ON A SINGULAR PERTURBED PROBLEM IN AN ANNULUS

SANJIBAN SANTRA AND JUNCHENG WEI

Abstract. In this paper, we extend the results obtained by Ruf–Srikanth [8]. We prove the existence of positive solution under Dirichlet and Neumann boundary conditions, which concentrate near the inner boundary and outer boundary respectively of an annulus as $\varepsilon \to 0$. In fact, our result is independent of the dimension of $\mathbb{R}^N$.

1. Introduction

There has been a considerable interest in understanding the behavoir of positive solutions of the elliptic problem

$$\begin{cases} \varepsilon^2 \Delta u - u + f(u) = 0 \quad \text{in } \Omega \\ \quad u > 0 \ \text{ in } \Omega, \\ \quad u = 0 \text{ or } \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is a parameter, f is a superlinear nonlinearity and Ω is a smooth bounded domain in $\mathbb{R}^N$. Let $F(u) = \int_0^u f(t)dt$. We consider the problems when $f(0) = 0$ and $f'(0) = 0$. This type of equations arises in various mathematical models derived from population theory, chemical reactor theory see Gidas-Ni-Nirenberg [6]. In the Dirichlet case, Ni – Wei showed in [13] that the least energy solutions of equation (1.1) concentrate, for $\varepsilon \to 0$, to single peak solutions, whose maximum points P_ε converge to a point P with maximal distance from the boundary $\partial\Omega$. In the Neumann case, Ni–Takagi [11] showed that for sufficiently small $\varepsilon > 0$, the least energy solution is a single boundary spike and has only one local maximum $P_\varepsilon \in \partial\Omega$. Moreover, in [12], they prove that $H(P_\varepsilon) \to \max_{P\in\partial\Omega} H(P)$ as $\varepsilon \to 0$ where $H(P)$ is the mean curvature of $\partial\Omega$ at P. A simplified proof was given by del Pino–Felmer in [3], for a wider class of nonlinearities using a method of symmetrisation.

Higher dimensional concentrating solutions was studied by Ambrosetti–Malchiodi – Ni in [1], [2]; they consider solutions which concentrate on spheres, i.e. on $(N-1)$-dimensional manifolds. They studied

$$\begin{cases} \varepsilon^2 \Delta u - V(r)u + f(u) = 0 \quad \text{in } A \\ \qquad u > 0 \text{ in } A, u = 0 \quad \text{on } \partial A \end{cases} \tag{1.2}$$

the problem, in an annulus $A = \{x \in \mathbb{R}^N : 0 < a < |x| < b\}$, $V(r)$ is a smooth radial potential bounded below by a positive constant. They introduced a modified potential $M(r) = r^{N-1}V^\theta(r)$, with $\theta = \frac{p+1}{p-1} - \frac{1}{2}$, satisfying $M'(b) < 0$ (respectively $M'(a) > 0$), then there exists a family of radial solutions which concentrates on

1991 *Mathematics Subject Classification.* Primary 35J10, 35J35, 35J65.

Key words and phrases. Dirichlet problem, Neumann problem, annulus, concentration.

The first author was supported by the Australian Research Council.

$|x| = r_\varepsilon$ with $r_\varepsilon \to b$ (respectively $r_\varepsilon \to a$) as $\varepsilon \to 0$. In fact, they conjectured that in $N \geq 3$ there could exist also solutions concentrating to some manifolds of dimension k with $1 \leq k \leq N-2$. Moreover, in $\mathbb{R}^2$, concentration of positive solutions on curves in the general case was proved by del Pino–Kowalczyk–Wei [4]. In [9], the asymptotic behavior of radial solutions for a singularly perturbed elliptic problem (1.2) was studied using the Morse index information on such solutions to provide a complete description of the blow-up behavior. As a consequence, they exhibit sufficient conditions which guarantees that radial ground state solutions blow-up and concentrate at the inner or outer boundary of the annulus.

In this paper, we consider the following two singular perturbed problems,

$$\begin{cases} \varepsilon^2 \Delta u - u + u^p = 0 & \text{in } A \\ u > 0 & \text{in } A \\ u = 0 & \text{on } \partial A, \end{cases} \tag{1.3}$$

$$\begin{cases} \varepsilon^2 \Delta u - u + u^p = 0 & \text{in } A \\ u > 0 & \text{in } A \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial A, \end{cases} \tag{1.4}$$

where A is an annulus in $\mathbb{R}^N = \mathbb{R}^M \times \mathbb{R}^K$ with $A = \{x \in \mathbb{R}^N : 0 < a < |x| < b\}$ and $\varepsilon > 0$ is a small number and ν denotes the unit normal to ∂A and $N \geq 2$. In this paper, we are interested in finding solution $u(x) = u(r,s)$ where $r = \sqrt{x_1^2 + x_2^2 + \cdots x_M^2}$ and $s = \sqrt{x_{M+1}^2 + x_{M+2}^2 + \cdots x_K^2}$.

Let us consider the conjecture due to Ruf and Srikanth:

Does there exist a solution for the problems (1.3) and (1.4), which concentrates on $\mathbb{R}^{M+K-1}$ dimensional subsets as $\varepsilon \to 0$?

Theorem 1.1. *For $\varepsilon > 0$ sufficiently small, there exists a solution of (1.3) which concentrates near the inner boundary of A.*

Theorem 1.2. *For $\varepsilon > 0$ sufficiently small, there exists a solution of (1.4) which concentrates near the outer boundary of A.*

2. SET UP FOR THE APPROXIMATION

Note that under symmetry assumptions, A can be reduced to a subset of $\mathbb{R}^2$ where $\mathcal{D} = \{(r,s) : r > 0, s > 0, a^2 < r^2 + s^2 < b^2\}$. Let $P_\varepsilon = (P_{1,\varepsilon}, P_{2,\varepsilon})$ be a point of maximum of u_ε in A, then $u_\varepsilon(P_\varepsilon) \geq 1$. From (1.3) we obtain

$$\varepsilon^2 u_{rr} + \varepsilon^2 u_{ss} + \varepsilon^2 \frac{(M-1)}{r} u_r + \varepsilon^2 \frac{(K-1)}{s} u_s - u + u^p = 0 \tag{2.1}$$

Let $\mathcal{D}_1, \mathcal{D}_2$ are the inner and outer boundary of $\mathcal{D}$ respectively and $\mathcal{D}_3, \mathcal{D}_4$ are the horizontal and vertical boundary of $\mathcal{D}$ respectively.

If $P = (P_1, P_2)$ be a point in $\mathcal{D}$ such that $dist(P, \mathcal{D}_1) = d$, then we can express,

$$P_1 = (a+d)\cos\theta; P_2 = (a+d)\sin\theta \tag{2.2}$$

where θ is the angle between the $x-$ axis and the line joining P. Furthermore, if $dist(P, \mathcal{D}_2) = d$, then we can express,

$$P_1 = (b-d)\cos\theta; P_2 = (b-d)\sin\theta. \tag{2.3}$$

See Figure 1 and Figure 2.

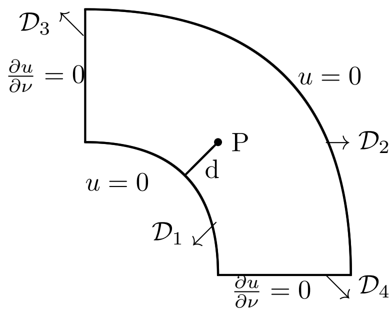

FIGURE 1. Dirichlet case

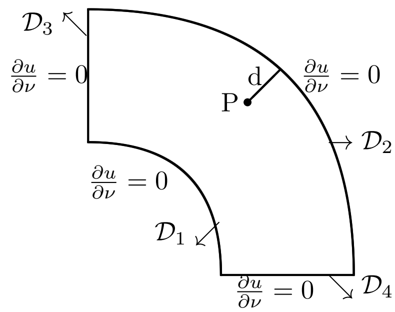

FIGURE 2. Neumann Case

The functional associated to the problem is

$$I_\varepsilon(u) = \int_{\mathcal{D}} r^{M-1} s^{K-1} \left(\frac{\varepsilon^2}{2} |\nabla u|^2 + \frac{1}{2} u^2 - \frac{1}{p+1} u^{p+1} \right) drds. \tag{2.4}$$

Moreover, (1.3) reduces to

$$\begin{cases} \varepsilon^2 u_{rr} + \quad \varepsilon^2 u_{ss} + \varepsilon^2 \dfrac{(M-1)}{r} u_r + \varepsilon^2 \dfrac{(K-1)}{s} u_s - u + u^p = 0 \text{ in } \mathcal{D} \\ \hfill u = 0 \text{ on } \mathcal{D}_1 \cup \mathcal{D}_2 \\ \hfill \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \mathcal{D}_3 \cup \mathcal{D}_4. \end{cases}$$

Re-scaling about the point P, we obtain in A_ε

$$u_{rr} + u_{ss} + \varepsilon \frac{(M-1)}{P_1 + \varepsilon r} u_r + \varepsilon \frac{(K-1)}{P_2 + \varepsilon s} u_s - u + u^p = 0. \tag{2.5}$$

The entire solution associated to (2.1) where U satisfies

$$\begin{cases} \Delta_{(r,s)} U - U + U^p = 0 & \text{in } \mathbb{R}^2 \\ U(r,s) > 0 & \text{in } \mathbb{R}^2 \\ U(r,s) \to 0 & \text{as } |(r,s)| \to \infty. \end{cases} \tag{2.6}$$

Let $z = (r,s)$. Moreover, $U(z) = U(|z|)$ and the asymptotic behavior of U at infinity is given by

$$\begin{cases} U(z) = A|z|^{-\frac{1}{2}} e^{-|z|} \left(1 + O\left(\dfrac{1}{|z|}\right)\right) \\ U'(z) = -A|z|^{-\frac{1}{2}} e^{-|z|} \left(1 + O\left(\dfrac{1}{|z|}\right)\right) \end{cases} \tag{2.7}$$

for some constant $A > 0$.

