Sample Solutions of Assignment 9 for MAT3270B

1. For the following ODEs, find the eigenvalues and eigenvectors, and classify the critical point (0,0) type and determine whether it is stable, asymptotically stable, or unstable.

a).
\[
x' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} x
\]

b).
\[
x' = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} x
\]

c).
\[
x' = \begin{pmatrix} 2 & -5 \\ 0 & 2 \end{pmatrix} x
\]

d).
\[
x' = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix} x
\]

**Answer:**

a).
\[
det(A - \lambda I) = \begin{vmatrix} 3 - \lambda & -2 \\ 2 & -2 - \lambda \end{vmatrix} = (\lambda - 2)(\lambda + 1) = 0
\]
The eigenvalues are
\[
\lambda_1 = 2, \quad \lambda_2 = -1
\]
The eigenvectors are
\[
X_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}
\]
and
\[
X_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}
\]
(0,0) is an unstable critical point of the system.
b). 
\[
\text{det}(A - \lambda I) = \begin{vmatrix} 1 - \lambda & -5 \\ 1 & -3 - \lambda \end{vmatrix} = \lambda^2 + 2\lambda + 2 = 0
\]

The eigenvalues are 
\[
\lambda_1 = -1 + i, \quad \lambda_2 = -1 - i
\]
The eigenvectors are 
\[
X_1 = \begin{pmatrix} 5 \\ -i \end{pmatrix}
\]
and
\[
X_2 = \begin{pmatrix} 5 \\ i \end{pmatrix}
\]
(0,0) is an asymptotically stable critical point of the system.

c). 
\[
\text{det}(A - \lambda I) = \begin{vmatrix} 2 - \lambda & -5 \\ 0 & 2 - \lambda \end{vmatrix} = (\lambda - 2)^2 = 0
\]
The eigenvalues are 
\[
\lambda = 2
\]
The eigenvectors are 
\[
X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]
and
\[
X_2 = t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]
(0,0) is an unstable critical point of the system.

d). 
\[
\text{det}(A - \lambda I) = \begin{vmatrix} 1 - \lambda & -2 \\ 1 & -1 - \lambda \end{vmatrix} = \lambda^2 + 1 = 0
\]
The eigenvalues are 
\[
\lambda_1 = i, \quad \lambda_2 = -i
\]
The eigenvectors are 
\[
X_1 = \begin{pmatrix} 2 \\ 1 - i \end{pmatrix}
\]
and

\[ X_2 = \begin{pmatrix} -2 \\ 1 - i \end{pmatrix} \]

(0,0) is a stable critical point of the system.

2. Transform the following problems into the form

\[ x' = \begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix} x \]

a).

\[ x' = \begin{pmatrix} 3 & -2 \\ 4 & -2 \end{pmatrix} x \]

b).

\[ x' = \begin{pmatrix} 2 & -2 \\ 3 & -2 \end{pmatrix} x \]

**Answer:** a).

\[ \det(A - \lambda I) = \begin{vmatrix} 3 - \lambda & -2 \\ 4 & -2 - \lambda \end{vmatrix} = \lambda^2 - \lambda + 2 = 0 \]

The eigenvalues are

\[ \lambda_1 = \frac{1}{2} + \frac{\sqrt{7}}{2} i, \quad \lambda_2 = \frac{1}{2} - \frac{\sqrt{7}}{2} i \]

The problem can be transformed to

\[ x' = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{7}}{2} \\ -\frac{\sqrt{7}}{2} & \frac{1}{2} \end{pmatrix} x \]

b).

\[ \det(A - \lambda I) = \begin{vmatrix} 2 - \lambda & -2 \\ 3 & -2 - \lambda \end{vmatrix} = \lambda^2 - \lambda + 2 = 0 \]
The eigenvalues are
\[ \lambda_1 = \frac{1}{2} + \frac{\sqrt{7}}{2} i, \quad \lambda_2 = \frac{1}{2} - \frac{\sqrt{7}}{2} i \]

The problem can be transformed to
\[ x' = \begin{pmatrix} 1 + \frac{\sqrt{7}}{2} & \frac{\sqrt{7}}{2} \\ -\frac{\sqrt{7}}{2} & \frac{1}{2} \end{pmatrix} x \]

