UNIQUENESS OF MULTIPLE-SPIKE SOLUTIONS VIA THE METHOD OF MOVING PLANES

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ABSTRACT. We study the uniqueness of multiple-spike solutions for some singularly perturbed Neumann problem in a ball. We are able to completely classify all two-peaked solutions. We can also classify all three-peaked solutions except some degenerate situations. Our main idea is by using the method of moving planes to show that two-peaks must be located on a line with the origin and the three-peaks must lie a two-dimensional hyperplane with the origin. Then we compute the degree of these solutions and show the uniqueness of such solutions.

1. INTRODUCTION

We consider the following singularly perturbed semilinear elliptic problem

$$\begin{cases} \epsilon^2 \Delta u - bu + f(u) = 0 \ \text{ in } \Omega, \\ u > 0 \ \text{ in } \ \Omega \ \text{ and } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$, $\epsilon > 0$ is a small constant, $\Delta := \sum_{j=1}^N \frac{\partial^2}{\partial x_j \partial x_j}$ denotes the Laplace operator in R^N, ν stands for the unit outer normal to $\partial\Omega$, $b > 0$ is a positive constant and $f(t)$ is a $C^{1+\sigma}(R) \cap C^2_{loc}(0, +\infty)$ function such that $f(0) = f^{'}(0) = 0$. Typical examples of the function $-bu + f(u)$ are

$$-au + f(u) = -u + u_+^p \text{ with } u_+ = \max(0, u), b = 1, \tag{1.2}$$

$$-bu + f(u) = u(u-a)(1-u) \text{ with } 0 < a < \frac{1}{2}, b = 1 + a, \tag{1.3}$$

where

$$1 < p < (\frac{N+2}{N-2})_+ (= \frac{N+2}{N-2} \ \text{ when } N \geq 3; = +\infty \ \text{ when } N = 1, 2).$$

Equation (1.1) with (1.2) or (1.3) arises in many branches of applied sciences. For example, it can be viewed as steady-state equation for the shadow

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system of Gierer-Meinhardt system in biological pattern formation ([8], [39]) or steady-state of parabolic equations in population dynamics ([27]).

Associated with (1.1) is the energy J_ϵ defined by

$$J_\epsilon[u] := \int_\Omega (\frac{\epsilon^2}{2}|\nabla u|^2 + \frac{b}{2}u^2 - F(u))dx \qquad \text{for } u \in H^1(\Omega), \tag{1.4}$$

where $F(u) = \int_0^u f(s)ds$.

It is known that any solution u of (1.1) is a critical point of J_ϵ and vice versa. In this paper, we restrict ourselves to families of solutions $\{u_\epsilon\}_{0<\epsilon<\epsilon_0}$ of (1.1) with *finite* energy, i.e.

$$\lim_{\epsilon \to 0} \epsilon^{-N} J_\epsilon[u_\epsilon] < +\infty. \tag{1.5}$$

It can be proved that for ϵ sufficiently small, any family of solutions of (1.1) satisfying (1.5) can have at most finite number of local maximum points (see [24], [36]). Let the local maximum points be $\{P_1^\epsilon, ..., P_K^\epsilon\} \subset \bar{\Omega}$. Then one can show that for ϵ sufficiently small, we have

(S1) $P_i^\epsilon \neq P_j^\epsilon$ if $i \neq j, P_i^\epsilon \to P_i^0 \in \bar{\Omega}$ as $\epsilon \to 0$, and u_ϵ attains a strict local maximum at $x = P_j^\epsilon$, for $j = 1, ..., K$,

(S2) $c_0^{-1} \geq u_\epsilon(P_j^\epsilon) \geq c_0 > 0$ for some constant c_0 independent of ϵ and

(S3) $u_\epsilon(x) \to 0$ as $\epsilon \to 0$ locally uniformly in $\bar{\Omega} \backslash \{P_1^0, ..., P_K^0\}$.

Such solutions satisfying (S1)-(S3) and (1.5) are called K spike solutions.

In the pioneering papers of [22], [21], [24] and [25], Lin, Ni and Takagi established the existence of least-energy solutions and showed that for ϵ sufficiently small the least-energy solution has only one local maximum point P^ϵ and $P^\epsilon \in \partial\Omega$. Moreover, $H(P^\epsilon) \to \max_{P \in \partial\Omega} H(P)$ as $\epsilon \to 0$, where $H(P)$ is the mean curvature of $\partial\Omega$ at P.

Since then many works have been devoted to finding solutions with multiple spikes for the Neumann problem as well as the Dirichlet problem. See [1], [6], [7], [4], [5], [11], [12], [13], [14], [15], [17], [18], [24], [25], [26], [28], [27], [32], [33], [34], [40], [41], and the references therein. (Recent surveys can be found in [29], [42].) It turns out that in the Neumann problem case, there are arbitrarily many multiple interior spike solutions. In particular, it was proved in [12] that given any bounded domain Ω and a positive integer K, there exists an ϵ_K such that for $0 < \epsilon < \epsilon_K$ problem (1.1) has at least one

$K-$interior spike solution with spikes located at $\{P_1^\epsilon, ..., P_K^\epsilon\}$. Moreover, as $\epsilon \to 0$,

$$\varphi_K(P_1^\epsilon, ..., P_K^\epsilon) \to \max_{(P_1,...,P_K)\in\Omega^K} \varphi_K(P_1, ..., P_K) \tag{1.6}$$

where

$$\varphi_K(P_1, ..., P_K) = \min_{i,j,k,i\neq j}(\frac{|P_i - P_j|}{2}, d(P_k, \partial\Omega)). \tag{1.7}$$

(1.6) shows that if $(P_1^\epsilon, ..., P_K^\epsilon) \to (P_1^0, ..., P_K^0)$ as $\epsilon \to 0$, then $(P_1^0, ..., P_K^0)$ attains the *sphere packing* positions in Ω:

$$\varphi_K(P_1^0, ..., P_K^0) = \max_{(P_1,...,P_K)\in\Omega^K} \varphi_K(P_1, ..., P_K). \tag{1.8}$$

In this paper, we consider mainly the case where the domain Ω is the unit ball $B = \{x \in R^N||x| < 1\}$. The results of [26] and [12] shows that there are arbitrary number of multiple boundary spikes or multiple interior spikes. In general, the limiting positions of the boundary spikes or interior spikes have certain symmetry. As an example, let us consider the interior three spikes constructed in [12] (see Fig. 1). The three locations P_1^0, P_2^0, P_3^0 form a perfect triangle and is symmetric under rotation by $\frac{2\pi}{3}$. Moreover, if $N \geq 3$, one can show that P_1^0, P_2^0, P_3^0 and the origin must lie on a two dimensional hyperplane. Naturally, one may ask: does the solution have the same *partial symmetry*?

The first result on the partial symmetry of spike solutions of (1.1) was due to Lin and Takagi [20]. (Independently, Grossi [10] obtained the symmetry of single interior spike solution.) In [20], Lin and Takagi showed that single boundary spike solution must be axially symmetric, single interior spike solution must be radially symmetric, and two boundary spikes $P_1^\epsilon \in \partial\Omega$, $P_2^\epsilon \in \partial\Omega$ must be anti-pole, namely, $P_1^\epsilon = -P_2^\epsilon$. By using these information, they showed the uniqueness of the single boundary spike solutions and two boundary spike solutions. (We remark that uniqueness of single boundary and single interior spike solutions in general domains is studied in [39], [37].)

In this paper, we study the partial symmetry for two spikes and three spikes (interior or boundary or mixed). The method of moving plane (MMP) gives us part of the partial symmetry but not the full. To obtain the full

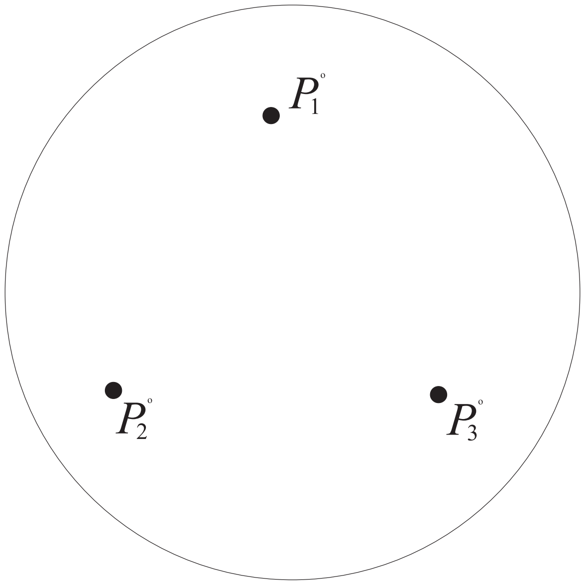

Figure 1

partial symmetry, we have to show the uniqueness of the solutions. To this end, we compute the degree of such solutions. To illustrate our idea, let us take a look again at the interior three spike solutions (see Fig. 1). Suppose we have three interior spikes $P_1^\epsilon, P_2^\epsilon, P_3^\epsilon$ with $P_j^\epsilon \to P_j^0, j = 1, 2, 3$ as $\epsilon \to 0$. To show the full partial symmetry, we have to show that $P_j^\epsilon = e^{\sqrt{-1}\frac{2(j-1)\pi}{3}} P_1^\epsilon, j = 2, 3$. Note that this is an equation with $3N$ variables and many symmetries (in other words, degeneracies). To overcome such difficulties, we proceed in two steps:

Step 1. We use MMP to show that $P_1^\epsilon, P_2^\epsilon, P_3^\epsilon$ and the origin must lie on a two-dimensional hyperplane and u_ϵ is axially symmetric with respect to the hyperplane. This reduces to a problem in R^2 with six variables. Now the rotational invariance reduces one more variable.

Step 2. We now show that $P_j^\epsilon = e^{\sqrt{-1}\frac{2(j-1)\pi}{3}} P_1^0$ when $P_j^\epsilon \in R^2, j = 2, 3$. To achieve this, we have to compute the degree of u_ϵ restricted to the symmetry class obtained in Step 1. We use Liapunov-Schmidt reduction method and asymptotic analysis to show that u_ϵ is nondegenerate and the degree at u_ϵ is exactly $(-1)^5$. This proves the uniqueness.

We now state our theorems. We always assume that $\Omega = B$ and $P_j^\epsilon \in \bar{\Omega}, j = 1..., K$ are the K spikes. Without loss of generality, we may assume that $b = 1$.

First we state the conditions on the function $f(t)$:

(f1) $f \in C^{1+\sigma}(R) \cap C^2_{loc}(0, +\infty)$ with $0 < \sigma \le 1, f(0) = 0, f'(0) = 0$ and $f(t) = 0$ for $t \le 0$.

(f2) There exists two positive constants α and β such that $0 < \alpha < \beta, (t-\alpha)(-t+f(t)) > 0$ for $0 < t < \alpha$ or $\alpha < t < \beta$, and $-\frac{1}{2}\beta^2 + F(\beta) > 0$ where $F(t) = \int_0^t f(s)ds$.

(f3) The problem in the whole space

$$\begin{cases} \Delta w - w + f(w) = 0, w > 0 \ \text{ in } R^N, \\ w(0) = \max_{y \in R^N} w(y), \lim_{|y| \to +\infty} w(y) = 0, \end{cases} \tag{1.9}$$

has a unique solution w, and it is nondegenerate, i.e.

$$\text{Kernel}(\Delta - 1 + f'(w)) = \text{span}\{\frac{\partial w}{\partial y_1}, ..., \frac{\partial w}{\partial y_N}\}. \tag{1.10}$$

By the well-known result of Gidas, Ni and Nirenberg [9], w is radially symmetric: $w(y) = w(|y|)$ and strictly decreasing: $w'(r) < 0$ for $r > 0, r = |y|$. Moreover, we have the following asymptotic behavior of w

$$w(r) = A_N r^{-\frac{N-1}{2}} e^{-r}(1 + O(\frac{1}{r})), w'(r) = -A_N r^{-\frac{N-1}{2}} e^{-r}(1 + O(\frac{1}{r})), \tag{1.11}$$

for r large, where $A_N > 0$ is a generic constant.

The uniqueness of w is proved in [16] for the case $f(u) = u^p$. For general nonlinearity, see [3]. For $f(u)$ defined by (1.3), the uniqueness of entire solution was proved by Peletier and Serrin [31].

In what follows we always assume that $f(t)$ satisfies (f1), (f2) and (f3).

Our first theorem concerns the case $K = 2$.

Theorem 1.1. *Let $K = 2$. Then for ϵ sufficiently small, $P_1^\epsilon, 0, P_2^\epsilon$ must lie on a line with P_1^ϵ and P_2^ϵ on each side. Without loss of generality, we may assume that P_1^ϵ and P_2^ϵ lie on the x_1-axis. Then u_ϵ is axially symmetric with respect to $x_j, j = 2, ..., N$.*

By Theorem 1.1 and some asymptotic analysis (Section 4 and Section 6), up to a rotation, there are three types of possibilities for the limiting positions of two peaks. The locations are listed below (see Fig. 2).

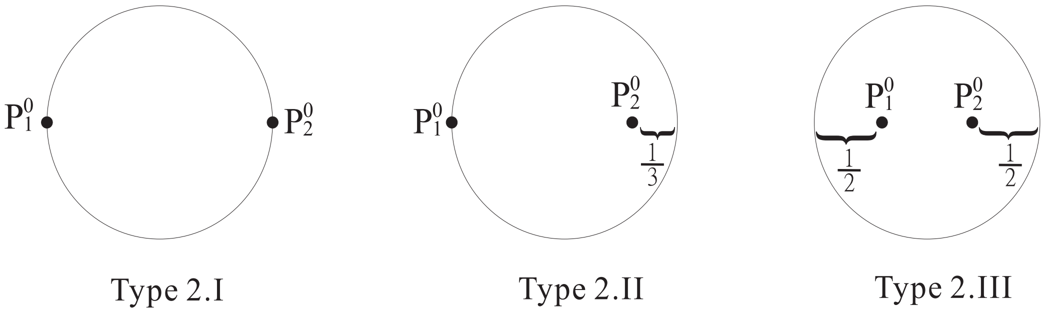

Figure 2

Our second theorem classifies all two-peaked solutions.

Theorem 1.2. *Let $K = 2$. Then for ϵ sufficiently small, up to a rotation, there are exactly three two-peaked solutions: $u_\epsilon^1, u_\epsilon^2, u_\epsilon^3$. The limiting locations of the three two-peaks are as above (Fig. 2).*

Moreover, $u_\epsilon^1, u_\epsilon^3$ are symmetric with respect to $x_j, j = 1, ..., N$, u_ϵ^2 is symmetric with respect to $x_j, j = 2, ..., N$.

We note that the existence of type 2.I solutions was proved in [26] and the existence of type 2.III solutions was proved in [12]. We remark that the existence and uniqueness of type 2.II solution is new. Combining Theorem 1.1 and Theorem 1.2, we have classified all two-peaked solutions.

Next we consider the three-peak case, which is more complicated.

Our third theorem shows that the three peaks and the origin must stay in a two-dimensional hyperplane.

Theorem 1.3. *Let $K = 3$ and $P_1^\epsilon, P_2^\epsilon, P_3^\epsilon$ be the three local maximum points of a three-peaked solution u_ϵ of (1.1). Then for ϵ sufficiently small, $P_1^\epsilon, P_2^\epsilon, P_3^\epsilon$ and 0 must lie on a two-dimensional hyperplane. Without loss of generality, we may assume that the hyperplane is $\Gamma := \{(x_1, ..., x_N) | x_3 = ... = x_N = 0\}$.*

Then u_ϵ is axially symmetric with respect to $x_j, j = 3, ..., N$. Moreover, the origin must be in the interior of the triangle formed by $P_1^\epsilon P_2^\epsilon P_3^\epsilon$.

Our last theorem concerns the uniqueness of three-peaked solutions. By Theorem 1.3 and some asymptotic analysis (Section 4 and Section 6), up to a rotation, there are seven types of possibilities for the limiting positions of three-peak. The locations are listed below (see Fig. 3).

We have

Theorem 1.4. *Let $K = 3$. Then for ϵ sufficiently small, there are at least seven types of three-peaked solutions. Among them, the first four types are unique. Each of the four type solutions is symmetric in $x_j, j = 3, ..., N$ and inherits the partial symmetry of the locations, e.g., type 3.II, type 3.III solutions are symmetric in x_2, type 3. I and type 3.IV are symmetric in $x_j, j = 1, ..., N$ and rotationally invariant by $\frac{2\pi}{3}$.*

The other three types of solutions are symmetric with respect to $x_j, j = 3, ..., N$.

The existence of type 3.I was proved in [26], and the existence of type 3.IV solution was proved in [12]. The existence of the other five solutions is new.

The uniqueness of other three types 3.V, 3.VI, 3.VII remains open. The main problem is that we can not show that they are symmetric with respect to the x_2-axis. Once this is shown, they are unique.

It is natural to ask what happens when $K \geq 4$. We pose the following conjecture at the end:

Conjecture: *If the limiting problem (1.8) has certain partial symmetry, then for ϵ sufficiently small, the solutions obtained in* [12] *inherits that partial symmetry.*

In this paper, we shall study the uniqueness of type 3.IV solutions of three-peaked solutions in detail, since it is the most complicated and it has the largest number of freedoms. The proofs of the uniqueness of other types will be mentioned at the last section.

