6}f'''(w)\phi_0^3 \\ (\partial_t^2 + f'(w))\phi_3 &= (\mathrm{Tr}(A^4)\,(t+h_0)^3 - J_\Gamma\,h_2)\,w' \\ & + \ \mathrm{Tr}(A^2)\,(t+h_0)\,\partial_t\phi_1 - \Delta_{g_0}\,\phi_1 \\ & - \ \frac{1}{2}\,f''(w)\phi_1^2 + \|dh_1\|^2_{g_0}\,w'' \\ & + \ (2\,\mathrm{Tr}(A^3)h_1 + [\partial_\zeta\Delta_{g_\zeta}h_1]_{|\zeta=0})(t+h_0)\,w' \\ & +\mathrm{Tr}(A^2)h_1\partial_t\phi_0 + \mathrm{Tr}(A^3)(t+h_0)^2\partial_t\phi_0 \\ & - \ f^{(2)}(w)\phi_0\phi_2 - \frac{f^{(3)}(w)}{3}\phi_0\phi_1 + \frac{f^{(4)}(w)}{4!}\phi_0^4. \end{cases} \tag{11.9}$$

We consider this system of equation as a system of ordinary differential equations (of the variable $t > 0$) which depends smoothly on parameters (namely $y \in \Gamma$) through the functions $\mathrm{Tr}(A^k)$, the metric g_0 on Γ or the functions $h_0, \dots, h_2$.

The next step relies on the solvability of a second order ordinary differential equation. We shall solve ϕ_0 and ϕ_1, ϕ_2, ϕ_3 differently. First we solve ϕ_0. Observe that a crucial fact is that since M is a CMC surface

$$\mathrm{Tr}(A) \equiv Constant \tag{11.10}$$

so that ϕ_0 can be chosen to be a function of t only. In fact we can choose

$$\phi_0(t) = -\mathrm{Tr}(A)p_0(t) \tag{11.11}$$

where $p_0(t)$ is the unique bounded solution to (7.6). Observe that $\phi_0(0) = 0$ and ϕ_0 decays exponentially in t.

Next we solve ϕ_1, ϕ_2 and ϕ_3 successively and at the meantime we also determine h_0, h_1, h_2. This is similar to the procedure done in Section §5.

As observed in Section §5 for a bounded function $q(t) \in L^\infty(0,\infty)$ a necessary and sufficient condition to obtain a bounded solution $p(t)$ to

$$p'' + f'(w)p = q(t), \quad p(0) = p'(0) = 0 \tag{11.12}$$

is the following

$$\int_0^\infty q(t)w'(t)dt = 0 \tag{11.13}$$

In fact the solution $p(t)$ is given by (5.39).

We now explain how the constant h_0 and the functions h_1 and h_2 are chosen. The idea is to use (11.13) in order to solve the system (11.9) for any given constant h_0 and any given set of functions $h_1, \dots, h_3$ and then we determine $h_0, \dots, h_3$ so that

$$\partial_t \phi_{1\,|t=0} = \partial_t \phi_{2\,|t=0} = \partial_t \phi_{3\,|t=0} + \varepsilon^4 \, \|dh_1\|^2_{g_0} = 0 \tag{11.14}$$

on Γ.

To begin with, observe that, thanks to when $t = 0$, we have

$$w'(0)\, \partial_t \phi_{1\,|t=0} = \left(\int_0^\infty (t + h_0)\, w'(t)\, dt \right) \mathrm{Tr}(A^2)$$
$$+ \int_0^\infty (\mathrm{Tr}(A)\partial_t\phi_0 + \frac{1}{2} f''(w)\phi_0^2) w'(t) dt$$

and hence, the first equation in (11.14) amounts to ask that the constant $h_0 \in \mathbb{R}$ is chosen so that

$$\int_0^\infty (t + h_0)\, w'^2\, dt = - \int_0^\infty (\mathrm{Tr}(A)\partial_t\phi_0 + \frac{1}{2} f''(w)\phi_0^2) w'(t) dt \tag{11.15}$$

which can be solved uniquely for h_0.

Next we choose h_1 so that $\partial_t \phi_2|_{t=0} = 0$. By (11.13) this amounts to choosing h_1 such that

$$\begin{aligned} J_\Gamma(h_1) \int_0^\infty (w'(t))^2 dt &= \mathrm{Tr}(A^3) \int_0^\infty (t + h_0)^2 (w'(t))^2 dt \\ &+ \int_0^\infty \Big(\mathrm{Tr}(A^2)(t + h_0)\partial_t\phi_0 + \frac{1}{2} f''(w)\phi_0\phi_1 + \frac{1}{6} f'''(w)\phi_0^3 \Big) w'(t) dt \end{aligned}$$

which has a unique solution h_1, thanks to the nondegeneracy assumption on M.

A similar argument as above can be used to solve ϕ_2 so that $\partial_t \phi_2|_{t=0} = 0$ and hence a unique h_2 can be found.

If we succeed in achieving these choices to determine h_1 and h_2, then according to Lemma 11.6 and Lemma 11.7, this will ensure that

$$\mathfrak{M}(w + \bar\phi, \bar h) = \mathcal{O}(\varepsilon^5)$$

in a neighborhood of Γ in $\Gamma \times [0, \infty)$ and

$$(w + \bar\phi)|_{t=0} = 0,$$
$$\mathfrak{N}(w + \bar\phi, \bar h) = w'(0) + \varepsilon \phi_0'(0) + \mathcal{O}(\varepsilon^5)$$

on Γ for

$$\bar\phi := \varepsilon\phi_0 + \varepsilon^2 \phi_1 + \varepsilon^3 \phi_2 + \varepsilon^4 \phi_3 \qquad \text{and} \qquad \bar h := \varepsilon\, h_0 + \varepsilon^2\, h_1 + \varepsilon^3\, h_2\,.$$

We could use this iteration scheme to solve the equations $\mathfrak{M}(v, h) = 0$ and $\mathfrak{N}(v, h) =$ constant to any order but it turns out that the above accuracy will be sufficient for our purpose.

