Topological Degree For Solutions of Fourth Order Mean Field Equations *

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Abstract

We consider the following fourth order mean field equation with Navier boundary condition

$$(*)\qquad \Delta^2 u = \rho \frac{h(x)e^u}{\int_\Omega he^u} \ \text{ in } \Omega, \quad u = \Delta u = 0 \ \text{ on } \partial\Omega,$$

where h is a $C^{2,\beta}$ positive function, Ω is a bounded and smooth domain in $\mathbb{R}^4$. We prove that for $\rho \in (32m\sigma_3, 32(m+1)\sigma_3)$ the degree-counting formula for (*) is given by

$$d(\rho) = \begin{cases} \frac{1}{m!}(-\chi(\Omega)+1)\cdots(-\chi(\Omega)+m) & \text{for } m > 0, \\ 1 \ \text{ for } m = 0 \end{cases}$$

where $\chi(\Omega)$ is the Euler characteristic of Ω. Similar result is also proved for the corresponding Dirichlet problem

$$(**)\qquad \Delta^2 u = \rho \frac{h(x)e^u}{\int_\Omega he^u} \ \text{ in } \Omega, \quad u = \nabla u = 0 \ \text{ on } \partial\Omega.$$

1 Introduction

This is a continuation of the previous paper Lin-Wei [18], where they studied the following fourth order mean field equation

$$\begin{cases} \Delta^2 u = \rho \frac{he^u}{\int_\Omega he^u} & \text{in} \quad \Omega, \\ u = \Delta u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{1.1}$$

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and computed the sharp estimates for the bubbling solutions. Our aim here is to compute the Leray-Schauder degree for solutions of (1.1).

In dimension two, the analogous problem

$$\begin{cases} -\Delta u = \rho \frac{he^u}{\int_\Omega he^u} & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a smooth and bounded domain in $\mathbb{R}^2$, has been extensively studied by many authors. We summarize the results for (1.2) and identify the difficulty in studying (1.1) now. Let (u_k, ρ_k) be a bubbling sequence to (1.2) with $\rho_k \leq C, \max_{x\in\Omega} u_k(x) \to +\infty$. Then it has been proved that

(P1) (**no boundary bubbles**) u_k is uniformally bounded near a neighborhood of $\partial\Omega$ (Ma-Wei [19]);

(P2) (**bubbles are simple**) $\rho_k \to 8m\pi$ for some $m \geq 1$ and $u_k(x) \to 8\pi \sum\limits_{j=1}^{m} G_2(\cdot, p_j)$ in $C^2(\Omega\backslash\{p_1, ..., p_m\})$ (Brezis-Merle [2], Nagasaki-Suzuki [22], Ma-Wei [19]), where G_2 is the Green function of $-\Delta$ with Dirichlet boundary condition;

(P3) (**sup + inf estimates**) at each bubble $p_{k,j}$ where $u_k(p_{k,j}) = \max_{x\in B_\delta(p_j)} u_k(x)$, the following refined estimate holds (Brezis-Li-Shafrir [5], Li [13], Li-Shafrir [14])

$$|u_k(x) - u_k(p_{k,j}) - \log \frac{1}{(1 + \frac{|x-x_{k,j}|^2}{\epsilon_{k,j}^2})^2}| \leq C \tag{1.3}$$

where $u_k(p_{k,j}) - \log(\int_\Omega he^{u_k}) = \log \frac{1}{\epsilon_{k,j}^2}$;

(P4) (**exact bubbling rate**) It holds then (Chen and Lin [7])

$$\rho_k - 8m\pi = c_0 \sum_{j=1}^{m} h(p_{k,j})^{-1} \Delta \log h(p_{k,j}) \epsilon_{k,j}^2 \log \frac{1}{\epsilon_{k,j}} + O(\sum_{j=1}^{m} \epsilon_{k,j}^2); \tag{1.4}$$

(P5) (**Leray-Schauder degree**) Li [13] initiated the program of computing the Leray-Schauder degree of solutions of (1.2). He showed that the Leray-Schauder degree remains a constant for $\rho \in (8\pi(m-1), 8\pi m)$ and that the degree depends only on the topology of Ω. Chen and Lin [8] obtained the exact Leray-Schauder degree counting formula as follows:

$$d(\rho) = \begin{cases} \frac{1}{m!}(-\chi(\Omega)+1)\cdots(-\chi(\Omega)+m) & \text{for } m > 0, \\ 1 \quad \text{for } m = 0 \end{cases} \tag{1.5}$$

where $\chi(\Omega)$ is the Euler characteristic of Ω.

In the previous paper [18], the same program was carried out for equation (1.1) and the properties (P1)-(P4) were established. Namely, we have the following theorem:

Theorem 1.1 *([18]) Let h be a positive $C^{2,\beta}$ function in Ω and u_k be a sequence of blowup solutions of (1.1) with $\rho = \rho_k \leq C$. Then (after extracting a subsequence), $\lim_{k\to+\infty}\rho_k = 32\sigma_3 m$ for some positive integer m. Furthermore,*

$$\rho_k - 32\sigma_3 m$$
$$= c_0\sum_{j=1}^{m}(h(p_{k,j}))^{-\frac{1}{2}}\epsilon_{k,j}^2\Big(\frac{1}{32\sigma_3}\Delta\log h(p_{k,j}) + \Delta R_4(p_{k,j},p_{k,j}) + \sum_{i\neq j}\Delta G_4(p_{k,j},p_{k,i})\Big) + o(\sum_{j=1}^{m}\epsilon_{k,j}^2) \tag{1.6}$$

where σ_3 is the area of the unit sphere in $\mathbb{R}^4$, i.e. $\sigma_3 = 2\pi^2$, $c_0 > 0$ is a generic constant, $G_4(\cdot,p)$ is the Green function of Δ^2 with Navier boundary condition $G_4(\cdot,p) = \Delta G_4(\cdot,p) = 0$ on $\partial\Omega$, R_4 is the regular part of G_4, $p_{k,j}$ are the local maximum points of u_k on $B_\delta(p_j)$, $\log\frac{384}{\epsilon_{k,j}^4} = u_k(p_{k,j}) - \log(\int_\Omega he^{u_k})$ and p_j satisfies

$$\nabla\Big(\frac{1}{32\sigma_3}\log h(p_j) + R_4(p_j,p_j) + \sum_{l\neq j}G_4(p_l,p_j)\Big) = 0, \quad j = 1,\ldots,m.$$

In this paper, we prove the following theorem, which completes the program (P5):

Theorem 1.2 *Let $32m\sigma_3 < \rho < 32(m+1)\sigma_3$ and $d(\rho)$ be the Leray-Schauder degree for equation (1.1). Then*

$$d(\rho) = \begin{cases} \frac{1}{m!}(-\chi(\Omega)+1)\cdots(-\chi(\Omega)+m) & \text{for } m > 0, \\ 1 & \text{for } m = 0 \end{cases} \tag{1.7}$$

where $\chi(\Omega)$ is the Euler characteristic of Ω.

Remark. We are informed by Prof. Malchiodi that he obtained a similar degree counting formula for the corresponding prescribing $Q-$curvature problem on a four dimensional compact manifold, [21]. He used Morse theory to obtain the formula. We remark that on compact manifolds, one doesn't need to prove Property (P1). On the other hand, one of the main difficulties in our proof is the property (P1).

As a consequence of Theorem 1.2, equation (1.1) always possesses a solution for $\rho \neq 32m\sigma_3$ whenever the Euler characteristic $\chi(\Omega) \leq 0$. (Here m can be made ≥ 2, by results of Lin-Wei [17].) In particular, we have

Corollary 1.3 *If for some integer $d \geq 1$ such that $H^d(\Omega) \neq 0$, then equation (1.1) always has a solution for $\rho \notin \{32m\sigma_3, m \geq 2\}$.*

Set $d_m^+ = \lim_{\rho\to 32m\sigma_3+} d(\rho)$ and $d_m^- = \lim_{\rho\to 32m\sigma_3-} d(\rho)$. One of the main steps in the proof of Theorem 1.2 is to calculate the gap $d_m^+ - d_m^-$ for any integer $m \geq 1$. Once this is known, $d(\rho)$ can be computed inductively on m. Clearly, the gap of $d_m^+ - d_m^-$ is due to the occurrence of blowup solutions when $\rho \to 32m\sigma_3$. Thus an important question is to analyze the blowup behavior of sequence of solutions u_k to (1.1) and to know the sign of $\rho_k - 32m\sigma_3$, which has been done in [18].

Remark 1.4 *Theorems 1.1-1.2 can be extended easily to the following $n-th$ order mean field type equation*

$$\begin{cases} (-\Delta)^n u = \rho \frac{he^u}{\int_\Omega he^u} & in \quad \Omega, \\ (-\Delta)^j u = 0 & on \quad \partial\Omega, j = 0, ..., n-1 \end{cases} \tag{1.8}$$

where Ω is a smooth and bounded domain in $\mathbb{R}^{2n}$.

Similar result can also be obtained for the corresponding Dirichlet problem

$$\begin{cases} \Delta^2 u = \rho \frac{he^u}{\int_\Omega he^u} & \text{in} \quad \Omega, \\ u = \nabla u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.9}$$

We summarize in the following theorem, whose proof will be given in Section 4.

Theorem 1.5 *(1) Let h be a positive $C^{2,\beta}$ function in Ω and u_k be a sequence of blowup solutions of (1.9) with $\rho = \rho_k \leq C$. Then (after extracting a subsequence), $\lim_{k\to+\infty} \rho_k = 32\sigma_3 m$ for some positive integer m. Furthermore,*

$$\rho_k - 32\sigma_3 m$$

$$= c_0 \sum_{j=1}^m (h(p_{k,j}))^{-\frac{1}{2}} \epsilon_{k,j}^2 \Big(\frac{1}{32\sigma_3} \Delta \log h(p_{k,j}) + \Delta R_4'(p_{k,j}, p_{k,j}) + \sum_{i\neq j} \Delta G_4'(p_{k,j}, p_{k,i}) \Big) + o(\sum_{j=1}^m \epsilon_{k,j}^2) \tag{1.10}$$

where σ_3 is the area of the unit sphere in $\mathbb{R}^4$, i.e. $\sigma_3 = 2\pi^2$, $c_0 > 0$ is a generic constant, $G_4'(\cdot, p)$ is the Green function of Δ^2 with Dirichlet boundary condition $G_4'(\cdot, p) = \nabla G_4'(\cdot, p) = 0$ on $\partial\Omega$, R_4' is the regular part of G_4', $p_{k,j}$ are the local maximum points of u_k on $B_\delta(p_j)$, $\log \frac{384}{\epsilon_{k,j}^4} = u_k(p_{k,j}) - \log(\int_\Omega he^{u_k})$ and p_j satisfies

$$\nabla \Big(\frac{1}{32\sigma_3} \log h(p_j) + R_4(p_j, p_j) + \sum_{l\neq j} G_4(p_l, p_j) \Big) = 0, \quad j = 1, \dots, m;$$

(2) Let $32m\sigma_3 < \rho < 32(m+1)\sigma_3$ and $d(\rho)$ be the Leray-Schauder degree for equation (1.9). Then

$$d(\rho) = \begin{cases} \frac{1}{m!}(-\chi(\Omega)+1)\cdots(-\chi(\Omega)+m) & for \ m > 0, \\ 1 \quad for \ m = 0. \end{cases} \tag{1.11}$$

Semilinear equations involving exponential nonlinearity and fourth order elliptic operator appear naturally in conformal geometry and in particular in prescribing $Q-$curvature on 4-dimensional Riemannian manifold M (see e.g. Chang-Yang [6])

$$P_g w + 2Q_g = 2\tilde{Q}_{g_w} e^{4w} \tag{1.12}$$

where P_g is the so-called Paneitz operator:

$$P_g = (\Delta_g)^2 + \delta(\frac{2}{3}R_g I - 2\mathrm{Ric}_g)d,$$

$g_w = e^{2w}g$, Q_g is $Q-$ curvature under the metric g, and $\tilde{Q}_{g_w}$ is the Q-curvature under the new metric g_w.