The paper is organized as follows: In Section 2, we shall give the proof of Theorem 1.1 and 1.3 by applying the well-known method of moving planes

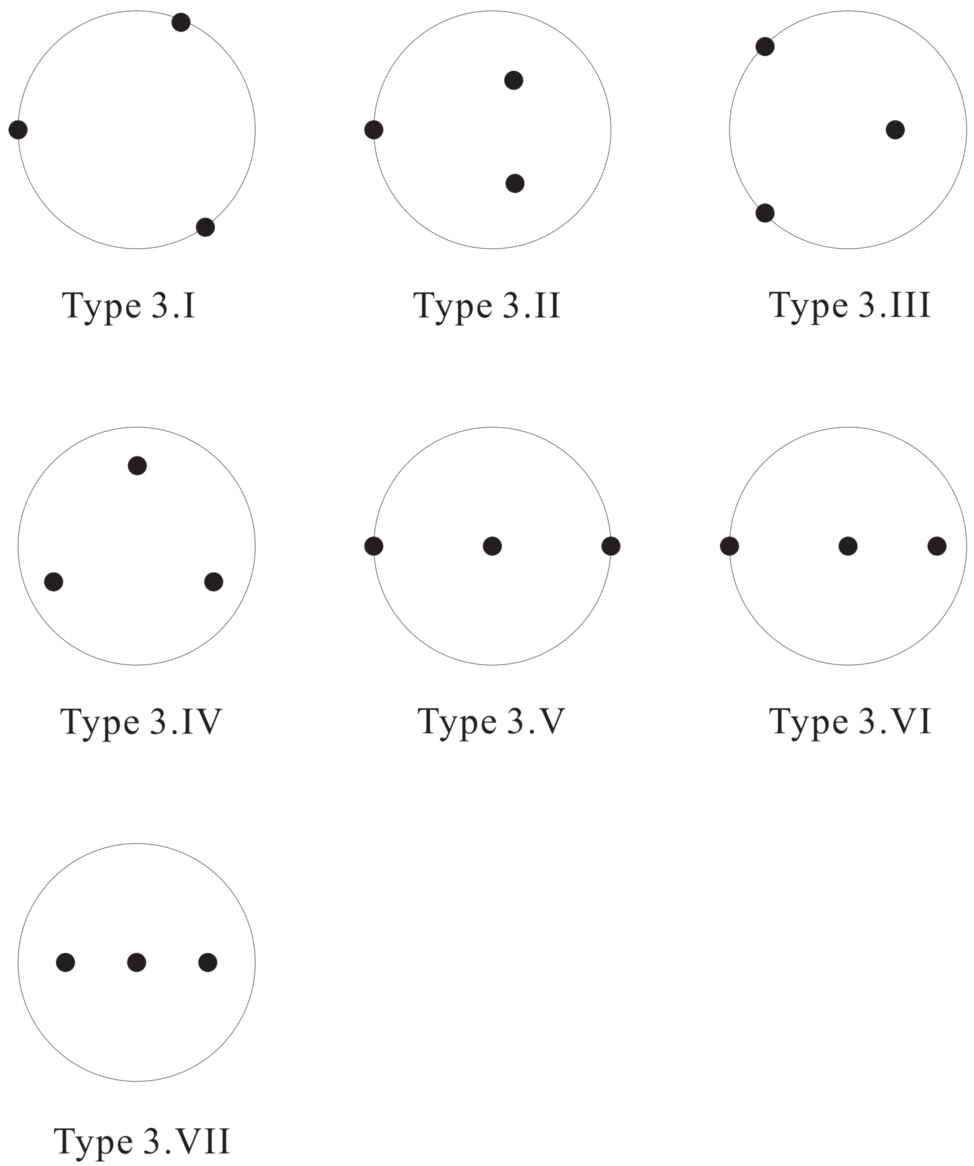

Figure 3

to Neumann problems. From Section 3 to Section 5, we compute the Morse index of type 3.IV solutions: In Section 3, we present some preliminaries on the reduction from infinite dimensional space $H^1(\Omega)$ to a finite dimensional problem on the space of spikes. In Section 4, we classify all types of limiting positions of two or three spikes. In Section 5, we show the uniqueness of type 3.IV solutions by computing its degree (restricted to certain symmetric class). Finally in Section 6, we show how similar ideas can be adopted to prove the uniqueness of other types of solutions. Several technical estimates are contained in Appendices A, B and C.

Throughout the paper, we use C to denote various constants independent of ϵ small. It is always assumed that $\epsilon > 0$ is small and $\delta > 0$ is a fixed but small constant.

2. Method of Moving Planes and the Proofs of Theorems 1.1 and 1.3

In this section, we apply the well-known method of moving planes to (1.1). We follow the proofs given in Section 3 of [20], where it is shown that for two boundary spikes $P_1^\epsilon, P_2^\epsilon$ there holds that $P_1^\epsilon = -P_2^\epsilon$, provided that ϵ is sufficiently small.

To describe the local structure of spike-layer solutions, we need to introduce a diffeomorphism $z = \Phi_\epsilon^j(x)$ which is defined in a neighborhood of each concentration point P_j^ϵ. If $P_j^\epsilon \in \Omega$, then $\Phi_\epsilon^j(x) = x$ and $\Omega_{\epsilon,R}^j = B_{\epsilon R}(P_j^\epsilon) = \{x||x - P_j^\epsilon| \leq \epsilon R\}$. If $P_j^\epsilon \in \partial\Omega$, then $z = \Phi_\epsilon^j(x)$ maps the boundary portion of $\partial\Omega$ at P_j^ϵ to $(0, ..., 0, 1)$. For details we refer to the reader to [24] and [20]. We also assume that Φ_ϵ^j maps the interior of Ω to the lower half-space $R_-^N = \{z|z_N < 0\}$. Let $B_R^- := \{z \in R_-^N||z| < R\}$. Set $\Omega_{\epsilon,R}^j = (\Phi_\epsilon^j)^{-1}(B_{\epsilon R}^-)$.

We state a general result on the asymptotic behavior of $K-$spikes.

Proposition 2.1. *Let $\{u_\epsilon\}$ be a family of solutions to (1.1) with K spikes $P_j^\epsilon \in \bar{\Omega}, j = 1, ..., K$. Suppose that*

$$\lim_{\epsilon \to 0} \frac{d(P_j^\epsilon, \partial\Omega)}{\epsilon} < +\infty, j = 1, ..., l,$$

and

$$\lim_{\epsilon \to 0} \frac{d(P_j^\epsilon, \partial\Omega)}{\epsilon} = +\infty, j = l + 1, ..., K.$$

Then $P_j^\epsilon \in \partial\Omega, j = 1, ..., l$ provided ϵ sufficiently small. Moreover, for any $\delta > 0$, there exists $R = R(\delta)$ and $\epsilon_1 = \epsilon(\delta)$ such that the following statements *hold if* $0 < \epsilon < \epsilon_1$

$$(i) \qquad \|u_\epsilon(x) - w(\Phi_j^\epsilon(x)/\epsilon)\|_{C^2(\bar{\Omega}^j_{\delta,R})} \leq \delta,$$

for $j = 1, ..., K$*;*

$$(ii) \qquad u_\epsilon(x) \leq C\delta e^{-\mu dist(x,\{P_1^\epsilon,...,P_K^\epsilon\})} \text{ for } x \in \bar{\Omega}\backslash \cup_{j=1}^K \Omega^j_{\epsilon,R}$$

where C *and* μ *are positive constants independent of* ϵ*.*

The proof of Proposition 2.1 is similar to that of Theorem 2.1 of [24]. We omit the details here.

Our main results in this section say that for two-peaks, they must lie a line with the origin. For three peaks, they must lie a two-dimensional hyperplane with the origin. Moreover, the corresponding solutions must be symmetric with respect to the line or the hyperplane. Since the proof of the two-peaked case is similar to that of [20], we focus our attention on three spikes. Without loss of generality, we assume that the three spikes $P_1^\epsilon, P_2^\epsilon, P_3^\epsilon$ lie on a two-dimensional hyperplane $\{(x_1, ..., x_N)|x_3 = \ldots = x_{N-1} = 0, x_N = t_\varepsilon\}$. If $t_\varepsilon = 0$, then $P_1^\epsilon, P_2^\epsilon, P_3^\epsilon$ and the origin are in a two-dimensional subspace. Hence, without loss of generality, we may assume that $t_\epsilon > 0$.

Set $e_\theta = (\sin\theta, 0, ..., 0, -\cos\theta)$ and Π^θ_{N-1} be the $(N-1)-$dimensional hyperplane perpendicular to the vector e_θ and x^θ denotes the reflection of x with respect to Π^θ_{N-1}. Set

$$w_\epsilon^\theta(x) = u_\epsilon(x) - u_\epsilon(x^\theta) \text{ for } x \in \Sigma_\theta$$

where Σ_θ is the connected component of $\Omega\backslash\Pi^\theta_{N-1}$ containing P_1^ϵ. Obviously w_ϵ^θ satisfies

$$\begin{cases} \epsilon^2\Delta w_\epsilon^\theta + c_\epsilon^\theta(x) w_\epsilon^\theta = 0 \text{ in } \Sigma_\theta, \\ \frac{\partial w_\epsilon^\theta}{\partial \nu}(x) = 0 \text{ on } \partial\Sigma_\theta\backslash\Pi^\theta_{N-1}, \\ w_\epsilon^\theta(x) = 0 \text{ on } \Sigma_\theta \cap \Pi^\theta_{N-1}, \end{cases} \tag{2.1}$$

where

$$c_\epsilon^\theta(x) = -1 + \frac{f(u_\epsilon(x)) - f(u_\epsilon(x^\theta))}{u_\epsilon(x) - u_\epsilon(x^\theta)}. \tag{2.2}$$

We prove our claim by a series of steps.

Step 1: We first prove that

$$w_\epsilon^0(x) > 0 \text{ for } x \in \Sigma_0 = \{x \in \Omega | x_N > 0\}. \tag{2.3}$$

Note that since $P_j^\epsilon, j = 1, 2, 3$ are the local maximum points of u_ϵ (and one of them must be global maximum point), we see that $w_\epsilon^0(P_j^\epsilon) > 0$ for some j. For contradiction, we assume that the set

$$E_\epsilon := \{x \in \Sigma_0 | w_\epsilon^0(x) < 0\}$$

is non-empty. We break the arguments into three cases. (All the following arguments are for a subsequence of $\epsilon_i \to 0$. For simplicity, we use the same notation ϵ to denote ϵ_i.)

Case 1: $\frac{t_\epsilon}{\epsilon} \to +\infty$ as $\epsilon \to 0$.

In this case, we see that for arbitrarily large $R > 0$, we have $E_\epsilon \subset (\cup_{j=1}^3 B_{\epsilon R}(P_j^\epsilon))^c$. Hence $u_\epsilon \leq \delta$ for $x \in E_\epsilon$ and ϵ small. Moreover $w_\epsilon^0(P_j^\epsilon) > 0$. Now that

$$c_\epsilon^\theta(x) \leq -\frac{1}{2}$$

for $x \in E_\epsilon$. By (2.1), the minimum value of w_ε, if it is negative, must be obtained on the boundary of Σ_0. Since $\frac{\partial w_\epsilon^\theta}{\partial \nu} = 0$ on $\partial\Omega \cap \Sigma_0$ and $w_\epsilon^\theta = 0$ on $\partial \Sigma_0$, by the Maximum Principle, $w_\epsilon^0 > 0$ and E_ϵ is empty.

Case 2: $C^{-1} \leq \frac{t_\epsilon}{\epsilon} \leq C$ for some $C > 1$ independent of ϵ.

Let $x_\epsilon \in \bar{E}_\epsilon$ be such that

$$w_\epsilon^0(x_\epsilon) = \inf_{E_\epsilon} w_\epsilon^0(x) < 0. \tag{2.4}$$

Assume for the moment that

$$\text{limsup}_{\epsilon \to 0}\{\frac{\min_{j=1,2,3} |x_\epsilon - P_j^\epsilon|}{\epsilon}\} \to +\infty.$$

Then by Proposition 2.1, $u_\epsilon(x_\epsilon) \to 0$ and $c_\epsilon(x_\epsilon) < -\frac{1}{2} < 0$ and $0 \leq \epsilon^2 \Delta w_\epsilon^0(x_\epsilon) = -c_\epsilon^0(x_\epsilon) w_\epsilon^0(x_\epsilon) < 0$, a contradiction. Therefore we conclude that

$$|x_\epsilon - P_j^\epsilon| \leq R\epsilon$$

for some $j = 1, 2, 3$. Without loss of generality, we may assume that $|x_\epsilon - P_1^\epsilon| \leq R\epsilon$. Let

$$x = \bar{P}_1^\epsilon + \epsilon y, v_\epsilon(y) = u_\epsilon(x), x_\epsilon = \bar{P}_1^\epsilon + \epsilon y_\epsilon.$$

Set $y_\epsilon = (y'_\epsilon, y_{\epsilon,N})$. Then $y_{\epsilon,N} \geq 0$ and let us assume that $y_{\epsilon,N} \to y_{0,N} \geq 0, y'_\epsilon \to y'$. We claim that $y_{0,N} > 0$. In fact, by our assumption, $\frac{P^\epsilon_{1,N}}{\epsilon} \to s > 0$. Then $v_\epsilon(y) \to w(y)$ in $C^2_{loc}(R^N)$. (If $P^\epsilon_1 \in \partial\Omega$, then we need to extend v_ϵ by reflection. We omit the details.) If $y_{0,N} = 0$, then $\frac{\partial w^0_\epsilon(\epsilon y + P^\epsilon_1)}{\partial y_N} \to 2\frac{\partial w}{\partial y_N}(y', s) < 0$ which contradicts to the fact that $\nabla w^0_\epsilon(x_\epsilon) = 0$. So $y_{0,N} > 0$. In this case, $\frac{\partial v_\epsilon}{\partial y_N} \to \frac{\partial w}{\partial y_N} > 0$ if $y_N > 0$. Hence $w^0_\epsilon(x_\epsilon) > 0$ for ϵ small. A contradiction again.

Case 3: $\frac{t_\varepsilon}{\varepsilon} = 0$ as $\varepsilon \downarrow 0$.

In order to yield a contradiction, as in the previous steps, we want to show

$$\lim_{\varepsilon\to 0}\left\{\frac{\min_{j=1,2,3}|x_\varepsilon - P^\varepsilon_j|}{\varepsilon}\right\} = +\infty, \tag{2.5}$$

wherever $w^0_\varepsilon(x_\varepsilon) < 0$. Set $N_\epsilon = \sup_{x\in\bar{B}_+}|w^0_\varepsilon(x)|$ and let $\bar{x}_\varepsilon \in \bar{B}_+$ be such that $|w^0_\varepsilon(\bar{x}_\varepsilon)| = N_\varepsilon$. By the maximum principle, it is easy to see

$$\min_{j=1,2,3}|\bar{x}_\varepsilon - P^\varepsilon_j| \leq C\,\varepsilon$$

for some constant $C > 0$. Without loss of generality, we may assume that $|\bar{x}_\varepsilon - P^\varepsilon_1| \leq C\,\varepsilon$. Now we want to prove

$$\lim_{\varepsilon\to 0}\frac{|x_\varepsilon - P^\varepsilon_1|}{\varepsilon} = +\infty \tag{2.6}$$

whenever $w^0_\varepsilon(x_\varepsilon) \leq 0$. Since the distance $|x_\varepsilon - P^\varepsilon_1|$ is invariant under an orthogonal transformation, we may assume $P^\varepsilon_1 - P^0_1 = (0, t_\varepsilon)$, after composing with a rotation. Scaling

$$\tilde{w}^0_\varepsilon(y) := \frac{1}{N_\varepsilon}w^0_\varepsilon(P^0_1 + \varepsilon y), \tag{2.7}$$

we see that $\tilde{w}^0_\varepsilon(y) \to \tilde{v}(y)$ in $C^2_{loc}(\bar{\mathbf{R}}^+_N)$, where

$$\tilde{v}(y) = c\,\frac{\partial w}{\partial y_N}(y) \quad\text{and}\quad c \neq 0. \tag{2.8}$$

Keeping the fact that u_ε attains its local maximum at P_1^ε, we apply the mean value theorem to see that

$$\begin{aligned}
\frac{\partial \tilde{w}_\varepsilon^0}{\partial y_N}(z_\varepsilon) &= \frac{\varepsilon}{N_\varepsilon}\left\{\frac{\partial u_\varepsilon}{\partial x_N}(P_1^\varepsilon) + \frac{\partial u}{\partial x_N}(\bar{P}_1^\varepsilon)\right\} \\
&= \frac{\varepsilon}{N_\varepsilon}\left\{\frac{\partial u}{\partial x_N}(\bar{P}_1^\varepsilon) - \frac{\partial u_\varepsilon}{\partial x_N}(P_1^\varepsilon)\right\} \\
&= -\frac{2\varepsilon^2}{N_\varepsilon}\frac{\partial^2 u_\varepsilon}{\partial x_N^2}(Q_1^\varepsilon)t_\varepsilon,
\end{aligned} \tag{2.9}$$

where $z_\varepsilon = \frac{P_1^\varepsilon - P_1^0}{\varepsilon}$, $\bar{P}_1^\varepsilon$ is the reflection of P_1^ε with respect to Σ_0 and $Q_1^\varepsilon \in (\bar{P}_1^\varepsilon, P_1^\varepsilon)$. Note that $\varepsilon^2 \frac{\partial^2 u_\varepsilon}{\partial x_N^2}(Q_1^\varepsilon) \to \frac{\partial^2 w}{\partial y_N^2}(0)$ as $\varepsilon \downarrow 0$ and $\frac{\partial \tilde{w}_\varepsilon^0}{\partial y_N}(z_\varepsilon) \to c\frac{\partial^2 w}{\partial y_N^2}(0)$ by (2.8). Therefore,

$$c = -2\lim_{\varepsilon\downarrow 0}\frac{t_\varepsilon}{N_\varepsilon} \le 0.$$

Since $c \neq 0$, we have

$$c = -2\lim_{\varepsilon\downarrow 0}\frac{t_\varepsilon}{N_\varepsilon} < 0. \tag{2.10}$$

Thus, (2.6) is proved.

To prove (2.5), say,

$$\lim_{\varepsilon\downarrow 0}\frac{|x_\varepsilon - P_2^\varepsilon|}{\varepsilon} = +\infty, \tag{2.11}$$

whenever $w_\varepsilon^0(x_\varepsilon) \le 0$. By composing with a rotation, we assume $P_2^\varepsilon = P_2^0 + t_\varepsilon e_N$ and set $w_\varepsilon^*(y) = \frac{1}{N_\varepsilon}w_\varepsilon^0(P_2^0 + \varepsilon y)$. Clearly, $w_\varepsilon^*(y)$ converges to $c^*\frac{\partial w}{\partial y_N}(y)$ in $C_{loc}^2(\bar{\mathbb{R}}_N^+)$ for some constant c^*. We claim $c^* < 0$. To see it, we apply the same computation as (2.8), that is,

$$\begin{aligned}
\frac{\partial w_\varepsilon^*(\tilde{z}_\varepsilon)}{\partial y_N} &= \frac{\varepsilon}{N_\varepsilon}\left\{\frac{\partial u_\varepsilon}{\partial x_N}(\bar{P}_2^\varepsilon) - \frac{\partial u_\varepsilon}{\partial x_N}(P_2^\varepsilon)\right\} \\
&= -\frac{2\varepsilon^2}{N_\varepsilon}\frac{\partial^2 u_\varepsilon}{\partial x_N^2}(\tilde{Q}_\varepsilon)t_\varepsilon,
\end{aligned}$$

where $\tilde{z}_\varepsilon = \frac{P_2^\varepsilon - P_2^0}{\varepsilon}$, $\bar{P}_2^\varepsilon$ is the reflection of P_2^ε and $\tilde{Q}_\varepsilon \in (\bar{P}_2^\varepsilon, P_2^\varepsilon)$. By (2.10),

$$\begin{aligned}
c^*\frac{\partial^2 w(0)}{\partial y_N^2} &= \lim_{\varepsilon\to 0}\frac{\partial w_\varepsilon^*(\tilde{z}_\varepsilon)}{\partial y_N} \\
&= c\,\frac{\partial^2 w}{\partial y_N^2}(0)
\end{aligned}$$

Thus, $c^* = c < 0$. Hence $w_\varepsilon^*(y)$ converges to a positive function $c\,\frac{\partial w}{\partial y_N}$ in $\mathbb{R}_+^N$ and then (2.11) follows. Thus, (2.6) is proved. By (2.6) and the maximum principle, we conclude $w_\varepsilon^0(y) > 0$ in B^+.