Proceeding as in the proof of Theorem 4, we look for true solutions of the form

$$\bar\phi := \varepsilon\phi_0 + \varepsilon^2 \phi_1 + \varepsilon^3 \phi_2 + \varepsilon^4 \phi_3 + \phi \qquad \text{and} \qquad \bar h := \varepsilon\, h_0 + \varepsilon^2\, h_1 + \varepsilon^3\, h_2 \, + h$$

where we use $\|\cdot\|_{C^{2,\sigma}_{0,\gamma}(\Gamma_\varepsilon\times(0,+\infty))}$ to measure ϕ and $\|\cdot\|_{C^{2,\sigma}(\Gamma)}$ to measure the function h. Since Γ is compact and nondegenerate, the rest of the proof goes exactly as those of Theorem 4. We omit the details.

12. Appendix: The BDG graph and its Jacobi operator

In this appendix we let Γ a fixed Bombieri-De Giorgi-Giusti minimal graph [5], as in the statement of Theorem 4.

12.1. Solvability for the Jacobi operator of the BDG graph.

We consider the linear problem

$$\mathcal{J}_\Gamma[h] := \Delta_\Gamma h + |A_\Gamma(y)|^2 h = \mathtt{g}(y) \quad \text{in } \Gamma. \tag{12.1}$$

In [8] the following result was established.

Proposition 12.1. *Let $4 < \nu < 5$. There exists a positive constant $C > 0$ such that if $\mathtt{g}$ satisfies*

$$\|\mathtt{r}^\nu\, g\|_{L^\infty(\Gamma)} \,<\, +\infty$$

then there is a unique solution of equation (12.1) such that $\|\mathtt{r}^{\nu-2}\, h\|_{L^\infty(\Gamma)} < +\infty$. This solution satisfies

$$\|\mathtt{r}^{\nu-2}\, h\|_{L^\infty(\Gamma)} \;\le\; C\, \|\mathtt{r}^\nu\, g\|_{L^\infty(\Gamma)}\,.$$

The proof of this result is based on the construction of explicit barriers, using the fact that the surfaces Γ and Γ_0 are uniformly close for r large. Barriers constitute an appropriate tool to solve Problem (12.1) since $\mathcal{J}_\Gamma$ satisfies maximum principle, as it follows from the presence of a positive bounded function in its kernel. In fact, we have that

$$\mathcal{J}_\Gamma[(1+|\nabla F|^2)^{-1/2}] = 0.$$

In the current setting we need to consider right hand sides with decay of order at most $O(r^{-4})$, the prototypes being $\mathtt{g} = \sum_{i=1}^8 k_i^3$ and $\mathtt{g} = \sum_{i=1}^8 k_i^4$. It is not possible in general to obtain a suitable barrier in the setting of the above proposition when $\nu \le 4$. We have however the validity of Proposition 12.2 below which will suffice for our purposes.

The closeness of the surfaces allows us to define a canonical correspondence between maps defined on Γ and functions on Γ_0 as follows. Let $p \in \Gamma$ with $\mathtt{r}(p) \gg 1$ and let $\nu(p)$ be the unit normal to Γ at p. Let $\pi(p) \in \Gamma_0$ be a point such that for some $t_p \in \mathbb{R}$ we have:

$$\pi(p) = p + t_p \nu(p). \tag{12.2}$$

As shown in [8], the point $\pi(p)$ exists and is unique when $r(p) \gg 1$, and the map $p \longmapsto \pi(p)$ is smooth, with uniformly bounded derivatives both for π and its inverse. The *approximate Jacobi operator* $\mathcal{J}_{\Gamma_0}$, corresponding to first variation of mean curvature at Γ_0, is given by

$$\mathcal{J}_{\Gamma_0}[h] := \Delta_{\Gamma_0} h + |A_{\Gamma_0}(y)|^2 h.$$

For large r, $\mathcal{J}_\Gamma$ is "close to" $\mathcal{J}_{\Gamma_0}$ in the sense of the following result, contained in [8].

Lemma 12.1. *Assume that h and h_0 are smooth functions defined respectively on Γ and Γ_0 for r large, and related through the formula*

$$h_0(\pi(y)) = h(y), \quad y \in \Gamma, \quad \mathtt{r}(y) > r_0.$$

There exists a $\sigma > 0$ such that

$$\mathcal{J}_\Gamma[h](y) = [\mathcal{J}_{\Gamma_0}[h_0] + O(r^{-2-\sigma})D^2_{\Gamma_0}h_0 + O(r^{-3-\sigma})D_{\Gamma_0}h_0 + O(r^{-4-\sigma})h_0\,]\,(\pi(y))\ . \tag{12.3}$$

We can compute explicitly the operator $\mathcal{J}_{\Gamma_0}$ as follows. Let us consider the first variation of mean curvature measured along vertical perturbations of the graph Γ_0, namely the linear operator $H'(F_0)$ defined by

$$H'(F_0)[\phi] \ := \ \frac{d}{dt}H(F_0 + t\phi)\,|_{t=0} \ = \ \nabla\cdot\left(\frac{\nabla\phi}{\sqrt{1+|\nabla F_0|^2}} - \frac{(\nabla F_0\cdot\nabla\phi)}{(1+|\nabla F_0|^2)^{\frac{3}{2}}}\nabla F_0\right).$$