Integrating (1.12) over M, we obtain

$$k_g := \int_M Q_g = \int_M (\tilde{Q}_{g_w})e^{4w}$$

where k_g is conformally-invariant. Thus, we can write (1.12) as

$$P_g w + 2Q_g = k_g \frac{\tilde{Q}_{g_w} e^{4w}}{\int_M \tilde{Q}_{g_w} e^{4w}} \tag{1.13}$$

In the special case, where the manifold is the Euclidean space, $P_g = \Delta^2$, and (1.13) becomes

$$\Delta^2 w = \rho \frac{h(x)e^{4w}}{\int_\Omega h(x)e^{4w}} \tag{1.14}$$

There is now an extensive literature about this problem, we refer to Adimurthi-Robert-Struwe [1], Baraket-Dammak-Ouni-Pacard [3], Clapp-Munoz-Musso [9], Druet-Robert [10], Hebey-Robert [11], Hebey-Robert-Wen [12], Malchiodi [20], Robert-Wei [24] and the references therein. In particular, we mention the two papers [3] and [9], where they constructed $m-$point blowing-up solutions for

$$\begin{cases} \Delta^2 u = \epsilon^4 h e^u & \text{in} \quad \Omega, \\ u = \Delta u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{1.15}$$

under either nondegenerate conditions ([3]) or topological nontrivial condition ([9]).

The organization of this paper is as follows: In Section 2, important preliminaries are presented. In Section 3, we prove Theorem 1.2 and the proof of a key lemma is in Appendix. Finally in Section 4, we give an outline of proof of Theorem 1.5.

Throughout this paper, unless otherwise stated, the letter C will always denote various generic constants which are independent of $k \geq 1$.

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2 Preliminaries

We state several results in this section, which will be used for the proof of Theorem 1.2 and Theorem 1.5.

Let G_4 denote the Green's function of Δ^2 under the Navier boundary condition, that is

$$\Delta^2 G_4(x,y) = \delta(x-y), \qquad G_4|_{\partial\Omega} = \Delta G_4|_{\partial\Omega} = 0. \tag{2.1}$$

We decompose

$$G_4(x,y) = \frac{1}{4\sigma_3} \log \frac{1}{|x-y|} + R_4(x,y). \tag{2.2}$$

It is easy to see that

$$\Delta_x G_4(x,y) < 0, \ \Delta_x R_4(x,y) > 0. \tag{2.3}$$

In [18], we have proved the following lemmas.

Lemma 2.1 *Let u_k be a bubbling sequence of (1.1) with $\rho_k \le C$. Then (after extracting a subsequence), $\rho_k \to 32\sigma_3 m$ and $u_k(x) \to 32\sigma_3 \sum_{j=1}^m G_4(\cdot, p_j)$ in $C^4_{loc}(\Omega \backslash \{p_1, \dots, p_m\})$, where $(p_1, ..., p_m)$ satisfies*

$$\nabla \left(\frac{1}{32\sigma_3} \log h(p_i) + R_4(p_i, p_i) + \sum_{j \ne i} G_4(p_i, p_j) \right) = 0, i = 1, ..., m. \tag{2.4}$$

Let δ_0 be a fixed small positive constant and $u_k(p_{k,j}) = \max_{x \in B_{\delta_0}(p_j)} u_k(x)$ and

$$e^{-c_k} = \frac{1}{\int_\Omega h(x) e^{u_k}}. \tag{2.5}$$

Define

$$l_{k,j} = u_k(p_{k,j}) - c_k, \quad e^{-\frac{l_{k,j}}{4}} = \frac{\epsilon_{k,j}}{\alpha_4^{\frac{1}{4}}}, \quad \epsilon_k = \min_{1 \le j \le m} \epsilon_{k,j}. \tag{2.6}$$

In fact, we can refine the Estimate A and Estimate B in [18]. That is, let

$$w_k(x) = u_k(x) - \sum_{i=1}^m \rho_{k,i} G_4(x, p_{k,i}) \quad \text{on} \quad \Omega \backslash \cup_{j=1}^m B_{\frac{\delta_0}{2}}(p_{k,j}) \tag{2.7}$$

and

$$G_j^*(x) = \rho_{k,j} R_4(x, p_{k,j}) + \sum_{l \ne j} \rho_{k,l} G_4(x, p_{k,l}), \tag{2.8}$$

where $\rho_{k,j} = \frac{\rho_k}{\int_\Omega h e^{u_k}} \int_{B_{\delta_0}(p_{k,j})} h(x) e^{u_k}$.

Lemma 2.2

$$|w_k(x)| + |\partial^\alpha w_k(x)| = O(\varepsilon_k^2) \qquad \textit{for} \quad |\alpha| \le 3 \quad \textit{in} \quad \Omega \backslash \cup_{j=1}^m B_{\frac{\delta_0}{2}}(p_{k,j}),$$

and

$$|\nabla(\log h(x) + G_j^*(x))| = O(\varepsilon_k^2) \qquad \textit{at} \quad x = p_{k,j}.$$

Using the Estimate C-F in [18], the proof is just the same as Lemma 5.3, Lemma 5.4 in [7]. We omit it here.

It has been proved ([16], [25]) that the solution to the following problem

$$\begin{cases} \Delta^2 U = e^U & \text{in} \quad \mathbb{R}^4, \\ \int_{\mathbb{R}^4} e^U < +\infty, \end{cases} \tag{2.9}$$

is given by

$$U_{\epsilon,a}(x) := \log \frac{\alpha_4 \epsilon^4}{(\epsilon^2 + |x-a|^2)^4}, \tag{2.10}$$

for any $\epsilon > 0, a \in \mathbb{R}^4, \alpha_4 = 384$, provided that

$$U(x) = o(|x|^2) \text{ as } \quad |x| \to +\infty. \tag{2.11}$$

Let $U = \log \frac{\alpha_4}{(1+|y|^2)^4}$ and $\tau \in (0,1)$ be a fixed constant. We have the following lemma which proves the nondegeneracy of U:

Lemma 2.3 *The solutions to the following linearized problem*

$$\Delta^2 \phi = e^U \phi, \quad |\phi(y)| \leq C(1+|y|)^\tau \tag{2.12}$$

is given by $\phi = \sum_{j=0}^4 c_j \psi_j$ *where*

$$\psi_0 = \frac{1-|y|^2}{1+|y|^2}, \quad \psi_j = \frac{y_j}{1+|y|^2}, \quad j = 1, ..., 4. \tag{2.13}$$

Recall that $G_4'(x,y)$ is defined in Theorem 1.5. In general, G_4' is not positive. We collect the property of G_4' in the following lemma which has been proved in [24].

Lemma 2.4 *Let* $d(x) = d(x, \partial\Omega)$. *Then we have*

$$|G_4'(x,y)| \leq C \log\Big(1 + (\frac{d(x)d(y)}{|x-y|^2})^2\Big),$$

and

$$|\nabla^i G_4'(x,y)| \leq C|x-y|^{-i} \min(1, \frac{d(y)}{|x-y|})^2, \qquad i \geq 1.$$

We also need to recall the well-known Pohozaev's identity for solutions of fourth-order equation

$$\Delta^2 u = h(x) e^u \text{ in } D.$$

Lemma 2.5 *Let* u *satisfy* $\Delta^2 u = h(x)e^u$ *in* D, *where* D *is a smooth and bounded domain in* $\mathbb{R}^4$. *Then we have*

$$\begin{aligned} \int_D (4h + \langle x-\xi, \nabla h\rangle) e^u \quad = \quad & \int_{\partial D} \langle x-\xi, \nu\rangle h(x) e^u + \int_{\partial D} \Big[\frac{1}{2}|\Delta u|^2 \langle x-\xi, \nu\rangle - 2\frac{\partial u}{\partial \nu}\Delta u \\ & - \langle x-\xi, \nabla u\rangle \frac{\partial \Delta u}{\partial \nu} - \langle x-\xi, \nabla\Delta u\rangle \frac{\partial u}{\partial \nu} + \langle x-\xi, \nu\rangle \langle \nabla u, \nabla \Delta u\rangle \Big], \end{aligned} \tag{2.14}$$

for any $\xi \in \mathbb{R}^4$.

Proof: In fact, multiplying $\Delta^2 u = h(x)e^u$ by $(x-\xi)\cdot\nabla u$ and integrating by parts, we obtain the lemma. $\square$

In the rest of the paper, we denote $H^2(\Omega)$ be the usual Sobolev space and $\mathcal{X} = H^2(\Omega)\cap\{u = \Delta u = 0 \text{ on } \partial\Omega\}$. On $\mathcal{X}$, we use the following inner product:

$$(u,v) = \int_\Omega (\Delta u \Delta v + uv).$$

3 The proof of Theorem 1.2

The purpose of this section is to prove Theorem 1.2. We follow the main steps used in [8]. Note that the notations $c_k, p_{k,j}, l_{k,j}, \epsilon_{k,j}$ given in (2.5), (2.6) have no relation with our definitions in this section and $\rho \in (32m\sigma_3, 32(m+1)\sigma_3)$. And from now on, we assume ρ is sufficiently close to $32m\sigma_3$.

Let $P = (p_1,\dots,p_m) \in \Omega^m, \Lambda = (\lambda_1,\dots,\lambda_m) \in \mathbb{R}^m, \varepsilon_j^4 = \alpha_4 e^{-\lambda_j}$ and $A = (a_1,\dots,a_m) \in \mathbb{R}^m$. Set

$$v_j(x) = \log\frac{e^{\lambda_j}}{(1+\sqrt{\frac{\rho h(p_j)}{\alpha_4}}e^{\frac{\lambda_j}{2}}|x-p_j|^2)^4}, \tag{3.1}$$

then $v_j(x)$ satisfies the equation $\Delta^2 v_j(x) = \rho h(p_j)e^{v_j}$. Let

$$H_j(x) = \exp\Big\{\log\frac{h(x)}{h(p_j)} + G_j^*(x) - G_j^*(p_j)\Big\} - 1 \tag{3.2}$$

on $B_{2\delta_0}(p_j)$, and

$$s_j = -\lambda_j - 2\log\frac{\rho h(p_j)}{\alpha_4} - 64\pi^2 R_4(p_j,p_j). \tag{3.3}$$

We assume λ_j large enough.

Let $\tau : \mathbb{R}\to\mathbb{R}$ be a smooth cutoff function satisfying $0\le\tau\le 1$ and $\tau(t) = 1$ for $|t|\le\delta_0$, $\tau(t) = 0$ for $|t|\ge 2\delta_0$. Set $\tau_j(x) = \tau(|x-p_j|)$ and let $\gamma_j(x)$ satisfy

$$\begin{cases} \Delta^2\gamma_j(x) = \rho h(p_j)e^{v_j}\Big(\gamma_j + (H_j(x) - \nabla H_j(p_j)\cdot(x-p_j))\tau_j\Big) & \text{on } \mathbb{R}^4 \\ \gamma_j(p_j) = 0, \quad \Delta\gamma_j(p_j) = 0, \quad \gamma_j(x)\to 0 \quad \text{as} \quad |x|\to+\infty. \end{cases} \tag{3.4}$$

As [8], γ_j is solvable and $|\gamma_j|, |\nabla_x\gamma_j|, |\partial_{\lambda_j}\gamma_j|, |\partial_{p_j}\gamma_j|, |\nabla_x\partial_{p_j}\gamma_j|, |\nabla_x\partial_{\lambda_j}\gamma_j|$ are of the order $O(\varepsilon_j^2)$. In fact, by Lemma 2.3, we know that $\gamma_j(x)$ is bounded in $\mathbb{R}^4\backslash B_{2\delta_0}(p_j)$ and hence is bounded in $\mathbb{R}^4$. Now we consider zero mode $\gamma_{j,0}$ of γ_j since the remaining terms are of small order with respect to $\gamma_{j,0}$.