This finishes Step 1.

Step 2. We claim

$$w_\varepsilon^\theta(x) = 0 \ \text{ in } \Sigma_\theta \text{ for some } \ \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}].$$

Once the claim is established, we see that all P_j^ϵ must be contained in $\prod_{N-1}^\theta$. Otherwise, say, $P_1^\epsilon \notin \prod_{N-1}^\theta$. Then the reflection point $\bar{P}^\epsilon$ of P_e^ϵ with respect to $\prod_{N-1}^\theta$ is not in Σ_θ and then $\bar{P}^\epsilon$ is not one of $\{P_1^\varepsilon, P_2^\varepsilon, P_3^\varepsilon\}$, a contradiction to our assumption. Therefore $P_j^\epsilon \in \prod_{N-1}^\theta \cap \{(x_1, \dots, x_N) \mid x_3 = \dots = x_{N-1} = 0, x_N = t_\varepsilon\}$ for all $j = 1, 2, 3$. Form here, $0, P_1^\epsilon, P_2^\epsilon$ and P_3^ϵ lies in a 2-dimensional subspace.

To prove $w_\varepsilon^\theta \equiv 0$ in Σ_θ for some θ, we set

$$\theta_0 = \sup\{\bar{\theta} \mid w_\varepsilon^\theta > 0 \ \text{ for } x \in \Sigma_\theta, \text{ and } 0 \le \theta \le \bar{\theta} \le \tfrac{\pi}{2}\}.$$

Note that by the definition of θ_0, $w_\epsilon^{\theta_0} \ge 0$ in Σ_{θ_0}, and if $w_\epsilon^{\theta_0}(x) > 0$ for some point $x \in \bar{\Sigma}_{\theta_0} \setminus \prod_{N-1}^{\theta_0}$, then $w_\epsilon^{\theta_0}(x) > 0$ for all $x \in \bar{\Sigma}_{\theta_0} \setminus \prod_{N-1}^{\theta_0}$. If $\theta_0 < \frac{\pi}{2}$, then by a standard argument of the method of moving planes, we can prove $w_\epsilon^{\theta_0}(x) \equiv 0$ in Σ_{θ_0}. See [20] for the detail. If $\theta_0 = \frac{\pi}{2}$ and $w_\epsilon^{\theta_0}(x) > 0$ in Σ_{θ_0}, then we can rotate the hyperplane $\prod_{N-1}^\theta$ in the reverse direction, that is,

$$\theta_1 = \inf\{\bar{\theta} \mid w_\varepsilon^\theta > 0 \ \text{ in } \Sigma_\theta \text{ for } -\tfrac{\pi}{2} \le \bar{\theta} \le 0\}.$$

Clearly, $\theta_1 > -\frac{\pi}{2}$ and by the standard argument of the method of moving planes, we have $w_\varepsilon^{\theta_1} \equiv 0$ in Σ_{θ_1}.

Step 3: By Step 2, $P_j^\epsilon, j = 1, 2, 3$ and the origin lie on a two-dimensional hyperplane. Without loss of generality, we may assume that the hyperplane is $\Gamma = \{x_3 = ... = x_N = 0\}$. Now we show that

$$w_\epsilon^0(x) \equiv 0 \text{ on } \Sigma_0.$$

Suppose that there is a sequence $\epsilon_i \to 0$ such that

$$N_i = \sup_{x \in \Sigma_0} |w_i^0(x)| > 0,$$

where w_i^0 stands for $w_{\epsilon_i}^0$. Choose an $x_i \in \bar{\Sigma}_0$ so that $|w_i^0(x_i)| = N_i$. As in Case 2 above, we can show that

$$\text{limsup}_{i\to+\infty} \min_{j=1,2,3} (\frac{|x_i - P_j^\epsilon|}{\epsilon}) < +\infty.$$

Without loss of generality, we may assume that $\frac{|x_i - P_1^\epsilon|}{\epsilon} \le C\epsilon_i$. As before, let

$$v_i(y) = \frac{1}{N_i} w_i^0(P_1^\epsilon + \epsilon y).$$

Then, along a subsequence, $\{v_i\}$ converges to v in C^2_{loc}, and similar as before

$$v(y) = c\frac{\partial w}{\partial y_N}(y)$$

for some constant $c \neq 0$. Since $u_\epsilon(P_1^\epsilon) = 0$, we have $\frac{\partial v_i}{\partial y_N}(0) = 0$, so that $\frac{\partial v}{\partial y_N}(0) = 0$ which implies that $c = 0$ since $\frac{\partial^2 w}{\partial y_N^2}(0) \neq 0$. A contradiction. Consequently, $w_\epsilon^0(x) \equiv 0$ on Σ_0, i.e., $u_\epsilon(x', x_N) = u_\epsilon(x', x_N)$. Similarly, we have u_ϵ is symmetric in $x_j, j = 3, ..., N$.

□

3. Preliminaries I: Reduction to Finite-Dimensional Problem

From this section until Section 5, we shall prove the uniqueness of type 3.IV solutions. (Existence is given in [12].) Our main idea is to show that type 3.IV solution is nondegenerate (in some symmetry class) and to compute the Morse index of such solutions.

We first introduce a general framework. We start with some approximate functions. Then we introduce the so-called Liapunov-Schmidt reduction process, which reduces the existence of multiple spike solutions into finding a critical point for a finite dimensional problem (Lemma 3.4).

Recall that $\Omega = B$ and let $P \in \Omega, P \neq 0$. It is easy to compute that

$$d_P := d(P, \partial\Omega) = 1 - |P|, \tag{3.1}$$

$$\nabla_P d_P = -\frac{P}{|P|}, \tag{3.2}$$

$$\frac{\partial^2}{\partial P_i \partial P_j} d_P = -\frac{1}{|P|}(\delta_{ij} - \frac{P_i P_j}{|P|^2}) \tag{3.3}$$

where $P = (P_1, ..., P_N)$. (We remark that d_P is not differentiable at $P = 0$.)

Let w be the unique solution of (1.9) and $P \in \Omega$. We need to introduce a useful function-the *"projection"* of w in Ω. This was first introduced in [36] and later studied in [35]. The idea of projecting a function has been used in other problems. See [2], [28], [30] and the references therein.

We define $w_{\epsilon,P}$ to be the unique solution of

$$\left\{ \begin{array}{l} \epsilon^2 \Delta w_{\epsilon,P} - w_{\epsilon,P} + f(w(\frac{x-P}{\epsilon})) = 0 \text{ in } \Omega, \\ w_{\epsilon,P} > 0 \text{ in } \Omega, \ \frac{\partial w_{\epsilon,P}}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{array} \right. \tag{3.4}$$

We set

$$\bar{w}_{\epsilon,P} = w(\frac{x-P}{\epsilon}), w_{\epsilon,P} = \bar{w}_{\epsilon,P}(x) + \varphi_{\epsilon,P}(x). \tag{3.5}$$

Then $\varphi_{\epsilon,P}$ satisfies

$$\left\{ \begin{array}{l} \epsilon^2 \Delta \varphi_{\epsilon,P} - \varphi_{\epsilon,P} = 0 \text{ in } \Omega, \\ \frac{\partial \varphi_{\epsilon,P}}{\partial \nu} = -\frac{\partial}{\partial \nu} \bar{w}_{\epsilon,P} \text{ on } \partial\Omega. \end{array} \right. \tag{3.6}$$

We state the following useful lemma about the properties of $\varphi_{\epsilon,P}$ and the computations of some integrals. The proof of it is delayed to Appendix A.

Lemma 3.1. *Let $\Omega = B$ and $P \in \Omega$.*

(1) For ϵ sufficiently small, we have

$$\varphi_{\epsilon,P}(x) = (1+o(1)) w(\frac{x-P}{\epsilon}), \ \textit{for} \ \ x \in \partial\Omega, \tag{3.7}$$

and

$$-\epsilon \log \varphi_{\epsilon,P}(P) \to 2 d_P, \quad \textit{as } \epsilon \to 0. \tag{3.8}$$

(2) If we further assume that $|P| \geq d_0$ for some $d_0 > 0$, then we have

$$\varphi_{\epsilon,P}(P+\epsilon y) = \varphi_{\epsilon,P}(P)(1+o(1)) e^{-<\nabla d_P, y>}, \ \textit{for } P+\epsilon y \in \bar{\Omega}, \tag{3.9}$$

$$\varphi_{\epsilon,P}(P) = (c_N + o(1))(d_P(1-d_P))^{-\frac{N-1}{2}} \epsilon^{N-1} e^{-2d_P/\epsilon}, \tag{3.10}$$

where $c_N > 0$ is a generic constant (depending on N only), and

$$\int_\Omega f'(\bar{w}_{\epsilon,P}) \frac{\partial \bar{w}_{\epsilon,P}}{\partial P_i} \varphi_{\epsilon,P}(x) dx$$

$$= (-\gamma_1 + o(1)) \epsilon^{N-1} \varphi_{\epsilon,P}(P) (\nabla d_P)_i + O(e^{-(2+\sigma) d_P/\epsilon}) \tag{3.11}$$

where $(\nabla d_P)_i$ *denotes the* $i-th$ *component of* ∇d_P *(which is* $-P_i/|P|$ *in our case) and*

$$\gamma_1 = \int_{R^N} f(w) e^{-y_1} dy > 0, \sigma = \min(p-1,1). \tag{3.12}$$

(3) For ϵ *sufficiently small and* $P_1, P_2 \in \Omega$, $\frac{|P_1-P_2|}{\epsilon} \to +\infty$, *we have*

$$\int_\Omega f'(\bar{w}_{\epsilon,P_1}) \bar{w}_{\epsilon,P_2} \frac{\partial \bar{w}_{\epsilon,P_1}}{\partial P_{1,i}}$$

(3.13)

$$= \epsilon^{N-1}(-\gamma_1 + o(1)) w(\frac{|P_1 - P_2|}{\epsilon})(\nabla_{P_1}(|P_1 - P_2|))_i + O(e^{-(1+\sigma)|P_1-P_2|/\epsilon})$$

where γ_1 *is given by (3.12).*

In the rest of this section, we will describe the so-called Liapunov-Schmidt procedure. Most of the material is from Sections 3,4 and 5 in [12]. See also Sections 4, 5 and 6 in [12].

We first introduce some notations.

We consider $K-$interior spikes. The case of boundary spikes or mixed boundary-interior spikes will be discussed in Section 6. Fix $\mathbf{P} = (P_1, ..., P_K) \in \Omega^K$. Let $\varphi_K(\mathbf{P}) = \varphi_K(P_1, ..., P_K)$ be defined at (1.7). We assume that

$$\mathbf{P} \in \Lambda_\delta = \{\mathbf{P} \in \Omega^K | \varphi_K(\mathbf{P}) \geq 2\delta\} \tag{3.14}$$

where δ is a small but fixed positive constant.

To solve (1.1), we see that it is equivalent to finding a zero of the following nonlinear equation:

$$S_\epsilon[u] := \Delta u - u + f(u_+) = 0, u \in H^2_\nu(\Omega_\epsilon), \tag{3.15}$$

where

$$\Omega_\epsilon = \{y | \epsilon y \in \Omega\},$$

$$H^2_\nu(\Omega_\epsilon) := \{u \in H^2(\Omega_\epsilon) | \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega_\epsilon\}. \tag{3.16}$$

For any $u, v \in H^1(\Omega)$, we define the inner product and the norm as follows:

$$< u, v >_\epsilon = \epsilon^{-N} \int_\Omega (\epsilon^2 \nabla u \cdot \nabla v + u \cdot v), \quad \|u\|_\epsilon = < u, v >_\epsilon^{\frac{1}{2}} .$$

To simplify notations, we use the following simplified symbols:

$$\partial_{j,i} := \frac{\partial}{\partial P_{j,i}}, w_{\epsilon,\mathbf{P}} = \sum_{j=1}^{K} w_{\epsilon,P_j}.$$

We remark that the variable of $w_{\epsilon,\mathbf{P}}$ is in Ω. Sometimes, we also consider $w_{\epsilon,\mathbf{P}}(\epsilon y)$ for $y \in \Omega_\epsilon$. We denote $w_{\epsilon,\mathbf{P}}(\epsilon y)$ as $w_{\epsilon,\mathbf{P}}$.

Now we define the approximate kernel and cokernel respectively as follows:

$$\mathcal{K}_{\epsilon,\mathbf{P}} := \text{span } \{\partial_{j,i} w_{\epsilon,\mathbf{P}} | j = 1, ..., K, i = 1, ..., N\} \subset H^2_\nu(\Omega_\epsilon), \tag{3.17}$$

$$\mathcal{C}_{\epsilon,\mathbf{P}} := \text{span } \{\partial_{j,i} w_{\epsilon,\mathbf{P}} | j = 1, ..., K, i = 1, ..., N\} \subset L^2(\Omega_\epsilon). \tag{3.18}$$

(Note that $\partial_{j,i} w_{\epsilon,\mathbf{P}} \in H^2_\nu(\Omega_\epsilon)$ as one can differentiate the equation (3.4).)

We also need the following spaces

$$\mathcal{K}^\perp_{\epsilon,\mathbf{P}} = \{u \in H^2_\nu(\Omega_\epsilon) | \int_{\Omega_\epsilon} u \partial_{j,i} w_{\epsilon,\mathbf{P}} = 0, j = 1, ..., K, i = 1, ..., N\}, \tag{3.19}$$

$$\mathcal{C}^\perp_{\epsilon,\mathbf{P}} = \{u \in L^2(\Omega_\epsilon) | \int_{\Omega_\epsilon} u \partial_{j,i} w_{\epsilon,\mathbf{P}} = 0, j = 1, ..., K, i = 1, ..., N\}. \tag{3.20}$$

Set

$$\tilde{L}_{\epsilon,\mathbf{P}}(\phi) = \Delta\phi - \phi + f'(w_{\epsilon,\mathbf{P}})\phi, \quad \mathcal{L}_{\epsilon,\mathbf{P}} = \pi^\perp_{\epsilon,\mathbf{P}} \circ \tilde{L}_{\epsilon,\mathbf{P}}, \tag{3.21}$$

for $\phi \in H^2_\nu(\Omega_\epsilon)$, where $\pi^\perp_{\epsilon,\mathbf{P}}$ is the projection from $L^2(\Omega_\epsilon)$ into $\mathcal{C}^\perp_{\epsilon,\mathbf{P}}$.

We recall the following result in [35] (see Propositions 3.1 and 3.2 in [35]).

Lemma 3.2. *For $\epsilon << 1$, $\mathcal{L}_{\epsilon,\mathbf{P}} : \mathcal{K}^\perp_{\epsilon,\mathbf{P}} \to \mathcal{C}^\perp_{\epsilon,\mathbf{P}}$ is one-to-one and onto. Moreover, the inverse of $\mathcal{L}_{\epsilon,\mathbf{P}}$ exists and bounded (independent of $\epsilon > 0$).*

Next, we have

Lemma 3.3. *(Lemma 2.4 of* [39]*) For ϵ sufficiently small, $\mathbf{P} \in \Lambda_\delta$, there exists a unique $v_{\epsilon,\mathbf{P}} \in \mathcal{K}^\perp_{\epsilon,\mathbf{P}}$ such that*

$$S_\epsilon(w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}}) \in \mathcal{C}_{\epsilon,\mathbf{P}}. \tag{3.22}$$

Moreover, $v_{\epsilon,\mathbf{P}}$ is C^2 in $\mathbf{P}$ and

$$\|v_{\epsilon,\mathbf{P}}\|_\epsilon \le Ce^{-(1+\sigma)\varphi_K(\mathbf{P})/\epsilon} \tag{3.23}$$

$$\|\partial_{j,i} v_{\epsilon,\mathbf{P}}\|_\epsilon \le C\epsilon^{-2} e^{-(1+\sigma)\varphi_K(\mathbf{P})/\epsilon} \tag{3.24}$$

where $\sigma = \min(1, p-1)$.

Proof: The existence of $v_{\epsilon,\mathbf{P}} \in \mathcal{K}_{\epsilon,\mathbf{P}}^{\perp}$ such that $S_\epsilon(w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}}) \in \mathcal{C}_{\epsilon,\mathbf{P}}$ follows from Section 3 in [12]. The C^2-smoothness of $v_{\epsilon,\mathbf{P}}$ in $\mathbf{P}$ follows from Lemma 3.5 in [12]. For estimate (3.23), please see Lemma 3.4 of [12]. It remains to estimate $\partial_{j,i} v_{\epsilon,\mathbf{P}}$ and prove (3.24). We decompose

$$\partial_{j,i} v_{\epsilon,\mathbf{P}} = \sum_{s=1,\dots,K,t=1,\dots,N} \alpha^{\epsilon}_{st,ji} \partial_{s,t} w_{\epsilon,\mathbf{P}} + v^{\perp}_{\epsilon,\mathbf{P}}, v^{\perp}_{\epsilon,\mathbf{P}} \in \mathcal{K}^{\perp}_{\epsilon,\mathbf{P}},$$

where $\alpha^{\epsilon}_{st,ji}$ are scalar constants.

We first note that by (3.23)

$$\int_{\Omega_\epsilon} \partial_{j,i} v_{\epsilon,\mathbf{P}} \partial_{l,m} w_{\epsilon,\mathbf{P}} = -\int_{\Omega_\epsilon} v_{\epsilon,\mathbf{P}} \partial_{j,i} \partial_{l,m} w_{\epsilon,\mathbf{P}} = O(\epsilon^{-2} e^{-(1+\sigma)\varphi_K(\mathbf{P})/\epsilon}).$$

Hence

$$\sum_{s,t} \alpha^{\epsilon}_{st,ji} \int_{\Omega_\epsilon} \partial_{l,m} w_{\epsilon,\mathbf{P}} \partial_{s,t} w_{\epsilon,\mathbf{P}} = O(\epsilon^{-2} e^{-(1+\sigma)\varphi_K(\mathbf{P})/\epsilon}).$$

Since

$$\int_{\Omega_\epsilon} \partial_{l,m} w_{\epsilon,\mathbf{P}} \partial_{s,t} w_{\epsilon,\mathbf{P}} = \epsilon^{-2}(\Gamma_0 + o(1)) \delta_{sl} \delta_{mt},$$

where $\Gamma_0 = \int_{R^N} (\frac{\partial w}{\partial y_1})^2 > 0$, we obtain $\alpha^{\epsilon}_{st,ji} = O(\epsilon^{-2} e^{-(1+\sigma)\varphi_K(\mathbf{P})/\epsilon})$.