Then we have the relation

$$\mathcal{J}_{\Gamma_0}[h] = H'(F_0)[\phi], \quad \text{where} \quad \phi(x') = \sqrt{1+|\nabla F_0(x')|^2}\,h(x', F(x')). \tag{12.4}$$

For vertical perturbations $\phi = \phi(r,\theta)$ of Γ_0, it is straightforward to compute

$$H'(F_0)[\phi] := \tilde{L} := \tilde{L}_0 + \tilde{L}_1, \tag{12.5}$$

with

$$\tilde{L}_0(\phi) \ = \frac{1}{r^7\sin^3(2\theta)}\left\{(9g^2\,\tilde{w}r^3\phi_\theta)_\theta + (r^5 g'^2\,\tilde{w}\phi_r)_r - 3(gg'\,\tilde{w}r^4\phi_r)_\theta - 3(gg'\,\tilde{w}r^4\phi_\theta)_r\right\}, \tag{12.6}$$

and

$$\tilde{L}_1(\phi) \ = \frac{1}{r^7\sin^3(2\theta)}\left\{(r^{-1}\,\tilde{w}\phi_\theta)_\theta + (r\tilde{w}\phi_r)_r\right\}, \tag{12.7}$$

$$\tilde{w}(r,\theta) := \frac{\sin^3 2\theta}{(r^{-4} + 9g^2 + g'^2)^{\frac{3}{2}}}. \tag{12.8}$$

We can expand

$$\tilde{w}(\theta, r) = \tilde{w}_0(\theta) + r^{-4}\,w_1(r,\theta),$$

where

$$\tilde{w}_0(\theta) := \frac{\sin^3(2\theta)}{(9g^2+g'^2)^{\frac{3}{2}}}, \quad w_1(r,\theta) = -\frac{3}{2}\frac{\sin^3(2\theta)}{(9g^2+g'^2)^{\frac{5}{2}}} + O(r^{-4}\sin^3(2\theta)).$$

We set

$$\begin{aligned} L_0(\phi) \ = \ \frac{1}{r^7\sin^3(2\theta)}\big\{&(9g^2\,\tilde{w}_0 r^3\phi_\theta)_\theta \ + \ (r^5 g'^2\,\tilde{w}_0\phi_r)_r \\ &- 3(gg'\,\tilde{w}_0 r^4\phi_r)_\theta - 3(gg'\,\tilde{w}_0 r^4\phi_\theta)_r\big\}. \end{aligned} \tag{12.9}$$

Crucial in the proof of Proposition 12.1, as in the arguments that follow below is the presence of explicit solutions that separate variables for the operator L_0. Let us consider the equation

$$L_0(r^\beta q(\theta)) = \frac{p(\theta)}{r^{4-\beta}}, \quad \theta \in (\frac{\pi}{4}, \frac{\pi}{2}). \tag{12.10}$$

By a direct computation we obtain

$$r^7\sin^3(2\theta)\,L_0(r^\beta q(\theta)) \ = \ r^{3+\beta}\,[\,9(g^2\,\tilde{w}_0 q')' - 3\beta(gg'q\,\tilde{w}_0)' + \,\tilde{w}_0(\beta+4)\,(\beta g'^2 q - 3gg'q')\,].$$

We see that $q = g^{\frac{\beta}{3}}$ annihilates the above operator. As a consequence, the operator takes a divergence form in the function $g^{-\frac{\beta}{3}}q$, namely,

$$r^7 \sin^3(2\theta)\, L_0(r^\beta q(\theta)) \;=\; 9r^{3+\beta}\, g^{\frac{\beta+4}{3}} \,\Big[\tilde{w}_0 g^{\frac{2}{3}}\,(\,g^{-\frac{\beta}{3}}q\,)'\Big]'.$$

Thus equation (12.10) becomes

$$\Big[\tilde{w}_0 g^{\frac{2}{3}}\,(\,g^{-\frac{\beta}{3}}q\,)'\Big]' = \frac{1}{9}p(\theta)g(\theta)^{-\frac{\beta+4}{3}}\,\sin^3(2\theta).$$

Provided that all quantities are well-defined, we get the following explicit formula for a solution $q(\theta)$, $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$.

$$q(\theta) = g^{\frac{\beta}{3}}(\theta)\,\big[A - \frac{1}{9}\int_{\frac{\pi}{4}}^{\theta} g^{-\frac{2}{3}}(\,9g^2 + g'^2\,)^{\frac{3}{2}}\,\frac{ds}{\sin^3(2s)}\int_s^{\frac{\pi}{2}} p(\tau)g^{-\frac{\beta+4}{3}}(\tau)\sin^3(2\tau)\,d\tau\,\big], \tag{12.11}$$

where A is an arbitrary constant.

Lemma 12.2. *(a) Let $p(\theta)$ be a smooth function, even with respect to $\pi/4$, namely*

$$p(\frac{\pi}{2} - \theta) = p(\theta) \quad \textit{for all} \quad \theta \in (0, \frac{\pi}{4}).$$

Then there exists a smooth function $h(r,\theta)$ with the same symmetry, that satisfies, for some $\mu > 0$,

$$\mathcal{J}_{\Gamma_0}[h] = \frac{p(\theta)}{r^4} + O(r^{-4-\mu}) \quad \textit{as } r \to +\infty, \tag{12.12}$$

and

$$\|\mathtt{r}^2(\log \mathtt{r})\, h\,\|_{L^\infty(\Gamma_0)} \;<\; +\infty.$$

(b) Let $p(\theta)$ be a smooth function, odd with respect to $\pi/4$, namely

$$p(\frac{\pi}{2} - \theta) = -p(\theta) \quad \textit{for all} \quad \theta \in (0, \frac{\pi}{4}).$$

Then there exists a smooth function $h(r,\theta)$ with the same symmetry, such that for some $\mu > 0$,

$$\mathcal{J}_{\Gamma_0}[h] = \frac{p(\theta)}{r^3} + O(r^{-4-\mu}) \quad \textit{as } r \to +\infty, \tag{12.13}$$

$$\|\mathtt{r}\, h\,\|_{L^\infty(\Gamma_0)} \;<\; +\infty,$$

and, in addition,

$$|\nabla_{\Gamma_0} h|^2 \;=\; \frac{\beta(\theta)}{r^4} + O(r^{-4-\mu})) \quad \textit{as } r \to +\infty, \tag{12.14}$$

where β is a positive function of class C^1, even with respect to $\frac{\pi}{4}$.