Let $K(z)$ denote the fundamental solution of $-\Delta_{(r,s)} + 1$ centered at 0. Then for $|z| \geq 1$, we have

$$\begin{cases} U(z) = \left(B + O\left(\dfrac{1}{|z|}\right)\right) K(z) \\ U'(z) = \left(-B + O\left(\dfrac{1}{|z|}\right)\right) K(z) \end{cases} \tag{2.8}$$

for some positive constant B.

Let $U_{\varepsilon,P}(z) = U(|\frac{z-P}{\varepsilon}|)$. Now we construct the projection map for the Dirichlet case as

$$\begin{cases} \varepsilon^2 \Delta_{(r,s)} PU_{\varepsilon,P} - PU_{\varepsilon,P} + U^p_{\varepsilon,P} = 0 & \text{in } \mathcal{D} \\ PU_{\varepsilon,P}(r,s) > 0 & \text{in } \mathcal{D} \\ PU_{\varepsilon,P}(r,s) = 0 & \text{on } \partial\mathcal{D}, \end{cases} \tag{2.9}$$

and the projection in the Neumann case as

$$\begin{cases} \varepsilon^2 \Delta_{(r,s)} QU_{\varepsilon,P} - QU_{\varepsilon P} + U^p_{\varepsilon,P} = 0 & \text{in } \mathcal{D} \\ QU_{\varepsilon,P}(r,s) > 0 & \text{in } \mathcal{D} \\ \dfrac{QU_{\varepsilon,P}}{\partial\nu}(r,s) = 0 & \text{on } \partial\mathcal{D}. \end{cases} \tag{2.10}$$

If $v_\varepsilon = U_{\varepsilon,P} - PU_{\varepsilon,P}$ and $w_\varepsilon = U_{\varepsilon,P} - QU_{\varepsilon,P}$. Then we have

$$\begin{cases} \varepsilon^2 \Delta_{(r,s)} v_\varepsilon - v_\varepsilon = 0 & \text{in } \mathcal{D} \\ v_\varepsilon = U_{\varepsilon,P} & \text{on } \partial\mathcal{D}, \end{cases} \tag{2.11}$$

$$\begin{cases} \varepsilon^2 \Delta_{(r,s)} w_\varepsilon - w_\varepsilon = 0 & \text{in } \mathcal{D} \\ \dfrac{\partial w_\varepsilon}{\partial\nu} = \dfrac{\partial U_{\varepsilon,P}}{\partial\nu} & \text{on } \partial\mathcal{D}. \end{cases} \tag{2.12}$$

Consider the function $s(\theta) = \cos^{M-1}\theta \sin^{K-1}\theta$ in $[0, \frac{\pi}{2}]$. Then neither $\theta_0 = 0$ nor $\theta_0 = \frac{\pi}{2}$ are points of maxima of s. But $s > 0$ and hence θ_0 lies in $(0, \frac{\pi}{2})$.

For any $\theta \in \Big[\theta_0 - \delta, \theta_0 + \delta\Big]$ we define the configuration space for the Dirichlet and Neumann case as

$$\Lambda_{\varepsilon,D} = \Big\{ P \in \mathcal{D} : dist(P, \mathcal{D}_1) \geq \frac{k}{2}\varepsilon \ln \frac{1}{\varepsilon} \Big\} \tag{2.13}$$

and

$$\Lambda_{\varepsilon,N} = \Big\{ P \in \mathcal{D} : dist(P, \mathcal{D}_2) \geq \frac{k}{2}\varepsilon \ln \frac{1}{\varepsilon} \Big\} \tag{2.14}$$

respectively for some $k > 0$ small.

We develop the following lemma similar to Lin, Ni and Wei [10].

Lemma 2.1. *Assume that $\frac{k}{2}\varepsilon|\ln\varepsilon| \leq d(P, \mathcal{D}_1) \leq \delta$, then we obtain*

$$v_\varepsilon(z) = (B + o(1))K\left(\frac{|z - P^\star|}{\varepsilon}\right) + O(\varepsilon^{2+\sigma}) \tag{2.15}$$

where $P^\star = P + 2d(P, \mathcal{D}_1)\nu_{\overline{P}}$ and $\overline{P} \in \mathcal{D}_1$ is a unique point, such that $d(P, \overline{P}) = 2d(P, \mathcal{D}_1)$ and σ is a small positive number; δ is the sufficiently small. Moreover, $\nu_{\overline{P}}$ is the outer unit normal at $\overline{P}$.

Proof. Define

$$\begin{cases} \varepsilon^2 \Delta_{(r,s)} \Psi_\varepsilon - \Psi_\varepsilon = 0 & \text{in } \mathcal{D} \\ \Psi_\varepsilon > 0 & \text{in } \mathcal{D} \\ \Psi_\varepsilon = 1 & \text{on } \partial\mathcal{D}, \end{cases} \tag{2.16}$$

Then for sufficiently small ε, Ψ_ε is uniformly bounded.

But for $z \in \partial\mathcal{D}$, we obtain

$$U_{\varepsilon,P}(z) = U\left(\frac{|z-P|}{\varepsilon}\right) = (A+o(1))\varepsilon^{\frac{1}{2}}|z-P|^{-\frac{1}{2}}e^{-\frac{|z-P|}{\varepsilon}}.$$

First, we have

$$U_{\varepsilon,P}(z) = (B+o(1))K\left(\frac{|z-P|}{\varepsilon}\right).$$

Hence by the comparison principle we obtain, for some $\sigma > 0$, small

$$v_\varepsilon \le C\varepsilon^{2+\sigma}\Psi_\varepsilon \text{ whenever } d(P,\mathcal{D}_1) \ge 2\varepsilon|\ln\varepsilon|.$$

Therefore, it remains to check whether (2.15) holds in

$$\frac{k}{2}\varepsilon|\ln\varepsilon| \le d(P,\mathcal{D}_1) \le 2\varepsilon|\ln\varepsilon|. \tag{2.17}$$

Define the function

$$\phi_1(z) = (B-\varepsilon^{\frac{1}{4}})K\left(\frac{|z-P^\star|}{\varepsilon}\right) + \varepsilon^{2+\sigma}\Psi_\varepsilon. \tag{2.18}$$

Then ϕ_1 satisfies

$$\varepsilon^2\Delta_{(r,s)}\phi_1 - \phi_1 = 0. \tag{2.19}$$

For any z in $\mathcal{D}_1$ with $|z-P| \le \varepsilon^{\frac{3}{4}}$ we have

$$\frac{|z-P|}{\varepsilon} = (1+O(\varepsilon^{\frac{1}{2}})|\ln\varepsilon|)\frac{|z-P^\star|}{\varepsilon} \tag{2.20}$$

and hence

$$v_\varepsilon \le \phi_1.$$

For any $z \in \mathcal{D}_1$ with $|z-P| \ge \varepsilon^{\frac{3}{4}}$ we have

$$v_\varepsilon(z) \le Ce^{-\varepsilon^{-\frac{1}{4}}} \le \varepsilon^{2+\sigma} \le \phi_1.$$

Summarizing, we obtain,

$$v_\varepsilon \le \phi_1 \text{ for all } z \in \mathcal{D}_1.$$

Similarly, we obtain the lower bound for $z \in \mathcal{D}_1$,

$$v_\varepsilon(z) \ge (B+\varepsilon^{\frac{1}{4}})K\left(\frac{|z-P^\star|}{\varepsilon}\right) - \varepsilon^{2+\sigma}\Psi_\varepsilon. \tag{2.21}$$

□

Corollary 2.1. *Assume that* $\frac{k}{2}\varepsilon|\ln\varepsilon| \le d(P,\mathcal{D}_2) \le \delta$ *where* δ *is sufficiently small. Then*

$$w_\varepsilon(z) = -(B+o(1))K\left(\frac{|z-P^\star|}{\varepsilon}\right) + O(\varepsilon^{2+\sigma}); \tag{2.22}$$

where $P^\star = P + 2d(P,\mathcal{D}_2)\nu_{\overline{P}}$ *where* $\overline{P} \in \mathcal{D}_2$ *is a unique point, such that* $d(P,\overline{P}) = 2d(P,\mathcal{D}_2)$ *and* σ *is a small positive number. Moreover,* $\nu_{\overline{P}}$ *is the outer unit normal at* $\overline{P}$.