3. Determine the critical points for each of the following systems

a. \( \frac{dx}{dt} = x(2x - 3y), \quad \frac{dy}{dt} = y(1 - x - y) \)
b. \( \frac{dx}{dt} = x - x^2 - xy, \quad \frac{dy}{dt} = \frac{1}{2}y - \frac{1}{4}y^2 - \frac{3}{4}xy \)
c. \( \frac{dx}{dt} = y, \quad \frac{dy}{dt} = \mu(1 - x^2)y - x, \quad \mu > 0 \)
d. \( \frac{dx}{dt} = y + x(1 - x^2 - y^2), \quad \frac{dy}{dt} = -x + y(1 - x^2 - y^2) \)

**Answer:**

a). The critical points of the system are (0,0), (0,1) and \( \left( \frac{3}{2}, \frac{2}{3} \right) \).
b). The critical points of the system are (0,0), (0,2) and (1,0).
c). The critical points of the system is (0,0).
d). The critical points of the system is (0,0).

4. Find out an equation for the form \( H(x, y) = c \) satisfies for the trajectories:

a. \( \frac{dx}{dt} = -x + y + x^2, \quad \frac{dy}{dt} = y - 2xy \)
b. \( \frac{dx}{dt} = 2x^2y - 3x^2 - 4y, \quad \frac{dy}{dt} = -2xy^2 + 6xy \)
Answer: a). We get
\[ \frac{dy}{dx} = \frac{y - 2xy}{-x + y + x^2} \]
The ODE can be rewritten as
\[ (y - 2xy)dx + (-x + y + x^2)dy = 0 \]
\[ \frac{\partial M(x, y)}{\partial y} = 1 - 2x \]
\[ \frac{\partial N(x, y)}{\partial x} = 1 - 2x \]
So this ODE is exact for \( \frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x} \). Thus there is a \( \psi(x, y) \) such that
\[ \frac{\partial \psi}{\partial x} = M(x, y) = y - 2xy \]
Integrating above equation, we obtain
\[ \psi(x, y) = xy - x^2y + f(y) \]
Setting \( \psi_y = N \) gives
\[ f'(y) = -y \]
then we get
\[ f(y) = -\frac{1}{2}y^2 + c \]
Hence,
\[ xy - x^2y - \frac{1}{2}y^2 = c \]

b). We get
\[ \frac{dy}{dx} = \frac{-2xy^2 + 6xy}{2x^2y - 3x^2 - 4y} \]
The ODE can be rewritten as
\[ (2xy^2 - 6xy)dx + (2x^2y - 3x^2 - 4y)dy = 0 \]
\[ \frac{\partial M(x, y)}{\partial y} = 4xy - 6x \]
\[ \frac{\partial N(x, y)}{\partial x} = 4xy - 6x \]
So this ODE is exact for \( \frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x} \). Thus there is a \( \psi(x, y) \) such that

\[
\frac{\partial \psi}{\partial x} = M(x, y) = 2xy^2 - 6xy
\]

Integrating above equation, we obtain

\[
\psi(x, y) = x^2y^2 - 3x^2y + f(y)
\]

Setting \( \psi_y = N \) gives

\[
f'(y) = -4y
\]

then we get

\[
f(y) = -2y^2 + c
\]

Hence,

\[
x^2y^2 - 3x^2y - 2y^2 = c
\]

5. Show that the trajectories of the nonlinear undamped pendulum equation

\[
\frac{d^2\theta}{dt^2} + \frac{g}{l} \sin \theta = 0
\]

are given by

\[
\frac{g}{l}(1 - \cos x) + \frac{y^2}{2} = c
\]

where \( x = \theta, \ y = \frac{d\theta}{dt} \).