Proof. We will prove next part (a). We consider first the case in which $p(\pi/4) = 0$. We will construct a smooth function $\phi_0(r,\theta)$ such that for all large r we have

$$\tilde{L}(\phi_0) = \frac{p(\theta)}{r^4} + O(r^{-4-\mu}) \tag{12.15}$$

for some $\mu > 0$.

Using Formula (12.11) with $\beta = 0$ and suitable constant A, we see that

$$L_0(q(\theta)) = \frac{p(\theta)}{r^4}, \quad \theta \in (\frac{\pi}{4}, \frac{\pi}{2}),$$

for

$$q(\theta) = -\frac{1}{9}\int_{\frac{\pi}{4}}^{\theta} g^{-\frac{2}{3}}(\,9g^2 + g'^2\,)^{\frac{3}{2}}\ \frac{ds}{\sin^3(2s)}\ \int_s^{\frac{\pi}{2}} p(\tau)g^{-\frac{4}{3}}(\tau)\sin^3(2\tau)\,d\tau\ .$$

Let us analyze the asymptotic behavior of $q(\theta)$ near $\theta = \pi/4$. Setting

$$x = \theta - \frac{\pi}{4}$$

we can expand

$$g(\theta) = g_1 x + O(x^3), \quad g_1 = g'(\pi/4), \quad p(\theta) = p_2 x^2 + O(x^4), \quad p_2 = p''(\pi/4).$$

Hence we have

$$\int_\theta^{\frac{\pi}{2}} p(\tau)g^{-\frac{4}{3}}(\tau)\sin^3(2\tau)\,d\tau\ = A_0 + O(x^{\frac{5}{3}})$$

where

$$A_0 = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} p(\tau)g^{-\frac{4}{3}}(\tau)\sin^3(2\tau)\,d\tau\ .$$

Thus, we have

$$q(\theta) = -{g_1}^{-\frac{11}{3}} A_0 \int_0^x s^{-\frac{2}{3}}\,ds + O(x^2).$$

Hence, for $A_2 = -3{g_1}^{-\frac{11}{3}} A_0$, we get the expansion

$$q(\theta)\ =\ A_2(\theta - \pi/4)^{\frac{1}{3}} + O(\theta - \pi/4)^2\,. \tag{12.16}$$

Now, let us consider Let $\eta(s)$ be a smooth cut-off function such that $\eta(s) = 1$ for $s < 1$ and $\eta(s) = 0$ for $s > 2$. We consider the interpolation

$$\phi_0(r,\theta)\ :=\ (1-\eta(s))q(\theta), \quad s := r^2 g(\theta).$$

Then, using that $p(\theta) \sim g(\theta)^2 = O(r^{-4})$ on the support of η, we get

$$\begin{aligned} L_0(\phi_0)\ &=\ \frac{p(\theta)}{r^4}\ +\ O(r^{-10})\ + \\ L_0(\eta)\,\psi\ +\ &\frac{\tilde{w}_0}{r^4\sin^3(2\theta)}\,3g\psi_\theta\,[3g\,\eta_\theta - g'\,r\eta_r]\,, \\ &\psi = -q(\theta)\,. \end{aligned} \tag{12.17}$$

Now, we compute

$$\begin{gathered} \eta_r = 2\eta' r g = O(r^{-1}), \quad \eta_\theta = \eta' r^2 g' = O(r^2), \\ \eta_{rr} = 4\eta'' r^2 g^2 + 2\eta' g = O(r^{-2}), \quad \eta_{r\theta} = 2\eta'' r^3 g g' + 2\eta' r g' = O(r), \\ \eta_{\theta\theta} = \eta'' r^4 g'^2 + \eta' r^2 g'' = O(r^4). \end{gathered}$$

Substituting these expressions in (12.9) we then get

$$L_0(\eta)\ =\ O(r^{-4}), \quad [3g\,\eta_\theta - g'\,r\eta_r]\ =\ O(1),$$

while on the other hand in the support of the derivatives of η we have

$$\psi = O(g(\theta)^{\frac{1}{3}}) = O(r^{-\frac{2}{3}})$$

and also

$$3g\psi_\theta = 3gq' = O(g(\theta)^{\frac{1}{3}}) = O(r^{-\frac{2}{3}}).$$

Thus, globally we get

$$L_0(\phi_0) = \frac{p(\theta)}{r^4} + O(r^{-4-\frac{2}{3}})\,.$$

Now, let us consider the full operator $\tilde{L}$ evaluated at this ϕ_0. On the one hand, it is straightforward to check that

$$\tilde{L}_0(\phi_0) - L_0(\phi_0) = O(r^{-8}).$$

Let us estimate now $\tilde{L}_1(\phi_0)$ in (12.7). We have that

$$\begin{aligned} L_1(\phi_0) &= \frac{(1-\eta)}{r^8\sin^3(2\theta)}(\,\tilde{w}_0 q_\theta)_\theta \\ &+ \frac{\psi}{r^7\sin^3(2\theta)}\left\{(r^{-1}\,\tilde{w}_0\eta_\theta)_\theta + (r\tilde{w}_0\eta_r)_r\right\} \\ &+ \frac{\tilde{w}_0\eta_\theta\psi_\theta}{r^8\sin^3(2\theta)} = I_1 + I_2 + I_3 \end{aligned}$$