3. Refinement of the projection

Define

$$H^1_0(\mathcal{D})=\Big\{u\in H^1: u(x)=u(r,s), u=0 \text{ in } \mathcal{D}_1 \text{ and } \mathcal{D}_2; \frac{\partial u}{\partial \nu}=0 \text{ in } \mathcal{D}_3 \text{ and } \mathcal{D}_4\Big\}.$$

Define a norm on $H^1_0(\mathcal{D})$ as

$$\|v\|^2_\varepsilon=\int_{\mathcal{D}} r^{M-1}r^{K-1}[\varepsilon^2|\nabla v|^2dx+v^2]drds \tag{3.1}$$

In this section, we will refine the projection, to incorporate the Neumann boundary condition on $\mathcal{D}_3$ and $\mathcal{D}_4$. We define a new projection as $V_{\varepsilon,P}=\eta PU_{\varepsilon,P}$ where $0\le\eta\le1$ is smooth cut off function

$$\eta(x)=\begin{cases}1 & \text{in } \mathcal{D}\cap B_d(P),\\ 0 & \text{in } \mathcal{D}\setminus B_{2d}(P).\end{cases} \tag{3.2}$$

Here $d=dist(P,\partial\mathcal{D})$ is dependent on ε. We will choose d at the end of the proof.

We define

$$u_\varepsilon=V_{\varepsilon,P_\varepsilon}+\varphi_{\varepsilon,P}. \tag{3.3}$$

Using this Ansatz, (1.3) reduces to

$$\begin{cases}\varepsilon^2\Delta_{(r,s)}\varphi_\varepsilon-\varphi_\varepsilon+\varepsilon^2\dfrac{(M-1)}{r}\varphi_{\varepsilon,s}+\varepsilon^2\dfrac{(K-1)}{s}\varphi_{\varepsilon,r} & +f'(V_{\varepsilon,P_\varepsilon})\varphi_\varepsilon=h \text{ in } \mathcal{D},\\ & \varphi_\varepsilon=0 \text{ on } \mathcal{D}_1\cup\mathcal{D}_2\\ & \dfrac{\partial\varphi_\varepsilon}{\partial\nu}=0 \text{ on } \mathcal{D}_3\cup\mathcal{D}_4;\end{cases}$$

where $h=-S_\varepsilon[V_{\varepsilon,P_\varepsilon}]+N_\varepsilon[\varphi_\varepsilon]$ and

$$\begin{aligned}S_\varepsilon[V_{\varepsilon,P}] &= \varepsilon^2\Delta_{(r,s)}V_{\varepsilon,P}+\varepsilon^2\frac{(M-1)}{r}V_{\varepsilon,P,r}+\varepsilon^2\frac{(K-1)}{s}V_{\varepsilon,P,s}\\ &- V_{\varepsilon,P}+f(V_{\varepsilon,P})\end{aligned} \tag{3.4}$$

and

$$N_\varepsilon[\varphi_\varepsilon]=\{f(V_{\varepsilon,P_\varepsilon}+\varphi_\varepsilon)-f(V_{\varepsilon,P_\varepsilon})-f'(V_{\varepsilon,P_\varepsilon})\varphi_\varepsilon\}.$$

Let

$$E_{\varepsilon,P}=\Big\{\omega\in H^1_0(\mathcal{D}),\Big\langle\omega,\frac{\partial V_{\varepsilon,P}}{\partial r}\Big\rangle_\varepsilon=\Big\langle\omega,\frac{\partial V_{\varepsilon,P}}{\partial s}\Big\rangle_\varepsilon=0\Big\}.$$

Lemma 3.1. *Then for any $z\in\mathcal{D}\setminus B_d(P)$*

$$V_{\varepsilon,P}(z)=\eta\bigg(U\Big(\frac{|z-P|}{\varepsilon}\Big)-v_{\varepsilon,P}(z)\bigg). \tag{3.5}$$

Moreover, we have

$$V_{\varepsilon,P}(z)=O(\varepsilon^k). \tag{3.6}$$

Proof. For any $z \in \mathcal{D} \setminus B_d(P)$ we have

$$
\begin{aligned}
V_{\varepsilon,P}(z) &\leq \left| U\left(\frac{|z-P|}{\varepsilon}\right) - v_{\varepsilon,P}(z) \right| \\
&= O(e^{-\frac{|x-P|}{\varepsilon}} + e^{-\frac{|x-P^\star|}{\varepsilon}} + \varepsilon^{3+\sigma}) \\
&= O(e^{-\frac{d(P,P^\star)}{\varepsilon}} + \varepsilon^{2+\sigma}) \\
&= O(e^{-\frac{2d(P,\partial\mathcal{D}_1)}{\varepsilon}} + \varepsilon^{2+\sigma}) = O(\varepsilon^k). \qquad (3.7)
\end{aligned}
$$

□

Moreover, $V_{\varepsilon,P}$ is zero outside $B_{2d}(P)$.

Lemma 3.2. *The energy expansion is given by*

$$
\begin{aligned}
I_\varepsilon(V_{\varepsilon,P}) &= \int_{\mathcal{D}} r^{M-1}s^{K-1}\left(\frac{\varepsilon^2}{2}|\nabla V_{\varepsilon,P}|^2 + \frac{1}{2}V_{\varepsilon,P}^2 - \frac{1}{p+1}V_{\varepsilon,P}^{p+1}\right)drds \\
&= \gamma\varepsilon^2 P_1^{M-1}P_2^{K-1} + \gamma_1\varepsilon^2 P_1^{M-1}P_2^{K-1}U\left(\frac{|P-P^\star|}{\varepsilon}\right) \\
&+ o(\varepsilon^{2+k})
\end{aligned}
$$

where $\gamma = \frac{p-1}{2(p+1)}\int_{\mathbb{R}^2} U^{p+1}drds$ *and* $\gamma_1 = \int_{\mathbb{R}^2} U^p e^{-r}drds$.

Proof. We obtain

$$
\begin{aligned}
I_\varepsilon(V_{\varepsilon,P}) &= \int_{\mathcal{D}} r^{M-1}s^{K-1}\left(\frac{\varepsilon^2}{2}|\nabla V_{\varepsilon,P}|^2 + \frac{1}{2}V_{\varepsilon,P}^2 - \frac{1}{p+1}V_{\varepsilon,P}^{p+1}\right)drds \\
&= \int_{\mathcal{D}} \eta^2 r^{M-1}s^{K-1}\left(\frac{\varepsilon^2}{2}|\nabla PU_{\varepsilon,P}|^2 + \frac{1}{2}PU_{\varepsilon,P}^2 - \frac{1}{p+1}PU_{\varepsilon,P}^{p+1}\right)drds \\
&+ \frac{1}{p+1}\int_{\mathcal{D}} r^{M-1}s^{K-1}\left(\eta^2 - \eta^{p+1}\right)PU_{\varepsilon,P}^{p+1}drds \\
&+ \varepsilon^2\int_{\mathcal{D}} r^{M-1}s^{K-1}\eta\nabla\eta PU_\varepsilon\nabla PU_\varepsilon drds + \varepsilon^2\int_{\mathcal{D}} r^{M-1}s^{K-1}|\nabla\eta|^2(PU_{\varepsilon,P})^2drds \\
&= J_1 + J_2 + J_3 + J_4. \qquad (3.8)
\end{aligned}
$$

Hence we have

$$\begin{aligned}
J_1 &= \int_{\mathcal{D}} r^{M-1}s^{K-1}\Big(\frac{\varepsilon^2}{2}|\nabla PU_{\varepsilon,P}|^2+\frac{1}{2}PU^2_{\varepsilon,P}-\frac{1}{p+1}PU^{p+1}_{\varepsilon,P}\Big)drds\\
&- \int_{\mathcal{D}}(1-\eta^2)r^{M-1}s^{K-1}\Big(\frac{\varepsilon^2}{2}|\nabla PU_{\varepsilon,P}|^2+\frac{1}{2}PU^2_{\varepsilon,P}-\frac{1}{p+1}PU^{p+1}_{\varepsilon,P}\Big)drds\\
&= \int_{\mathcal{D}} r^{M-1}s^{K-1}\Big(\frac{1}{2}U^p_{\varepsilon,P}PU_{\varepsilon,P}-\frac{1}{p+1}PU^{p+1}_{\varepsilon,P}\Big)drds\\
&+ \varepsilon^2\int_{\partial B_{2d}(P)} r^{M-1}s^{K-1}\Big(\frac{\partial PU_{\varepsilon,P}}{\partial r}+\frac{\partial PU_{\varepsilon,P}}{\partial s}\Big)PU_{\varepsilon,P}drds\\
&- \varepsilon^2\int_{\partial B_{d}(P)} r^{M-1}s^{K-1}\Big(\frac{\partial PU_{\varepsilon,P}}{\partial r}+\frac{\partial PU_{\varepsilon,P}}{\partial s}\Big)PU_{\varepsilon,P}drds\\
&= \varepsilon^2\Big(\frac{1}{2}-\frac{1}{p+1}\Big)\int_{\mathcal{D}_\varepsilon}(P_1+\varepsilon r)^{M-1}(P_2+\varepsilon s)^{K-1}U^{p+1}(z)drds\\
&+ \frac{1}{2}\int_{\mathcal{D}} U^p_{\varepsilon,P}v_\varepsilon r^{M-1}s^{K-1}drds\\
&= \Big(\frac{1}{2}-\frac{1}{p+1}\Big)\varepsilon^2P_1^{M-1}P_2^{K-1}\int_{\mathbb{R}^2}U^{p+1}drds\\
&+ \frac{1}{2}\int_{\mathcal{D}} U^p_{\varepsilon,P}v_\varepsilon r^{M-1}s^{K-1}drds\\
&+ \varepsilon^2\int_{\partial B_{2d}(P)} r^{M-1}s^{K-1}\Big(\frac{\partial PU_{\varepsilon,P}}{\partial r}+\frac{\partial PU_{\varepsilon,P}}{\partial s}\Big)PU_{\varepsilon,P}drds\\
&- \varepsilon^2\int_{\partial B_{d}(P)} r^{M-1}s^{K-1}\Big(\frac{\partial PU_{\varepsilon,P}}{\partial r}+\frac{\partial PU_{\varepsilon,P}}{\partial s}\Big)PU_{\varepsilon,P}drds\\
&+ \int_{\mathcal{D}\setminus B_d} r^{M-1}s^{K-1}\Big(\frac{\varepsilon^2}{2}|\nabla PU_{\varepsilon,P}|^2+\frac{1}{2}PU^2_{\varepsilon,P}-\frac{1}{p+1}PU^{p+1}_{\varepsilon,P}\Big)drds+o(\varepsilon^2). \qquad (3.9)
\end{aligned}$$