Let

$$\Psi(x) = \frac{\varepsilon_j^2 - \sqrt{\rho h(p_j)}|x-p_j|^2}{\varepsilon_j^2 + \sqrt{\rho h(p_j)}|x-p_j|^2}.$$

Then it is easy to see that $\Psi(x)$ satisfies

$$\Delta^2\Psi(x) = \rho h(p_j)e^{v_j(x)}\Psi(x). \tag{3.5}$$

Using (3.4) and (3.5), we obtain

$$\int_{B_{r_0}(p_j)} [\Delta^2\gamma_{j,0}(x)\Psi(x) - \Delta^2\Psi(x)\gamma_{j,0}(x)] = \int_{B_{r_0}(p_j)} \rho h(p_j)e^{v_j} H_{j,0}\Psi(x), \tag{3.6}$$

where $H_{j,0}$ is zero mode of $H_j(x) - \nabla H_j(p_j)\cdot(x-p_j) = O(|x-p_j|^2)$ and $B_{r_0}(p_j)\subset\Omega$.

The left-hand-side of (3.6) equals

$$\begin{aligned}
&= \int_{\partial B_{r_0}(p_j)} \left[\frac{\partial\Delta\gamma_{j,0}}{\partial\nu}\Psi(x) - \Delta\gamma_{j,0}\frac{\partial\Psi}{\partial\nu} + \Delta\Psi\frac{\partial\gamma_{j,0}}{\partial\nu} - \gamma_{j,0}\frac{\partial\Delta\Psi}{\partial\nu}\right] \\
&= \int_{\partial B_{r_0}(p_j)} \frac{\partial\Delta\gamma_{j,0}}{\partial\nu}(\Psi(x)+1) - \int_{\partial B_{r_0}(p_j)} \frac{\partial\Delta\gamma_{j,0}}{\partial\nu} + O(\varepsilon_j^2) \\
&= \int_{\partial B_{r_0}(p_j)} \frac{\partial\Delta\gamma_{j,0}}{\partial\nu}\frac{2\varepsilon_j^2}{\varepsilon_j^2+\sqrt{\rho h(p_j)}\delta_0^2} - \int_{\partial B_{r_0}(p_j)} \frac{\partial\Delta\gamma_{j,0}}{\partial\nu} + O(\varepsilon_j^2) \\
&= -\int_{\partial B_{r_0}(p_j)} \frac{\partial\Delta\gamma_{j,0}}{\partial\nu} + O(\varepsilon_j^2)
\end{aligned}$$

The right-hand-side of (3.6) equals $O(\varepsilon_j^2)$ since after scaling the integration is finite. Thus

$$\int_{\partial B_{r_0}(p_j)} \frac{\partial\Delta\gamma_{j,0}}{\partial\nu} = O(\varepsilon_j^2).$$

On the other hand, taking integration of (3.4) under zero mode and using Green's Formula, we have

$$\int_{\partial B_{r_0}(p_j)} \frac{\partial\Delta\gamma_{j,0}}{\partial\nu} = \int_{B_{r_0}(p_j)} \rho h(p_j)e^{v_j}\gamma_{j,0} + O(\varepsilon_j^2).$$

Hence $\gamma_{j,0} = O(\varepsilon_j^2)$, so does γ_j. By elliptic estimates we get that $|\partial^\alpha\gamma_j| = O(\varepsilon_j^2)$ for $|\alpha|\le 3$.

Now we can refine the estimate of the left hand side of (3.6). That is,

$$\int_{B_{\delta_0}(p_j)} [\Delta^2\gamma_j(x)\Psi(x) - \Delta^2\Psi(x)\gamma_j(x)] = \int_{B_{\delta_0}(p_j)} \rho h(p_j)e^{v_j}(H_j(x) - \nabla H_j(p_j)\cdot(x-p_j))\Psi(x). \tag{3.7}$$

The left-hand-side of (3.7) equals

$$\begin{aligned}
&= \int_{\partial B_{\delta_0}(p_j)} \left[\frac{\partial\Delta\gamma_j}{\partial\nu}\Psi(x) - \Delta\gamma_j\frac{\partial\Psi}{\partial\nu} + \Delta\Psi\frac{\partial\gamma_j}{\partial\nu} - \gamma_j\frac{\partial\Delta\Psi}{\partial\nu}\right] \\
&= \int_{\partial B_{\delta_0}(p_j)} \frac{\partial\Delta\gamma_j}{\partial\nu}\frac{2\varepsilon_j^2}{\varepsilon_j^2+\sqrt{\rho h(p_j)}\delta_0^2} - \int_{\partial B_{\delta_0}(p_j)} \frac{\partial\Delta\gamma_j}{\partial\nu} + O(\varepsilon_j^4) \\
&= -\int_{\partial B_{\delta_0}(p_j)} \frac{\partial\Delta\gamma_j}{\partial\nu} + O(\varepsilon_j^4).
\end{aligned}$$

Since

$$H_j(x) - \nabla H_j(p_j)\cdot(x-p_j) = \sum_{l,m} \frac{\partial^2}{\partial x_l \partial x_m}[G_j^* + \log h](x_l - p_{j,l})(x_m - p_{j,m}) + O(|x-p_j|^3),$$

the right-hand-side of (3.7) equals

$$\begin{aligned}
&= \frac{\alpha_4}{\sqrt{\rho_k h(p_j)}} \int_{\mathbb{R}^4} \frac{1}{(1+|z|^2)^4}\left[\varepsilon_j^2 \sum_{l,m} \frac{\partial^2}{\partial x_l \partial x_m}[G_j^* + \log h] z_l z_m + O(\varepsilon_j^3 |z|^3)\right] \frac{1-|z|^2}{1+|z|^2} dz \\
&= \frac{\alpha_4}{\sqrt{\rho_k h(p_j)}} \varepsilon_j^2 \Delta(G_j^* + \log h)(p_j) \frac{1}{4} \int_{\mathbb{R}^4} \frac{|z|^2}{(1+|z|^2)^4} \frac{1-|z|^2}{1+|z|^2} dz + o(\varepsilon_j^2),
\end{aligned}$$

and

$$\frac{\alpha_4}{4} \int_{\mathbb{R}^4} \frac{|z|^2}{(1+|z|^2)^4} \frac{1-|z|^2}{1+|z|^2} dz = \frac{\alpha_4}{4} \int_{\mathbb{R}^4} \frac{|z|^2-1}{(1+|z|^2)^4} \frac{1-|z|^2}{1+|z|^2} dz < 0. \tag{3.8}$$

Thus, we obtain

$$\int_{\partial B_{\delta_0}(p_j)} \frac{\partial \Delta \gamma_j}{\partial \nu} = l_j(Q) \varepsilon_j^2 + o(\varepsilon_j^2), \tag{3.9}$$

where $l_j(Q)$ is defined in (3.13).

Define

$$\begin{cases} u_j = [v_j + \gamma_j + 32\sigma_3 (R_4(x,p_j) - R_4(p_j,p_j)) - s_j]\tau_j + 32\sigma_3 G_4(x,p_j)(1-\tau_j), \\ u_{P,\Lambda,A}(x) = \sum_{j=1}^m a_j u_j. \end{cases} \tag{3.10}$$

We will see that when λ_j, p_j, a_j are suitably chosen, $u_{P,\Lambda,A}$ is considered an approximate solution to equation (1.1).

Claim: Letting $\xi_j(x) = u_j - 32\sigma_3 G_4(x,p_j)$, then $\nabla_{p_j}\xi_j(x), \partial_{\lambda_j}\xi_j(x), \nabla_x \xi_j(x), \Delta_x^2 \xi_j(x)$ are of the order $O(\varepsilon_j^2)$ in $B_{2\delta_0}(p_j)\backslash B_{\delta_0}(p_j)$.

Proof: First, note that $u_j - 32\sigma_3 G_4(x,p_j) = 0$ on $\Omega \backslash B_{2\delta_0}(p_j)$. On $B_{2\delta_0}(p_j)\backslash B_{\delta_0}(p_j)$, we have

$$\begin{aligned}
u_j - 32\sigma_3 G_4(x,p_j) &= \Big(v_j - s_j + \gamma_j + 64\pi^2(R_4(x,p_j) - R_4(p_j,p_j) - G_4(x,p_j))\Big)\tau_j \\
&= \Big[2\lambda_j - 4\log\Big(1 + \sqrt{\frac{\rho h(p_j)}{\alpha_4}}e^{\frac{\lambda_j}{2}}|x-p_j|^2\Big) + 2\log\frac{\rho h(p_j)}{\alpha_4} \\
&\quad +64\pi^2(R_4(x,p_j) - G_4(x,p_j))\Big]\tau_j + O(\varepsilon_j^2) \\
&= \Big(-8\log|x-p_j| + 64\pi^2(R_4(x,p_j) - G_4(x,p_j))\Big)\tau_j \\
&\quad +O\Big(\varepsilon_j^2 + \frac{1}{1+\sqrt{\frac{\rho h(p_j)}{\alpha_4}}e^{\frac{\lambda_j}{2}}\delta_0^2}\Big) \\
&= O(\varepsilon_j^2).
\end{aligned}$$

It is not hard to see that all derivatives of ξ_j is bounded by ε_j^2. □

Now we start to define $S_\rho(Q)$ such that $u_{P,\Lambda,A}$ is a good approximate solution to our problem.

Let Γ_m be a subset of Ω^m defined by

$$\Gamma_m = \{(x_1,\ldots,x_m) \in \Omega^m : x_i = x_j \quad \text{for some} \quad i \neq j\}.$$

For any function $h(x)$ on Ω, we set f_h to be a function on $\Omega^m \backslash \Gamma_m$:

$$f_h(x_1,\ldots,x_m) = \sum_{j=1}^m (\log h(x_j) + 16\sigma_3 R_4(x_j,x_j) + 32\sigma_3 \sum_{i\neq j} G_4(x_i,x_j)). \tag{3.11}$$

Clearly,

$$\nabla H_j(p_j) = 0 \quad \text{if and only if} \quad \nabla_{x_j} f_h(p_1,\ldots,p_m) = 0. \tag{3.12}$$

For any critical $Q = (q_1,\ldots,q_m)$ of f_h, we set

$$\begin{cases} l_j(Q) = c_0 h(p_j)^{-\frac{1}{2}}\Big(\frac{1}{32\sigma_3}\Delta \log h(q_j) + \Delta R_4(q_j,q_j) + \sum\limits_{i\neq j} G_4(q_i,q_j)\Big), \\ l(Q) = \sum\limits_{j=1}^m l_j(Q) h(p_j) e^{\frac{G_j^*(p_j)}{2}}, \end{cases} \tag{3.13}$$

where c_0 is defined in Theorem 1.1.

From now on, h is assumed to satisfy the following two conditions:

(i) The function f_h is a Morse function on $\Omega^m \backslash \Gamma_m$ with critical points $Q_1,\ldots,Q_N$;

(ii) The quantity $l(Q)$ doesn't vanish for any critical point of f_h.
It is not difficult to construct a function h on Ω such that both conditions (i) and (ii) hold. For $\rho \neq 32m\sigma_3$ and each critical point of f_h, we set $\lambda_j(Q), j = 1, \ldots, m$ to satisfy

$$\rho - 32\sigma_3 m = \alpha_4^{\frac{1}{2}} \frac{l(Q)}{h(p_j)} e^{\frac{-G_j^*(p_j) - \lambda_j(Q)}{2}}, \tag{3.14}$$

which implies that $\lambda_j(Q) \to +\infty$ as $\rho \to 32m\sigma_3$.

Next we will decompose a solution u as $u = u_{P,\Lambda,A} + w$ for some P, Λ, A and $w \in O_{P,\Lambda}$, where

$$O_{P,\Lambda} = \Big\{ w \in \mathcal{X} : \int_\Omega \Delta w \Delta u_j = \int_\Omega \Delta w \Delta \partial_{\lambda_j} u_j = \int_\Omega \Delta w \Delta \partial_{p_j} u_j = 0, \quad j = 1, \ldots, m \Big\}. \tag{3.15}$$

The triplet (P, Λ, A) is chosen according to each critical point Q of f_h, we define

$$\begin{aligned} S_\rho(Q) = \Big\{ \quad & u = u_{P,\Lambda,A} + w : |p_j - q_j| \le C\varepsilon_j^2(Q), \quad |\lambda_1 - \lambda_1(Q)| \le C, \quad |t_j - t_1| \le C\varepsilon_j^2(Q), \\ & \text{for} \quad 2 \le j \le m, \quad |a_j - 1| \le C\varepsilon_j^2(Q), \quad \|w\|_{\mathcal{X}} \le C\varepsilon_j^2(Q), \quad w \in O_{P,\Lambda} \Big\}, \end{aligned} \tag{3.16}$$

where t_j is defined by

$$t_j = \lambda_j + G_j^*(q_j) + 2 \log \frac{\rho h(q_j)}{\alpha_4},$$

and the constant C is large.