We observe that where $1-\eta$ is supported we have at worst

$$q_{\theta\theta} = O(g^{-\frac{5}{3}}) = O(r^{\frac{10}{3}})$$

and hence we find

$$I_1 = O(r^{-8+\frac{10}{3}}) = O(r^{-4-\frac{4}{3}}).$$

We also compute

$$I_2 = O(r^{-4-\frac{2}{3}}), \quad I_3 = O(r^{-4-\frac{4}{3}}).$$

Hence,

$$L_1(\phi_0) = O(r^{-4-\frac{2}{3}}).$$

We also readily see that $(\tilde{L}_1 - L_1)\phi_0$ is even smaller than the above bound. We conclude

$$\tilde{L}(\phi_0) = \frac{p(\theta)}{r^4} + O(r^{-4-\frac{2}{3}})\,. \tag{12.18}$$

where ϕ_0 is a symmetric, smooth bounded function. We recall that we have obtained this under the assumption that $p(0) = 0$.

We consider next the case $p(0) \neq 0$.

Let us compute $L_0(\log r)$. We get

$$\begin{aligned} L_0(\log r) &= \frac{1}{r^7\sin^3(2\theta)}\left\{4r^3 g'^2(\theta) - 3r^3(gg'\,\tilde{w}_0)_\theta\right\} \\ &= \frac{1}{r^4\sin^3(2\theta)}\left\{3g'^2(\theta) - 3gg''\,\tilde{w}_0 - 3gg'(\tilde{w}_0)_\theta\right\} \end{aligned}$$

$$= \frac{1}{r^4}\Big\{\frac{g'^2}{(9g^2+g'^2)^{3/2}} - \frac{3gg''}{(9g^2+g'^2)^{3/2}} - \frac{3g'g}{\sin^3(2\theta)}(\tilde{w}_0)_\theta\Big\}. \tag{12.19}$$

Then we observe that we can decompose

$$L_0(\log r) = \frac{g_1^{-1}}{r^4} + \frac{b(\theta)}{r^4}\,, \quad g_1 = g'(\pi/4) \tag{12.20}$$

where $b(\theta)$ is symmetric, smooth and with $b(\pi/4) = 0$. In addition, we readily check that

$$\tilde{L}_1(\log r) = O(r^{-12}), \quad (\tilde{L}_0 - L_0)(\log r) = O(r^{-11}),$$

hence

$$L(\log r) \;=\; \frac{g_1^{-1}}{r^4} \;+\; \frac{b(\theta)}{r^4} \;+\; O(r^{-11}). \tag{12.21}$$

Hence, if we let

$$A := g_1 p(\pi/4),$$

then we have that

$$L(A\log r) \;=\; \frac{p(\theta)}{r^4} \;-\; \frac{p_1(\theta)}{r^4} \;+\; O(r^{-11}) \tag{12.22}$$

where

$$p_1(\theta) := -Ab(\theta) + p(\theta) - p(\pi/4).$$

Now, let us consider a bounded approximate solution $\phi_0(r,\theta)$ as built above where p is replaced by p_1. We see then that

$$\phi_1 := A\log r + \phi_0$$

satisfies

$$L(\phi_1) \;=\; \frac{p(\theta)}{r^4} \;+\; O(r^{-4-\frac{2}{3}}). \tag{12.23}$$

Observe that then the function

$$h := (1-\eta(r))\,(1+|\nabla F_0|^2)^{-1/2}\phi_1$$

is smooth, symmetric, and satisfies (12.12) The proof of part (a) is concluded.

We prove now part (b). Let us consider Formula (12.11) for $\beta = 1$ We have now that

$$\tilde{L}(r\,q(\theta)) = \frac{p(\theta)}{r^3}, \theta \in (\frac{\pi}{4}, \frac{\pi}{2}) \tag{12.24}$$

for

$$q(\theta) = g^{\frac{1}{3}}(\theta)\int_{\frac{\pi}{4}}^{\theta} g^{-\frac{2}{3}}(\,9g^2 + g'^2\,)^{\frac{3}{2}}\,\frac{ds}{\sin^3(2s)}\int_s^{\frac{\pi}{2}} p(\tau)g^{-\frac{5}{3}}(\tau)\sin^3(2\tau)\,d\tau\,. \tag{12.25}$$

Since $p(0) = 0$ and p is smooth, we have that the asymptotic behavior of $q(\theta)$ near $\theta = \pi/4$ is now given by

$$q(\theta) \;= A_1(\theta - \pi/4)^{\frac{2}{3}} + O(\theta-\pi/4)^{\frac{5}{3}}.$$

Then we define

$$\phi_0(r,\theta) = (1-\eta(s))\,r\,q(\theta), \quad \theta \in (\frac{\pi}{4}, \frac{\pi}{2}), \quad s = r^2 g(\theta).$$

Similar computations as those for the first case of the lemma, lead us now to

$$L_0(\phi_0) = \frac{p(\theta)}{r^3} + O(r^{-4-\frac{1}{3}}), \quad L_1(\phi_0) = O(r^{-4-\frac{1}{3}}),$$

and consistently to

$$\tilde{L}(\phi_0) = \frac{p(\theta)}{r^3} + O(r^{-4-\frac{1}{3}}).$$

Finally, the function

$$h = \frac{\phi_0}{\sqrt{1+|\nabla F_0|^2}}$$

extended oddly through $\theta = \frac{\pi}{4}$ satisfies (12.13).