Now we estimate

$$\begin{aligned}
&\varepsilon^2\Big(\frac{1}{2}-\frac{1}{p+1}\Big)\int_{\mathcal{D}_\varepsilon}(P_1+\varepsilon r)^{M-1}(P_2+\varepsilon s)^{K-1}U^{p+1}(z)drds\\
&= \frac{p-1}{2(p+1)}\varepsilon^2P_1^{M-1}P_2^{K-1}\int_{\mathbb{R}^2}U^{p+1}drds+O(\varepsilon^4)P_1^{M-1}P_2^{K-1} \qquad (3.10)
\end{aligned}$$

From Lemma 3.1, we compute the interaction term

$$\begin{aligned}
\int_{\mathcal{D}} U^p_{\varepsilon,P}v_\varepsilon r^{M-1}s^{K-1}drds &= \varepsilon^2\int_{\mathcal{D}_\varepsilon}U^pU\Big(\Big|z-\frac{P-P^\star}{\varepsilon}\Big|\Big)(P_1+\varepsilon r)^{M-1}(P_2+\varepsilon s)^{K-1}drds\\
&+ O(\varepsilon^4)\\
&= \varepsilon^2P_1^{M-1}P_2^{K-1}U\Big(\Big|\frac{P-P^\star}{\varepsilon}\Big|\Big)(\gamma+o(1))+O(\varepsilon^4)\\
&= \varepsilon^2P_1^{M-1}P_2^{K-1}U\Big(\frac{2d(P,\partial\mathcal{D}_1)}{\varepsilon}\Big)(\gamma+o(1))+O(\varepsilon^4). \qquad (3.11)
\end{aligned}$$

Also we have

$$J_2=\int_{\mathcal{D}} r^{M-1}s^{K-1}\Big(\eta^2-\eta^{p+1}\Big)PU^{p+1}_{\varepsilon,P}drds=O(\varepsilon^2)\varepsilon^{\frac{(p+1)k}{2}},$$

Furthermore, we have

$$\varepsilon^2\int_{\partial B_d(P)} r^{M-1}s^{K-1}\Big(\frac{\partial PU_{\varepsilon,P}}{\partial r}+\frac{\partial PU_{\varepsilon,P}}{\partial s}\Big)PU_{\varepsilon,P}drds=O(\varepsilon^{2+\frac{1}{3}+k});$$

$$J_3=\varepsilon^2\int_{\mathcal{D}} r^{M-1}s^{K-1}\eta\nabla\eta PU_\varepsilon\nabla PU_\varepsilon drds=o(\varepsilon^{k+2}),$$

and

$$J_4=\varepsilon^2\int_{\mathcal{D}} r^{M-1}s^{K-1}|\nabla\eta|^2(PU_{\varepsilon,P})^2drds=o(\varepsilon^{k+2}).$$

Hence we obtain the result. □

4. The reduction

In this section, we will reduce the proof of Theorem 1.1 to finding a solution of the form $V_{\varepsilon,P}+\varphi_{\varepsilon,P}$ for (1.3) to a finite dimensional problem. We will prove that for each $P\in\Lambda_{\varepsilon,D}$, there is a unique $\varphi_{\varepsilon,P}\in E_\varepsilon$ such that

$$\Big\langle I'_\varepsilon\Big(V_{\varepsilon,P}+\varphi_{\varepsilon,P}\Big),\eta\Big\rangle_\varepsilon=0;\ \forall\eta\in E_{\varepsilon,P}.$$

Let

$$J_\varepsilon(\varphi)=I_\varepsilon\Big(V_{\varepsilon,P}+\varphi_{\varepsilon,P}\Big).$$

From now on we consider $\varphi_{\varepsilon,P}=\varphi$. We expand $J_\varepsilon(\varphi)$ near $\varphi_{\varepsilon,P}=0$ as

$$J_\varepsilon(\varphi)=J_\varepsilon(0)+l_{\varepsilon,P}(\varphi)+\frac{1}{2}Q_{\varepsilon,P}(\varphi,\varphi)+R_\varepsilon(\varphi)$$

where

$$\begin{aligned} l_{\varepsilon,P}(\varphi) &= \int_{\mathcal{D}} r^{M-1}s^{K-1}\Big[\varepsilon^2\nabla V_{\varepsilon,P}\nabla\varphi+V_{\varepsilon,P}\varphi-V^p_{\varepsilon,P}\varphi\Big]drds\\ &= \int_{\mathcal{D}} r^{M-1}s^{K-1}S_\varepsilon[V_{\varepsilon,P}]\varphi drds, \end{aligned}\tag{4.1}$$

$$Q_{\varepsilon,P}(\varphi,\psi) = \int_{\mathcal{D}} r^{M-1}s^{K-1}\Big[\varepsilon^2\nabla\varphi\nabla\psi+\varphi\psi-pV^{p-1}_{\varepsilon,P}\varphi\psi\Big]drds,\tag{4.2}$$

and

$$\begin{aligned} R_\varepsilon(\varphi) &= \frac{1}{p+1}\int_{\mathcal{D}} r^{M-1}s^{K-1}\Big[\Big(V_{\varepsilon,P}+\varphi\Big)^{p+1}-\Big(V_{\varepsilon,P}\Big)^{p+1}\\ &- (p+1)\Big(V_{\varepsilon,P}\Big)^p\phi-\frac{p(p+1)}{2}\Big(V_{\varepsilon,P}\Big)^{p-1}\varphi^2\Big]drds. \end{aligned}\tag{4.3}$$

We will prove in Lemma 4.1 that $l_{\varepsilon,P}(\varphi)$ is a bounded linear functional in $E_{\varepsilon,P}$. Hence by the Riesz representation theorem, there exists $l_{\varepsilon,P}\in E_{\varepsilon,P}$ such that

$$\langle l_{\varepsilon,P},\varphi\rangle_\varepsilon=l_{\varepsilon,P}(\varphi)\ \forall\,\varphi\in E_{\varepsilon,P}.$$

In Lemma 4.2 we will prove that $Q_{\varepsilon,P}(\varphi,\eta)$ is a bounded linear operator from $E_{\varepsilon,P}$ to $E_{\varepsilon,P}$ such that

$$\langle Q_{\varepsilon,P}\varphi,\eta\rangle_\varepsilon=Q_{\varepsilon,P}(\varphi,\eta)\ \forall\,\varphi,\eta\in E_{\varepsilon,P}.$$

Thus finding a critical point of $J_\varepsilon(\varphi)$ is equivalent to solving the problem in $E_{\varepsilon,P}$:

$$l_{\varepsilon,P} + Q_{\varepsilon,P}\varphi + R'_\varepsilon(\varphi) = 0. \tag{4.4}$$

We will prove in Lemma 4.3 that the operator $Q_{\varepsilon,P}$ is invertible in $E_{\varepsilon,P}$. In Lemma 4.5, we will prove that if φ belongs to a suitable set, $R'_\varepsilon(\varphi)$ is a small perturbation term in (4.4). Thus we can use the contraction mapping theorem to prove that (4.4) has a unique solution for each fixed $P \in \Lambda_{\varepsilon,D}$.