**Answer:** Let

\[
x = \theta, \ y = \frac{d\theta}{dt}
\]

then

\[
\frac{dx}{dt} = \frac{d\theta}{dt} = y
\]

\[
\frac{dy}{dt} = \frac{d^2\theta}{dt^2} = -\frac{g}{l} \sin \theta = -\frac{g}{l} \sin x
\]
We get
\[ \frac{dy}{dx} = -\frac{g}{l} \sin x \]
Hence,
\[ \frac{g}{l} (1 - \cos x) + \frac{1}{2} y^2 = c. \]

6. Prove that for the system
\[ \frac{dX}{dt} = f(X) \]
there is at most one trajectory passing through a given point \( X_0 \).

**Answer:** Let \( \phi_1(t) \), \( \phi_2(t) \) be two trajectories passing through \( X_0 \), i.e. \( \phi_1(t_1) = X_0 \), \( \phi_2(t_2) = X_0 \), then
\[ \frac{d\phi_1}{dt} = f(\phi_1), \ \phi_1(t_1) = X_0 \]
\[ \frac{d\phi_2}{dt} = f(\phi_2), \ \phi_2(t_2) = X_0 \]
Set
\[ \hat{\phi}_2 = \phi_2(t + t_2 - t_1) \]
then
\[ \frac{d\hat{\phi}_2}{dt} = \frac{d\phi_2}{dt} f(\hat{\phi}_2), \ \phi_1(t_1) = X_0 \]
By uniqueness,
\[ \phi_1(t) = \hat{\phi}_2(t) = \phi_2(t + t_2 - t_1) \]
Hence, \( \phi_1 \) and \( \phi_2 \) are the same trajectories.

7. Prove that if a trajectory starts at a noncritical point of the system
\[ \frac{dX}{dt} = f(X) \]
the it can not reach a critical point in a finite length of time.

**Answer:** Assume the contrary. we suppose that \((x_0, y_0)\) is a critical point of the system, and the solution \(x = \phi(t), y = \psi(t)\) satisfies \(\phi(a) = x_0, \psi(a) = y_0\).

On the other hand, \(x = x_0, y = y_0\) is a solution of the given system satisfying the initial condition \(x = x_0, y = y_0\) at \(t = a\). By the uniqueness, \(\phi(t) = x_0, \psi(t) = y_0, \forall t \geq 0\), hence, \(\phi(0) = x_0, \psi(t) = y_0\). This contradict the assumption of the problem.

8. Assuming the trajectory corresponding to solution \(x = \phi(t), y = \psi(t), -\infty < t < +\infty\), of an autonomous system is closed, show that the solution must be periodic.

**Answer:** Suppose that \(\phi(t_1) = \phi(t_2) = X_0, t_1 \neq t_2\). By the same proof as in Problem 6, we get

\[\phi(t + t_2 - t_1) = \phi(t), \forall t\]

Let \(T = t_2 - t_1\), the

\[\phi(t + T) = \phi(t), \forall t\]

Hence \(\phi(t)\) is periodic with period \(T\).

9. Write the following spring-mass system as a system of equations by introducing \(x = u, u = \frac{du}{dt}\)

\[m \frac{d^2u}{dt^2} + c \frac{du}{dt} + ku = 0\]

Find out the critical point and analyze the stability of the critical point.
Answer: Let 
\[ x = u, \ y = \frac{du}{dt} \]
then
\[ \frac{dx}{dt} = \frac{du}{dt} = y \]
\[ \frac{dy}{dt} = \frac{d^2u}{dt^2} = \frac{c}{m}y - \frac{k}{m}x \]
(0,0) is a critical point of the system.

\[ \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ \frac{k}{m} & \frac{c}{m} - \lambda \end{vmatrix} = m\lambda^2 + c\lambda + k = 0 \]

Case (i) \( c^2 - 4mk > 0 \) and \( k \neq 0 \)
In this case, there are two distinct real eigenvalues, say \( r_1, r_2 \).
Since \( m, c, k > 0 \), then
\[ r_1, r_2 < 0 \]
(0,0) is asymptotically stable critical point of the system.

Case (ii) \( c^2 - 4mk = 0 \) and \( k \neq 0 \)
In this case, there are repeated eigenvalues, say \( r \).
Since \( m, c, k > 0 \), then
\[ r < 0 \]
(0,0) is asymptotically stable critical point of the system.