Next we observe that

$$S_\epsilon(w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}}) = \sum_{s,t} \beta^{\epsilon}_{s,t}(\mathbf{P}) \partial_{s,t} w_{\epsilon,\mathbf{P}}, \tag{3.25}$$

where $\beta^{\epsilon}_{s,t}(\mathbf{P}) \in C^1$ and $\beta^{\epsilon}_{s,t}(\mathbf{P}) = O(\epsilon^{-2} e^{-(1+\sigma)\varphi_K(\mathbf{P})/\epsilon})$.

Differentiating the equation (3.25) by $\partial_{l,m}$, we have,

$$S'_\epsilon(w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}})(\partial_{l,m} w_{\epsilon,\mathbf{P}} + \partial_{l,m} v_{\epsilon,\mathbf{P}}) - \sum_{s,t} \beta^{\epsilon}_{s,t}(\mathbf{P}) \partial_{l,m} \partial_{s,t} w_{\epsilon,\mathbf{P}} \in \mathcal{C}_{\epsilon,\mathbf{P}}.$$

Substituting the decomposition of $\partial_{j,i} v_{\epsilon,\mathbf{P}}$ into the above equation, we obtain that

$$S'_\epsilon(w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}})(v^{\perp}_{\epsilon,\mathbf{P}}) + S'_\epsilon(w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}})(\partial_{l,m} w_{\epsilon,\mathbf{P}} + \sum_{s,t} \alpha^{\epsilon}_{st,ji} \partial_{s,t} w_{\epsilon,\mathbf{P}})$$

$$- \sum_{s,t} \beta^{\epsilon}_{s,t}(\mathbf{P}) \partial_{s,t} \partial_{l,m} w_{\epsilon,\mathbf{P}} \in \mathcal{C}_{\epsilon,\mathbf{P}}.$$

It is easy to see that

$$\pi^{\perp}_{\epsilon,\mathbf{P}} \circ S'_\epsilon(w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}}) : \mathcal{K}^{\perp}_{\epsilon,\mathbf{P}} \to \mathcal{C}^{\perp}_{\epsilon,\mathbf{P}}$$

is again invertible for ϵ sufficiently small. Hence (since $v^{\perp}_{\epsilon,\mathbf{P}} \in \mathcal{K}^{\perp}_{\epsilon,\mathbf{P}}$)

$$\|v^{\perp}_{\epsilon,\mathbf{P}}\|_{H^2(\Omega_\epsilon)} \leq C\|\pi^{\perp}_{\epsilon,\mathbf{P}} \circ S'_\epsilon(w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}})(\partial_{l,m}w_{\epsilon,\mathbf{P}} + \sum_{s,t} \alpha^{\epsilon}_{st,ji}\partial_{s,t}w_{\epsilon,\mathbf{P}})|\|_{L^2(\Omega_\epsilon)}$$

$$+C\sum_{s,t}|\beta^{\epsilon}_{s,t}(\mathbf{P})|\|\partial_{s,t}\partial_{l,m}w_{\epsilon,\mathbf{P}}\|_{L^2(\Omega_\epsilon)}$$

$$\leq C\epsilon^{-2}e^{-(1+\sigma)\varphi_K(\mathbf{P})/\epsilon}.$$

(3.24) is thus proved.

□

Now we define

$$M_\epsilon(\mathbf{P}) = M_\epsilon(P_1, ..., P_K) := \epsilon^{-N}J_\epsilon[w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}}] \tag{3.26}$$

where J_ϵ is the energy functional introduced in (1.4) of Section 1 and $v_{\epsilon,\mathbf{P}}$ is given by Lemma 3.3.

Then we have the following reduction theorem.

Lemma 3.4. *(Proposition 4.1 of* [12]*)* $u_\epsilon = w_{\epsilon,\mathbf{P}^\epsilon} + v_{\epsilon,\mathbf{P}^\epsilon}, \mathbf{P}^\epsilon \in \Lambda_{2\delta}$ *is a critical point of* J_ϵ *if and only if* P^ϵ *is a critical point of* $M_\epsilon(\mathbf{P})$.

Therefore, to prove the existence and uniqueness of solutions of (1.1), we just need to concentrate on the study of $M_\epsilon(\mathbf{P})$, which is a finite-dimensional problem. We shall compute $\nabla M_\epsilon(\mathbf{P})$ and $\nabla^2 M_\epsilon(\mathbf{P})$ in the next two sections.

4. Preliminaries II: computations of $\nabla M_\epsilon(\mathbf{P})$ and $\nabla^2 M_\epsilon(\mathbf{P})$

In this section, we first obtain a general formula for the locations of K interior spikes $(P_1^\epsilon, ..., P_K^\epsilon)$ and classify all types of limiting locations. Then we compute the (first and second order) derivatives of $M_\epsilon(\mathbf{P})$.

The following theorem shows that there will be no spike collapsing to the boundary or with each other.

Lemma 4.1. *Let* $(P_1^\epsilon, ..., P_K^\epsilon)$ *be the* K *local maximum points of a* $K-$*peaked solution* u_ϵ *of (1.1). Suppose that* $K \leq 3$. *Let* $P_j^\epsilon \to P_j^0$ *for* $j = 1, ..., K$. *Then we have*

$$P_i^0 \neq P_j^0, \text{ for } i \neq j, \tag{4.1}$$

and if $P_i^0 \in \partial\Omega$, *then* $P_i^\epsilon \in \partial\Omega$ *for* ϵ *sufficiently small.*

Lemma 4.1 eliminates the collision of spikes or collision of interior spikes with the boundary. Note that if the mean curvature of the domain is not constant, one can construct multiple spikes concentrating at one local minimum point of the mean curvature (see [14]). The proof of Lemma 4.1 is technical and thus we delay the proof of it to the appendix C.

By Lemma 4.1, if $P_j^\epsilon, j = 1, 2, 3$ are three interior spikes, then $\varphi_K(P_1^\epsilon, \ldots, P_K^\epsilon) \geq \delta_0$ for some $\delta_0 > 0$. Now we choose $\delta = \frac{\delta_0}{4}$. By Lemma 3.4, $u_\epsilon = w_{\epsilon,\mathbf{P}^\epsilon} + v_{\epsilon,\mathbf{P}^\epsilon}$ is a solution with three interior spikes if and only if $\mathbf{P}^\epsilon$ is a critical point of M_ϵ, since $\mathbf{P}^\epsilon \in \Lambda_{2\delta}$.

The asymptotic expansion of $M_\epsilon(\mathbf{P})$ in Λ_δ is given in Proposition 4.1 of [12]

Lemma 4.2. *(Proposition 4.1 of* [12]*.) For ϵ sufficiently small and $\mathbf{P} \in \Lambda_\delta$, we have*
(4.2)
$$M_\epsilon(\mathbf{P}) = KI(w) - \frac{1}{2}(\gamma_1 + o(1))(\sum_{i=1}^{K} \varphi_{\epsilon,P_i}(P_i)) - (\gamma_1 + o(1)) \sum_{k \neq l} w(|P_k - P_l|/\epsilon)$$
where
$$I(w) = \frac{1}{2}\int_{R^N} |\nabla w|^2 + \frac{1}{2}\int_{R^N} w^2 - \int_{R^N} F(w) \tag{4.3}$$
and γ_1 is given by (3.12).

We now show that the asymptotic expansion in (4.2) holds true in C^2 sense.

We define
$$\tilde{M}_\epsilon(\mathbf{P}) := -\frac{\gamma_1}{2}\sum_{j=1}^{K} \varphi_{\epsilon,P_j}(P_j) - \gamma_1 \sum_{k \neq l} w(|P_k - P_l|/\epsilon). \tag{4.4}$$

By (3.10) of Lemma 3.1 and (1.11), we see that if $|P_j| \geq \frac{1}{10}, j = 1, ..., K$, then we have
$$\tilde{M}_\epsilon(\mathbf{P}) := -\frac{c_N(\gamma_1 + o(1))}{2}\epsilon^{\frac{N-1}{2}} \sum_{j=1}^{N} c(P_j) e^{-2d_{P_j}/\epsilon} \tag{4.5}$$
$$-A_N(\gamma_1 + o(1))\epsilon^{\frac{N-1}{2}} \sum_{k \neq l} (|P_k - P_l|)^{-\frac{N-1}{2}} e^{-|P_k - P_l|/\epsilon},$$

where c_N is given in (3.10) of Lemma 3.1, $A_N > 0$ is given by (1.11), and

$$c(P) = (d_P(1-d_P))^{-\frac{N-1}{2}}. \tag{4.6}$$

The following lemma is the key estimate we need.

Lemma 4.3. *Suppose that* $\mathbf{P} \in \Lambda_\delta$ *and* ϵ *is sufficiently small.*

(1) If $|P_j| \geq d_0$ *for some* j *and* $d_0 > 0$*, then we have*

$$\partial_{j,i} M_\epsilon(\mathbf{P}) = \partial_{j,i} \tilde{M}_\epsilon(\mathbf{P}) + O(\tilde{M}_\epsilon(\mathbf{P})), i = 1, ..., N. \tag{4.7}$$

(2). If $|P_j| \leq \frac{1}{10}$ *for some* j*, then we have*

$$\partial_{j,i} M_\epsilon(\mathbf{P}) = -(\gamma_1 + o(1)) \sum_{k \neq j} \partial_{j,i}(w(|P_k - P_j|/\epsilon)) + O(\tilde{M}_\epsilon(\mathbf{P})), i = 1, ..., N. \tag{4.8}$$

(3) Suppose that $\mathbf{P}^\epsilon$ *is a critical point of* $M_\epsilon(\mathbf{P})$ *such that* $|P_j^\epsilon| \geq d_0, j = 1, ..., N$ *for some* $d_0 > 0$*. Then we have*

$$\partial_{l,m}\partial_{j,i} M_\epsilon(\mathbf{P})\big|_{\mathbf{P}=\mathbf{P}^\epsilon} = \partial_{l,m}\partial_{j,i} \tilde{M}_\epsilon(\mathbf{P})\big|_{\mathbf{P}=\mathbf{P}^\epsilon} + O(\epsilon^{-1}\tilde{M}_\epsilon(\mathbf{P}^\epsilon)), l = 1, ..., K, m = 1, ..., N. \tag{4.9}$$

More precisely, we have

$$\begin{aligned} &\partial_{l,m}\partial_{j,i} M_\epsilon(\mathbf{P})\big|_{\mathbf{P}=\mathbf{P}^\epsilon} \\ &= \epsilon^{N-2}(\gamma_1 + o(1)) w(|P_j^\epsilon - P_l^\epsilon|/\epsilon) e_{jl,m}^\epsilon e_{jl,i}^\epsilon (1 - \delta_{jl}) \\ &\quad - \epsilon^{N-2}(\gamma_1 + o(1)) \varphi_{\epsilon, P_j^\epsilon}(P_j^\epsilon) e_{j,i}^\epsilon e_{l,m}^\epsilon \delta_{jl} \\ &\quad - \epsilon^{N-2}(\gamma_1 + o(1)) \sum_{k \neq j} w(|P_j^\epsilon - P_k^\epsilon|/\epsilon) e_{jk,i}^\epsilon e_{jk,m}^\epsilon \delta_{jl}, \end{aligned} \tag{4.10}$$

where

$$e_j^\epsilon = \frac{P_j^\epsilon}{|P_j^\epsilon|}, e_{jk}^\epsilon = \frac{P_j^\epsilon - P_k^\epsilon}{|P_j^\epsilon - P_k^\epsilon|}, j \neq k, \tag{4.11}$$

and $e_{j,i}^\epsilon$ *and* $e_{jk,i}^\epsilon$ *denote the* $i-th$ *component of the vectors* e_j^ϵ *and* e_{jk}^ϵ*, respectively.*

Remark: The reason that we have to consider cases when P_j^ϵ is close to the origin is that the function d_P is not differentiable at the origin. However if we know the rate of P_j^ϵ approaching the origin, we may still be able to compute the derivatives (see [39]). This is a delicate issue that needs further investigation. For the purpose of this paper, Lemma 4.3 is good enough.

The proof of Lemma 4.3 is very technical and we will present it in Appendix B.

Let $\mathbf{P}^\epsilon = (P_1^\epsilon, ..., P_K^\epsilon)$ be a critical point of $M_\epsilon(\mathbf{P})$. Namely, we have

$$\partial_{j,i} M_\epsilon(\mathbf{P})|_{\mathbf{P}=\mathbf{P}^\epsilon} = 0, j = 1, ..., K, i = 1, ..., N. \tag{4.12}$$

(4.10) shows that if $\mathbf{P}^\epsilon$ is a critical point of M_ϵ and $|P_j^\epsilon| \geq \frac{1}{10}, j = 1, ..., K$, then we have

$$\begin{aligned} & c_N(\gamma_1 + o(1)) c(P_j^\epsilon) e^{-2d_{P_j^\epsilon}/\epsilon} (\nabla_{P_j^\epsilon} d_{P_j^\epsilon})_i \\ & + A_N(\gamma_1 + o(1)) \sum_{l \neq j} (|P_l^\epsilon - P_j^\epsilon|)^{-\frac{N-1}{2}} e^{-|P_l^\epsilon - P_j^\epsilon|/\epsilon} (\nabla_{P_j^\epsilon} |P_j^\epsilon - P_l^\epsilon|)_i \\ & + O(\epsilon (\sum_{j=1}^K e^{-2d_{P_j^\epsilon}/\epsilon} + \sum_{l \neq k} e^{-|P_l^\epsilon - P_k^\epsilon|/\epsilon})) = 0, j = 1, ..., K, i = 1, ..., N. \end{aligned} \tag{4.13}$$

(4.8) and (4.13) enable us to classify all types of locations for the limiting interior two or interior three peaks. Moreover, it also gives us the estimate on the speed of $\mathbf{P}^\epsilon$ approaching the limiting positions (Lemma 5.1 in the next section).

Lemma 4.4. *Let $K = 2$ or 3 and $\tilde{P}_j^\epsilon \in \Omega, j = 1, ..., K$ be the K local maximum points of u_ϵ. Suppose that $\tilde{P}_j^\epsilon \to P_j^0, j = 1, ..., K$, then up to a rotation, $\mathbf{P}^0 = (P_1^0, ..., P_K^0)$ must belong to type 2.III (for $K = 2$, Fig. 2) and either type 3.IV or type 3.VII (for $K = 3$, Fig. 3).*

Proof: Let u_ϵ be a solution of (1.1) with two interior spikes or three interior spikes. By Lemma 3.4, we have $u_\epsilon = w_{\epsilon,\mathbf{P}^\epsilon} + v_{\epsilon,\mathbf{P}^\epsilon}$, where $\mathbf{P}^\epsilon \in \Lambda_\delta$. Let $\tilde{P}_j^\epsilon, j = 1, 2, ..., K$ be the K local maximum points of u_ϵ. Then, up to a rotation, we have $\tilde{P}_j^\epsilon - P_j^\epsilon = o(1), j = 1, ..., K$. By Lemma 4.1, $P_j^0 \neq P_i^0$ for $i \neq j$.

Let us consider $K = 2$ first. From (4.8), it is easy to see that both $|P_1^\epsilon|$ and $|P_2^\epsilon|$ must be larger than $\frac{1}{10}$ (as otherwise, (4.8) is not balanced). Thus we may assume that $|P_1^\epsilon| \geq \frac{1}{10}, |P_2^\epsilon| \geq \frac{1}{10}$.

Since both $\tilde{P}_1^\epsilon$ and $\tilde{P}_2^\epsilon$ are in the interior, (4.13) implies that we must have

$$|P_1^\epsilon - P_2^\epsilon| = 2d_{P_1}^\epsilon + o(1) = 2d_{P_2^\epsilon} + o(1)$$

and that P_1^0 and P_2^0 must be anti-pole. This shows that the limiting position must be type 2.III.

For $K = 3$, we proceed similarly. By Lemma 4.1, $P_j^\epsilon \to P_j^0, j = 1,2,3$, where $P_j^0 \in \Omega, P_k^0 \neq P_l^0$ if $k \neq l$. We consider two cases.

Case 1: Suppose one of $P_j^0, j = 1,2,3$ is the origin, say $P_2^0 = 0$. Then we consider (4.13) at $j = 1$ and conclude that $2d_{P_1^0} = |P_1^0 - P_2^0|$. Similarly we have $2d_{P_2^0} = |P_1^0 - P_2^0|$. From the equation (4.8) at $j = 2$, we see that

$$P_1^0 + P_2^0 = 0$$

which shows that (P_1^0, P_2^0, P_3^0) is type 3.VII.

Case 2: Suppose $P_j^0 \neq 0, j = 1,2,3$. Then $|P_j^\epsilon| \geq d_0$ for $j = 1,2,3$ and some $d_0 > 0$. We claim

Step 1. $|P_j^0 - P_k^0| \geq 2\max(d_{P_j^0}, d_{P_k^0})$ for $j \neq k$.

Suppose step 1 does not hold. Without loss of generality, we may assume

$$|P_2^0 - P_1^0| < 2d_{P_2^0}. \tag{4.14}$$

By (4.13) for $j = 2$, we have $|P_3^0 - P_2^0| = |P_2^0 - P_1^0|$ and vectors $\overrightarrow{P_1^0P_2^0}$ and $\overrightarrow{P_2^0P_3^0}$ are parallel. Thus, P_1^0, P_2^0 and P_3^0 must be on the same line. Since $|P_2^0 - P_3^0| = |P_2^0 - P_1^0|, P_2^0$ lies between P_1^0 and P_3^0. We claim $|P_1^0 - P_2^0| \geq 2d_{P_1}$ and $|P_3^0 - P_2^0| \geq 2d_{P_3}$. Otherwise, say, $|P_1^0 - P_2^0| < 2d_{P_1}$. By the argument above, $P_1^0 \in (P_2^0, P_3^0)$, which yields a contradiction to $P_2^0 \in (P_1^0, P_3^0)$. Applying (4.13) for $j = 1$ and $j = 3$, we obtain $2d_{P_1} = |P_1^0 - P_2^0|$ and $2d_{P_3} = |P_3^0 - P_2^0|$, because $|P_1^0 - P_3^0| = |P_1^0 - P_2^0| + |P_2^0 - P_3^0| > 2\max(d_{P_1}, d_{P_3})$. Since $2d_{P_1^0} = |P_1^0 - P_2^0| < |P_1^0 - P_3^0|$, (4.13) also yields 0 lies on the line of $\overleftrightarrow{P_1^0P_3^0}$. Applying the equalities $2d_{P_1} = |P_1^0 - P_2^0|, 2d_{P_3} = |P_2^0 - P_3^0|$ and $|P_3^0 - P_2^0| = |P_2^0 - P_1^0|$, we conclude $P_2^0 = 0$, which yields a contradiction to our assumption. Therefore, step 1 is proved.

Step 2. $|P_j^0 - P_i^0| = 2d_{P_i^0}$ for $i \neq j$.