Lemma 4.1. *The functional $l_{\varepsilon,P} : H_0^1(\mathcal{D}) \to \mathbb{R}$ defined in (4.1) is a bounded linear functional. Moreover, we have*

$$\|l_{\varepsilon,P}\|_\varepsilon = O(\varepsilon^2).$$

Proof. We have $l_{\varepsilon,P}$

$$\begin{aligned}
l_{\varepsilon,P}(\varphi) &= \int_{\mathcal{D}} r^{M-1}s^{K-1}S_\varepsilon[V_{\varepsilon,P}]\varphi drds \\
&= \int_{\mathcal{D}} r^{M-1}s^{K-1}\Big[\varepsilon^2\Delta_{(r,s)}V_{\varepsilon,P} + \varepsilon^2\frac{(M-1)}{r}V_{\varepsilon,P,r} + \varepsilon^2\frac{(K-1)}{s}V_{\varepsilon,P,s} - V_{\varepsilon,P} + f(V_{\varepsilon,P})\Big]\varphi \\
&= \int_{\mathcal{D}} r^{M-1}s^{K-1}\Big[\varepsilon^2\Delta_{(r,s)}\eta PU_{\varepsilon,P} + \varepsilon^2\frac{(M-1)}{r}(\eta PU_{\varepsilon,P})_r + \varepsilon^2\frac{(K-1)}{s}(\eta PU_{\varepsilon,P})_s \\
&- \eta PU_{\varepsilon,P} + f(\eta PU_{\varepsilon,P})\Big]\varphi \\
&= \int_{\mathcal{D}} \eta r^{M-1}s^{K-1}\Big[\varepsilon^2\Delta_{(r,s)}PU_{\varepsilon,P} + \varepsilon^2\frac{(M-1)}{r}PU_{\varepsilon,P,r} + \varepsilon^2\frac{(K-1)}{s}PU_{\varepsilon,P,s} \\
&- PU_{\varepsilon,P} + f(PU_{\varepsilon,P})\Big]\varphi + \varepsilon^2\int_{\mathcal{D}} r^{M-1}s^{K-1}[PU_{\varepsilon,P}\Delta_{(r,s)}\eta + \nabla PU_{\varepsilon,P}\nabla\eta]\varphi \\
&+ \int_{\mathcal{D}} r^{M-1}s^{K-1}(\eta-\eta^p)PU^p_{\varepsilon,P}\varphi \\
&= \varepsilon^2\int_{\mathcal{D}} r^{M-1}s^{K-1}\Big[\frac{(M-1)}{r}PU_{\varepsilon,P,r} + \frac{(K-1)}{s}PU_{\varepsilon,P,s}\Big]\varphi \\
&+ \varepsilon^2\int_{\mathcal{D}} r^{M-1}s^{K-1}\Big[\eta_r\frac{(M-1)}{r}PU_{\varepsilon,P,r} + \eta_s\frac{(K-1)}{s}PU_{\varepsilon,P}\Big]\varphi \\
&+ \int_{\mathcal{D}} \eta r^{M-1}s^{K-1}\Big[f(PU_{\varepsilon,P}) - f(U_{\varepsilon,P})\Big]\varphi \\
&= \varepsilon^2\int_{\mathcal{D}} \eta r^{M-1}s^{K-1}\Big[\frac{(M-1)}{r}PU_{\varepsilon,P,r} + \frac{(K-1)}{s}PU_{\varepsilon,P,s}\Big]\varphi drds \\
&+ \int_{\mathcal{D}} \eta r^{M-1}s^{K-1}\Big[f(PU_{\varepsilon,P}) - f(U_{\varepsilon,P})\Big]\varphi \\
&+ \int_{\mathcal{D}} r^{M-1}s^{K-1}(\eta-\eta^p)PU^p_{\varepsilon,P}\varphi drds \\
&+ \varepsilon^2\int_{\mathcal{D}} r^{M-1}s^{K-1}[PU_{\varepsilon,P}\Delta_{(r,s)}\eta + \nabla PU_{\varepsilon,P}\nabla\eta]\varphi drds.
\end{aligned}$$

In order to estimate all the terms we decompose the domain into $\mathcal{D} = (\mathcal{D}\backslash B_{2d}(P)) \cup (B_{2d}(P) \setminus B_d(P)) \cup B_d(P)$. We obtain

$$\begin{aligned} \int_{\mathcal{D}} \eta r^{M-1} s^{K-1}\Big[f(PU_{\varepsilon,P}) - f(U_{\varepsilon,P})\Big]\varphi dx &= \int_{\mathcal{D}} r^{M-1} s^{K-1}\Big[f(PU_{\varepsilon,P}) - f(U_{\varepsilon,P})\Big]\varphi dx \\ &+ \int_{\mathcal{D}} (1-\eta) r^{M-1} s^{K-1}\Big[f(PU_{\varepsilon,P}) - f(U_{\varepsilon,P})\Big]\varphi dx \\ &= I_1 + I_2. \end{aligned}$$

From I_1, we obtain

$$\begin{aligned} I_1 &\leq \int_{B_d(P)} \Big(U_{\varepsilon,P}\Big)^{p-1} v_\varepsilon \varphi dx + \int_{B_{2d}(P)\backslash B_d(P)} \Big(U_{\varepsilon,P}\Big)^{p-1} v_\varepsilon \varphi dx \\ &+ \int_{\mathcal{D}\backslash B_{2d}(P)} \Big(U_{\varepsilon,P}\Big)^{p-1} v_\varepsilon \varphi dx \\ &\leq C\varepsilon^2 \bigg(\int_{B_d(P)} |\varphi|^2 r^{M-1} s^{k-1} drds\bigg)^{\frac{1}{2}} + C\varepsilon^{2+k}\|\phi\|_\varepsilon + o(1)\varepsilon^{2+k}\|\phi\|_\varepsilon \\ &= O(\varepsilon^2)\|\varphi\|_\varepsilon. \end{aligned}$$

Furthermore,

$$I_2 \leq \int_{B_{2d}(P)\backslash B_d(P)} \Big(PU_{\varepsilon,P}\Big)^{p-1} v_\varepsilon \varphi = O(\varepsilon^2)\|\varphi\|_\varepsilon.$$

Also it is easy to check using the decay estimates in (2.15), all the other terms are of order $\varepsilon^2\|\varphi\|_\varepsilon$. Hence we obtain

$$|l_{\varepsilon,P}(\varphi)| \quad = \quad O(\varepsilon^2)\|\varphi\|_\varepsilon.$$

and as a result

$$\|l_{\varepsilon,P}\|_\varepsilon = O(\varepsilon^2).$$

□

Lemma 4.2. *The bilinear form $Q_{\varepsilon,P}(\varphi,\eta)$ defined in (4.2) is a bounded linear. Furthermore,*

$$|Q_{\varepsilon,P}(\varphi,\eta)| \leq C\|\varphi\|_\varepsilon\|\eta\|_\varepsilon$$

where C is independent of ε.

Proof. Using the Hölder's inequality, there exists $C > 0$, such that

$$\int_{\mathcal{D}} r^{M-1} s^{K-1} V_{\varepsilon,P}^{p-1} \varphi\eta \, drds \leq C \int_{\mathcal{D}} r^{M-1} s^{K-1} |\varphi||\eta| \leq C\|\varphi\|_\varepsilon\|\eta\|_\varepsilon$$

and

$$\bigg|\int_{\mathcal{D}} r^{M-1} s^{K-1}[\varepsilon^2\nabla\varphi\nabla\eta + \varphi\eta] drds\bigg| \leq C\|\varphi\|_\varepsilon\|\eta\|_\varepsilon.$$

□

Lemma 4.3. *There exists $\rho > 0$ independent of ε, such that*

$$\|Q_{\varepsilon,P}\varphi\|_\varepsilon \geq \rho\|\varphi\|_\varepsilon \ \forall\, \varphi \in E_{\varepsilon,P},\ P \in \Lambda_{\varepsilon,P}.$$

Proof. Suppose there exists a sequence $\varepsilon_n \to 0$, $\varphi_n \in E_{\varepsilon_n,P}$, $P \in \Lambda_{\varepsilon,P}$ such that $\|\varphi_n\|_{\varepsilon_n} = \varepsilon_n$ and

$$\|Q_{\varepsilon_n}\varphi_n\|_{\varepsilon_n} = o(\varepsilon_n).$$

Let $\tilde{\varphi}_{i,n} = \varphi_n(\varepsilon_n z + P)$ and $\mathcal{D}_n = \{y : \varepsilon_n z + P \in \mathcal{D}\}$ such that

(4.5)
$$\int_{\mathcal{D}_n} r^{M-1}s^{K-1}[|\nabla\tilde{\varphi}_{i,n}|^2 + \tilde{\varphi}_{i,n}|^2] = \varepsilon_n^{-2}\int_{\mathcal{D}} r^{M-1}s^{K-1}[\varepsilon^2|\nabla\varphi_{i,n}|^2 + \varphi_{i,n}|^2] = 1.$$

Hence there exists $\varphi \in H^1(\mathbb{R}^2)$ such that $\tilde{\varphi}_n \rightharpoonup \varphi \in H^1(\mathbb{R}^2)$ and hence $\tilde{\varphi}_n \to \varphi \in L^2_{loc}(\mathbb{R}^2)$. We claim that

$$\Delta_{(r,s)}\varphi - \varphi + pU^{p-1}\varphi = 0 \quad \text{in } \mathbb{R}^2$$

that is for all $\eta \in C_0^\infty(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} r^{M-1}s^{K-1}\nabla\varphi\nabla\eta + \int_{\mathbb{R}^2} r^{M-1}s^{K-1}\varphi\eta = p\int_{\mathbb{R}^2} r^{M-1}s^{K-1}U^{p-1}\varphi\eta. \tag{4.6}$$