Similarly as [8], we obtain that any blowup solution must be contained in $S_\rho(Q)$ for some critical point Q of f_h provided that ρ is sufficiently close to $32m\sigma_3$. The proof is omitted here.

Set

$$T(\rho) = \rho \Delta^{-2} \frac{he^u}{\int_\Omega he^u}.$$

Since each solution in $S_\rho(Q)$ has a representation (P, Λ, A, w), the nonlinear operator $u + T(\rho)u$ of (1.1) can be split according to this representation. Thus, our problem of counting degree can be reduced to a finite-dimensional problem. At each point p_j, the nonlinear term ρhe^u is linearly expressed, up to higher order, in terms of $a_j - 1, \gamma_j, H_j$ and w. More precisely, we let

$$\tilde{\beta}_j = \lambda_j |a_j - 1| + \sum_{l \neq j} |a_l - 1| + |\gamma_j| + |H_j| + |w|.$$

Then on $B_{\delta_0}(p_j)$,

$$\begin{aligned} \rho he^u &= \rho he^{u-w+w} = \rho he^{u-w}(1 + w) + (e^w - 1 - w)\rho he^{u-w} \\ &= \rho h(p_j) e^{v_j + t_j} \Big[1 + (a_j - 1)(v_j - s_j) + \sum_{l \neq j} 32\sigma_3 (a_l - 1) G(p_l, p_j) \\ &\quad + H_j(x) + \gamma_j(x) + w + \sum_{l=1}^{m} (a_l - 1) O(|x - p_j|) \Big] + \tilde{E}, \end{aligned} \tag{3.17}$$

where

$$\tilde{E} = (e^w - 1 - w)\rho h e^{u-w} + \rho h(p_j) e^{v_j + t_j}\Big[O(\tilde{\beta}_j^{\,2}) + O(w^2)\Big].$$

Using the expression for $\rho h e^u$ above, we obtain an estimate for $\int_\Omega \rho h e^u$.

Lemma 3.1 *Let $u = u_{P,\Lambda,A} + w \in S_\rho(Q)$. Then as $\rho \to 32m\sigma_3$,*

$$\begin{aligned} \int_\Omega \rho h e^u \;&=\; \sum_{j=1}^m \Big(64\pi^2 e^{t_j} + 128\pi^2(a_j - 1)\lambda_j e^{t_j} + \alpha_4^{\frac12} l_j(Q) e^{t_j - \frac{\lambda_j}{2}}\Big) \\ &\quad + O\big(1 + \sum_{l=1}^m |a_l - 1| e^{t_l}\big) + o\big(e^{t_j - \frac{\lambda_j}{2}}\big). \end{aligned} \tag{3.18}$$

Proof: Note that $t_j = \lambda_j + O(1)$ and $v_j - s_j = 2\lambda_j - 4\log(1 + \sqrt{\frac{\rho h(p_j)}{\alpha_4}} e^{\frac{\lambda_j}{2}} |x - p_j|^2) + O(1)$. By (3.17),

$$\begin{aligned} \int_\Omega \rho h e^u &= \sum_{j=1}^m \int_{B_{\delta_0}(p_j)} \rho h e^u + \int_{\Omega \setminus \cup_{j=1}^m B_{\delta_0}(p_j)} \rho h e^u \\ &= \sum_{j=1}^m \int_{B_{\delta_0}(p_j)} \Big(\rho h(p_j) e^{v_j + t_j}\big(1 + \ldots\big) + \tilde{E}\Big) + O(1). \end{aligned} \tag{3.19}$$

By the explicit expression of v_j, we have

$$\int_{B_{\delta_0}(p_j)} \rho h(p_j) e^{v_j + t_j} = 64\pi^2 e^{t_j} + O(1), \tag{3.20}$$

$$\int_{B_{\delta_0}(p_j)} \rho h(p_j) e^{v_j + t_j}(a_j - 1)(v_j - s_j) = 128\pi^2(a_j - 1)\lambda_j e^{t_j} + O(|a_j - 1| e^{t_j}), \tag{3.21}$$

$$\int_{B_{\delta_0}(p_j)} \rho h(p_j) e^{v_j + t_j}\Big(\sum_{l \neq j}(a_l - 1) 32\sigma_3 G_4(p_l, p_j) + \sum_{l=1}^m |a_l - 1| O(|x - p_j|)\Big) = O\big(\sum_{l=1}^m |a_l - 1| e^{t_j}\big). \tag{3.22}$$

Since these computations are straightforward, we skip the details. By (3.9) and the fact $\nabla H_j(p_j) \cdot (x - p_j)$ is an odd function in $B_{\delta_0}(p_j)$, we have

$$\begin{aligned} \int_{B_{\delta_0}(p_j)} \rho h(p_j) e^{v_j + t_j}(\gamma_j + H_j) &= \int_{B_{\delta_0}(p_j)} \rho h(p_j) e^{v_j + t_j}(\gamma_j + H_j - \nabla H_j(p_j) \cdot (x - p_j)) \\ &= \int_{B_{\delta_0}(p_j)} e^{t_j} \Delta^2 \gamma_j = e^{t_j} \int_{\partial B_{\delta_0}(p_j)} \frac{\partial(\Delta \gamma_j)}{\partial \nu} \\ &= \alpha_4^{\frac12} e^{t_j - \frac{\lambda_j}{2}} l_j(Q) + o(e^{t_j - \frac{\lambda_j}{2}}). \end{aligned} \tag{3.23}$$

To estimate the terms involving w, we use (3.10) for u_j at each p_j. Then

$$\begin{aligned}\int_{B_{\delta_0}(p_j)} \rho h(p_j) e^{v_j+t_j} w &= e^{t_j}\Big(\int_\Omega w\Delta^2 u_j - \int_{B_{\delta_0}(p_j)} w\Delta^2\gamma_j - \int_{\Omega\setminus B_{\delta_0}(p_j)} w\Delta^2(u_j - 32\sigma_3 G_4(x,p_j))\Big)\\ &= e^{t_j}\Big(\int_\Omega \Delta w\Delta u_j - \int_{B_{\delta_0}(p_j)} w\Delta^2\gamma_j - \int_{\Omega\setminus B_{\delta_0}(p_j)} w\Delta^2(u_j - 32\sigma_3 G_4(x,p_j))\Big)\\ &\le Ce^{t_j}\varepsilon_j^2\|w\|_{H_0^2(\Omega)}\end{aligned} \tag{3.24}$$

since $w \in O_{P,\Lambda}$.

As to $\int_{B_{\delta_0}(p_j)} \rho h(p_j) e^{v_j+t_j}\tilde{E}$, using the same method in [8], we can get

$$\int_{B_{\delta_0}(p_j)} \rho h(p_j) e^{v_j+t_j}|\tilde{E}| = O(1). \tag{3.25}$$

By (3.20)-(3.25), we finish the proof of Lemma 3.1. □

Now, we want to express $u + T(\rho)u$ in a formula similar to (3.17). For simplicity, we use T to denote $T(\rho)$. We then have by (3.17) that

$$\begin{aligned}\frac{\rho h e^u}{\int_\Omega h e^u} &= (1+w)\frac{\rho h e^{u_{P,\Lambda,A}}}{\int_\Omega h e^u} + (e^w - 1 - w)\frac{\rho h e^{u_{P,\Lambda,A}}}{\int_\Omega h e^u}\\ &= \rho h(p_j) e^{v_j}\Big[1 + \Big(\frac{e^{t_j}}{\int_\Omega h e^u} - 1\Big) + (a_j - 1)(v_j - s_j) + \sum_{l\neq j} 32\sigma_3(a_l - 1)G_4(p_j,p_l)\\ &\quad + \sum_{l=1}^m (a_l - 1)O(|x-p_j|) + H_j(x) + \gamma_j + w + O(\beta_j^2)\Big] + E,\end{aligned} \tag{3.26}$$

where

$$\beta_j = \tilde{\beta}_j + \Big|\frac{e^{t_j}}{\int_\Omega h e^u} - 1\Big|. \tag{3.27}$$

Thus, in $B_{\delta_0}(p_j)$, we have

$$\begin{aligned}\Delta^2(u+Tu) &= \Delta^2 u - \frac{\rho h e^u}{\int_\Omega h e^u} = a_j(\Delta^2 v_j + \Delta^2 \gamma_j) + \Delta^2 w - \frac{\rho h e^u}{\int_\Omega h e^u} \\ &= a_j \rho h(p_j) e^{v_j}\Big(1 + \gamma_j + H_j - \nabla H_j(p_j)\cdot(x-p_j)\Big) + \Delta^2 w - \frac{\rho h e^u}{\int_\Omega h e^u} \\ &= \Delta^2 w - \rho h(p_j) e^{v_j}\Big[(a_j-1)(v_j - s_j - 1) + a_j \nabla H_j(p_j)\cdot(x-p_j) + \Big(\frac{e^{t_j}}{\int_\Omega h e^u} - 1\Big) \\ &\quad + \sum_{l\neq j} 32\sigma_3(a_l-1)G_4(p_l,p_j) + \sum_{l=1}^m (a_l-1)O(|x-p_j|) + w\Big] - E,\end{aligned} \tag{3.28}$$

where

$$E = (e^w - 1 - w)\frac{\rho h e^{u_{P,\Lambda,A}}}{\int_\Omega h e^u} + \rho h(p_j)e^{v_j}\big(O(w^2) + O(\beta_j^2)\big).$$

On $B_{2\delta_0}(p_j)\backslash B_{\delta_0}(p_j)$, since $u_j - 32\sigma_3 G_4(x,p_j)$ is small, we write $\Delta^2(u+Tu)$ as

$$\begin{aligned}\Delta^2(u+Tu) &= \Delta^2 w + a_j \Delta^2(u_j - 32\sigma_3 G_4(x,p_j)) \\ &\quad - \frac{\rho h}{\int_\Omega h e^u} e^{a_j(u_j - 32\sigma_3 G_4(x,p_j)) + \sum_{l=1}^m a_l 32\sigma_3 G(x,p_l) + w}\end{aligned} \tag{3.29}$$

and on $\Omega\backslash \cup_{j=1}^m B_{2\delta_0}(p_j)$, we have

$$\Delta^2(u+Tu) = \Delta^2 w - \frac{\rho h}{\int_\Omega h e^u} e^{\sum_{l=1}^m a_l 32\sigma_3 G(x,p_l) + w}. \tag{3.30}$$

According to the above expression of $\Delta^2(u+Tu)$ in different regions, the dominate terms of $u + Tu \in S_\rho(Q)$ can be obtained as follows:

Lemma 3.2 *Let $u = u_{P,\Lambda,A} + w \in S_\rho(Q)$, then as $\rho \to 32m\sigma_3$,*

(i) $$\langle \Delta(u+Tu), \Delta w_1\rangle = B(w,w_1) + O(\varepsilon^2 \|w_1\|_{H_0^2}),$$

where

$$B(w,w_1) := \int_\Omega \Delta w \Delta w_1 - \sum_{j=1}^m \int_{B_{\delta_0}(p_j)} \rho h(p_j) e^{v_j} w w_1$$

is a positive, symmetric, bilinear form satisfying $B(w,w) \geq c\|w\|_{H_0^2}$ for some constant $c > 0$;