We first note that P_1^0, P_2^0 and P_3^0 can not be on the same line. Because if P_1^0, P_2^0 and P_3^0 are on the same line and P_2^0 lies between P_1^0 and P_3^0. By step 1,

$$|P_3^0 - P_1^0| > |P_2^0 - P_1^0| \geq 2d_{P_1^0}, \quad \text{and}$$
$$|P_3^0 - P_1^0| > |P_3^0 - P_2^0| \geq 2d_{P_3^0}.$$

Applying (4.13), we obtain that 0 is on the line $\overleftrightarrow{P_3^0 P_1^0}$,

$$|P_2^0 - P_1^0| = 2d_{P_1^0} \quad \text{and} \quad |P_2^0 - P_3^0| = 2d_{P_3^0}.$$

If $|P_3^0 - P_2^0| = |P_2^0 - P_1^0|$, then P_2^0 is the origin and it yields a contradiction. So, we may assume $|P_3^0 - P_2^0| > |P_2^0 - P_1^0|$. Applying (4.13) to $j = 2$, we have

$$|P_2^0 - P_1^0| = 2d_{P_2^0}.$$

Thus, $d_{P_1^0} = d_{P_2^0}$, which is impossible. Hence we have proved that P_1^0, P_2^0 and P_3^0 could not be on the same line.

Suppose

$$|P_1^0 - P_2^0| > 2d_{P_2^0}.$$

Then (4.13) gives us that 0 is on the line P_2^0 and P_3^0 and $|P_3^0 - P_2^0| = 2d_{P_2^0}$. Since P_1^0 is not on the line of $\overleftrightarrow{P_2^0 P_3^0}$, $|P_1^0 - P_2^0| = |P_1^0 - P_3^0| = 2d_{P_1^0}$ by (4.13). Thus, we have

$$|P_1^0 - P_2^0| = |P_1^0 - P_3^0| > |P_2^0 - P_3^0|.$$

Applying (4.13) again to $j = 2$,

$$|P_2^0 - P_3^0| = 2d_{P_3^0}.$$

Therefore, $2d_{P_3^0} = |P_2^0 - P_3^0| = 2d_{P_2^0}$. Since $2d_{P_3^0} = 2d_{P_2^0}$, we have $P_2^0 = -P_3^0$. From here and $|P_1^0 - P_2^0| > |P_2^0 - P_3^0|$, we conclude that $|P_1^0| > |P_2^0| = |P_3^0|$, but it yields a contradiction to $d_{P_1^0} > d_{P_3^0}$. This proves step 2 and concludes that (P_1^0, P_2^0, P_3^0) must be type 3.IV. □

5. UNIQUENESS OF THE TYPE 3.IV SOLUTIONS

In this section, we prove the uniqueness of the type 3.IV solution for ϵ sufficiently small. Let P_1^0, P_2^0, P_3^0 be the limiting positions as shown in Fig. 3.

Let u_ϵ be a three-peaked solution with its local maximum points $\tilde{P}_j^\epsilon, j = 1,2,3$. By MMP (Section 2), the solution u_ϵ is symmetric in $x_j, j = 3, ..., N$. Let

(5.1)
$$H^2_{\nu,s}(\Omega_\epsilon) = \{u \in H^2_\nu(\Omega_\epsilon) | u \text{ is symmetric with respect to } \ x_j, j = 3, ..., N\}.$$

Consider the following minimization problem

$$\min_{P_j \in \hat{\Lambda}_{2\delta}} \|u_\epsilon - \sum_{j=1}^3 w_{\epsilon,P_j}\|_{L^2(\Omega_\epsilon)} \tag{5.2}$$

where

$$\hat{\Lambda}_{2\delta} = \{(P_1, P_2, P_3) \mid |P_j - P_j^0| \le 2\delta, P_{j,i} = 0, j = 1,2,3, i \ge 3\}. \tag{5.3}$$

It is easy to see that (5.2) can be attained and thus we have

$$u_\epsilon = w_{\epsilon,\mathbf{P}^\epsilon} + \phi_\epsilon \tag{5.4}$$

where $\mathbf{P}^\epsilon \in \hat{\Lambda}_\delta, \phi_\epsilon \in H^2_{\nu,s}(\Omega)$. Moreover, $\phi_\epsilon \in \mathcal{K}^\perp_{\epsilon,\mathbf{P}^\epsilon}$. Since

$$S[w_{\epsilon,\mathbf{P}^\epsilon} + \phi_\epsilon] = 0 \in \mathcal{C}_{\epsilon,\mathbf{P}^\epsilon}, \phi_\epsilon \in \mathcal{K}^\perp_{\epsilon,\mathbf{P}^\epsilon},$$

by Lemma 3.3, we see that

$$\phi_\epsilon = v_{\epsilon,\mathbf{P}^\epsilon} \tag{5.5}$$

where $v_{\epsilon,\mathbf{P}^\epsilon}$ is defined by Lemma 3.3. (Note that P_j^ϵ may not be a local maximum point of u_ϵ. But it is easy to show that up to a permutation, $P_j^\epsilon = \tilde{P}_j^\epsilon + o(1), j = 1,2,3.$)

For $\mathbf{P} \in \hat{\Lambda}_\delta$, we may define $\hat{P}_j = (P_{j,1}, P_{j,2}), \hat{\mathbf{P}} = (\hat{P}_1, \hat{P}_2, \hat{P}_3)$ and

$$\hat{M}_\epsilon(\hat{\mathbf{P}}) = M_\epsilon(\mathbf{P}). \tag{5.6}$$

Similar to Lemma 3.4, we have that $\hat{\mathbf{P}}^\epsilon$ is a critical point of $\hat{M}_\epsilon(\hat{\mathbf{P}})$ if and only if $u_\epsilon = w_{\epsilon,\mathbf{P}^\epsilon} + v_{\epsilon,\mathbf{P}^\epsilon}$ is a critical point of J_ϵ.

To avoid clumsy notation, we drop the hat now. Thus our problem is reduced to a six-dimensional problem. Moreover, by rotation, we may fix $P_{11} = 0$. Let $\mathbf{p} = (P_{12}, P_{21}, P_{22}, P_{31}, P_{32})$. Then if $\mathbf{P}^\epsilon$ is a critical point, the corresponding $\mathbf{p}^\epsilon$ is also a critical point of $M_\epsilon(\mathbf{P})$. Thus all we need to prove is the uniqueness of the critical point of $M_\epsilon(\mathbf{P})$ for $\mathbf{P}$ in the set

$$\omega = \{(P_{12}, P_{21}, P_{22}, P_{31}, P_{32}) | \mathbf{P} \in \hat{\Lambda}_\delta, P_{11} = 0\},$$

which is a five-dimensional problem.

We begin with the following important lemma which computes the speed of $\mathbf{P}^\epsilon$ approaching $\mathbf{P}^0$:

Lemma 5.1. *Let $P_j^\epsilon, j = 1, 2, 3$ be as above. Then there exists a unique vector $\vec{a} \in R^2$ such that*

$$P_j^\epsilon = e^{\sqrt{-1}\frac{2(j-1)\pi}{3}}(P_1^0 + \epsilon\vec{a}) + o(\epsilon), j = 1, 2, 3 \tag{5.7}$$

where $\vec{a} = (0, a_2)$ for some fixed number a_2.

Proof: Our main tool is equation (4.13). Since P_1^0, P_2^0, P_3^0 are not on a line, by (4.13), we have

$$2d_{P_j^\epsilon} = |P_i^\epsilon - P_j^\epsilon| + O(\varepsilon) \ \text{ for } \ j \neq i.$$

Adding all the three equations in (4.13) together, we obtain that

$$\sum_{j=1}^{3} e^{-|P_j^\epsilon|/\epsilon}\frac{P_j^\epsilon}{|P_j^\epsilon|} + o(\sum_{j=1}^{3} e^{-|P_j^\epsilon|/\epsilon}) = 0 \tag{5.8}$$

Since

$$\frac{P_1^\epsilon}{|P_1^\epsilon|} = \vec{e}_2 = (0,1)$$

and

$$\frac{P_j^\epsilon}{|P_j^\epsilon|} = e^{\sqrt{-1}\frac{2(j-1)\pi}{3}}\vec{e}_2 + o(1), j = 2, 3,$$

we deduce from (4.13) that

$$|P_j^\epsilon| = |P_1^\epsilon| + o(\epsilon), d_{P_j^\epsilon} = d_{P_1^\epsilon} + o(\epsilon), j = 2, 3. \tag{5.9}$$

Next we examine the equation (4.13) at $j = 1$. We have

$$\frac{P_1^\epsilon}{|P_1^\epsilon|} + a_1^0 e^{(2d_{P_1^\epsilon} - |P_2^\epsilon - P_1^\epsilon|)/\epsilon}\frac{P_2^\epsilon - P_1^\epsilon}{|P_2^\epsilon - P_1^\epsilon|} + a_1^0 e^{(2d_{P_1^\epsilon} - |P_3^\epsilon - P_1^\epsilon|)/\epsilon}\frac{P_3^\epsilon - P_1^\epsilon}{|P_3^\epsilon - P_1^\epsilon|} = o(1)$$

where a_1^0 is a generic constant. Since the following decomposition is unique,

$$\frac{P_1^0}{|P_1^0|} = -\frac{P_2^0 - P_1^0}{|P_2^0 - P_1^0|} - \frac{P_3^0 - P_1^0}{|P_3^0 - P_1^0|}$$

we see that

$$a_1^0 e^{(2d_{P_1^\epsilon} - |P_2^\epsilon - P_1^\epsilon|)/\epsilon} = 1 + o(1), a_1^0 e^{(2d_{P_1^\epsilon} - |P_3^\epsilon - P_1^\epsilon|)/\epsilon} = 1 + o(1),$$

which implies that

$$|P_2^\epsilon - P_1^\epsilon| = 2d_{P_1^\epsilon} + \epsilon a_0 + o(\epsilon), |P_3^\epsilon - P_1^\epsilon| = 2d_{P_1^\epsilon} + \epsilon a_0 + o(\epsilon) \tag{5.10}$$

where a_0 is a generic constant. Similarly, with the same a_0, we have

$$|P_i^\epsilon - P_j^\epsilon| = 2d_{P_i^\epsilon} + \epsilon a_0 + o(\epsilon), i \neq j, i, j = 1, 2, 3. \tag{5.11}$$

From (5.9) and (5.11), we see that $d_{P_i^\epsilon} = d_{P_j^\epsilon} + o(\varepsilon)$ for $i \neq j$. Thus, by (5.11) again, we have

$$|P_i^\epsilon - P_j^\epsilon| = 2d_{P_k^\epsilon} + \epsilon a_0 + o(\varepsilon), \quad \text{and}$$
$$|P_i^\epsilon - P_j^\epsilon| = |P_k^\epsilon - P_l^\epsilon| + o(\varepsilon), i \neq j \quad \text{and} \quad k \neq l.$$

By (5.9), we set $P_2^\epsilon = e^{i\theta_1^\varepsilon} P_1^\epsilon + o(\varepsilon)$ and $P_3^\epsilon = e^{i\theta_2^\varepsilon} P_2^\epsilon + o(\varepsilon)$. Since $|P_2^\epsilon - P_3^\epsilon| = |P_1^\epsilon - P_2^\epsilon|$ and $|P_1^\epsilon| = |P_2^\epsilon| + o(\varepsilon)$, we obtain $\theta_2^\epsilon = \theta_1^\epsilon + o(\varepsilon)$. By using $|P_1^\epsilon - P_2^\epsilon| = |P_1^\epsilon - P_3^\epsilon| + o(\varepsilon)$, we can deduce

$$\theta_1^\varepsilon = \frac{2\pi}{3} + o(\varepsilon), \quad \text{that is,}$$

$$P_j^\epsilon = e^{\sqrt{-1}\frac{2(j-1)\pi}{3}} P_1^\epsilon + o(\epsilon), j = 2, 3. \tag{5.12}$$

Substituting (5.12) into (5.10), we see that

$$P_1^\epsilon = P_1^0 + \epsilon \vec{a} + o(\epsilon) \tag{5.13}$$

for some unique $\vec{a}$. (Note that $\vec{a} = a_2 \vec{e}_2$ for some a_2.) By (5.13) and (5.12), (5.7) holds.

$\square$

By Lemma 5.1, any critical point $\mathbf{P}^\epsilon$ of $M_\epsilon(\mathbf{P})$ in $B_\delta(\mathbf{P}^0)$ must satisfy $\mathbf{P}^\epsilon = \mathbf{P}^0 + \epsilon \mathbf{a} + o(\epsilon)$ for some fixed $\mathbf{a}$. Let $\mathbf{Q}^\epsilon = \mathbf{P}^0 + \epsilon \mathbf{a}$.

Our next lemma shows that every critical point $\mathbf{P}^\epsilon$ must be nondegenerate.

Lemma 5.2. *Let $\mathbf{P}^\epsilon \in B_{\delta\epsilon}(\mathbf{Q}^\epsilon)$ be a critical point of $M_\epsilon(\mathbf{P})$. Then for ϵ sufficiently small, we have*

$$-\sum_{j,l,m,i} \partial_{l,m}\partial_{j,i} M_\epsilon(\mathbf{P})\big|_{\mathbf{P}=\mathbf{P}^\epsilon} \eta_{l,m}\eta_{j,i} \geq C\epsilon^{\frac{3N-5}{2}} e^{-2\varphi_K(\mathbf{P}^\epsilon)/\epsilon} |\eta|^2 \tag{5.14}$$

where C is independent of $\epsilon, \eta = (\eta_{11}, \eta_{12}, \eta_{21}, \eta_{22}, \eta_{31}, \eta_{32}) \in R^6, \eta_{11} = 0$ and $|\eta|^2 = \sum_{i,j} \eta_{ij}^2$.

Proof: We have by Lemma 4.3 (3),

$$-\sum_{j,l,m,i} \partial_{l,m}\partial_{j,i} M_\epsilon(\mathbf{P})\big|_{\mathbf{P}=\mathbf{P}^\epsilon} \eta_{l,m}\eta_{j,i}$$

$$= (\gamma_1 + o(1))\epsilon^{N-2} \sum_j \varphi_{\epsilon,P_j^\epsilon}(P_j^\epsilon) \sum_l < e_l^\epsilon, \eta_l >^2 \tag{5.15}$$

$$+(\gamma_1 + o(1))\epsilon^{N-2} w(|P_1^\epsilon - P_2^\epsilon|/\epsilon)(1+o(1)) \sum_{j\neq l} < e_{jl}^\epsilon, \eta_j - \eta_l >^2 .$$

Since $\mathbf{P}^\epsilon \in B_{\delta\epsilon}(\mathbf{Q}^\epsilon)$, $|P_i^\epsilon - P_j^\epsilon| = |P_1^\epsilon - P_2^\epsilon| + o(\epsilon)$ for $i \neq j$ and $d_{P_j^\epsilon} = d_{P_1^\epsilon} + o(\epsilon)$. Hence

$$\varphi_{\epsilon,P_j^\epsilon}(P_j^\epsilon) \sim w(|P_1^\epsilon - P_2^\epsilon|), j = 1, 2, 3.$$

(5.15) shows that

$$-\sum_{j,l,m,i} \partial_{l,m}\partial_{j,i} M_\epsilon(\mathbf{P})\big|_{\mathbf{P}=\mathbf{P}^\epsilon} \eta_{l,m}\eta_{j,i}$$

$$\geq C\epsilon^{N-2} e^{-2\varphi_K(\mathbf{P}^\epsilon)/\epsilon} \Big(\sum_l < e_l^\epsilon, \eta_l >^2 + \sum_{j\neq l} < e_{jl}^\epsilon, \eta_j - \eta_l >^2\Big) \tag{5.16}$$

for some $C > 0$ independent of ϵ.

We now show that

$$-\sum_{j,l,m,i} \partial_{l,m}\partial_{j,i} M_\epsilon(\mathbf{P})\big|_{\mathbf{P}=\mathbf{P}^\epsilon} \eta_{l,m}\eta_{j,i} \geq C\epsilon^{N-2} e^{-2\varphi_K(\mathbf{P}^\epsilon)/\epsilon} |\eta|^2$$

where C is independent of ϵ and $|\eta|^2 = \sum_{i,j} \eta_{ij}^2$.

To this end, it is enough to show that

$$\sum_l < e_l^\epsilon, \eta_l >^2 + \sum_{j\neq l} < e_{jl}^\epsilon, \eta_j - \eta_l >^2 \geq C|\eta|^2 \tag{5.17}$$

if $\eta_{11} = 0$. In fact, the left hand side of (5.17) is equal to 0 if and only if

$$< e_l^\epsilon, \eta_l >= 0, < e_{jl}^\epsilon, \eta_l - \eta_j >= 0$$

For $j = 1, < e_1^\epsilon, \eta_1 >= 0$, then $\eta_{12} = 0$. Hence $\eta_1 = 0$, which implies that $< e_{12}^\epsilon, \eta_2 >=< e_2^\epsilon, \eta_2 >= 0$. Hence $\eta_2 = 0$. Similarly, we have $\eta_3 = 0$. Thus (5.17) holds true.

□

(5.14) shows that the matrix $(\partial_{l,m}\partial_{j,i} M_\epsilon(\mathbf{P})\big|_{\mathbf{P}=\mathbf{P}^\epsilon})$ is negatively definite if we restrict to the space $\{\eta_{11} = 0\}$. Thus the Morse index is 5.

Finally we have

Lemma 5.3. *For $\delta > 0$ small, there exists a unique critical point of $M_\epsilon(\mathbf{P})$ over $B_{\delta\epsilon}(\mathbf{Q}^\epsilon)$.*

Proof:

First, by restricting to the symmetric class of functions, we can adopt the arguments of [12] to show there exists a critical point $\mathbf{P}^\epsilon$ of $M_\epsilon(\mathbf{P})$. By Lemma 5.1, $\mathbf{P}^\epsilon = P_0 + \epsilon\mathbf{a} + o(\epsilon)$ and any other critical point of $M_\epsilon(\mathbf{P})$ is in $B_{\delta\epsilon}(\mathbf{Q}^\epsilon)$.

We now show that $\mathbf{P}^\epsilon$ is unique.

By Lemma 5.2, there are only finite number of critical points of $M_\epsilon(\mathbf{P})$ in $B_{\delta\epsilon}(\mathbf{Q}^\epsilon)$ (since each critical point is nondegenerate). Let k_ϵ be the number of critical points. At each critical point, we have by Lemma 5.2,

$$deg(\nabla M_\epsilon, B_{\delta_i\epsilon}(\mathbf{Q}_i^\epsilon), 0) = (-1)^5 = -1$$

where $\delta_i > 0$ are small constants so that $B_{\delta_i\epsilon}(\mathbf{Q}_i^\epsilon)$ contains only one critical point (i.e. $\mathbf{Q}_i^\epsilon$) of $M_\epsilon(\mathbf{P})$.