Now

$$\begin{aligned}\int_{\mathcal{D}} r^{M-1}s^{K-1}\Big[\varepsilon^2 D\varphi_\varepsilon D\eta + \varphi_\varepsilon\eta - pV_{\varepsilon,P}^{p-1}\varphi_\varepsilon\eta\Big] &= \langle Q_{\varepsilon_n,P}\varphi_n,\eta\rangle_\varepsilon \\ &= o(\varepsilon_n)\|\eta\|_{\varepsilon_n}\end{aligned}$$

which implies

$$\int_{\mathcal{D}_\varepsilon} r^{M-1}s^{K-1}\Big[\nabla\tilde{\varphi}_\varepsilon\nabla\tilde{\eta} + \tilde{\varphi}_\varepsilon\tilde{\eta} - p\tilde{V}_{\varepsilon,P}^{p-1}\tilde{\varphi}_\varepsilon\tilde{\eta}\Big] = o(1)\|\tilde{\eta}\|,$$

where

$$\tilde{V}_{\varepsilon_n,P_n} = V_{\varepsilon_n,P_n}(\varepsilon_n y + P),$$

$$\|\tilde{\eta}\|^2 = \int_{\mathcal{D}_n} r^{M-1}s^{K-1}\Big[|\nabla\tilde{\eta}|^2 + |\tilde{\eta}|^2\Big],$$

$$\begin{aligned}\tilde{E}_{\varepsilon_n,P} = \Big\{\tilde{\eta} : &\int_{\mathcal{D}_n} r^{M-1}s^{K-1}\nabla\tilde{\eta}\nabla\tilde{W}_{n,r} + r^{M-1}s^{K-1}\tilde{\eta}\tilde{W}_{n,r} \\ &= 0 = \int_{\mathcal{D}_n} r^{M-1}s^{K-1}\nabla\tilde{\eta}\nabla\tilde{W}_{n,s} + r^{M-1}s^{K-1}\tilde{\eta}\tilde{W}_{n,s}\Big\},\end{aligned}$$

and $\tilde{W}_{n,r} = \varepsilon_n\frac{\partial\tilde{V}_{\varepsilon_n}(\varepsilon_n y+P_n)}{\partial r}$, $\tilde{W}_{n,s} = \varepsilon_n\frac{\partial\tilde{V}_{\varepsilon_n}(\varepsilon_n y+P)}{\partial s}$. Let $\eta \in C_0^\infty(\mathbb{R}^2)$. Then we can choose $a_1, a_2 \in \mathbb{R}$ such that

$$\tilde{\eta}_n = \eta - [a_1\tilde{W}_{n,r} + a_2\tilde{W}_{n,s}].$$

Note that $\tilde{W}_{n,r}$ satisfies the problem

$$\begin{cases} -\Delta_{(r,s)}\tilde{W}_{n,r} + \tilde{W}_{n,r} = p\eta U^{p-1}(y)\dfrac{\partial U}{\partial r} + \Phi_n(y) & \text{in } \mathcal{D}_n \\ \tilde{W}_{n,r} = 0 & \text{on } \mathcal{D}_{1,n}\cup\mathcal{D}_{2,n} \\ \dfrac{\partial\tilde{W}_{n,r}}{\partial\nu} = 0 & \text{on } \mathcal{D}_{3,n}\cup\mathcal{D}_{4,n}\end{cases} \tag{4.7}$$

where $\Phi_n(y) = \varepsilon_n\frac{\partial\eta}{\partial r}U^p + \frac{\partial}{\partial r}\Big[\nabla\eta\nabla\tilde{P}U_{\varepsilon,P} + \Delta\eta\tilde{P}U_{\varepsilon,P}\Big]$.

Then we claim that $\tilde{W}_{n,r}$ is bounded in $H_0^1(\mathcal{D}_n)$. Using the Hölder's inequality, we have

$$
\begin{aligned}
\int_{\mathcal{D}_n} r^{M-1}s^{N-1}[\nabla\tilde{W}_{n,r}|^2+\tilde{W}_{n,r}^2] &= p\int_{\mathcal{D}_n} r^{M-1}s^{N-1}\eta U^{p-1}\frac{\partial U}{\partial r}\tilde{W}_{n,r}\\
&+ \int_{\mathcal{D}_n} r^{M-1}s^{N-1}\Phi_n W_{n,r}\\
&\leq C\Big(\int_{\mathcal{D}_n} r^{M-1}s^{k-1}\tilde{W}_{n,r}^2\Big)^{\frac{1}{2}}\\
&\leq C\Big(\int_{\mathcal{D}_n} r^{M-1}s^{N-1}[\nabla\tilde{W}_{n,r}|^2+\tilde{W}_{n,r}^2]\Big)^{\frac{1}{2}}. \qquad (4.8)
\end{aligned}
$$

Hence $\int_{\mathcal{D}_n} r^{M-1}s^{N-1}\Big[|\nabla\tilde{W}_{n,r}|^2+\tilde{W}_{n,r}^2\Big]$ is uniformly bounded and as a result there exists W_r such that

$$\tilde{W}_{n,r} \rightharpoonup W_r \text{ in } H^1(\mathbb{R}^2)$$

up to a subsequence. Hence

$$\tilde{W}_{n,r} \to W_r \text{ in } L^2_{loc}.$$

Note that W_r satisfies the problem,

$$
(4.9)\quad \begin{cases} -\Delta_{(r,s)}W_r + W_r = pU^{p-1}\dfrac{\partial U}{\partial r} & \text{in } \mathbb{R}^2\\ \displaystyle\int_{\mathbb{R}^2} r^{M-1}s^{K-1}[|\nabla W_r|^2+|W_r|^2] = p\int_{\mathbb{R}^2} r^{M-1}s^{K-1}U^{p-1}\frac{\partial U}{\partial r}W_r & . \end{cases}
$$

We claim that $\tilde{W}_{n,r} \to W_r$ in $H^1(\mathbb{R}^2)$. First note that

$$
\begin{aligned}
\int_{\mathcal{D}_n} r^{M-1}s^{K-1}[|\nabla\tilde{W}_{n,r}|^2+|\tilde{W}_{n,r}|^2] &= p\int_{\mathcal{D}_n} r^{M-1}s^{K-1}U^{p-1}\frac{\partial U}{\partial r}\tilde{W}_{n,r}\\
&+ \int_{\mathcal{D}_n} r^{M-1}s^{K-1}\Phi_n\tilde{W}_{n,r}\\
&\to p\int_{\mathbb{R}^2} r^{M-1}s^{K-1}U^{p-1}\frac{\partial U}{\partial r}W_r\\
&= \int_{\mathbb{R}^2} r^{M-1}s^{K-1}[|\nabla W_r|^2+|W_r|^2]drds. \qquad (4.10)
\end{aligned}
$$

Here we have used that $\tilde{W}_{n,r}$ converges weakly in L^2. Hence $\tilde{W}_{n,r} \to W_r = \frac{\partial U}{\partial r}$ in H^1 strongly. Similarly, we can show that $\tilde{W}_{n,s} \to W_s = \frac{\partial U}{\partial s}$ in H^1 strongly. Now if we plug the value η_n in (4.7) we obtain and letting $n \to \infty$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^2} r^{M-1}s^{K-1}\Big[\nabla\varphi\nabla\eta - pU^{p-1}\varphi\eta+\varphi\eta\Big]\\
=\;& a_1\Big(\int_{\mathbb{R}^2} r^{M-1}s^{K-1}\Big[\nabla\varphi\nabla\frac{\partial U}{\partial r}+\varphi\frac{\partial U}{\partial r} - pU^{p-1}\varphi\frac{\partial U}{\partial r}\Big]\Big)\\
+\;& a_2\Big(\int_{\mathbb{R}^2} r^{M-1}s^{K-1}\Big[\nabla\varphi\nabla\frac{\partial U}{\partial s}+\varphi\frac{\partial U}{\partial s} - pU^{p-1}\varphi\frac{\partial U}{\partial s}\Big]\Big).
\end{aligned}
$$

Using the non-degeneracy condition we obtain

$$\int_{\mathbb{R}^N} r^{M-1}s^{K-1}\Big[\nabla\varphi\nabla\eta+\varphi\eta-pU^{p-1}\varphi\eta\Big]=0.$$

Hence we have (4.6).
Since $\varphi\in H^1(\mathbb{R}^2)$, it follows by non-degeneracy

$$\varphi=b_1\frac{\partial U}{\partial r}+b_2\frac{\partial U}{\partial s}.$$

Since $\tilde\varphi_n\in\tilde E_{\varepsilon_n,P}$, letting $n\to\infty$ in (4.7), we have

$$\int_{\mathbb{R}^2} r^{M-1}s^{K-1}\nabla\varphi\nabla\frac{\partial U}{\partial r}=0$$

$$\int_{\mathbb{R}^2} r^{M-1}s^{K-1}\nabla\varphi\nabla\frac{\partial U}{\partial s}=0,$$

which implies $b_1=b_2=0$. Hence $\varphi=0$ and for any $R>0$ we have

$$\int_{B_{\varepsilon_n R}(P)} r^{M-1}s^{K-1}\varphi_n^2\,drds=o(\varepsilon_n^2).$$

Hence

$$\begin{aligned}o(\varepsilon_n^2)\geq\langle Q_{\varepsilon_n,P}(\varphi_n),\varphi_n\rangle_{\varepsilon_n} &\geq \|\varphi_n\|^2_{\varepsilon_n}-p\int_{\mathcal{D}}(V_{\varepsilon_n,P})^{p-1}\varphi_n^2\\ &\geq \varepsilon_n^2-o(1)\varepsilon_n^2\end{aligned}$$

which implies a contradiction. □

Lemma 4.4. *Let $R_\varepsilon(\varphi)$ be the functional defined by (4.3). Let $\varphi\in H_0^1(\mathcal{D})$, then*

$$|R_\varepsilon(\varphi)| \leq o(1)\|\varphi\|^2_\varepsilon+o(1)\varepsilon^{\frac{(p-1)k}{2}}\|\varphi\|^2_\varepsilon=\varepsilon^\tau\|\varphi\|^2_\varepsilon \tag{4.11}$$

and

$$\|R'_\varepsilon(\varphi)\|_\varepsilon \leq o(1)\|\varphi\|_\varepsilon+o(1)\varepsilon^{\frac{(p-1)k}{2}}\|\varphi\|_\varepsilon=\varepsilon^\tau\|\varphi\|_\varepsilon. \tag{4.12}$$

for some $\tau>0$ small.