Finally, let us estimate the quantity $|\nabla_{\Gamma_0} h|^2$. The following general formula is easy to derive, to relate Euclidean derivatives of a function ψ with those along a graph of a function F_0. We have

$$|\nabla_{\Gamma_0}\psi|^2 \;=\; \left|\nabla\psi - \frac{\nabla\psi\cdot\xi}{|\xi|^2}\xi\right|^2 \;+\; \frac{1}{1+|\xi|^2}\,\frac{|\nabla\psi\cdot\xi|^2}{|\xi|^2}, \quad \xi = \nabla F_0. \tag{12.26}$$

Since $h = \psi(r,\theta)$ we obtain the following expression

$$|\nabla_{\Gamma_0}h|^2 = \frac{1}{(9g^2+g'^2)}(3gr^{-1}h_\theta - g'h_r)^2 + \frac{1}{r^4}\frac{1}{(9g^2+g'^2+r^{-2})}\frac{(3gh_r+g'r^{-1}h_\theta)^2}{9g^2+g'^2}.$$

Using the explicit expression for h, we see that the second term in the sum is always $O(r^{-4-\mu})$ with $\mu>0$. On the other hand, in the first term, the contribution of the cut-off region is of higher order. In fact just dropping the cut-off we finally see that

$$|\nabla_{\Gamma_0}h|^2 = \frac{\beta(\theta)}{r^4} + O(r^{-4-\mu})$$

where β is a positive function of class C^1, even with respect to $\frac{\pi}{4}$. The proof is concluded. □

Proposition 12.2. *(a) Problem (12.1) has a solution h with $\|\mathtt{r}^2(\log\mathtt{r})\,h\|_{L^\infty(\Gamma)} < +\infty$ if*

$$\mathtt{g} = \sum_{i=1}^{8} k_i^4 \quad \text{or} \quad \mathtt{g} = \left[\sum_{i=1}^{8} k_i^2\right]^2.$$

(b) If $\mathtt{g} = \sum_{i=1}^{8} k_i^3$, then Problem (12.1) has a solution h with $\|\mathtt{r}\,h\|_{L^\infty(\Gamma)} < +\infty$.

Proof. Let us prove (a). Let k_i^0 denote the principal curvatures of Γ_0. Then we compute directly that the functions

$$\sum_{i=1}^{8}|k_i^0|^4 \quad\text{and}\quad \left[\sum_{i=1}^{8}|k_i^0|^2\right]^2$$

are both of the form (for large r)

$$\mathtt{g}(y) = \frac{p(\theta)}{r^4}$$

with p symmetric and smooth. In addition, we have that away from the origin,

$$\sum_{i=1}^{8} k_i^4(y) \;=\; \sum_{i=1}^{8}|k_i^0(\pi(y))|^4 \;+\; O(\mathtt{r}(y)^{-6}).$$

Let h_0 be the approximate solution predicted by Part (a) of Lemma 12.2 in Γ_0, so that for instance

$$\Delta_{\Gamma_0}h_0 + |A_{\Gamma_0}|^2 h_0 = \sum_{i=1}^{8}|k_i^0|^4 + O(\mathtt{r}^{-4-\mu})$$

where

$$\|\mathtt{r}^2\log\mathtt{r}\,h_0\|_{L^\infty(\Gamma_0)} < +\infty.$$

Let $h_1(y) := h_0(\pi(y))$. Then, according to Lemma 12.1 and a direct computation we find that

$$\mathcal{J}_\Gamma[h_1](y) = \mathcal{J}_{\Gamma_0}[h_0](\pi(y)) + O(\mathtt{r}(y)^{-4-\mu}).$$

Hence

$$\mathcal{J}_\Gamma[h_1](y) = \sum_{i=1}^{8} k_i^4(y) + \zeta(y)$$

where $\zeta = O(\mathtt{r}^{-4-\mu})$. By Proposition 12.1 there exists a solution h_2 of

$$\mathcal{J}_\Gamma[h_2] = -\zeta$$

with $\|\mathtt{r}^{2+\mu}\, h_1\|_{L^\infty(\Gamma)} < +\infty$. The desired result follows by simply setting $h := h_1 + h_2$. The proof for the other right hand side is the same. For part (b) the argument is similar, taking into account Part (b) Lemma 12.2. □

12.2. **Weighted Schauder estimates.** We have the following result, that controls the decay of the first two derivatives of solutions of equation (12.1).

Lemma 12.3. *Let $\nu \ge 2$. There exists a constant $C > 0$ such that the following holds. Let h be a solution of equation (12.1) such that*

$$\|\mathtt{g}\|_{\sigma,\nu,\Gamma} + \|\mathtt{r}^{\nu-2}h\|_{L^\infty(\Gamma)} < +\infty.$$

Then

$$\|D^2_\Gamma h\|_{\sigma,\nu,\Gamma} + \|h\|_{\sigma,\nu-2,\Gamma} \;\le\; C\,[\,\|\mathtt{g}\|_{\sigma,\nu,\Gamma} + \|\mathtt{r}^{\nu-2}h\|_{L^\infty(\Gamma)}]. \tag{12.27}$$

Proof. We use the local coordinates (4.7). Then, around a point p with $r(p) = R$, for any sufficiently large R, the equation reads on $B(0, 2\theta R)$ for a small, fixed $\theta > 0$ as

$$a^0_{ij}(\mathtt{y})\partial_{ij}h + b^0_i(\mathtt{y})\partial_i h + |A_\Gamma(\mathtt{y})|^2 h = \mathtt{g}(\mathtt{y}) \quad \text{in } B(0, 2\theta R).$$