(ii) $$\langle \Delta(u+Tu), \Delta\partial_{p_j} u_j\rangle = 64\pi^2 \nabla H_j(p_j) + O\Big(|a_j - 1|\lambda_j + |\frac{e^{t_j}}{\int_\Omega h e^u} - 1| + \varepsilon_j^2\Big);$$

$$(iii)\quad \langle \Delta(u+Tu), \Delta\partial_{\lambda_j}u_j\rangle = -(a_j-1)\Big(128\pi^2\lambda_j - \tfrac{640}{3}\pi^2 + 128\pi^2\log\tfrac{\rho h(p_j)}{\alpha_4} + (64\pi^2)^2R_4(p_j,p_j)\Big) - \sum_{l\neq j}(a_l-1)(64\pi^2)^2G_4(p_l,p_j) - 64\pi^2\Big(\frac{e^{t_j}}{\int_\Omega he^u}-1\Big) + O(\varepsilon_j^3);$$

$$(iv)\quad \langle \Delta(u+Tu), \Delta u_j\rangle = \Big(2\lambda_j - \tfrac{10}{3} + 2\log\tfrac{\rho h(p_j)}{\alpha_4} + 64\pi^2R_4(p_j,p_j)\Big)\langle\Delta(u+Tu),\Delta\partial_{\lambda_j}u_j\rangle + 64\pi^2\sum_{l\neq j}G_4(p_l,p_j)\langle\Delta(u+Tu),\Delta\partial_{\lambda_j}u_j\rangle + 128\pi^2(a_j-1)\lambda_j + O(\varepsilon_j^2).$$

We prove Lemma 3.2 in Appendix because its proof involves a lot of computations. Using Lemma 3.2, we can deform T to a simpler operator on $S_\rho(Q)$. For $u = u_{P,\Lambda,A} + w \in S_\rho(Q)$ and $0 \le t \le 1$, define T_t by the following inner products:

$$\langle \Delta(u+T_tu), \Delta w_1\rangle = t\langle\Delta(u+Tu),\Delta w_1\rangle + (1-t)\langle\Delta w,\Delta w_1\rangle$$

for $w_1 \in S_\rho(Q)$,

$$\langle \Delta(u+T_tu), \Delta\partial_{p_j}u_j\rangle = t\langle\Delta(u+Tu),\Delta\partial_{p_j}u_j\rangle + (1-t)(64\pi^2\nabla H_j(p_j)),$$

$$\begin{aligned}\langle \Delta(u+T_tu), \Delta\partial_{\lambda_j}u_j\rangle &= t\langle\Delta(u+Tu),\Delta\partial_{\lambda_j}u_j\rangle + (1-t)\Big\{-(a_j-1)\Big(128\pi^2\lambda_j - \tfrac{640}{3}\pi^2 + \\ &\quad 128\pi^2\log\tfrac{\rho h(p_j)}{\alpha_4} + (64\pi^2)^2R_4(p_j,p_j)\Big) - \sum_{l\neq j}(a_l-1)(64\pi^2)^2G_4(p_l,p_j) \\ &\quad -64\pi^2\Big(\frac{e^{t_j}}{\int_\Omega he^u}-1\Big)\Big\},\end{aligned}$$

$$\begin{aligned}\langle \Delta(u+T_tu), \Delta u_j\rangle &= t\Big(2\lambda_j - \tfrac{10}{3} + 2\log\tfrac{\rho h(p_j)}{\alpha_4} + 64\pi^2R_4(p_j,p_j)\Big)\langle\Delta(u+Tu),\Delta\partial_{\lambda_j}u_j\rangle \\ &\quad + t64\pi^2\sum_{l\neq j}G_4(p_l,p_j)\langle\Delta(u+Tu),\Delta\partial_{\lambda_j}u_j\rangle + t\Big(O(\varepsilon_j^2)\Big) \\ &\quad +(1-t)128\pi^2(a_j-1)\lambda_j.\end{aligned}$$

Then T_0 is simpler than T, and it is trivial in the direction $w_1 \in O_{P,\Lambda}$. Then, we have the following:

Lemma 3.3 *Assume $(\rho - 32m\sigma_3)l(Q) > 0$. There is $\varepsilon_1 > 0$ such that if $|\rho - 32m\sigma_3| < \varepsilon_1$, then $u + T_tu \neq 0$ for any $u \in \partial S_\rho(Q)$ and $0 \le t \le 1$.*

Although the proof is very similar as the proof of Lemma 4.3 in [8], here we prove it again for completeness.

Proof: Assume $u = u_{P,\Lambda,A} + w \in \bar{S}_\rho(Q)$ and $u + T_tu = 0$ for some $0 \le t \le 1$. We will show $u \notin \partial S_\rho(Q)$.

From $\langle\Delta(u+T_tu),\Delta w\rangle=0$, we have by Lemma 3.2

$$\|w\|^2_{\mathcal{X}}\le O(\varepsilon_j^2)\|w\|_{H_0^2},$$

which implies that

$$\|w\|_{\mathcal{X}}=O(\varepsilon_j^2)<C\varepsilon_j^2 \tag{3.31}$$

provided that C is large.

Using $\langle\Delta(u+T_tu),\Delta\partial_{\lambda_j}u_j\rangle=0$, $\langle\Delta(u+T_tu),\Delta u_j\rangle=0$, (3.31) and part (iv) of Lemma 3.2 yields

$$128\pi^2\lambda_j(a_j-1)=O(e^{-\frac{\lambda_j}{2}})\quad\text{for}\quad j=1,\dots,m,$$

that is, when ρ is close to $32m\sigma_3$,

$$|a_j-1|=O(\lambda_j^{-1}e^{-\frac{\lambda_j}{2}})<Ce^{-\frac{\lambda_1(Q)}{2}}\qquad\text{for}\quad 1\le j\le m. \tag{3.32}$$

Next we estimate the term $\dfrac{e^{t_j}}{\int_\Omega he^u}-1$. Since ρ is close to $32m\sigma_3$, we have that λ_j is large,

$$|\lambda_j-\lambda_j(Q)|=O(1)\quad\text{and}\quad\lambda_1=\lambda_j+O(1).$$

$$\begin{aligned}e^{-\frac{\lambda_j}{2}}&=e^{-\frac{t_j}{2}}\frac{\rho h(p_j)}{\alpha_4}e^{\frac{G_j^*(p_j)}{2}}=e^{-\frac{t_1}{2}}\frac{\rho h(p_j)}{\alpha_4}e^{\frac{G_j^*(p_j)}{2}}+O(\varepsilon_j^2)\\&=e^{-\frac{\lambda_1}{2}}\frac{h(p_j)}{h(p_1)}e^{\frac{G_j^*(p_j)-G_1^*(p_1)}{2}}+O(\varepsilon_j^2).\end{aligned} \tag{3.33}$$

By $\langle\Delta(u+T_tu),\Delta\partial_{\lambda_j}u_j\rangle=0$, (iii) of Lemma 3.2 and (3.32) yield

$$e^{t_j}(\int_\Omega he^u)^{-1}-1=O(e^{-\frac{\lambda_1(Q)}{2}}), \tag{3.34}$$

$$e^{t_j}(\int_\Omega he^u)^{-1}-1+2\lambda_j(a_j-1)=O(e^{-\frac{\lambda_1(Q)}{2}}). \tag{3.35}$$

By $\langle\Delta(u+T_tu),\Delta\partial_{p_j}u_j\rangle=0$ and (3.32), (ii) of Lemma 3.2 yields

$$\begin{aligned}|\nabla H_j(p_j)|&=O\Big(\lambda_j|a_j-1|+e^{t_j}(\int_\Omega he^u)^{-1}-1+e^{-\frac{\lambda_j}{2}}\Big)\\&\le O(1)e^{-\frac{\lambda_1(Q)}{2}}.\end{aligned} \tag{3.36}$$

Also we have

$$|P-Q|\le c|\nabla H_j(p_j)|\le O(1)e^{-\frac{\lambda_1(Q)}{2}}<Ce^{-\frac{\lambda_1(Q)}{2}}. \tag{3.37}$$

It remains to estimate $t_j - t_1$ and $\lambda_1 - \lambda_1(Q)$. By $|t_j - t_1| \le Ce^{-\frac{\lambda_1(Q)}{2}}$,

$$\begin{aligned} e^{t_l - t_j} &= 1 + (t_l - t_j) + O(C^2 e^{-\lambda_1(Q)}) \\ &= 1 + (t_l - t_j) + O(e^{-\frac{\lambda_1(Q)}{2}}). \end{aligned}$$

By Lemma 3.1, we have

$$\begin{aligned} e^{-t_j}\int_\Omega he^u &= \sum_{l=1}^m \Big(64\pi^2 + 128\pi^2(a_l - 1)\lambda_l + \alpha_4^{\frac{1}{2}} l_l(Q)e^{-\frac{\lambda_l}{2}}\Big)e^{t_l - t_j} \\ & +O(\sum_{l=1}^m |a_l - 1|e^{t_l - t_j}) + o(\sum_{l=1}^m e^{t_l - t_j - \frac{\lambda_l}{2}}) \\ &= \sum_{l=1}^m \Big(64\pi^2(1 + t_l - t_j) + 128\pi^2(a_l - 1)\lambda_l + \alpha_4^{\frac{1}{2}} l(Q_l)e^{-\frac{\lambda_l}{2}}\Big) \\ & +O(\sum_{l=1}^m |a_l - 1|) + o(\sum_{l=1}^m e^{-\frac{\lambda_l}{2}}) \end{aligned} \tag{3.38}$$

and

$$\frac{e^{t_j}}{\rho\int_\Omega he^u} = \frac{1}{64\pi^2 m}\Big(1 + o(\sum_{l=1}^m \varepsilon_l)\Big). \tag{3.39}$$

Thus

$$\begin{aligned} \frac{e^{t_j}}{\int_\Omega he^u} - 1 &= \frac{e^{t_j}}{\rho\int_\Omega he^u}(\rho - \frac{\rho\int_\Omega he^u}{e^{t_j}}) \\ &= \frac{1}{64\pi^2 m}\Big\{\rho - 64\pi^2 m - \sum_{l=1}^m \Big[64\pi^2(t_l - t_j) + 128\pi^2\lambda_l(a_l - 1) + \alpha_4^{\frac{1}{2}} l_l(Q)e^{-\frac{\lambda_l}{2}}\Big]\Big\} \\ & +O(\sum_{l=1}^m |a_l - 1|) + o(\sum_{l=1}^m e^{-\frac{\lambda_1(Q)}{2}}). \end{aligned} \tag{3.40}$$

Now taking the summation of (3.40) from $j = 1, \ldots, m$ and by (3.35), we obtain

$$
\begin{aligned}
O(1)e^{-\frac{\lambda_1(Q)}{2}} &= \sum_j \Big[\Big(\frac{e^{t_j}}{\int_\Omega he^u} - 1\Big) + 2\lambda_j(a_j - 1)\Big] \\
&= \frac{1}{64\pi^2}\Big(\rho - 32m\sigma_3 - \sum_l \alpha_4^{\frac{1}{2}} l_l(Q) e^{-\frac{\lambda_l}{2}}\Big).
\end{aligned}
$$

Hence

$$
O(e^{-\frac{\lambda_1(Q)}{2}}) = \frac{\alpha_4^{\frac{1}{2}}}{64\pi^2}\frac{l(Q)}{h(p_1)} e^{-\frac{G_1^*(p_1)}{2}}\Big(e^{-\frac{\lambda_1(Q)}{2}} - e^{-\frac{\lambda_1}{2}}\Big),
$$

which implies that

$$
|\lambda_1(Q) - \lambda_1| = O(1) < C. \tag{3.41}
$$

To obtain estimates for $\lambda_j - \lambda_1, j \geq 2$, or equivalently for $t_j - t_1, j \geq 2$, we have by part (iii) of Lemma 3.2 and (3.40)

$$
\begin{aligned}
0 &= \langle \Delta(u + T_t u), \Delta\partial_{\lambda_j} u_j\rangle = O(e^{-\frac{\lambda_1}{2}}) - 64\pi^2\Big(\frac{e^{t_j}}{\int_\Omega he^u} - 1\Big) \\
&= O(e^{-\frac{\lambda_1}{2}}) - \frac{1}{m}(\rho - 32m\sigma_3) - \frac{64\pi^2}{m}\sum_l (t_l - t_j).
\end{aligned}
$$

Therefore,

$$
|t_j - \frac{1}{m}\sum_l t_l| = O(e^{-\frac{\lambda_1}{2}})
$$

and

$$
\begin{aligned}
|t_j - t_1| &\leq |t_j - \tfrac{1}{m}\sum_l t_l| + |\tfrac{1}{m}\sum_l t_l - t_1| \\
&= O(e^{-\frac{\lambda_1}{2}}) < Ce^{-\frac{\lambda_1(Q)}{2}} \quad \text{for} \quad j \geq 2.
\end{aligned} \tag{3.42}
$$