Proof. We have

$$\begin{aligned}|R_\varepsilon(\varphi)| &\leq o\Big(\int_{\mathcal{D}} r^{M-1}s^{K-1}V^{p-1}_{\varepsilon,P}\varphi^2\Big)\\ &\leq o(1)\int_{B_d(P)} r^{M-1}s^{K-1}V^{p-1}_{\varepsilon,P}\varphi^2+o\Big(\int_{\mathcal{D}\setminus B_d(P)}V^{p-1}_{\varepsilon,P}\varphi^2\Big)\end{aligned}$$

Moreover, by the exponential decay of $V_{\varepsilon,P}$ we obtain,

$$o\Big(\int_{\mathcal{D}\setminus B_d(P)} r^{M-1}s^{K-1}V^{p-1}_{\varepsilon,P}\varphi^2\Big)\leq Co(1)\varepsilon^{\frac{p-1}{2}k}\int_{\mathcal{D}} r^{M-1}s^{K-1}\varphi^2\leq o(1)\varepsilon^{\frac{p-1}{2}k}\|\varphi\|^2_\varepsilon.$$

The second estimate follows in a similar way. □

Lemma 4.5. *There exists $\varepsilon_0>0$ such that for $\varepsilon\in(0,\varepsilon_0]$, there exists a C^1 map $\varphi_{\varepsilon,P}:E_{\varepsilon,P}\to H$, such that $\varphi_{\varepsilon,P}\in\Lambda_{\varepsilon,D}$ satisfying*

$$\Big\langle I'_\varepsilon\Big(V_{\varepsilon,P}+\varphi_{\varepsilon,P}\Big),\eta\Big\rangle_\varepsilon=0,\ \ \forall\eta\in\Lambda_{\varepsilon,D}.$$

Moreover, we have

$$\|\varphi_{\varepsilon,P}\|_\varepsilon = O(\varepsilon^2).$$

Proof. We have $l_{\varepsilon,P} + Q_{\varepsilon,P}\varphi + R'_\varepsilon(\varphi) = 0$. As $Q^{-1}_{\varepsilon,P}$ exists, the above equation is equivalent to solving

$$Q^{-1}_{\varepsilon,P} l_{\varepsilon,P} + \varphi + Q^{-1}_{\varepsilon,P} R'_\varepsilon(\varphi) = 0.$$

Define

$$\mathcal{G}(\varphi) = -Q^{-1}_{\varepsilon,P} l_{\varepsilon,P} - Q^{-1}_{\varepsilon,P} R'_\varepsilon(\varphi) \ \forall \varphi \in \Lambda_{\varepsilon,D}.$$

Hence the problem is reduced to finding a fixed point of the map $\mathcal{G}$.
For any $\varphi_1 \in \Lambda_\varepsilon$ and $\varphi_2 \in E_\varepsilon$ with $\|\varphi_1\|_\varepsilon \le \varepsilon^{2-\tau}$, $\|\varphi_2\|_\varepsilon \le \varepsilon^{2-\tau}$

$$\|\mathcal{G}(\varphi_1) - \mathcal{G}(\varphi_2)\|_\varepsilon \le C\|R'_\varepsilon(\varphi_1) - R'_\varepsilon(\varphi_2)\|_\varepsilon.$$

From Lemma 4.4, we have

$$\langle R'_\varepsilon(\varphi_1) - R'_\varepsilon(\varphi_2), \eta\rangle_\varepsilon \le o(1)\|\varphi_1 - \varphi_2\|_\varepsilon \|\eta\|_\varepsilon.$$

Hence we have

$$\|R'_\varepsilon(\varphi_1) - R'_\varepsilon(\varphi_2)\|_\varepsilon \le o(1)\|\varphi_1 - \varphi_2\|_\varepsilon.$$

Hence $\mathcal{G}$ is a contraction as

$$\|\mathcal{G}(\varphi_1) - \mathcal{G}(\varphi_2)\|_\varepsilon \le Co(1)\|\varphi_1 - \varphi_2\|_\varepsilon.$$

Also for $\varphi \in E_\varepsilon$ with $\|\varphi\|_\varepsilon \le \varepsilon^{2-\tau}$, and $\tau > 0$ sufficiently small

$$\begin{aligned} \|\mathcal{G}(\varphi)\|_\varepsilon &\le C\|l_{\varepsilon,P}\|_\varepsilon + C\|R'_\varepsilon(\varphi)\|_\varepsilon \\ &\le C\varepsilon^2 + C\varepsilon^{2-\tau+\tau} \\ &\le C\varepsilon^2. \end{aligned} \tag{4.13}$$

Hence

$$\mathcal{G} : \Lambda_{\varepsilon,D} \cap B_{\varepsilon^{2-\tau}}(0) \to \Lambda_{\varepsilon,D} \cap B_{\varepsilon^{2-\tau}}(0)$$

is a contraction map. Hence by the contraction mapping principle, there exists a unique $\varphi \in \Lambda_{\varepsilon,D} \cap B_{\varepsilon^k}(0)$ such that $\varphi_{\varepsilon,P} = \mathcal{G}(\varphi_{\varepsilon,P})$ and

$$\|\varphi_{\varepsilon,P}\|_\varepsilon = \|\mathcal{G}(\varphi_{\varepsilon,P})\|_\varepsilon \le C\varepsilon^2.$$

□

We write $u_\varepsilon = V_{\varepsilon,P} + \varphi_{\varepsilon,P}$. Then we have

$$\begin{aligned} I_\varepsilon(u_\varepsilon) &= I_\varepsilon(V_{\varepsilon,P}) \\ &+ \int_D r^{M-1}s^{K-1}(\varepsilon^2 \nabla V_{\varepsilon,P}\nabla\varphi_\varepsilon - V_{\varepsilon,P}\varphi_\varepsilon + f(V_{\varepsilon,P})\varphi_\varepsilon) drds \\ &+ \frac{1}{2}\bigg(\int_D r^{M-1}s^{K-1}\Big[\varepsilon^2|\nabla\varphi_\varepsilon|^2 - \varphi_\varepsilon^2 + f'(V_{\varepsilon,P})\varphi^2_{\varepsilon,P}\Big] drds\bigg) \\ &- \int_D r^{M-1}s^{K-1}\Big[F(V_{\varepsilon,P} + \varphi_\varepsilon) - F(V_{\varepsilon,P}) - \varepsilon f(V_{\varepsilon,P})\varphi_{\varepsilon,P} - \frac{1}{2}f'(V_{\varepsilon,P})\varphi^2_{\varepsilon,P}\Big] drds \end{aligned}$$

which can be expressed as

$$
\begin{aligned}
I_\varepsilon(u_\varepsilon) &= I_\varepsilon(V_{\varepsilon,P}) \\
&+ \int_{\mathcal{D}} E_\varepsilon(V_{\varepsilon,P})\varphi_{\varepsilon,P} r^{M-1}s^{K-1}drds \\
&+ \frac{1}{2}\bigg(\int_{\mathcal{D}}[\varepsilon^2|\nabla\varphi_\varepsilon|^2dx - f'(V_{\varepsilon,P})\varphi_\varepsilon^2]r^{M-1}s^{K-1}drds\bigg) \\
&- \int_{\mathcal{D}} r^{M-1}s^{K-1}\bigg[F(V_{\varepsilon,P}+\varphi_\varepsilon) - F(V_{\varepsilon,P}) - f(V_{\varepsilon,P})\varphi_\varepsilon - \frac{1}{2}f'(V_{\varepsilon,P})\varphi_\varepsilon^2\bigg]drds \\
&= I_\varepsilon\bigg(V_{\varepsilon,P}\bigg) + O(\|l_{\varepsilon,P}\|_\varepsilon\|\varphi_{\varepsilon,P}\|_\varepsilon + \|\varphi_\varepsilon\|_\varepsilon^2 + R_\varepsilon(\varphi_{\varepsilon,P})) \\
&= I_\varepsilon\bigg(V_{\varepsilon,P}\bigg) + O(\varepsilon^4). \qquad (4.14)
\end{aligned}
$$

5. The reduced problem: min-max procedure

Proof of Theorem 1.1. Let $\mathcal{G}_\varepsilon(P) = \mathcal{G}_\varepsilon(d,\theta) = I_\varepsilon(u_\varepsilon)$. Consider the problem

$$\min_{d\in\Lambda_{\varepsilon,P}} \max_{\theta_0-\delta\le\theta\le\theta_0+\delta} \mathcal{G}_\varepsilon(d,\theta).$$