Consider the scalings

$$\tilde h(\mathtt{y}) = R^{\nu-2}h(R\mathtt{y}), \quad \tilde{\mathtt{g}}(\mathtt{y}) = R^\nu \mathtt{g}(R\mathtt{y}).$$

Then we obtain the following equation.

$$\tilde a^0_{ij}(\mathtt{y})\partial_{ij}\tilde h + \tilde b^0_i(\mathtt{y})\tilde\partial_i h + \tilde b_0(\mathtt{y})h = \tilde{\mathtt{g}} \quad \text{in } B(0, 2\theta),$$

where

$$\tilde a_{ij}(\mathtt{y}) = a^0_{ij}(R\mathtt{y}), \quad \tilde b_i(\mathtt{y}) = Rb^0_i(R\mathtt{y}), \quad \tilde b_0(\mathtt{y}) = R^2|A_\Gamma(R\mathtt{y})|^2.$$

We will apply interior elliptic estimates to this equation. First, let us notice that from the estimates obtained for the metric, we have that the coefficients above are all uniformly bounded an elliptic in $B(0, 2\theta)$. Besides, we have that their first derivatives are also bounded in this region, with bounds uniform on the point p and on R.

Elliptic estimates then yield

$$\|D^2_{\mathtt{y}}\tilde h\|_{C^{0,\sigma}(B(0,\theta))} + \|\tilde h\|_{C^{0,\sigma}(B(0,\theta))} \;\le\; C[\|\tilde{\mathtt{g}}\|_{C^{0,\sigma}(B(0,2\theta))} + \|\tilde h\|_{L^\infty(B(0,2\theta))}]. \tag{12.28}$$

Let us observe that for any $\mathtt{y}_1, \mathtt{y}_2 \in B(0, 2\theta)$ we have

$$|\tilde{\mathtt{g}}(\mathtt{y}_1)| \;=\; |R^\nu \mathtt{g}(R\mathtt{y})| \;\le\; C\|\mathtt{r}^\nu \mathtt{g}\|_{L^\infty(\Gamma)},$$

and

$$\frac{|\tilde{\mathtt{g}}(\mathtt{y}_1) - \tilde{\mathtt{g}}(\mathtt{y}_2)|}{|\mathtt{y}_1 - \mathtt{y}_2|^\sigma} = R^{\nu+\sigma}\frac{|\mathtt{g}(R\mathtt{y}_1) - \mathtt{g}(R\mathtt{y}_2)|}{|R\mathtt{y}_1 - R\mathtt{y}_2|^\sigma} \;\le\; C[\mathtt{g}]_{\sigma,\nu,\Gamma}.$$

Therefore, we have the inequalities

$$\|\tilde{\mathtt{g}}\|_{C^{0,\sigma}(B(0,2\theta))} \;\le\; C\,\|\mathtt{g}\|_{\sigma,\nu,\Gamma}, \quad \|\tilde h\|_{L^\infty(B(0,2\theta))} \le C\,\|\mathtt{r}^{\nu-2}h\|_{L^\infty(\Gamma)}. \tag{12.29}$$

Now, we have that

$$D^2\tilde h(\mathtt{y}) = R^\nu[D^2 h](R\mathtt{y}).$$

Hence for $\mathtt{y}_1, \mathtt{y}_2 \in B(0, \theta R)$ we have

$$R^{\nu} \frac{D^2 h(\mathtt{y}_1) - D^2 h(\mathtt{y}_2)}{|\mathtt{y}_1 - \mathtt{y}_2|^{\sigma}} = \frac{D^2 \tilde{h}(R^{-1}\mathtt{y}_1) - D^2 \tilde{h}(R^{-1}\mathtt{y}_2)}{|\mathtt{y}_1 - \mathtt{y}_2|^{\sigma}} \leq C\, R^{-\sigma}\, \|\tilde{h}\|_{C^{0,\sigma}(B(0,\theta))}.$$

It follows that if $\Lambda = Y_p(B(0,\theta))$ then

$$[D^2_\Gamma h]_{\nu,\sigma,\Lambda} \leq C\, \|D^2 \tilde{h}\|_{C^{0,\sigma}(B(0,\theta))}.$$

Similarly we have that

$$[h]_{\nu-2,\sigma,\Lambda} \leq C\, \|\tilde{h}\|_{C^{0,\sigma}(B(0,\theta))},$$

while clearly, also,

$$\|\mathtt{r}^{\nu-2} h\|_{L^\infty(\Lambda)} + \|\mathtt{r}^{\nu} D^2_\Gamma h\|_{L^\infty(\Lambda)} \leq C\, [\, \|\tilde{h}\|_{C^{0,\sigma}(B(0,\theta))} + \|D^2 \tilde{h}\|_{C^{0,\sigma}(B(0,\theta))}\,].$$

Hence from inequalities (12.28) and (12.29) we obtain

$$\|D^2_\Gamma h\|_{\sigma,\nu,\Lambda} + \|h\|_{\sigma,\nu-2,\Gamma} \leq C\, [\, \|\mathtt{g}\|_{\sigma,\nu,\Gamma} + \|\mathtt{r}^{\nu-2} h\|_{L^\infty(\Gamma)}],$$

where C is uniform in p con $\mathtt{r}(p) >> 1$. Using this and an interior estimate for the equation on a bounded region, the desired estimate (12.27) follows. □

Corollary 12.1. 1. *The solution h predicted by Proposition 12.1 satisfies the estimate*

$$\|D^2_\Gamma h\|_{\sigma,\nu,\Gamma} + \|h\|_{\sigma,\nu-2,\Gamma} \leq C\, \|\mathtt{g}\|_{\sigma,\nu,\Gamma}.$$

2. *The solution in Part (a) of Proposition 12.2 satisfies that for any small $\tau > 0$,*

$$\|D^2_\Gamma h\|_{\sigma,4-\tau,\Gamma} + \|h\|_{\sigma,2-\tau,\Gamma} < +\infty.$$

3. *The solution in Part (b) of Proposition 12.2 satisfies*

$$\|D^2_\Gamma h\|_{\sigma,3,\Gamma} + \|h\|_{\sigma,1,\Gamma} < +\infty$$

while for some $\mu > 0$

$$\|D_\Gamma h\|_{\sigma,2+\mu,\Gamma} < +\infty.$$

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References

[1] A. D. Alexandrov. *Uniqueness theorems for surfaces in the large. I*, (Russian) Vestnik Leningrad Univ. Math. 11, 5-17 (1956).