From (3.31), (3.32), (3.37), (3.41) and (3.42), we obtain $u \notin \partial S_\rho(Q)$. The proof is complete. □

Next we start to derive our degree counting formulas. We denote

$$
B_R = \{u \in \mathcal{X} : \quad \|u\|_{\mathcal{X}} < R\}.
$$

Then as in [8], for $\rho > 32m\sigma_3$, the degree $d(\rho)$ of the nonlinear map $u + Tu$ can be counted by

$$
d_m^+ - d_m^- = \frac{1}{m!}\sum_{l(Q_i)>0} \deg(u + Tu; S_\rho(Q_i), 0) - \frac{1}{m!}\sum_{l(Q_i)<0} \deg(u + Tu; S_\rho(Q_i), 0),
$$

where the summation is over all the critical points Q_i of f_h such that $l(Q_i) > 0 (l(Q_i) < 0)$. To computer $\deg(u + Tu; S_\rho(Q_i), 0)$, we set $\Phi_i := (\Phi_{i,1}, \Phi_{i,2}, \Phi_{i,3}) : S^*(Q_i) \to \mathbb{R}^{4m} \times \mathbb{R}^m \times \mathbb{R}^m$ by

$$\Phi_{i,1}^{(j)} = \langle\ \Delta(u + T_0 u), \Delta\partial_{p_j} u_j\rangle = 64\pi^2 \nabla H_j(p_j),$$

$$\Phi_{i,2}^{(j)} = \langle\ \Delta(u + T_0 u), \Delta\partial_{\lambda_j} u_j\rangle,$$

$$\Phi_{i,3}^{(j)} = \langle\ \Delta(u + T_0 u), \Delta u_j\rangle = 128\pi^2 (a_j - 1)\lambda_j,$$

where $S^*(Q_i) = \{(P, \Lambda, A) : u_{P,\Lambda,A} + w \in S_\rho(Q_i)\}$ and $\Phi_{i,l}^{(j)}, j = 1, \ldots, m$ are components of $\Phi_{i,l}, l = 1, 2, 3$. Clearly,

$$\deg(u + Tu; S_\rho(Q_i), 0) = \deg(\Phi_i; S^*(Q_i), 0). \tag{3.43}$$

Now we state the degree-counting formula for (3.43).

Lemma 3.4 *Assume h satisfy the conditions (i) and (ii). Then*

$$deg(\Phi_i; S^*(Q_i), 0) = sgn(\rho - 32m\sigma_3)(-1)^{m+\mathrm{ind}(Q_i)},$$

where

$$sgn(\rho - 32m\sigma_3) = \begin{cases} 1, & if \quad \rho > 32m\sigma_3, \\ -1, & if \quad \rho < 32m\sigma_3, \\ 0, & if \quad \rho = 32m\sigma_3. \end{cases}$$

Note that the sign for $\rho - 32m\sigma_3$ at a critical point Q is completely determined by the quantity $l(Q)$.

Proof: Let

$$\theta_j = \frac{1}{64\pi^2 m}\Big\{\rho - 64\pi^2 m - 64\pi^2 \sum_{l=1}^m \Big[(t_l - t_j) + 2\lambda_l(a_l - 1)\Big] - \sum_{l=1}^m \alpha_4^{\frac{1}{2}} l_l(Q) e^{-\frac{\lambda_l}{2}}\Big\}. \tag{3.44}$$

We note that in $\langle\Delta(u + T_0 u), \Delta\partial_{\lambda_j} u_j\rangle$, the term

$$\frac{e^{t_j}}{\int_\Omega he^u} - 1 = \theta_j + o(\theta_j)$$

can be further deformed such that $\dfrac{e^{t_j}}{\int_\Omega he^u} - 1$ is replaced by θ_j. For simplicity, we still denote our new map by Φ_i.

To compute the degree, we can simplify Φ_i further by setting $\hat{\Phi}_{i,1} = \Phi_{i,1}, \hat{\Phi}_{i,3} = \Phi_{i,3}$, and

$$\begin{aligned}
\hat{\Phi}^{(j)}_{i,2} &= \Phi^{(j)}_{i,2} + \Big(1 - \frac{5}{3\lambda_j} + \frac{1}{\lambda_j}\log\frac{\rho h(p_j)}{\alpha_4} + \frac{32\pi^2}{\lambda_j}R_4(p_j,p_j)\Big)\hat{\Phi}^{(j)}_{i,3} \\
&\quad + \sum_{l\neq j}\frac{32\pi^2}{\lambda_l}G_4(p_l,p_j)\hat{\Phi}^{(l)}_{i,3} - \frac{1}{m}\sum_{l=1}^{m}\hat{\Phi}^{(l)}_{i,3} \qquad (3.45)\\
&= -\frac{1}{m}\Big(\rho - 32m\sigma_3 - 64\pi^2\sum_{l=1}^{m}(t_l - t_j) - \alpha_4^{\frac{1}{2}}\sum_{l=1}^{m} l_l(Q)e^{-\frac{\lambda_l}{2}}\Big).
\end{aligned}$$

Clearly, we have

$$\frac{\partial\hat{\Phi}_{i,1}}{\partial\Lambda} = \frac{\partial\hat{\Phi}_{i,1}}{\partial A} = \frac{\partial\hat{\Phi}_{i,2}}{\partial A} = 0, \qquad (3.46)$$

and

$$\Phi_i(P,\Lambda,A) = 0 \quad \text{if and only if} \quad \hat{\Phi}_i(P,\Lambda,A) = 0,$$
$$\deg(\Phi_i; S^*(Q_i), 0) = \deg(\hat{\Phi}_i; S^*(Q_i), 0).$$

Note that $\hat{\Phi}_{i,1} = 64\pi^2\nabla H_j(p_j)$. Therefore $\hat{\Phi}_i(P,\Lambda,A) = 0$ if and only if

$$P = Q_i, \qquad A = (1,\dots,1) \quad \text{and}$$

$$\begin{cases} \rho - 32m\sigma_3 = \alpha_4^{\frac{1}{2}}\sum\limits_{l=1}^{m} l_l(Q)e^{-\frac{\lambda_l}{2}}, \\ t_1 = \cdots = t_m. \end{cases} \qquad (3.47)$$

It is not difficult to see that if $|\rho - 32m\sigma_3|$ is sufficiently small, equation (3.47) possesses a unique solution $(\lambda_1,\dots,\lambda_m)$ up to permutation. Let $\Lambda_i(\rho) = (\lambda_1,\dots,\lambda_m)$ denote the solution of (3.47). Hence $(Q_i, \Lambda_i(\rho), A_1)$ is the unique solution of $\hat{\Phi}_i = 0$, where $A_1 = (1,\dots,1)$. By (3.46), the degree of $\hat{\Phi}_i$ at $(Q_i,\Lambda_i(\rho),A_1)$ is the sign of the product of $\det\frac{\partial\hat{\Phi}_{i,1}}{\partial P}\cdot\det\frac{\partial\hat{\Phi}_{i,2}}{\partial\Lambda}\cdot\det\frac{\partial\hat{\Phi}_{i,3}}{\partial A}$. Thus,

$$\deg(\hat{\Phi}_i; S^*(Q_i), 0) = (-1)^{\operatorname{ind}(Q_i)}\operatorname{sgn}\det\Big(\frac{\partial\hat{\Phi}_{i,2}}{\partial\Lambda}\Big).$$

By (3.45), we have

$$\frac{1}{64\pi^2}\frac{\partial\hat{\Phi}^{(j)}_{i,2}}{\partial\lambda_j} = -\frac{m-1}{m}\frac{\partial t_j}{\partial\lambda_j} - \frac{1}{128m\pi^2}l_j(Q_i)\varepsilon_j^2 + o(\varepsilon_j^2),$$

and for $l \neq j$,

$$\frac{1}{64\pi^2}\frac{\partial\hat{\Phi}^{(j)}_{i,2}}{\partial\lambda_l} = \frac{1}{m}\frac{\partial t_l}{\partial\lambda_l} - \frac{1}{128m\pi^2}l_l(Q_i)\varepsilon_l^2 + o(\varepsilon_l^2).$$

Thus,

$$\frac{1}{64\pi^2}\sum_{l=1}^{m}\frac{\partial\hat{\Phi}_{i,2}^{(j)}}{\partial\lambda_l}=-\frac{\partial t_j}{\partial\lambda_j}+\frac{1}{m}\sum_{l=1}^{m}\frac{\partial t_l}{\partial\lambda_l}-\frac{1}{128m\pi^2}\sum_{l=1}^{m}l_l(Q_i)\varepsilon_l^2+o(\sum_{l=1}^{m}\varepsilon_l^2)$$

and

$$\frac{1}{64\pi^2}\sum_{j=1}^{m}\sum_{l=1}^{m}\frac{\partial\hat{\Phi}_{i,2}^{(j)}}{\partial\lambda_l}=-\frac{1}{128\pi^2}\sum_{l=1}^{m}l_l(Q_i)\varepsilon_l^2+o(\sum_{l=1}^{m}\varepsilon_l^2).$$

Therefore,

$$\begin{aligned}\det\Big[\frac{\partial\hat{\Phi}_{i,2}}{\partial\Lambda}\Big] &= \det\left[\begin{pmatrix} -m+1 & 1 & \dots & 1 & a_{1m} \\ 1 & -m+1 & \dots & 1 & a_{2m} \\ 1 & 1 & \dots & -m+1 & a_{m-1,m} \\ a_{m1} & a_{m2} & \dots & \dots & -\frac{1}{128\pi^2}\sum\limits_{l=1}^{m}l_l(Q_i)\varepsilon_l^2 \end{pmatrix}+o(\sum_{l=1}^{m}\varepsilon_l^2)\right] \\ &= (-1)^m\tau_0\sum_{l=1}^{m}l_l(Q_i)\varepsilon_l^2+o(\sum_{l=1}^{m}\varepsilon_l^2) \\ &= (-1)^m\tau_0(\rho-32m\sigma_3)+o(\sum_{l=1}^{m}\varepsilon_l^2),\end{aligned}$$

where

$$\tau_0=\frac{1}{128\pi^2}\det\begin{pmatrix} m-1 & 1 & \dots & 1 \\ 1 & m-1 & \dots & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & m-1 \end{pmatrix}_{(m-1)\times(m-1)} > 0$$

and $a_{ij}=O(\sum\limits_{l=1}^{m}\varepsilon_l^2)$.

Thus we have

$$\text{sgn}\ \det\Big(\frac{\partial\hat{\Phi}_{i,2}}{\partial\Lambda}\Big)=(-1)^m\text{sgn}(\rho-32m\sigma_3).$$

This proves Lemma 3.4. $\square$

Finally we are ready to complete the proof of Theorem 1.2.

Completion of the proof of Theorem 1.2: By Lemma 3.4 and the Hopf theorem, we get by hand that

$$\begin{aligned} d_m^+-d_m^- &= \tfrac{(-1)^m}{m!}\sum_{i=1}^{N}(-1)^{ind(Q_i)} \\ &= \tfrac{1}{m!}\Big(-\chi(\Omega)(-\chi(\Omega)+1)\cdots(-\chi(\Omega)+m-1)\Big)\end{aligned}$$

where $Q_1, \ldots, Q_N$ are all critical points of f_h and $\chi(\Omega)$ is the Euler characteristic of Ω. Then for any $\rho \in (32m\sigma_3, 32(m+1)\sigma_3)$, we have

$$d(\rho) = 1 + \sum_{j=1}^{m}(d_j^+ - d_j^-) = \frac{1}{m!}(-\chi(\Omega)+1)\ldots(-\chi(\Omega)+m).$$

The detail can be founded in the proof of Lemma 5.2 in [8]. $\square$

4 Proof of Theorem 1.5

In [24], Robert and Wei have given a complete proof of $(P1)-(P3)$ when $h(x) \equiv 1$ for problem (1.9). Now let $h(x)$ be a positive $C^{2,\beta}$ function. Since the proof is very similar, we only give an outline of proof of Theorem 1.5.