Hence by the additivity of the degree we have

$$deg(\nabla M_\epsilon, B_{\delta\epsilon}(\mathbf{Q}^\epsilon), 0) = k_\epsilon(-1)^5. \tag{5.18}$$

On the other hand, it is easy to see that $\tilde{M}_\epsilon(\mathbf{P})$ has only one critical point in $B_{\delta\epsilon}(\mathbf{Q}^\epsilon)$ (because of the nondegeneracy of $(\nabla^2\tilde{M}_\epsilon(\mathbf{P}))$). For $\mathbf{P} \in B_{\delta\epsilon}(\mathbf{Q}^\epsilon)$, we have

$$e^{-2d_{P_i}/\epsilon} = (1+O(\delta))e^{-2d_{Q_i^\epsilon}}, w(|P_i - P_j|/\epsilon) = (1+O(\delta))w(|Q_i^\epsilon - Q_j^\epsilon|/\epsilon),$$

$$M_\epsilon(\mathbf{P}) = (1+O(\delta))M_\epsilon(\mathbf{Q}^\epsilon).$$

By (1) of Lemma 4.3, we have $\nabla M_\epsilon(\mathbf{P}) = \nabla\tilde{M}_\epsilon(\mathbf{P}) + O(\tilde{M}_\epsilon(\mathbf{P}))$. Note that $\nabla M_\epsilon(\mathbf{P}) \neq 0$ and $\nabla\tilde{M}_\epsilon(\mathbf{P}) \neq 0$ on $\partial B_{\delta\epsilon}(\mathbf{Q}^\epsilon)$. By a continuity argument, we obtain that

$$deg(\nabla M_\epsilon, B_{\delta\epsilon}(\mathbf{Q}^\epsilon), 0) = deg(\nabla\tilde{M}_\epsilon(\mathbf{P}), B_{\delta\epsilon}(\mathbf{Q}^\epsilon), 0) = -1. \tag{5.19}$$

Comparing (5.18) and (5.19), we deduce that $k_\epsilon = 1$.

$\square$

Lemma (5.3) shows that type 3.IV solution is unique, up to a rotation, provided that ϵ is sufficiently small.

6. Existence and uniqueness of other types of solutions

In the previous sections, we have proved the uniqueness of type 3.IV solutions. It is easy to see that same techniques (with much simpler computations) show that the type 2.III solutions are unique. It remains to deal with boundary spikes and mixed-interior-boundary spikes.

To be able to deal with the boundary peak case, we use different approximate functions for boundary spikes. (Here the fact that $\Omega = B$ plays an important role.)

We begin with Theorem 1.1 of [20]: for every $P \in \partial\Omega$, there exists a unique boundary spike solution $u_{\epsilon,P}$ which concentrates at P. Moreover $u_{\epsilon,P}$ is axially symmetric with respect to the straight line joining 0 and P. So for $P \in \partial\Omega$, we may choose our approximate function as follows

$$w_{\epsilon,P} = u_{\epsilon,P}, \text{ for } P \in \partial\Omega. \tag{6.1}$$

For $P \in \partial\Omega$, we denote the $i-$th tangential derivative at P_j as $\partial_{j,i} = \frac{\partial}{\partial \tau_{P_j,i}}, i = 1, ..., N-1$.

In this case, we see that

$$S_\epsilon(w_{\epsilon,P}) = 0, \text{ if } P \in \partial\Omega. \tag{6.2}$$

Let $\mathbf{P} = (P_1, ..., P_K)$ be such that $P_j \in \partial\Omega, i = 1, ..., K_1, P_j \in \Omega, j = K_1+1, ..., K$. We define a new function (which was introduced first in [13])

$$\tilde{\varphi}_K(\mathbf{P}) = \min_{i=1,...,K_1, j=1,...,K, i\neq j} (\frac{1}{2}|P_i - P_j|, \varphi_{K-K_1}(P_{K_1+1}, ..., P_K)) \tag{6.3}$$

We can also define $w_{\epsilon,\mathbf{P}}, \mathcal{K}_{\epsilon,\mathbf{P}}, \mathcal{C}_{\epsilon,\mathbf{P}}, \mathcal{K}^{\perp}_{\epsilon,\mathbf{P}}, \mathcal{C}^{\perp}_{\epsilon,\mathbf{P}}, \mathcal{L}_{\epsilon,\mathbf{P}}, v_{\epsilon,\mathbf{P}}, M_\epsilon(\mathbf{P})$, etc.

Then similar to Section 3, we will obtain the following

$$S_\epsilon(w_{\epsilon,\mathbf{P}}) = O(e^{-2\tilde{\varphi}_K(\mathbf{P})/\epsilon}) \tag{6.4}$$

$$\|v_{\epsilon,\mathbf{P}}\|_\epsilon \leq Ce^{-(1+\sigma)\tilde{\varphi}_K(\mathbf{P})/\epsilon} \tag{6.5}$$

$$\|\partial_{j,i} v_{\epsilon,\mathbf{P}}\|_\epsilon \leq C\epsilon^{-2} e^{-(1+\sigma)\tilde{\varphi}_K(\mathbf{P})/\epsilon} \tag{6.6}$$

where $\sigma = \min(1, p-1)$.

Moreover, we have the following equations for the equilibrium positions: for $j = 1, ..., K_1$, we have

$$(6.7) \qquad (\gamma_1 + o(1)) \sum_{l \neq j} (|P_l^\epsilon - P_j^\epsilon|)^{-\frac{N-1}{2}} e^{-|P_l^\epsilon - P_j^\epsilon|/\epsilon} \partial_{j,i}(|P_j^\epsilon - P_l^\epsilon|)$$

$$+O(\epsilon(\sum_{j=1}^{K} e^{-2d_{P_j^\epsilon}/\epsilon} + \sum_{l \neq k} e^{-|P_l^\epsilon - P_k^\epsilon|/\epsilon})) = 0, j = 1, ..., K_1, i = 1, ..., N-1,$$

(here $\partial_{j,i}$ means the $i-th$ tangential derivative at P_j^ϵ) and for $j = K_1+1, ..., K$ (if $|P_j^\epsilon| \geq \delta_0$), we have

$$(6.8) (\gamma_1 + o(1)) c_N c(P_j^\epsilon) e^{-2d_{P_j^\epsilon}/\epsilon} (\nabla_{P_j^\epsilon} d_{P_j^\epsilon})_i$$
$$+(\gamma_1 + o(1)) A_N \sum_{l \neq j} (|P_l^\epsilon - P_j^\epsilon|)^{-\frac{N-1}{2}} e^{-|P_l^\epsilon - P_j^\epsilon|/\epsilon} (\nabla_{P_j^\epsilon} |P_j^\epsilon - P_l^\epsilon|)_i$$
$$+O(\epsilon(\sum_{j=1}^{K} e^{-2d_{P_j^\epsilon}/\epsilon} + \sum_{l \neq k} e^{-|P_l^\epsilon - P_k^\epsilon|/\epsilon})) = 0, j = K_1 + 1, ..., K, i = 1, ..., N.$$

The next lemma is an extension of Lemma 4.4.

Lemma 6.1. *Let $K = 2$ or 3 and $\tilde{P}_j^\epsilon \in \Omega, j = 1, ..., K$ be the K local maximum points of u_ϵ. Suppose that $\tilde{P}_j^\epsilon \to P_j^0, j = 1, ..., K$, then up to a rotation, $\mathbf{P}^0 = (P_1^0, ..., P_K^0)$ must belong to one of the three types (for $K = 2$, Fig. 2) and of the seven types (for $K = 3$, Fig. 3).*

Proof: As in the proof of Lemma 4.4, by Lemma 4.1, $P_j^0 \neq P_i^0$ for $i \neq j$.

By Lemma 4.4, we only need to consider boundary spikes or mixed interior and boundary spikes.

Let us consider $K = 2$ first. By Theorem 1.1, we may assume that $P_1^\epsilon, P_2^\epsilon$ lie on the x_1-axis. We may further assume that $P_j^\epsilon = (l_j^\epsilon, 0, ..., 0), j = 1, 2, l_1^\epsilon < l_2^\epsilon$. If both P_1^ϵ and P_2^ϵ lie on the boundary, then necessarily by [20], $l_1^\epsilon = -1, l_2^\epsilon = 1$, which is type 2.I. Suppose that $P_1^\epsilon \in \partial\Omega$, we may assume that $l_1^\epsilon = -1$. In this case, (6.8) implies that $|P_1^\epsilon - P_2^\epsilon| = 2d_{P_2^\epsilon} + o(1)$ and therefore $l_1^\epsilon \to \frac{1}{3}$, which is type 2.II.

For $K = 3$, by Theorem 1.3, we may assume that $P_1^\epsilon, P_2^\epsilon, P_3^\epsilon$ lie on a two-dimensional plane, say $(x_1, x_2)-$plane. That is we have $P_j^\epsilon = (P_{j,1}^\epsilon, P_{j,2}^\epsilon, 0, ..., 0)$, $j = 1, 2, 3$.

Suppose first we have three boundary spikes: $P_j^\epsilon \in \partial\Omega, j = 1,2,3$. We may assume that $P_1^\epsilon = (-1,0,...,0)$. We examine the equation (6.7) for $j=1$ and $\partial_{1,i} = \frac{\partial}{\partial P_{1,2}^\epsilon}$. Then it is easy to see that we must have

(6.9)

$$e^{-|P_2^\epsilon - P_1^\epsilon|/\epsilon}(\frac{P_{2,2}^\epsilon - P_{1,2}^\epsilon}{|P_2^\epsilon - P_1^\epsilon|} + e^{-|P_3^\epsilon - P_1^\epsilon|/\epsilon}(\frac{P_{3,2}^\epsilon - P_{1,2}^\epsilon}{|P_3^\epsilon - P_1^\epsilon|} + o(e^{-|P_2^\epsilon - P_1^\epsilon|/\epsilon} + e^{-|P_3^\epsilon - P_1^\epsilon|/\epsilon}) = 0$$

which shows that

$$|P_2^\epsilon - P_1^\epsilon| = |P_3^\epsilon - P_1^\epsilon| + o(1)$$

Similarly, we will have

$$|P_1^\epsilon - P_2^\epsilon| = |P_3^\epsilon - P_2^\epsilon| + o(1),$$

$$|P_1^\epsilon - P_3^\epsilon| = |P_2^\epsilon - P_3^\epsilon| + o(1).$$

As $\epsilon \to 0$, it shows that P_1^0, P_2^0 and P_3^0 must form a perfect triangle, which is type 3.I.

The other cases are similar. We omit the details.

□

As for the existence, we have

Lemma 6.2. *For ϵ sufficiently small and $K=2$, up to a rotation, there are at least three solutions. The limiting positions are shown in Fig. 2. Moreover, these solutions inherit the partial symmetries of their limiting positions.*

Similarly for ϵ sufficiently small and $K=3$, up to a rotation, there are at least seven solutions. The limiting positions are shown in Fig. 3. Moreover, these solutions inherit the partial symmetries of their limiting positions.

Proof: The existence of all the above solutions follow a general procedure: we first restrict to a symmetric class and then use the idea in [12] and [13]. As an example, we consider the existence of type 2.II solutions. We fix $P_1 = (-1,0,...,0)$ and let $P_2 = (l_2, 0, ..., 0), |l_2 - \frac{1}{3}| < \delta$. Then we solve the following problem

$$S_\epsilon(w_{\epsilon,P_1} + w_{\epsilon,P_2} + v) \in \mathcal{C}_{\epsilon,P_2}, \ \in \mathcal{K}_{\epsilon,P_2} \cap H^2_{\nu,s}(\Omega_\epsilon), \tag{6.10}$$

where

$$H^2_{\nu,s}(\Omega_\epsilon) = \{u \in H^2_\nu(\Omega_\epsilon) | u \text{ is symmetric with respect to } x_j, j = 2, ..., N\}.$$

Problem (6.10) can be solved since the degeneracy at P_1 is eliminated by restricting to $H^2_{\nu,s}(\Omega_\epsilon)$.

Then we define a reduced energy functional

$$\tilde{M}_\epsilon(P_2) = J_\epsilon[w_{\epsilon,P_1} + w_{\epsilon,P_2} + v_\epsilon]$$

where v_ϵ is obtained in (6.10). It is easy to compute that asymptotically we have

(6.11)

$$\tilde{M}_\epsilon(P_2) = \epsilon^N[2I(w) - (c_1 + o(1))w(|P_1 - P_2|/\epsilon) - (\frac{\gamma_1}{2} + o(1))\varphi_{\epsilon,P_2}(P_2)]$$

where c_1 is some positive constant, $I(w) = \frac{1}{2}\int_{R^N}(|\nabla w|^2 + w^2) - \int_{R^N} F(w)$ is the energy of the ground state w.

Similar to [12], we now maximize the reduced energy

$$\max_{|l_2 - \frac{1}{3}| \leq \delta} \tilde{M}_\epsilon(P_2), P_2 = (l_2, 0, ..., 0) \tag{6.12}$$

Then by the energy expansion of (6.11), it is easy to see that the maximization problem is attained by some $l_2^\epsilon \to \frac{1}{3}$ and the corresponding solution $u_\epsilon = w_{\epsilon,P_1} + w_{\epsilon,P_2^\epsilon}$ is a solution of type 2.II.

The proof of the existence of other types of solutions is similar. We omit the details.

□

As for uniqueness, we consider case by case.

The uniqueness of type 2.I solutions is given in [20].

For type 2.II solutions, by Theorem 1.1, we may assume that $P_1^\epsilon = (-1, 0, ..., 0), P_2^\epsilon = (l^\epsilon, 0, ..., 0)$. By rotation, we may fix P_1^ϵ. Moreover, we may consider solutions which are axially symmetric with respect to the x_1-axis. This reduces the total degree of freedoms into one. Now it is easy to show that $M_\epsilon(\mathbf{P})$ has a nedegenarte local maximum at P_2^ϵ. This shows the Morse index is 1 and uniqueness follows.

The uniqueness of type 2.III follows the same proof as the uniqueness of type 3.IV.

Next we consider $K = 3$.

For type 3.I solutions, by Theorem 1.3, $P_j^\epsilon, j = 1, 2, 3$ and the origin must lie a hyperplane. This becomes a two-dimensional problem. Now we fix $P_1^\epsilon = (-1, 0, ..., 0)$. Then it has two degree of freedoms. It is easy to see that the Morse index is 2.

For type 3.II, type 3.III, the proofs are similar.

7. Appendix A: Proof of Lemma 3.1

In this appendix, we prove Lemma 3.1 in Section 3.

Recall that $\varphi_{\epsilon,P}(x)$ satisfies

$$\begin{cases} \epsilon^2 \Delta \varphi_{\epsilon,P}(x) - \varphi_{\epsilon,P}(x) = 0 \text{ in } \Omega, \\ \frac{\partial \varphi_{\epsilon,P}}{\partial \nu} = -\frac{\partial}{\partial \nu} w(\frac{x-P}{\epsilon}) \text{ on } \partial\Omega. \end{cases} \tag{7.1}$$

On $\partial\Omega$, we have

$$\frac{\partial}{\partial \nu}\varphi_{\epsilon,P}(x) = (-w^{'}(\frac{x-P}{\epsilon}))\frac{< x-P, \nu >}{|x-P|}\frac{1}{\epsilon}$$

$$= \epsilon^{-1}(1+O(\epsilon))(\frac{\epsilon}{|x-P|})^{\frac{N-1}{2}} e^{-\frac{|x-P|}{\epsilon}} \frac{< x-P, \nu >}{|x-P|} \tag{7.2}$$

We consider an auxiliary problem (considered in [28] and [38])

$$\begin{cases} \epsilon^2 \Delta \varphi^D_{\epsilon,P}(x) - \varphi^D_{\epsilon,P}(x) = 0 \text{ in } \Omega, \\ \varphi^D_{\epsilon,P} = w(\frac{x-P}{\epsilon}) \text{ on } \partial\Omega. \end{cases} \tag{7.3}$$

Let $\varphi^D_{\epsilon,P} = e^{-\Psi_{\epsilon,P}(x)/\epsilon}$, where $\Psi_{\epsilon,P}(x)$ satisfies

$$\begin{cases} \epsilon^2 \Delta \Psi_{\epsilon,P}(x) - |\nabla \Psi_{\epsilon,P}(x)|^2 + 1 = 0 \text{ in } \Omega, \\ \Psi_{\epsilon,P} = -\epsilon \log w(\frac{x-P}{\epsilon}) \text{ on } \partial\Omega. \end{cases} \tag{7.4}$$

By Lemma 3.6 of [28], we see that as $\epsilon \to 0$,

$$\Psi_{\epsilon,P}(P) \to 2d_P, \quad \text{as } \epsilon \to 0. \tag{7.5}$$

It is proved in Lemma 3.7 of [38] that

(7.6)
$$\frac{\partial \Psi_{\epsilon,P}(x)}{\partial \nu} = (-1+O(\epsilon))\frac{\partial}{\partial \nu}|x-P| = (-1+O(\epsilon))\frac{< x-P, \nu >}{|x-P|} \text{ on } \partial\Omega$$

Thus comparing (7.1), (7.2) and (7.6), we obtain that

$$\varphi_{\epsilon,P}(x) = (1+O(\epsilon))e^{-\Psi_{\epsilon,P}(x)/\epsilon}, \ x \in \bar{\Omega}. \tag{7.7}$$

Hence $\varphi_{\epsilon,P}(x) = (1+O(\epsilon))w(\frac{x-P}{\epsilon})$ on $\partial\Omega$. (3.7) of Lemma 3.1 is thus proved.

(3.8) follows from (7.5).

To compute the exact asymptotic expansion of $\varphi_{\epsilon,P}(P)$, we use the Green's function. Let $G_\epsilon(x,z)$ be the Green's function of

$$\begin{cases} \epsilon^2\Delta G_\epsilon(x,z) - G_\epsilon(x,z) + \delta(z-x) = 0 \text{ in } \Omega, \\ \frac{\partial G_\epsilon(x,z)}{\partial \nu_x} = 0 \text{ on } \partial\Omega. \end{cases} \tag{7.8}$$

Then we have

$$\varphi_{\epsilon,P}(x) = \int_{\partial\Omega} G_\epsilon(x,z)\frac{\partial \varphi_{\epsilon,P}}{\partial\nu}(z)dz. \tag{7.9}$$

We decompose

$$G_\epsilon(x,z) = K_\epsilon(|x-z|) + H_\epsilon(x,z)$$

where $K_\epsilon(r)$ is the fundamental solution of $\epsilon^2\Delta - 1$ in $R^N\backslash\{0\}$.