[2] L. Ambrosio and X. Cabré, *Entire solutions of semilinear elliptic equations in $\mathbb{R}^3$ and a conjecture of De Giorgi*, J. Amer. Math. Soc. 13 (2000), 725–739.

[3] H. Berestycki, L.A. Caffarelli, L. Nirenberg, *Monotonicity for elliptic equations in unbounded Lipschitz domains.* Comm. Pure Appl. Math. 50(11), 1089-1111 (1997)

[4] J.L. Barbosa and M.P. do Carmo, and Jost Eschenburg, *Stability of hypersurfaces of constant mean curvature in Riemannian manifolds*, Math. Z. 197 (1988), no. 1, 123-138.

[5] E. Bombieri, E. De Giorgi, E. Giusti, *Minimal cones and the Bernstein problem*, Invent. Math. 7 (1969) 243-268.

[6] C.J. Costa, *Example of a complete minimal immersions in $\mathbb{R}^3$ of genus one and three embedded ends,* Bol. Soc. Bras. Mat. 15(1-2) (1984), 47-54.

[7] E. De Giorgi, *Convergence problems for functionals and operators*, Proc. Int. Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978), 131-188, Pitagora, Bologna (1979).
[8] M. del Pino, M. Kowalczyk and J. Wei, *On De Giorgi's Conjecture in Dimensions $N \geq 9$,* Annals of Mathematics 174 (2011), no. 3, 1485-1569.
[9] M. del Pino, M. Kowalczyk and J. Wei, *Entire solutions of the Allen-Cahn equation and complete embedded minimal surfaces of finite total curvature in $\mathbb{R}^3$.* J. Differential Geom. 93 (2013), no. 1, 67131.
[10] A. Farina, *1d symmetry for solutions of quasilinear elliptic equations.* Trans. Amer. Math. Soc 363 (2011) no. 2, 579–609.
[11] A. Farina and E. Valdinoci, *Flattening results for elliptic PDEs in unbounded domains with applications to overdetermined problems.* Arch. Ration. Mech. Anal. 195 (2010), no. 3, 1025-1058.
[12] A. Farina, L. Mari and E. Valdinoci, *Splitting theorems, symmetry results and overdetermined problems for Riemannian manifolds.* Comm. Partial Differential Equations 38 (2013), no. 10, 1818-1862.
[13] B. Gidas, W.-M. Ni and L. Nirenberg. *Symmetry and related properties via the maximum principle.* Commun. Math. Phys. 68, 209–243 (1979).
[14] N. Ghoussoub and C. Gui, *On a conjecture of De Giorgi and some related problems*, Math. Ann. 311 (1998), 481–491.
[15] L. Hauswirth, F. Helein and F. Pacard, *On an overdetermined elliptic problem.* Pacific J. Math. 250 (2011), no. 2, 319-334
[16] D. Hoffman, W.H. Meeks III, *Embedded minimal surfaces of finite topology*, Ann. of Math. 131 (1990), 1-34.
[17] R. Mazzeo and F. Pacard, *Constant mean curvature surfaces with Delaunay ends.* Comm. Analysis and Geometry. 9, 1, (2001), 169-237.
[18] F. Morabito, *Index and nullity of the Gauss map of the Costa-Hoffman-Meeks surfaces,* Indiana Univ. Math. J. 58 (2009), no. 2, 677-707.
[19] S. Nayatani, *Morse index and Gauss maps of complete minimal surfaces in Euclidean $3-$space*, Comm. Math. Helv. 68(4)(1993), 511–537.
[20] F. Pacard and M. Ritoré, *From constant mean curvature hypersurfaces to the gradient theory of phase transitions*, J. Differential Geom. 64 (2003), 359–423.
[21] A. Ros, P. Sicbaldi, *Geometry and topology of some overdetermined elliptic problems,* J. Differential Equations 255 (2013) 951-977.
[22] O. Savin, *Regularity of at level sets in phase transitions,* Ann. of Math. 169 (2009), 41–78.
[23] P. Sicbaldi *New extremal domains for the first eigenvalue of the Laplacian in flat tori.* Calc. Var. Partial Differential Equations 37 (2010), no. 3-4, 329-344.
[24] F. Schlenk, P. Sicbaldi, *Bifurcating extremal domains for the first eigenvalue of the Laplacian.* Adv. Math. 229 (2012), no. 1, 602-632.
[25] J. Serrin. *A symmetry problem in potential theory.* Arch.Rat.Mech. Anal. 43 (1971), 304-318.
[26] J. Simons, *Minimal varieties in Riemannian manifolds.* Ann. of Math. 88 (1968), 62-105.
[27] M. Traizet, *Classification of the solutions to an overdetermined elliptic problem in the plane.* Geom. Funct. Anal. 24 (2014), no. 2, 690720.
[28] B. White, *The space of minimal submanifolds for varying Riemannian metrics*, Indiana Univ. Math. J. 40 (1991), no. 1, 161200

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