Let u_k be a sequence of solution of (1.9). We assume that

$$\max_{x\in\Omega} u_k(x) \to +\infty \qquad \text{as} \quad k \to +\infty.$$

Let $\alpha_k = \log \int_\Omega h(x)e^{u_k}$ and

$$\hat{u}_k = u_k - \alpha_k.$$

Define

$$u_k(p_k) = \max_{x\in\Omega} u_k(x), \qquad \frac{\mu_k}{\alpha_4^{\frac{1}{4}}} = e^{-\frac{1}{4}\hat{u}_k(p_k)}.$$

Then using Lemma 2.4 and Green's representation, along the line of [24], we can prove the following claims step by step. The proof is omitted here.

Claim 1: If $\max_{x\in\Omega} u_k(x) \to +\infty$, then $\alpha_k \geq C$ and $\lim_{k\to+\infty} \dfrac{d(p_k, \partial\Omega)}{\mu_k} = +\infty$.

Claim 2: $\lim_{k\to+\infty} u_k(p_k + \mu_k x) - u_k(p_k) = \log \dfrac{1}{(1+\sqrt{\rho_k h(p_k)}|x|^2)^4}$ in $C^4_{loc}(\mathbb{R}^4)$.

Claim 3: Assume that $\alpha_k \to +\infty$. Since $\rho_k \leq C$, there exists $S = \{p_1, \ldots, p_m\} \subset \bar{\Omega}$ for some positive integer m and $a_1, \ldots, a_m \geq 32\sigma_3$ such that $\lim_{k\to+\infty} u_k(x) = \sum\limits_{j=1}^{m} a_j G_4'(\cdot, p_j)$ in $C^4_{loc}(\bar{\Omega}\backslash S)$.

Next we prove $(P1)$, that is, we exclude the boundary blow-ups.

Claim 4: Assume that $\alpha_k \to +\infty$. Let $x_0 \in \partial\Omega$. Then $\lim_{r\to 0}\lim_{k\to+\infty}\int_{B_r(x_0)\cap\Omega} \rho_k h(x) e^{\hat{u}_k} dx = 0$. In particular, $S \cap \partial\Omega = \emptyset$.

Since we use different form of Pohozaev's identity in the proof with respect to [24], we give the proof of this claim for completeness.

Proof: Let $x_0 \in \partial\Omega \cap S$. Then $\lim_{r\to 0}\lim_{k\to+\infty}\int_{B_r(x_0)\cap\Omega} \rho_k h(x) e^{\hat{u}_k} dx \geq 32\sigma_3$. Thus there exists $\delta_0 > 0$ small such that for any $\delta \leq \delta_0$, $\int_{B_\delta(x_0)\cap\Omega} \rho_k h(x) e^{\hat{u}_k} dx \geq 16\sigma_3$. Furthermore, we may assume that $S \cap B_{\delta_0}(x_0) = \{x_0\}$.

Let $y_k = x_0 + \rho_{k,r}\nu(x_0)$ with

$$\rho_{k,r} = \frac{\int_{\partial\Omega\cap B_r(x_0)} (x - x_0, \nu)(\Delta u_k)^2 dx}{\int_{\partial\Omega\cap B_r(x_0)} (\nu(x_0)\cdot\nu)(\Delta u_k)^2 dx}, \tag{4.1}$$

where $r < \delta_0$ such that $\frac{1}{2} \le \nu(x_0)\cdot\nu \le 1$ for $x \in \bar{B}_r(x_0)\cap\partial\Omega$. Then it is easy to see that $|\rho_{k,r}| \le 2r$ and

$$\int_{\partial\Omega\cap B_r(x_0)} (x - y_k, \nu)(\Delta u_k)^2 = 0. \tag{4.2}$$

Now applying the Pohozaev's identity in $\Omega\cap B_r(x_0)$ with $\xi = y_k$ and using Dirichlet boundary condition and (4.2), we obtain that

$$\begin{aligned}\int_{\Omega\cap B_r(x_0)} \rho_k(4h + \langle x - y_k, \nabla h\rangle) \quad e^{\hat{u}_k} &= \int_{\Omega\cap\partial B_r(x_0)} \rho_k\langle x - y_k, \nu\rangle h(x)e^{u_k-\alpha_k} - 2\int_{\Omega\cap\partial B_r(x_0)} \frac{\partial u_k}{\partial\nu}\Delta u_k \\ &+ \int_{\Omega\cap\partial B_r(x_0)} \left[\frac{1}{2}(\Delta u_k)^2\langle x - y_k, \nu\rangle + \langle x - y_k, \nabla u_k\rangle\frac{\partial(-\Delta u_k)}{\partial\nu}\right] \\ &+ \int_{\Omega\cap\partial B_r(x_0)} \left[-\langle x - y_k, \nabla\Delta u_k\rangle\frac{\partial u_k}{\partial\nu} + \langle x - y_k, \nu\rangle\langle\nabla u_k, \nabla\Delta u_k\rangle\right].\end{aligned}$$

Note that $u_k \to \sum_{j=1}^{m} a_j G_4'(x, p_j)$ in $C^3(\bar{\Omega}\backslash S)$, where $G_4'(x, x_0) = 0$. Thus we obtain that all the terms in the last three integrals are of the term

$$\int_{\Omega\cap\partial B_r(x_0)} O(1) = O(r^3)$$

while

$$\int_{\Omega\cap\partial B_r(x_0)} \langle x - y_k, \nu\rangle\rho_k h(x)e^{u_k-\alpha_k}d\sigma = O(r^4).$$

On the other hand,

$$\int_{\Omega\cap B_r(x_0)} \rho_k\langle x - y_k, \nabla h(x)\rangle e^{\hat{u}_k}dx = \int_{\Omega\cap B_r(x_0)} \rho_k\langle x - y_k, \nabla\log h(x)\rangle h(x)e^{\hat{u}_k}dx = O(r).$$

Thus we obtain that

$$|\int_{\Omega\cap B_r(x_0)} 4\rho_k h(x)e^{\hat{u}_k}dx| \le Cr. \tag{4.3}$$

□

Claim 5: We show that $\alpha_k \to +\infty$. This is exactly the same as Claim 12 of [24].

Claim 6: $a_j = 32\sigma_3, j = 1, \ldots, m$ and p_j satisfies

$$\nabla\Big(\log h(p_j) + 32\sigma_3 R_4^{'}(p_j, p_j) + 32\sigma_3 \sum_{i\neq j} G_4^{'}(p_i, p_j)\Big) = 0, \qquad j = 1, \ldots, m. \tag{4.4}$$

Once we prove that there are no boundary bubbles for (1.9), to prove Theorem 1.5, we just need to change the Navier boundary condition to Dirichlet boundary condition. By almost the same computation in the proof of Theorem 1.1 and Theorem 1.2, we finish the proof of Theorem 1.5.

5 Appendix : Proof of Lemma 3.2

This final section is devoted to the proof of Lemma 3.2. According to Appendix B in [18], we get part (i) and the readers can also refer to Section 6 in [8].

Next we prove part (iii) at first. On $B_{2\delta_0}(p_j)$, we have

$$\partial_{\lambda_j} u_j = \partial_{\lambda_j}(v_j - s_j)\sigma_j + O(\varepsilon_j^2) = \frac{2}{1 + \frac{\rho h(p_j)}{\alpha_4} e^{\frac{\lambda_j}{2}} |x - p_j|^2}\sigma_j + O(\varepsilon_j^2) \tag{5.1}$$

and on $\Omega \backslash B_{2\delta_0}(p_j)$,

$$\partial_{\lambda_j} u_j = 0.$$

We compute $\langle \Delta(u + Tu), \Delta\partial_{\lambda_j} u_j\rangle = \langle \Delta^2(u + Tu), \partial_{\lambda_j} u_j\rangle$ by using (3.28)-(3.30).

Since $w \in O_{P,\Lambda}$, we have

$$\int_\Omega \Delta w \Delta \partial_{\lambda_j} u_j = 0. \tag{5.2}$$

By (5.1),

$$\begin{aligned} \int_{B_{\delta_0}(p_j)} \rho h(p_j) e^{v_j} \partial_{\lambda_j} u_j &= \alpha_4 \int_{\mathbb{R}^4} \frac{2}{(1 + |z|^2)^5} dz + O(\varepsilon_j^2) \\ &= 64\pi^2 + O(\varepsilon_j^2) \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} &\int_{B_{\delta_0}(p_j)} \rho h(p_j) e^{v_j} \Big[-4 \log\Big(1 + \frac{\rho h(p_j)}{\alpha_4} e^{\frac{\lambda_j}{2}} |x - p_j|^2\Big)\Big] \partial_{\lambda_j} u_j \\ = \;& \alpha_4 \int_{\mathbb{R}^4} \frac{8}{(1 + |z|^2)^5} [-\log(1 + |z|^2)] dz + O(\varepsilon_j^2) \\ = \;& -\frac{448}{3}\pi^2 + O(\varepsilon_j^2). \end{aligned} \tag{5.4}$$

Using $v_j - s_j = 2\lambda_j - 4\log\Big(1 + \frac{\rho h(p_j)}{\alpha_4} e^{\frac{\lambda_j}{2}} |x-p_j|^2\Big) + 2\log\frac{\rho h(p_j)}{\alpha_4} + 64\pi^2 R_4(p_j,p_j)$, we have

$$\begin{aligned} &-\int_{B_{\delta_0}(p_j)} \rho h(p_j) e^{v_j}(a_j-1)(v_j-s_j-1)\partial_{\lambda_j} u_j \\ =\ & (a_j-1)\Big[-128\pi^2\lambda_j + \frac{640}{3}\pi^2 - 128\pi^2 \log\frac{\rho h(p_j)}{\alpha_4} - (64\pi^2)^2 R_4(p_j,p_j) + O(\varepsilon_j^2)\Big]. \end{aligned} \tag{5.5}$$

and

$$\begin{aligned} &-\int_{B_{\delta_0}(p_j)} \rho h(p_j) e^{v_j} \sum_{l\neq j} 64\pi^2 (a_l-1) G_4(p_l,p_j)\partial_{\lambda_j} u_j \\ =\ & -\sum_{l\neq j}(64\pi^2)^2(a_l-1)G_4(p_l,p_j) + O(\sum_{l\neq j}|a_l-1|\varepsilon_j^2). \end{aligned} \tag{5.6}$$

Since $\nabla H_j(p_j)\cdot(x-p_j)$ is an odd function in $B_{\delta_0}(p_j)$,

$$\begin{aligned} &\int_{B_{\delta_0}(p_j)} \rho h(p_j) e^{v_j} \nabla H_j(p_j)\cdot(x-p_j)\partial_{\lambda_j} u_j \\ =\ & \int_{B_{\delta_0}(p_j)} \rho h(p_j) e^{v_j} \nabla H_j(p_j)\cdot(x-p_j)\Big(\frac{2}{1+\frac{\rho h(p_j)}{\alpha_4}e^{\frac{\lambda_j}{2}}|x-p_j|^2}\sigma_j + O(\varepsilon_j^2)\Big) \\ =\ & O(\varepsilon_j^3). \end{aligned} \tag{5.7}$$

To estimate the term with $w\partial_{\lambda_j} u_j$, we use the condition $w \in O_{P,\Lambda}$, that is

$$\begin{aligned} 0 &= \int_\Omega \Delta w \Delta \partial_{\lambda_j} u_j = \int_\Omega w\Delta^2 \partial_{\lambda_j} u_j \\ &= \int_{B_{\delta_0}(p_j)} w\Delta^2\partial_{\lambda_j} u_j + O(\varepsilon_j^2 \|w\|_{H_0^2(\Omega)}) \\ &= \int_{B_{\delta_0}(p_j)} w\Delta^2\partial_{\lambda_j} v_j + O(\varepsilon_j^4) \\ &= \int_{B_{\delta_0}(p_j)} w\rho h(p_j) e^{v_j}\partial_{\lambda_j} v_j + O(\varepsilon_j^4). \end{aligned} \tag{5.8}$$