To prove that $\mathcal{G}_\varepsilon(P) = I_\varepsilon\bigg(V_{\varepsilon,P}+\varphi_{\varepsilon,P}\bigg)$ is a solution of (1.1), we need to prove that P is a critical point of $\mathcal{G}_\varepsilon$, in other words we are required to show that P is a interior point of $\Lambda_{\varepsilon,D}$.
For any $P\in\Lambda_{\varepsilon,P}$, from Lemma 4.3 we obtain

$$
\begin{aligned}
\mathcal{G}_\varepsilon(P) &= I_\varepsilon\bigg(V_{\varepsilon,P}\bigg) + O(\|l_{\varepsilon,P}\|_\varepsilon\|\varphi_{\varepsilon,P}\|_\varepsilon + \|\varphi_\varepsilon\|_\varepsilon^2 + R_\varepsilon(\varphi_{\varepsilon,P})) \\
&= I_\varepsilon\bigg(V_{\varepsilon,P}\bigg) + o(1)\varepsilon^{k+2} \\
&= \varepsilon^2\gamma P_1^{M-1}P_2^{K-1} + \varepsilon^2\gamma_1 P_1^{M-1}P_2^{K-1}U\bigg(\frac{2d(P,\mathcal{D}_1)}{\varepsilon}\bigg) + o(\varepsilon^{k+2}). \qquad (5.1)
\end{aligned}
$$

We have the expansion

$$
\begin{aligned}
\mathcal{G}_\varepsilon(d,\theta) &= \gamma\varepsilon^2[a^{M+K-2} + a^{M+K-1}d + \gamma^{-1}\gamma_1 a^{M+K-2}U\bigg(\frac{2d(P,\mathcal{D}_1)}{\varepsilon}\bigg) \\
&+ O(d^2)]\cos^{M-1}\theta\sin^{K-1}\theta + o(\varepsilon^{2+k}).
\end{aligned}
$$

It is clear that the maximum is attained at some interior point of $\theta'\in(\theta_0-\delta,\theta_0+\delta)$. Now we prove that for that θ' the minimum is attained at a critical point of $\Lambda_{\varepsilon,P}$.

Let $P\in\Lambda_{\varepsilon,P}$, be a point of minimum of $\mathcal{G}_\varepsilon(d,\theta')$, then we obtain

$$\mathcal{G}_\varepsilon(d,\theta') = \gamma\varepsilon^2[a^{M+K-2} + a^{M+K-1}d + O(d^2)]\cos^{M-1}\theta'\sin^{K-1}\theta' + O(\varepsilon^{2+k}).$$

Choose $\tilde{P}$ such that the $d' = d(\tilde{P},\partial\mathcal{D}_1) \ge \frac{k}{2}\varepsilon|\ln\varepsilon|$. Then $\tilde{P}\in\Lambda_{\varepsilon,P}$.
But by definition, we have

$$\mathcal{G}_\varepsilon(d,\theta') \le \mathcal{G}_\varepsilon(d',\theta'). \qquad (5.2)$$

From this we obtain

$$\begin{aligned} & \gamma[a^{M+K-2}+a^{M+K-1}d+O(d^2)]\cos^{M-1}\theta'\sin^{K-1}\theta'+O(\varepsilon^k) \\ \leq\ & \gamma\left[a^{M+K-2}+a^{M+K-1}d'+\gamma_1\gamma^{-1}e^{\frac{d'}{\varepsilon}}+O(d^2)\right]\cos^{M-1}\theta'\sin^{K-1}\theta' \\ +\ & o(\varepsilon^k) \end{aligned}$$

Hence this implies that $d\sim\varepsilon|\ln\varepsilon|$. Hence $d\to 0$. This finishes the proof. □

6. The reduced problem: max-max procedure

Proof of Theorem 1.2. Here we obtain the critical point using a max-max procedure. The projection in the Neumann case is just $Q_{\varepsilon,P}$. Hence the reduced problem

$$\mathcal{R}_\varepsilon(P)=\varepsilon^2\gamma P_1^{M-1}P_2^{K-1}-\varepsilon^2\gamma_1P_1^{M-1}P_2^{K-1}U\left(\frac{2d(P,\mathcal{D}_2)}{\varepsilon}\right)+o(\varepsilon^{k+2}). \tag{6.1}$$

Consider

$$\max_{d\in\Lambda_{\varepsilon,N}}\ \max_{\theta_0-\delta\leq\theta\leq\theta_0+\delta}\mathcal{R}_\varepsilon(d,\theta). \tag{6.2}$$

We have the expansion

$$\begin{aligned} \mathcal{R}_\varepsilon(d,\theta) &= \gamma\varepsilon^2[a^{M+K-2}+a^{M+K-1}d-\gamma^{-1}\gamma_1a^{M+K-2}U\left(\frac{2d(P,\mathcal{D}_2)}{\varepsilon}\right) \\ &+ O(d^2)]\cos^{M-1}\theta\sin^{K-1}\theta+o(\varepsilon^{2+k}). \end{aligned}$$

It is clear that the maximum in θ is attained at some interior point of $\theta'\in(\theta_0-\delta,\theta_0+\delta)$. Now we prove that for that θ' the minimum is attained at a critical point of $\Lambda_{\varepsilon,N}$.

Let $P\in\Lambda_{\varepsilon,N}$, be a point of maximum of $\mathcal{R}_\varepsilon(d,\theta')$, then we obtain

$$\mathcal{R}_\varepsilon(d,\theta') = \gamma\varepsilon^2[a^{M+K-2}+a^{M+K-1}d+O(d^2)]\cos^{M-1}\theta'\sin^{K-1}\theta'+o(\varepsilon^{2+k}).$$

Choose $\tilde{P}$ such that the $d'=d(\tilde{P},\partial\mathcal{D}_1)\geq\frac{k}{2}\varepsilon|\ln\varepsilon|$. Then $\tilde{P}\in\Lambda_{\varepsilon,P}$.
But by definition, we have

$$\mathcal{R}_\varepsilon(d',\theta')\leq\mathcal{R}_\varepsilon(d,\theta'). \tag{6.3}$$

This implies

$$\begin{aligned} & \gamma[a^{M+K-2}+a^{M+K-1}d+O(d^2)]\cos^{M-1}\theta'\sin^{K-1}\theta'+o(\varepsilon^k) \\ \geq\ & \gamma\left[a^{M+K-2}+a^{M+K-1}d'-\gamma_1\gamma^{-1}e^{\frac{d'}{\varepsilon}}+O(d^2)\right]\cos^{M-1}\theta'\sin^{K-1}\theta' \\ +\ & o(\varepsilon^k) \end{aligned}$$

Hence $d\sim\varepsilon|\ln\varepsilon|$. Hence $d\to 0$. Theorem 1.2 is proved. □

References

[1] A. Ambrosetti, A. Malchiodi, W. Ni; Singularly perturbed elliptic equations with symmetry: existence of solutions concentrating on spheres. I. *Comm. Math. Phys.* 235 (2003), no. 3, 427–466.

[2] A. Ambrosetti, A. Malchiodi, W. Ni; Singularly perturbed elliptic equations with symmetry: existence of solutions concentrating on spheres. II. *Indiana Univ. Math. J.* 53 (2004), no. 2, 297–329.

[3] M. Del Pino, P. Felmer; Spike-layered solutions of singularly perturbed elliptic problems in a degenerate setting. *Indiana Univ. Math. J.* **48** (1999), no. 3, 883–898.

[4] M. Del Pino, M. Kowalczyk, J. Wei; Concentraton on curves for nonlinear Schrödinger equations. *Comm. Pure Appl. Math.* 60 (2007), no. 1, 113–146.
[5] M. Esteban, P. Lions; Existence and nonexistence results for semilinear elliptic problems in unbounded domains. *Proc. Roy. Soc. Edinburgh Sect.* A 93 (1982/83), 1–14.
[6] B. Gidas, W. Ni, L. Nirenberg; Symmetry and related properties via the maximum principle. *Comm. Math. Phys.* 68 (1979), no. 3, 209–243.
[7] F. Pacella, P.N.Srikanth; A reduction method for semilinear elliptic equations and solutions concentrating on sphere. (Preprint 2012).
[8] B. Ruf, P. Srikanth; Singularly perturbed elliptic equations with solutions concentrating on a 1-dimensional orbit. *J. Eur. Math. Soc.* 12 (2010), no. 2, 413–427.
[9] P. Esposito, G.Mancini, S.Santra, P. Srikanth; Asymptotic behavior of radial solutions for a semilinear elliptic problem on an annulus through Morse index. *J. Diff. Equations* 239 (2007), no. 1, 1-15.
[10] F. Lin, W. M. Ni, J. Wei; On the number of interior peak solutions for a singularly perturbed Neumann problem. *Comm. Pure Appl. Math.* 60 (2007), no. 2, 252–281.
[11] W. M. Ni, I. Takagi; On the shape of least-energy solutions to a semilinear Neumann problem. *Comm. Pure Appl. Math.* **4** (1991), no. 7, 819–851.
[12] W. M. Ni, I. Takagi; Locating the peaks of least-energy solutions to a semilinear Neumann problem. *Duke Math. J.* 70 (1993), no. 2, 247–281
[13] W. M. Ni, J. Wei; On the location and profile of spike-layer solutions to singularly perturbed semilinear Dirichlet problems. *Comm. Pure Appl. Math.* **48** (1995), no. 7, 731–768.

Sanjiban Santra, School of Mathematics and Statistics, The University of Sydney, NSW 2006, Australia.
E-mail address: sanjiban.santra@sydney.edu.au

Juncheng Wei, Department of Mathematics, The University of British Columbia, Vancouver.
E-mail address: jcwei@math.ubc.ca