Case (iii) \( c^2 - 4mk < 0 \) and \( k \neq 0 \)
In this case, there are two complex eigenvalues, say \( r_1 = \lambda + \mu i, \ r_2 = \lambda - \mu i \) where \( \lambda = -\frac{c}{m} \). Since \( m, c > 0 \), (0,0) is asymptotically stable critical point of the system if \( c \neq 0 \). Similarly, (0,0) is stable critical point of the system if \( c = 0 \).
10. Case (iv) $k = 0$
In this case, $r_1 = 0$, $r_2 = -\frac{c}{m}$. $(0,0)$ is stable critical point of the system.

10. Show that the system is almost linear and $(0,0)$ is a stable critical point of the system

\[
\frac{dx}{dt} = -x - xy^2, \quad \frac{dy}{dt} = -y - x^2y
\]

**Answer:**

\[
\left(\begin{array}{c}
\frac{dx}{dt} \\
\frac{dy}{dt}
\end{array}\right) = \left(\begin{array}{cc}
-1 & 0 \\
0 & -1
\end{array}\right) \left(\begin{array}{c}
x \\
y
\end{array}\right) + \left(\begin{array}{c}
-xy^2 \\
-x^2y
\end{array}\right)
\]

Let $x = r \cos \theta$, $y = r \sin \theta$ then

\[
\lim \frac{-xy^2}{r} = \lim (-r^2 \cos \theta \sin^2 \theta) = 0
\]

\[
\lim \frac{-x^2y}{r} = 0
\]

det $A = 1$, the system is almost linear.

\[
\text{det}(A - \lambda I) = \begin{vmatrix}
-1 - \lambda & 0 \\
0 & -1 - \lambda
\end{vmatrix} = (\lambda + 1)^2 = 0
\]

The eigenvalues are

\[
\lambda = -1
\]

$(0,0)$ is stable critical point of the system.

11. Show that the system is almost linear and $(0,0)$ is an asymptotically stable critical point of the system

\[
\frac{dx}{dt} = -\frac{1}{2}x^3 + 2xy^2, \quad \frac{dy}{dt} = -y^3
\]
Answer:

Let \( x = r \cos \theta, \ y = r \sin \theta \) then

\[
\lim -\frac{1}{2} x^3 + 2xy^2 \frac{r}{r} = \lim (-\frac{1}{2} r^2 \cos^3 \theta + r^2 \cos \theta \sin^2 \theta) = 0
\]

\[
\lim -\frac{y^3}{r} = \lim (r^2 \sin^3 \theta) = 0
\]

Hence, the system is almost linear.

As for the type of the critical point, we will give the solution on Problem 5 of Assignment 10.

12. Determine all real critical points of the following system of equations

a) \( \frac{dx}{dt} = x + y^2, \ \frac{dy}{dt} = x + y \)
b) \( \frac{dx}{dt} = 1 - y, \ \frac{dy}{dt} = x^2 - y^2 \)

Answer: a) The critical points of the system are (0,0) and (1,-1).

b) The critical points of the system are (1,1) and (1,-1).

13. Consider the following problem. Determine the eigenvalues and critical points, classify the type of the critical point and determine whether it is stable, asymptotically stable, or unstable.

\[
x' = \begin{pmatrix} \varepsilon & 1 \\ -1 & \varepsilon \end{pmatrix} x
\]

Here \( \varepsilon \) is a real number.
Answer: Since \( \det A = \varepsilon^2 + 1 > 0 \), (0,0) is the only critical point of the system.

\[
\det(A - \lambda I) = \begin{vmatrix} \varepsilon - \lambda & 1 \\ -1 & \varepsilon - \lambda \end{vmatrix} = \lambda^2 - 2\varepsilon\lambda + \varepsilon^2 + 1 = 0
\]

Case A: \( \varepsilon = 0 \)

The eigenvalues are 

\[ \lambda_1 = i, \lambda_2 = -i \]

(0,0) is a stable center.

Case B: \( \varepsilon > 0 \)

The eigenvalues are 

\[ \lambda_1 = \varepsilon + i, \lambda_2 = \varepsilon - i \]

(0,0) is an unstable spiral point.