Then H_ϵ satisfies

$$\begin{cases} \epsilon^2\Delta H_\epsilon - H_\epsilon = 0 \text{ in } \Omega, \\ \frac{\partial H_\epsilon(x,z)}{\partial\nu} = -\frac{\partial K_\epsilon(|x-z|)}{\partial\nu} \text{ on } \partial\Omega. \end{cases} \tag{7.10}$$

On $\partial\Omega$, we have

$$\frac{\partial}{\partial\nu}H_\epsilon(x) = (-K^{'}(\frac{x-P}{\epsilon}))\frac{< x-P, \nu >}{|x-P|}\frac{1}{\epsilon}$$

$$= \epsilon^{-1}(d_N + O(\epsilon))(\frac{\epsilon}{|x-P|})^{\frac{N-1}{2}}e^{-\frac{|x-P|}{\epsilon}}\frac{< x-P,\nu >}{|x-P|} \tag{7.11}$$

for some generic number $d_N > 0$.

As before, we have

$$H_\epsilon(x,z) = (d_N + o(1))\varphi_{\epsilon,x}(z) = (d_N+o(1))w(\frac{z-P}{\epsilon}) \text{ for } z\in\partial\Omega. \tag{7.12}$$

So we have

$$\begin{aligned} &\varphi_{\epsilon,P}(P) \\ = &\int_{\partial\Omega}(2+O(\epsilon))K_\epsilon(|z-P|)\frac{\partial\varphi_{\epsilon,P}}{\partial\nu}dz \\ = &(c_N+o(1))\epsilon^{-1}\int_{\partial\Omega}(\frac{\epsilon}{|z-P|})^{N-1}e^{-2|z-P|/\epsilon}\frac{< z-P,\nu >}{|z-P|}dz \\ = &(c_N+o(1))\epsilon^{N-2}\int_{\partial\Omega}(\frac{1}{|z-P|})^{N-1}e^{-2|z-P|/\epsilon}\frac{< z-P,\nu >}{|z-P|}dz \end{aligned} \tag{7.13}$$

Let P be such that $|P| \geq d_0$ for some $d_0 > 0$. Then the integral in (7.13) is a typical Laplace integral and can computed by classical Laplace method: namely, we let $z = \sqrt{\epsilon}y$ and then obtain

$$\varphi_{\epsilon,P}(P) = (c_N + o(1))(d_P(1-d_P))^{-\frac{N-1}{2}}\epsilon^{\frac{3N}{2}-\frac{5}{2}}e^{-2d_P/\epsilon}$$

for some positive constant $c_N > 0$. This proves (3.10) of Lemma 3.1.

Next we prove (3.9) of Lemma 3.1. To this end, we note that for $x = P + \epsilon y$

$$\begin{aligned}\varphi_{\epsilon,P}(x) &= \int_{\partial\Omega} G_\epsilon(x,z)\frac{\partial\varphi_{\epsilon,P}}{\partial\nu}(z)dz \\ &= \epsilon^{-1}(c_N + o(1))\int_{\partial\Omega}(\frac{\epsilon}{|z-x|})^{-\frac{N-1}{2}}(\frac{\epsilon}{|z-P|})^{-\frac{N-1}{2}} \\ &\quad e^{-\frac{|z-x|+|z-P|}{\epsilon}}\frac{< z-P,\nu >}{|z-P|}dz \\ &= \epsilon^{-1}(c_N + o(1))\int_{\partial\Omega}(\frac{\epsilon}{|z-x|})^{-\frac{N-1}{2}}(\frac{\epsilon}{|z-P|})^{-\frac{N-1}{2}} \\ &\quad e^{-\frac{2|z-P|}{\epsilon}}e^{-\frac{<z-P,y>}{|z-P|}}\frac{< z-P,\nu >}{|z-P|}dz \\ &= (1+o(1))\varphi_{\epsilon,P}(P)e^{-<\nabla d_P, y>}\end{aligned}$$

which proves (3.9) of Lemma 3.1.

Finally we prove (3.11) and (3.13) of Lemma 3.1.

For $P \in \Omega$, we define

$$\Omega_{\epsilon,P} := \{y | \epsilon y + P \in \Omega\}. \tag{7.14}$$

If $P = 0$, we denote $\Omega_{\epsilon,P}$ as Ω_ϵ.

For $P \in \Omega$, we have

$$\begin{aligned}&\int_\Omega f'(\bar{w}_{\epsilon,P})\frac{\partial\bar{w}_{\epsilon,P}}{\partial P_i}\varphi_{\epsilon,P}(x)dx \\ &= (-1+o(1))\varphi_{\epsilon,P}(P)\epsilon^{N-1}\int_{R^N} f'(w)\frac{\partial w}{\partial y_i}e^{-<\nabla d_P,y>}dy \\ &\quad \text{(by Lemma 3.1 (2))} \\ &= (-\gamma_1 + o(1))\epsilon^{N-1}\varphi_{\epsilon,P}(P)(\nabla d_P)_i + O(e^{-(2+\sigma)d_P/\epsilon})\end{aligned} \tag{7.15}$$

where γ_1 is given in (3.12). This prove (3.11).

For $P_1, P_2 \in \Omega$ with $|P_1 - P_2|/\epsilon \to +\infty$, we have

$$
\begin{aligned}
(7.16)\ & \int_{\Omega} f'(\bar{w}_{\epsilon,P_1})\bar{w}_{\epsilon,P_2}\frac{\partial \bar{w}_{\epsilon,P_1}}{\partial P_{1,i}} \\
= \ & (-1+o(1))\epsilon^{N-1}\int_{\Omega_{\epsilon,P_1}} f^{'}(w(y))\frac{\partial w}{\partial y_i}w(y+\frac{P_1-P_2}{\epsilon})dy \\
& +O(e^{-(1+\sigma)|P_1-P_2|/\epsilon}) \\
= \ & \epsilon^{N-1}\int_{R^N} f(w)\frac{\partial}{\partial y_i}w(y+\frac{P_1-P_2}{\epsilon})dy + O(e^{-(1+\sigma)|P_1-P_2|/\epsilon}) \\
= \ & \epsilon^{N-1}(-\gamma_1+o(1))w(\frac{|P_1-P_2|}{\epsilon})(\nabla_{P_1}(|P_1-P_2|))_i + O(e^{-(1+\sigma)|P_1-P_2|/\epsilon})
\end{aligned}
$$

which proves (3.13). □

8. Appendix B: Proof of Lemma 4.3

In this appendix, we prove Lemma 4.3.

Proof of (1) and (2) of Lemma 4.3: Observe that

$$
\begin{aligned}
(8.1)\quad & \nabla_{j,i}M_\epsilon(\mathbf{P}) \\
= \ & < w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}}, \partial_{j,i}(w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}}) >_\epsilon - \epsilon^{-N}\int_\Omega f(w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}})\partial_{j,i} \\
& (w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}}) \\
= \ & < w_{\epsilon,\mathbf{P}}, \partial_{j,i}(w_{\epsilon,\mathbf{P}}) >_\epsilon - \epsilon^{-N}\int_\Omega f(w_{\epsilon,\mathbf{P}})\partial_{j,i}(w_{\epsilon,\mathbf{P}}) \\
& + < v_{\epsilon,\mathbf{P}}, \partial_{j,i}(w_{\epsilon,\mathbf{P}}) >_\epsilon - \epsilon^{-N}\int_\Omega f^{'}(w_{\epsilon,\mathbf{P}})v_{\epsilon,\mathbf{P}}\partial_{j,i}(w_{\epsilon,\mathbf{P}}) \\
& + < w_{\epsilon,\mathbf{P}}, \partial_{j,i}(v_{\epsilon,\mathbf{P}}) >_\epsilon - \epsilon^{-N}\int_\Omega f(w_{\epsilon,\mathbf{P}})\partial_{j,i}(v_{\epsilon,\mathbf{P}}) \\
& + < v_{\epsilon,\mathbf{P}}, \partial_{j,i}(v_{\epsilon,\mathbf{P}}) >_\epsilon - \epsilon^{-N}\int_\Omega f(w_{\epsilon,\mathbf{P}})\partial_{j,i}(v_{\epsilon,\mathbf{P}}) + O(e^{-(2+\sigma)\varphi_K(\mathbf{P})/\epsilon})
\end{aligned}
$$

$$
\begin{aligned}
&= \; < w_{\epsilon,\mathbf{P}}, \partial_{j,i} w_{\epsilon,\mathbf{P}} >_\epsilon - \epsilon^{-N} \int_\Omega f(w_{\epsilon,\mathbf{P}}) \partial_{j,i}(w_{\epsilon,\mathbf{P}}) + O(e^{-(2+\sigma)\varphi_K(\mathbf{P})/\epsilon}) \\
&= \; \epsilon^{-N} \int_\Omega \left[\sum_{l=1}^K f(\bar{w}_{\epsilon,P_l}) - f(\sum_{l=1}^K w_{\epsilon,P_l}) \right] (\partial_{j,i} w_{\epsilon,\mathbf{P}}) + O(e^{-(2+\sigma)\varphi_K(\mathbf{P})/\epsilon}) \\
&\quad \text{(by Lemma 3.1)} \\
&= \; \epsilon^{-N} \int_\Omega \left[\sum_{l=1}^K (f(\bar{w}_{\epsilon,P_l}) - f(w_{\epsilon,P_l})) - \sum_{l \neq j} f'(w_{\epsilon,P_j}) w_{\epsilon,P_l} \right] \partial_{j,i} w_{\epsilon,P_j} \\
&\quad + O(e^{-(2+\sigma)\varphi_K(\mathbf{P})/\epsilon}) \\
&= \; \epsilon^{-N} \int_\Omega \left[f(\bar{w}_{\epsilon,P_j}) - f(w_{\epsilon,P_j}) \right] \partial_{j,i} w_{\epsilon,P_j} \\
&\quad - \epsilon^{-N} \sum_{l \neq j} \int_\Omega f'(w_{\epsilon,P_j}) w_{\epsilon,P_l} \partial_{j,i} w_{\epsilon,P_j} + O(e^{-(2+\sigma)\varphi_K(\mathbf{P})/\epsilon}) \\
&= \; \int_{\Omega_{\epsilon,P_j}} f'(\bar{w}_{\epsilon,P_j})(-\varphi_{\epsilon,P_j}) \partial_{j,i} \bar{w}_{\epsilon,P_j} \\
&\quad - \sum_{l \neq j} \int_{\Omega_{\epsilon,P_j}} f'(\bar{w}_{\epsilon,P_j}) \bar{w}_{\epsilon,P_l} \partial_{j,i} \bar{w}_{\epsilon,P_j} + O(e^{-(2+\sigma)\varphi_K(\mathbf{P})/\epsilon})
\end{aligned}
$$

We discuss two cases.

Case 1: If $|P_j^\epsilon| \leq \frac{1}{10}$, then the first term in (8.1)

$$
\int_{\Omega_{\epsilon,P_j}} f'(\bar{w}_{\epsilon,P_j})(-\varphi_{\epsilon,P_j}) \partial_{j,i} \bar{w}_{\epsilon,P_j} = O(e^{-2d_{P_j^\epsilon}}/\epsilon)
$$

which is a higher order term, comparing with the second order term (since $|P_k^\epsilon - P_j^\epsilon| \leq \frac{11}{10} < 2d_{P_j^\epsilon}$). This proves (4.8) of Lemma 4.3.

Case 2: If $|P_j^\epsilon| \geq d_0$ for some $d_0 > 0$, then (8.1) equals

$$
\begin{aligned}
&\epsilon^{N-1}(\gamma_1 + o(1)) \varphi_{\epsilon,P_j}(P_j)(\nabla d_{P_j})_i \\
&+ \epsilon^{N-1}(\gamma_1 + o(1)) \sum_{l \neq j} w(|P_j - P_l|/\epsilon)(\nabla |P_j - P_l|)_i
\end{aligned}
\tag{8.2}
$$

by (3.11) and (3.13) of Lemma 3.1.

By using Lemma 3.1, we see that (4.7) holds. $\square$

Proof of (3) of Lemma 4.3: Let $\mathbf{P}^\epsilon$ be a critical point of $M_\epsilon(\mathbf{P})$ in Λ_δ such that $|P_j^\epsilon| \geq d_0, j=1,...,K$ for some $d_0>0$. We now expand,

$$\begin{aligned}
& \partial_{l,m}\partial_{j,i}M_\epsilon(\mathbf{P})\big|_{\mathbf{P}=\mathbf{P}^\epsilon} \\
= \; & < \partial_{l,m}(w_{\epsilon,\mathbf{P}}+v_{\epsilon,\mathbf{P}}), \partial_{j,i}(w_{\epsilon,\mathbf{P}}+v_{\epsilon,\mathbf{P}}) >_\epsilon |_{\mathbf{P}=\mathbf{P}^\epsilon} \\
& + < w_{\epsilon,\mathbf{P}}+v_{\epsilon,\mathbf{P}}, \partial_{l,m}\partial_{j,i}(w_{\epsilon,\mathbf{P}}+v_{\epsilon,\mathbf{P}}) >_\epsilon |_{\mathbf{P}=\mathbf{P}^\epsilon} \\
& -\epsilon^{-N}\int_\Omega f'(w_{\epsilon,\mathbf{P}^\epsilon}+v_{\epsilon,\mathbf{P}^\epsilon})\partial_{l,m}(w_{\epsilon,\mathbf{P}}+v_{\epsilon,\mathbf{P}})|_{\mathbf{P}=\mathbf{P}^\epsilon}\partial_{j,i}(w_{\epsilon,\mathbf{P}}+v_{\epsilon,\mathbf{P}})|_{\mathbf{P}=\mathbf{P}^\epsilon} \\
& -\epsilon^{-N}\int_\Omega f(w_{\epsilon,\mathbf{P}^\epsilon}+v_{\epsilon,\mathbf{P}^\epsilon})\partial_{l,m}\partial_{j,i}(w_{\epsilon,\mathbf{P}}+v_{\epsilon,\mathbf{P}})|_{\mathbf{P}=\mathbf{P}^\epsilon} \\
= \; & < \partial_{l,m}(w_{\epsilon,\mathbf{P}}+v_{\epsilon,\mathbf{P}}), \partial_{j,i}(w_{\epsilon,\mathbf{P}}+v_{\epsilon,\mathbf{P}}) >_\epsilon |_{\mathbf{P}=\mathbf{P}^\epsilon} \\
& -\epsilon^{-N}\int_\Omega f'(w_{\epsilon,\mathbf{P}^\epsilon}+v_{\epsilon,\mathbf{P}^\epsilon})\partial_{l,m}(w_{\epsilon,\mathbf{P}}+v_{\epsilon,\mathbf{P}})|_{\mathbf{P}=\mathbf{P}^\epsilon}\partial_{j,i}(w_{\epsilon,\mathbf{P}}+v_{\epsilon,\mathbf{P}})|_{\mathbf{P}=\mathbf{P}^\epsilon} \\
& (\text{since } \mathbf{P}^\epsilon \text{ is a critical point of } M_\epsilon(\mathbf{P})) \\
= \; & < \partial_{l,m}w_{\epsilon,\mathbf{P}}, \partial_{j,i}w_{\epsilon,\mathbf{P}} >_\epsilon |_{\mathbf{P}=\mathbf{P}^\epsilon} - \epsilon^{-N}\int_\Omega f'(w_{\epsilon,\mathbf{P}^\epsilon}+v_{\epsilon,\mathbf{P}^\epsilon})\partial_{l,m}w_{\epsilon,\mathbf{P}}|_{\mathbf{P}=\mathbf{P}^\epsilon}\partial_{j,i}w_{\epsilon,\mathbf{P}}|_{\mathbf{P}=\mathbf{P}^\epsilon} \\
& + < \partial_{l,m}w_{\epsilon,\mathbf{P}}, \partial_{j,i}v_{\epsilon,\mathbf{P}} >_\epsilon |_{\mathbf{P}=\mathbf{P}^\epsilon} - \epsilon^{-N}\int_\Omega f'(w_{\epsilon,\mathbf{P}^\epsilon}+v_{\epsilon,\mathbf{P}^\epsilon})\partial_{l,m}w_{\epsilon,\mathbf{P}}|_{\mathbf{P}=\mathbf{P}^\epsilon}\partial_{j,i}v_{\epsilon,\mathbf{P}}|_{\mathbf{P}=\mathbf{P}^\epsilon} \\
& + < \partial_{l,m}v_{\epsilon,\mathbf{P}}, \partial_{j,i}w_{\epsilon,\mathbf{P}} >_\epsilon |_{\mathbf{P}=\mathbf{P}^\epsilon} - \epsilon^{-N}\int_\Omega f'(w_{\epsilon,\mathbf{P}^\epsilon}+v_{\epsilon,\mathbf{P}^\epsilon})\partial_{l,m}v_{\epsilon,\mathbf{P}}|_{\mathbf{P}=\mathbf{P}^\epsilon}\partial_{j,i}w_{\epsilon,\mathbf{P}}|_{\mathbf{P}=\mathbf{P}^\epsilon} \\
& + < \partial_{l,m}v_{\epsilon,\mathbf{P}}, \partial_{j,i}v_{\epsilon,\mathbf{P}} >_\epsilon |_{\mathbf{P}=\mathbf{P}^\epsilon} - \epsilon^{-N}\int_\Omega f'(w_{\epsilon,\mathbf{P}^\epsilon}+v_{\epsilon,\mathbf{P}^\epsilon})\partial_{l,m}v_{\epsilon,\mathbf{P}}|_{\mathbf{P}=\mathbf{P}^\epsilon}\partial_{j,i}v_{\epsilon,\mathbf{P}}|_{\mathbf{P}=\mathbf{P}^\epsilon} \\
= \; & I_1+I_2+I_3+I_4
\end{aligned}$$

where $I_i, i=1,...,4$ are defined at the last equality.

We now estimate each term. Certainly the estimate of I_2 is the same as that of I_3. By Lemma 3.4,

$$I_4 = O(\|\partial_{l,m}v_{\epsilon,\mathbf{P}^\epsilon}\|_\epsilon\|\partial_{j,i}v_{\epsilon,\mathbf{P}^\epsilon}\|_\epsilon) = O(e^{-(2+\sigma)\varphi_K(\mathbf{P}^\epsilon)/\epsilon}). \tag{8.3}$$

Next we consider I_2:

(8.4)

$$I_2 = \epsilon^{-N}\int_\Omega\left[f(\bar{w}_{\epsilon,P_l^\epsilon})\partial_{l,m}\bar{w}_{\epsilon,P_l^\epsilon} - f(w_{\epsilon,\mathbf{P}^\epsilon}+v_{\epsilon,\mathbf{P}^\epsilon})\partial_{l,m}w_{\epsilon,P_l^\epsilon}\right]\partial_{j,i}w_{\epsilon,P_j^\epsilon} = O(e^{-(2+\sigma)\varphi_K(\mathbf{P}^\epsilon)/\epsilon}).$$

Similarly, we have

$$I_3 = O(e^{-(2+\sigma)\varphi_K(\mathbf{P}^\epsilon)/\epsilon}). \tag{8.5}$$

Hence it remains to compute I_1 only. We divide it into two cases: $j \neq l$ and $j = l$.