Case C: \( \varepsilon < 0 \)

The eigenvalues are 

\[ \lambda_1 = \varepsilon + i, \lambda_2 = \varepsilon - i \]

(0,0) is an asymptotical stable spiral point.

14. Consider the following Lienard equation

\[
\frac{d^2 x}{dt^2} + c(x) \frac{dx}{dt} + g(x) = 0
\]

where \( g(0) = 0 \).

a). Write it as a system of two first order equations by introducing 
\[ y = \frac{dx}{dt}. \]

b). Show that (0,0) is a critical point and that the system is almost linear in the neighborhood of (0,0).
c). Show that if $c(0) > 0$, $g'(0) > 0$ then the critical point is asymptotically stable, and that if $c(0) < 0$ or $g'(0) < 0$, then the critical point is unstable.

**Answer:** a). Let 
\[ y = \frac{dx}{dt} \]
then 
\[ \frac{dy}{dt} = \frac{d^2x}{dt^2} \]
From the original equation, we get 
\[ \frac{dy}{dt} = -c(x)y - g(x). \]
Let 
\[ X = \begin{pmatrix} x \\ y \end{pmatrix} \]
We get 
\[ X' = \begin{pmatrix} 0 & 1 \\ 0 & -c(x) \end{pmatrix} X + \begin{pmatrix} 0 \\ -g(x) \end{pmatrix} \]

b). Since $g(0) = 0$, then $(0,0)$ is the critical point of the system. Let 
\[ c(x) = c(0) + \eta_1(x) \]
where $\eta_1(x) \to 0, x \to 0$
\[ g(x) = g'(0)x + \eta_2(x) \]
where $\eta_1(x)/x \to 0, x \to 0$
The system can written as 
\[ X' = \begin{pmatrix} 0 & 1 \\ -g'(0) & -c(0) \end{pmatrix} X + \begin{pmatrix} 0 \\ -\eta_1(x)y - \eta_2(x) \end{pmatrix} \]
Take
\[ G^T = (0, -\eta_1(x)y - \eta_2(x)) \]
Then \(|G| \rightarrow 0,(x,y) \rightarrow (0,0)\). Hence the system is almost linear in the neighborhood of (0,0).

c).
\[
\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -g'(0) & -c(0) - \lambda \end{vmatrix} = \lambda^2 + c(0)\lambda + g'(0) = 0
\]
Hence
\[
\lambda_1 = \frac{-c(0) + \sqrt{c^2(0) - 4g'(0)}}{2} \\
\lambda_1 = \frac{-c(0) - \sqrt{c^2(0) - 4g'(0)}}{2}
\]
Case A: \(c(0) > 0, g'(0) > 0\)
If \(\triangle = c^2(0) - 4g' > 0\), then \(\lambda_2 < \lambda_1 < 0\) and (0,0) is an asymptotical stable critical point.

If \(\triangle = c^2(0) - 4g' < 0\), then \(\lambda_2 = \alpha + \beta i, \lambda_1 = \alpha - \beta i\) where \(\alpha = -\frac{c(0)}{2} < 0\) and (0,0) is an asymptotical stable critical point.

If \(\triangle = c^2(0) - 4g' = 0\), then \(\lambda_2 = \lambda_1 = -\frac{c(0)}{2} < 0\) and (0,0) is an asymptotical stable critical point.

Case B: \(c(0) < 0\)
If \(\triangle = c^2(0) - 4g' > 0\), then \(\lambda_1 > \lambda_2 > 0\) and (0,0) is an unstable critical point.

If \(\triangle = c^2(0) - 4g' < 0\), then \(\lambda_2 = \alpha + \beta i, \lambda_1 = \alpha - \beta i\) where \(\alpha = -\frac{c(0)}{2} > 0\) and (0,0) is an unstable critical point.
If \( \Delta = c^2(0) - 4g' = 0 \), then \( \lambda_2 = \lambda_1 = -\frac{c(0)}{2} > 0 \) and \((0,0)\) is unstable critical point.

Case C: \( g'(0) < 0 \)
In this case, \( \lambda_1 > 0 > \lambda_2 > 0 \) and \((0,0)\) is an unstable critical point.