When $j \neq l$, we have by Lemma 3.1

$$
\begin{aligned}
(8.6)\quad I_1 &= \epsilon^{-N} \int_\Omega \left[f'(\bar{w}_{\epsilon,P_l^\epsilon}) \partial_{l.m} \bar{w}_{\epsilon,P_l^\epsilon} - f'(w_{\epsilon,\mathbf{P}^\epsilon}) \partial_{l,m} w_{\epsilon,P_l^\epsilon} \right] \partial_{j,i} w_{\epsilon,P_j^\epsilon} \\
&= \epsilon^{-N} \int_\Omega \Big[f'(\bar{w}_{\epsilon,P_l^\epsilon}) \partial_{l,m} \bar{w}_{\epsilon,P_l^\epsilon} - (f'(w_{\epsilon,P_l^\epsilon}) \\
&\quad + f'(w_{\epsilon,P_j^\epsilon})) \partial_{l,m} w_{\epsilon,P_l^\epsilon} \Big] \partial_{j,i} \bar{w}_{\epsilon,P_j^\epsilon} + O(e^{-(2+\sigma)\varphi_K(\mathbf{P}^\epsilon)/\epsilon}) \\
&= \epsilon^{-N} \int_\Omega \left[f'(\bar{w}_{\epsilon,P_l^\epsilon}) \partial_{l,m} \bar{w}_{\epsilon,P_l^\epsilon} - f'(w_{\epsilon,P_l^\epsilon}) \partial_{l,m} w_{\epsilon,P_l^\epsilon} \right] \partial_{j,i} \bar{w}_{\epsilon,P_j^\epsilon} \\
&\quad - \epsilon^{-N} \int_\Omega f'(w_{\epsilon,P_j^\epsilon}) \partial_{l,m} w_{\epsilon,P_l^\epsilon} \partial_{j,i} \bar{w}_{\epsilon,P_j^\epsilon} + O(e^{-(2+\sigma)\varphi_K(\mathbf{P}^\epsilon)/\epsilon}) \\
&= -\epsilon^{-N} \int_\Omega f'(w_{\epsilon,P_j^\epsilon}) \partial_{l,m} w_{\epsilon,P_l^\epsilon} \partial_{j,i} \bar{w}_{\epsilon,P_j^\epsilon} + O(e^{-(2+\sigma)\varphi_K(\mathbf{P}^\epsilon)/\epsilon}) \\
&= \epsilon^{-2} w\left(\frac{|P_j^\epsilon - P_l^\epsilon|}{\epsilon}\right) \int_{\Omega_{\epsilon,P_j^\epsilon}} f'(w) \frac{\partial w}{\partial y_i} e^{-<e_{jl}^\epsilon, y>} (e_{jl}^\epsilon)_m \\
&\quad + O(e^{-(2+\sigma)\varphi_K(\mathbf{P}^\epsilon)/\epsilon}) \\
&= \epsilon^{-2} w(|P_j^\epsilon - P_l^\epsilon|/\epsilon)(\gamma_1 + o(1))(e_{jl}^\epsilon)_m (e_{jl}^\epsilon)_i \\
&\quad + O(e^{-(2+\sigma)\varphi_K(\mathbf{P}^\epsilon)/\epsilon})
\end{aligned}
$$

For $j = l$, we have

$$
\begin{aligned}
I_1 &= \epsilon^{-N} \int_\Omega \left[f'(\bar{w}_{\epsilon,P_j^\epsilon}) \partial_{j,m} \bar{w}_{\epsilon,P_j^\epsilon} - f'(w_{\epsilon,\mathbf{P}^\epsilon}) \partial_{j,m} w_{\epsilon,P_j^\epsilon} \right] \partial_{j,i} w_{\epsilon,P_j^\epsilon} \\
&= \epsilon^{-N} \int_\Omega \left[f'(\bar{w}_{\epsilon,P_j^\epsilon}) \partial_{j,m} \bar{w}_{\epsilon,P_j^\epsilon} - f'(w_{\epsilon,P_j^\epsilon}) \partial_{j,m} w_{\epsilon,P_j^\epsilon} \right] \partial_{j,i} w_{\epsilon,P_j^\epsilon} \\
&\quad - \epsilon^{-N} \sum_{k \neq j} \int_\Omega f''(\bar{w}_{\epsilon,P_j^\epsilon}) w_{\epsilon,P_k^\epsilon} \partial_{j,m} w_{\epsilon,P_j^\epsilon} \partial_{j,i} w_{\epsilon,P_j^\epsilon} + O(e^{-(2+\sigma)\varphi_K(\mathbf{P}^\epsilon)/\epsilon}) \\
&= I_{1,1} - I_{1,2}
\end{aligned}
$$

For $I_{1,1}$, we have

$$
\begin{aligned}
(8.7)\qquad & I_{1,1} \\
= \ & \epsilon^{-N}\int_\Omega \left[\partial_{j,m} f(\bar{w}_{\epsilon,P_j^\epsilon}) - \partial_{j,m} f(w_{\epsilon,P_j^\epsilon})\right]\partial_{j,i}\bar{w}_{\epsilon,P_j^\epsilon} \\
= \ & \epsilon^{-N}\int_\Omega \left[(-\frac{\partial}{\partial x_m} f(\bar{w}_{\epsilon,P_j^\epsilon}) - f(w_{\epsilon,P_j^\epsilon}))\right] \\
& (-\frac{\partial}{\partial x_i}\bar{w}_{\epsilon,P_j^\epsilon} + O(e^{-(2+\sigma)\varphi_K(\mathbf{P}^\epsilon)/\epsilon}) \\
= \ & \epsilon^{-N}\int_\Omega (f(w_{\epsilon,P_j^\epsilon}) - f(\bar{w}_{\epsilon,P_j^\epsilon})(\frac{\partial^2}{\partial x_i \partial x_m}\bar{w}_{\epsilon,P_j^\epsilon}) + O(e^{-(2+\sigma)\varphi_K(\mathbf{P}^\epsilon)/\epsilon}) \\
= \ & \epsilon^{-2}\int_{\Omega_{\epsilon,P_j^\epsilon}} f'(w)(y)\varphi_{\epsilon,P_j^\epsilon}(P_j^\epsilon+\epsilon y)\frac{\partial^2 w}{\partial y_i \partial y_m} + O(e^{-(2+\sigma)\varphi_K(\mathbf{P}^\epsilon)/\epsilon}) \\
= \ & \epsilon^{-2}\varphi_{\epsilon,P_j^\epsilon}(P_j^\epsilon)\int_{R^N} f'(w)e^{-<\nabla_{P_j^\epsilon} d_{P_j^\epsilon}, y>}\frac{\partial^2 w}{\partial y_i \partial y_m}dy \\
& +O(e^{-(2+\sigma)\varphi_K(\mathbf{P}^\epsilon)/\epsilon}) \\
& \text{(by (3.9) of Lemma 3.1)} \\
= \ & \epsilon^{-2}\varphi_{\epsilon,P_j^\epsilon}(P_j^\epsilon)\int_{R^N} (-f''(w)\frac{\partial w}{\partial y_i}\frac{\partial w}{\partial y_m}e^{-<\nabla_{P_j^\epsilon} d_{P_j^\epsilon}, y>}dy \\
& +O(e^{-(2+\sigma)\varphi_K(\mathbf{P}^\epsilon)/\epsilon}) \\
= \ & \epsilon^{-2}\varphi_{\epsilon,P_j^\epsilon}(P_j^\epsilon)e_{j,i}^\epsilon e_{j,m}^\epsilon(-\gamma_1 + o(1)).
\end{aligned}
$$

For $I_{1,2}$, we have

$$
\begin{aligned}
(8.8)\qquad & I_{1,2} \\
= \ & \epsilon^{-N}\sum_{k\not j}\int_\Omega f''(\bar{w}_{\epsilon,P_j^\epsilon})\bar{w}_{\epsilon,P_k^\epsilon}\partial_{j,m}\bar{w}_{\epsilon,P_j^\epsilon}\partial_{j,i}\bar{w}_{\epsilon,P_j^\epsilon} + O(e^{-(2+\sigma)\varphi_K(\mathbf{P}^\epsilon)/\epsilon}) \\
= \ & \epsilon^{-2}\sum_{k\neq j}\int_{R^N} f''(w)\frac{\partial w}{\partial y_i}\frac{\partial w}{\partial y_m}w(y+\frac{P_j^\epsilon - P_k^\epsilon}{\epsilon})dy + O(e^{-(2+\sigma)\varphi_K(\mathbf{P}^\epsilon)/\epsilon}) \\
= \ & \epsilon^{-2}\sum_{k\neq j} w(\frac{|P_j^\epsilon - P_k^\epsilon|}{\epsilon})e_{jk,i}^\epsilon e_{jk,m}^\epsilon(\gamma_1 + o(1)).
\end{aligned}
$$

Combining all together, we have

$$\begin{aligned}
& \partial_{l,m}\partial_{j,i} M_\epsilon(\mathbf{P})\big|_{\mathbf{P}=\mathbf{P}^\epsilon} \\
= \ & \epsilon^{-2}(\gamma_1+o(1))w(\frac{|P_j^\epsilon-P_k^\epsilon|}{\epsilon})e_{jl,m}^\epsilon e_{jl,i}^\epsilon(1-\delta_{jl}) \\
& -\epsilon^{-2}(\gamma_1+o(1))\varphi_{\epsilon,P_j^\epsilon}(P_j^\epsilon)e_{j,i}^\epsilon e_{l,m}^\epsilon \delta_{jl} \\
& -\epsilon^{-2}(\gamma_1+o(1))\sum_{k\neq j} w(\frac{|P_j^\epsilon-P_k^\epsilon|}{\epsilon})e_{jk,i}^\epsilon e_{jk,m}^\epsilon \delta_{jl},
\end{aligned}$$

which is exactly (4.9). □

9. Appendix C: Proof of Lemma 4.1

In this appendix, we prove Lemma 4.1.

We first exclude the case when multiple interior spikes collapse to the boundary. In fact this follows from the same proof of Lemma 2.3 of [36]. We prove by contradiction.

Suppose now that $P_j^\epsilon \in \Omega, \frac{d(P_j^\epsilon,\partial\Omega)}{\epsilon} \to +\infty, P_j^\epsilon \to P_0 \in \partial\Omega, j=1,...,n_1 \leq K$. Suppose that all the other spikes stay away from P_0. That is $|P_j^\epsilon - P_0| \geq \delta_0 > 0$ for $j = n_1+1,...,K$.

We recall the following Pohozaev identity: suppose that u satisfies $\epsilon^2\Delta u - u + f(u) = 0$ in a domain Ω, then for any $y \in R^N$, we have

$$\begin{aligned}
& \int_{\Omega_0}\left[(NF(u)-\frac{N-2}{2}uf(u))-u^2\right] \\
= \ & \int_{\partial\Omega_0}\Bigg[(\epsilon^2(x-y,\nabla u)\frac{\partial u}{\partial\nu}-\epsilon^2<x-y,\nu>\frac{|\nabla u|^2}{2}) \\
& +<x-y,\nu>(\frac{u^2}{2}-F(u))+\epsilon^2\frac{N-2}{2}u\frac{\partial u}{\partial \nu}\Bigg].
\end{aligned}$$

Since y is arbitrary, we deduce that

$$\int_{\partial\Omega_0}\left[\nu(\epsilon^2\frac{|\nabla u|^2}{2}+\frac{u^2}{2}-F(u))-\epsilon^2\nabla u\frac{\partial u}{\partial\nu}\right]=0. \tag{9.1}$$

Now we choose $\Omega_0 = \{x \in \Omega | |x - P_0| \leq \frac{1}{10}\}$ so that $|\nu(x) - \nu(P_0)| \leq \frac{1}{10}$ where $\nu(x)$ is the normal derivative at $x \in \partial\Omega \cap \partial\Omega_0$. Let

$$\rho_\epsilon = \min_{j,k,l=1,\dots,n_1,k\neq l} (d(P_j^\epsilon, \partial\Omega), |P_k^\epsilon - P_l^\epsilon|)/\epsilon.$$

Then similar to the proof of (2.8) and (2.12) of [36], we have

$$u_\epsilon \geq Ce^{-\rho_\epsilon} \quad \text{on } \partial\Omega_0 \cap \partial\Omega \tag{9.2}$$

and

$$u_\epsilon, |\nabla u_\epsilon| \leq Ce^{-\frac{\delta}{\epsilon}} \quad \text{on } \partial\Omega_0 \backslash \partial\Omega. \tag{9.3}$$

Substituting (9.2) and (9.3) into (9.1), we have

$$\begin{aligned} 0 &= \int_{\partial\Omega_0 \cap \partial\Omega} \left[\nu(\epsilon^2 \frac{|\nabla u_\epsilon|^2}{2} + \frac{u_\epsilon^2}{2} - F(u_\epsilon)) \right] + O(e^{-\delta/\epsilon}) \\ &\geq C \int_{\partial\Omega_0 \cap \partial\Omega} u_\epsilon^2 + O(e^{-\delta/\epsilon}) \geq Ce^{-2\rho_\epsilon} \end{aligned} \tag{9.4}$$

which is contradiction.

Second, we exclude the case when multiple interior spikes collapse in the interior. Namely, we have $P_j^\epsilon, j = 1, ..., n_2, 2 \leq n_2 \leq K$ where $P_j^\epsilon \to P_0 \in \Omega, j = 1, ..., n_2$. Suppose that all the other spikes stay away from P_0. That is $|P_j^\epsilon - P_0| \geq \delta_0 > 0, j = n_2 + 1, ..., K$. Similar to the proof of Lemma 3.1 of [34], we see that equations (6.7) remains true: so we have (the boundary terms all are higher order terms)

$$\sum_{k\neq j} w(|P_k^\epsilon - P_j^\epsilon|/\epsilon) \nabla_{P_j^\epsilon}(|P_k^\epsilon - P_j^\epsilon|) + o(\sum_{k\neq j} w(|P_k^\epsilon - P_j^\epsilon|/\epsilon)) = 0, j = 1, ..., n_2. \tag{9.5}$$

Since $2 \leq n_2 \leq K \leq 3$, we have either $n_2 = 2$ or $n_2 = 3$.

If $n_2 = 2$, (9.5) becomes

$$w(|P_1^\epsilon - P_2^\epsilon|/\epsilon) \nabla_{P_1^\epsilon}(|P_1^\epsilon - P_2^\epsilon|) + o(w(|P_1^\epsilon - P_2^\epsilon|/\epsilon)) = 0$$

which is impossible.

If $n_2 = 3$, we let $|P_1^\epsilon - P_2^\epsilon| = \min_{i\neq j, i,j=1,2,3} |P_i^\epsilon - P_j^\epsilon|$. We consider two cases. The first case is that $\lim_{\epsilon\to 0} \frac{w(|P_1^\epsilon - P_2^\epsilon|/\epsilon)}{w(|P_3^\epsilon - P_2^\epsilon|/\epsilon)} < +\infty$. In this case, let us assume that $< P_3^\epsilon - P_1^\epsilon, P_2^\epsilon - P_1^\epsilon >\geq 0$. Then for equation (9.5) at $j=1$, we have

$$w(|P_1^\epsilon - P_2^\epsilon|/\epsilon) + w(|P_3^\epsilon - P_1^\epsilon|/\epsilon) < \nabla_{P_1^\epsilon}(|P_3^\epsilon - P_1^\epsilon|), \nabla_{P_1^\epsilon}(|P_2^\epsilon - P_1^\epsilon|) >$$
$$+o(w(|P_1^\epsilon - P_2^\epsilon|/\epsilon)) = 0,$$

which is impossible. If $\lim_{\epsilon\to 0}\frac{w(|P_1^\epsilon - P_2^\epsilon|/\epsilon)}{w(|P_3^\epsilon - P_1^\epsilon|/\epsilon)} < +\infty$, the proof is similar.

The second case is that $\lim_{\epsilon\to 0}\frac{w(|P_3^\epsilon - P_2^\epsilon|/\epsilon)}{w(|P_1^\epsilon - P_2^\epsilon|/\epsilon)} = 0$, then this is reduced to the $n_2 = 2$ case.

Finally, we need to exclude the case when multiple interior spikes collapse to a boundary spike or multiple boundary spikes collapse. That is we have $|P_j^\epsilon - P_1^\epsilon| \to 0, j = 2, ..., n_3 \leq K$, where $P_1^\epsilon \in \partial\Omega$. Without loss of generality, we may assume that $P_1^\epsilon = P_0 = (-1, ..., 0)$. If $K = 2$, by Theorem 1.1, P_2^ϵ is on the other side of P_1^ϵ. Thus, $|P_2^\epsilon - P_1^\epsilon| \geq 1$. So we may assume that $K = 3$. By Theorem 1.3, we have $|P_3^\epsilon - P_1^\epsilon| \geq 1$ and $|P_2^\epsilon - P_1^\epsilon| \to 0$. There are two cases to be considered.

Case 1: $P_2^\epsilon \in \Omega, P_2^\epsilon \to P_0 = P_1^\epsilon$. Let $\tilde\epsilon = \frac{\epsilon}{\delta_\epsilon}$, where $\delta_\epsilon = |P_1^\epsilon - P_2^\epsilon|$.

Equations (6.7) and (6.8) remains true as long as $\tilde\varphi_K(P_1^\epsilon, ..., P_K^\epsilon)/\epsilon \to +\infty$. Now we look at the equation (6.8) for $j = 2, i = 1$. We have

$$2(\gamma_1 + o(1))c(P_2^\epsilon)e^{-2d_{P_2^\epsilon}/\epsilon}(\nabla_{P_2^\epsilon} d_{P_2^\epsilon})_1$$
$$+(\gamma_1 + o(1))(|P_1^\epsilon - P_2^\epsilon|)^{-\frac{N-1}{2}} e^{-|P_1^\epsilon - P_2^\epsilon|/\epsilon}(\nabla_{P_2^\epsilon}|P_2^\epsilon - P_1^\epsilon|)_1 = 0 \tag{9.6}$$

Note that both $\nabla_{P_2^\epsilon} d_{P_2^\epsilon}$ and $\nabla_{P_2^\epsilon}(|P_2^\epsilon - P_1^\epsilon|)$ are pointing in the same direction. Equation (9.6) can not hold.

Case 2: $P_1^\epsilon \in \partial\Omega, P_2^\epsilon \in \partial\Omega, |P_2^\epsilon - P_1^\epsilon| \to 0$. In this case, we apply (6.7) to $j = 1, \frac{\partial}{\partial\tau_{P_1^\epsilon}} = \frac{\partial}{\partial x_{1,2}^\epsilon}$ and we obtain

$$(\gamma_1 + o(1))(|P_1^\epsilon - P_2^\epsilon|)^{-\frac{N-1}{2}} e^{-|P_1^\epsilon - P_2^\epsilon|/\epsilon}(\nabla_{P_2^\epsilon}|P_2^\epsilon - P_1^\epsilon|)_2 = 0 \tag{9.7}$$

which is impossible.

□

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