Hence, by (3.24), we have

$$\begin{aligned} \int_{B_{\delta_0}(p_j)} w\rho h(p_j) e^{v_j}\partial_{\lambda_j} u_j &= \int_{B_{\delta_0}(p_j)} w\rho h(p_j) e^{v_j}\Big(1 + \partial_{\lambda_j} v_j + O(\varepsilon_j^2)\Big) \\ &= O(\varepsilon_j^4). \end{aligned} \tag{5.9}$$

Just by the similar computation in [8], we can obtain that

$$\int_{B_{\delta_0}(p_j)} E\partial_{\lambda_j} u_j = O(\varepsilon_j^3).$$

On $\Omega\backslash \cup_{j=1}^m B_{\delta_0}(p_j)$,

$$e^u = O(e^w), \quad \frac{\rho h e^u}{\int_\Omega h e^u} = O(e^{-\lambda_j})e^w, \qquad \partial_{\lambda_j} u_j = O(\varepsilon_j^2).$$

Thus,

$$\int_{\Omega\backslash\cup_{j=1}^m B_{\delta_0}(p_j)} \frac{\rho h e^u}{\int_\Omega h e^u}\partial_{\lambda_j} u_j = o(\varepsilon_j^4) \tag{5.10}$$

and

$$\int_{B_{2\delta_0}(p_j)\backslash B_{\delta_0}(p_j)} \Delta^2 u\partial_{\lambda_j} u_j = \int_{B_{2\delta_0}(p_j)\backslash B_{\delta_0}(p_j)} \Delta^2(u - 32\sigma_3 G_4(x,p_j))\partial_{\lambda_j} u_j = O(\varepsilon_j^4). \tag{5.11}$$

Hence

$$\begin{aligned}\langle \Delta(u+Tu), \Delta\partial_{\lambda_j} u_j\rangle \;&=\; -(a_j-1)\Big(128\pi^2\lambda_j - \frac{640}{3}\pi^2 + 128\pi^2\log\frac{\rho h(p_j)}{\alpha_4} + (64\pi^2)^2 R_4(p_j,p_j)\Big) \\ &\quad -\sum_{l\neq j}(64\pi^2)^2(a_l-1)G_4(p_l,p_j) - 64\pi^2\Big(\frac{e^{t_j}}{\int_\Omega he^u} - 1\Big) + O(\varepsilon_j^3).\end{aligned} \tag{5.12}$$

This proves part (iii).

For the proof of part (iv), since $\int_\Omega he^u \sim e^{\lambda_j}$, we write

$$\begin{aligned}\langle \Delta(u+Tu), \Delta u_j\rangle &= \langle \Delta^2(u+Tu), u_j\rangle + \int_{\partial\Omega} \Delta(u+Tu)\nabla u_j \\ &= \langle \Delta^2(u+Tu), u_j\rangle + \int_{\partial\Omega} \Delta Tu\nabla u_j \\ &= \langle \Delta^2(u+Tu), u_j\rangle + O(\varepsilon_j^4).\end{aligned}$$

Since $w \in S_\rho(Q)$,

$$\int_\Omega \Delta w\Delta u_j = 0.$$

By (3.10), on $B_{\delta_0}(p_j)$ we have

$$u_j = 2\lambda_j + 2\log\frac{\rho h(p_j)}{\alpha_4} + 64\pi^2 R_4(p_j,p_j) - 4\log\Big(1+\sqrt{\frac{\rho h(p_j)}{\alpha_4}}e^{\frac{\lambda_j}{2}}|x-p_j|^2\Big) + O(\varepsilon_j^2)$$

and

$$v_j - s_j - 1 = 2\lambda_j + 2\log\frac{\rho h(p_j)}{\alpha_4} + 64\pi^2 R_4(p_j,p_j) - 4\log\Big(1+\sqrt{\frac{\rho h(p_j)}{\alpha_4}}e^{\frac{\lambda_j}{2}}|x-p_j|^2\Big) - 1.$$

By direct computation, we have

$$\int_{B_{\delta_0}(p_j)} \rho h(p_j)e^{v_j} = 64\pi^2 + O(\varepsilon_j^4)$$

and

$$\int_{B_{\delta_0}(p_j)} \rho h(p_j)e^{v_j}\log\Big(1+\sqrt{\frac{\rho h(p_j)}{\alpha_4}}e^{\frac{\lambda_j}{2}}|x-p_j|^2\Big) = \frac{160\pi^2}{3} + O(\varepsilon_j^4).$$

Then

$$\begin{aligned}
&\int_{B_{\delta_0}(p_j)} \rho h(p_j)e^{v_j}(v_j - s_j - 1)u_j \\
= &\int_{B_{\delta_0}(p_j)} \rho h(p_j)e^{v_j}\Big\{4\lambda_j^2 + 2\lambda_j\Big[4\log\frac{\rho h(p_j)}{\alpha_4} + 128\pi^2 R_4(p_j,p_j) \\
&-8\log(1+\sqrt{\frac{\rho h(p_j)}{\alpha_4}}e^{\frac{\lambda_j}{2}}|x-p_j|^2) - 1\Big] + O(1)\Big\} \\
= &\ 256\pi^2\lambda_j^2 + 512\pi^2\lambda_j\Big(\log\frac{\rho h(p_j)}{\alpha_4} + 32\pi^2 R_4(p_j,p_j) - \frac{23}{12}\Big) + O(1).
\end{aligned} \tag{5.13}$$

Similarly,

$$\int_{B_{\delta_0}(p_j)} \rho h(p_j)e^{v_j}u_j = 128\pi^2\lambda_j + 128\pi^2\Big(\log\frac{\rho h(p_j)}{\alpha_4} + 32\pi^2 R_4(p_j,p_j) - \frac{5}{3}\Big) + O(\varepsilon_j^2) \tag{5.14}$$

and

$$\begin{aligned}
&\int_{B_{\delta_0}(p_j)} \rho h(p_j)e^{v_j}\nabla H_j(p_j)\cdot(x-p_j)u_j \\
= &\int_{B_{\delta_0}(p_j)} \rho h(p_j)e^{v_j}|\nabla H_j(p_j)\cdot(x-p_j)| \times O(\varepsilon_j^2) \\
= &\ O(\varepsilon_j^3)
\end{aligned} \tag{5.15}$$

since $\nabla H_j(p_j)\cdot(x-p_j)$ is an odd function in $B_{\delta_0}(p_j)$.

$$\int_{B_{\delta_0}(p_j)} \rho h(p_j)e^{v_j}\sum_{l=1}^m (a_l - 1)O(|x-p_j|)u_j = O(\sum_{l=1}^m |a_l - 1|\lambda_j\varepsilon_j). \tag{5.16}$$

Using (3.24) and Hölder's inequality, we have

$$
\begin{aligned}
&\int_{B_{\delta_0}(p_j)} \rho h(p_j) e^{v_j} w u_j \\
= \;& \int_{B_{\delta_0}(p_j)} \rho h(p_j) e^{v_j} w \Big[\lambda_j - s_j - 4\log\Big(1+\sqrt{\frac{\rho h(p_j)}{\alpha_4}} e^{\frac{\lambda_j}{2}} |x-p_j|^2\Big) + O(\varepsilon_j^2)\Big] \\
\le \;& O(\lambda_j \varepsilon_j^2 \|w\|_{H_0^2(\Omega)}) + C\Big(\int_{B_{\delta_0}(p_j)} \rho h(p_j) e^{v_j} w^2\Big)^{\frac{1}{2}} \qquad (5.17)\\
& \times \Big\{\int_{B_{\delta_0}(p_j)} \rho h(p_j) e^{v_j} \Big[-4\log\Big(1+\sqrt{\frac{\rho h(p_j)}{\alpha_4}} e^{\frac{\lambda_j}{2}} |x-p_j|^2\Big) + O(\varepsilon_j^2)\Big]^2\Big\}^{\frac{1}{2}} \\
= \;& O(1)\|w\|_{H_0^2(\Omega)}.
\end{aligned}
$$

By a similar argument as in [8], we can get

$$
\int_{B_{\delta_0}(p_j)} |E| u_j = o(\varepsilon_j^2). \tag{5.18}
$$

For the integration over $B_{\delta_0}(p_l), l \neq j$, the dominated term is

$$
\begin{aligned}
& -\int_{B_{\delta_0}(p_l)} \rho h(p_l) e^{v_l} \Big[(a_l - 1)(v_l - s_l - 1) + \Big(\frac{e^{t_l}}{\int_\Omega h e^u} - 1\Big)\Big] u_j \\
= \;& -(64\pi^2)^2 G_4(p_l, p_j) \Big[2(a_l - 1)\lambda_l + \Big(\frac{e^{t_l}}{\int_\Omega h e^u} - 1\Big)\Big] + O(\varepsilon_j^2) \qquad (5.19)\\
= \;& 64\pi^2 \sum_{l\neq j} G_4(p_l, p_j) \langle \Delta(u + Tu), \Delta \partial_{\lambda_j} u_j \rangle + O(\varepsilon_j^2).
\end{aligned}
$$

It is easy to see that the other terms are bounded by $O(\varepsilon_j^2)$.

For the integration outside of $\cup_{j=1}^m B_{\delta_0}(p_j)$, we have that $\int_\Omega h e^u = O(e^{t_j})$ and $u_j = O(1)$. Then

$$
\int_{B_{2\delta_0}(p_j)\setminus B_{\delta_0}(p_j)} \frac{\rho h e^u}{\int_\Omega h e^u} u_j = O(e^{-t_j}) = O(\varepsilon_j^4) \tag{5.20}
$$

and

$$
\int_{B_{2\delta_0}(p_j)\setminus B_{\delta_0}(p_j)} u_j \Delta^2 u = \int_{B_{2\delta_0}(p_j)\setminus B_{\delta_0}(p_j)} \Delta^2(u - 32\sigma_3 G_4(x, p_j)) u_j = O(\varepsilon_j^2). \tag{5.21}
$$

Similarly,

$$
\int_{\Omega\setminus\cup_{j=1}^m B_{2\delta_0}(p_j)} \frac{\rho h e^u}{\int_\Omega h e^u} u_j = O(\varepsilon_j^4) \tag{5.22}
$$

and

$$\int_{\Omega\setminus\cup_{j=1}^m B_{2\delta_0}(p_j)} u_j\Delta^2 u = O(\varepsilon_j^2). \tag{5.23}$$

So, according to the above estimates we have

$$\begin{aligned}
& \langle\Delta(u+Tu),\Delta u_j\rangle \\
= \ & -(a_j-1)\Big[256\pi^2\lambda_j^2+512\pi^2\lambda_j\Big(\log\frac{\rho h(p_j)}{\alpha_4}+32\pi^2R_4(p_j,p_j)-\frac{23}{12}\Big)\Big] \\
& -2(64\pi^2)^2\sum_{l\neq j}(a_l-1)G_4(p_l,p_j)\lambda_l-128\pi^2\Big(\lambda_j+\log\frac{\rho h(p_j)}{\alpha_4}+32\pi^2R_4(p_j,p_j)-\frac{5}{3}\Big) \\
& \times\Big(\frac{e^{t_j}}{\int_\Omega he^u}-1\Big)+64\pi^2\sum_{l\neq j}G_4(p_l,p_j)\langle\Delta(u+Tu),\Delta\partial_{\lambda_j}u_j\rangle+O(\varepsilon_j^2) \\
= \ & \Big[2\lambda_j-\frac{10}{3}+2\log\frac{\rho h(p_j)}{\alpha_4}+64\pi^2R_4(p_j,p_j)\Big]\langle\Delta(u+Tu),\Delta\partial_{\lambda_j}u_j\rangle \\
& +64\pi^2\sum_{l\neq j}G_4(p_l,p_j)\langle\Delta(u+Tu),\Delta\partial_{\lambda_j}u_j\rangle+128\pi^2(a_j-1)\lambda_j+O(\varepsilon_j^2).
\end{aligned}$$

Finally we prove part (ii). From $w\in O_{P,\Lambda}$, we have

$$\int_\Omega \Delta w\Delta\partial_{p_j}u_j=0. \tag{5.24}$$

On $B_{\delta_0}(p_j)$,

$$\begin{aligned}
\partial_{p_j}u_j &= -\partial_x v_j+\frac{\partial_{p_j}h(p_j)}{h(p_j)}\partial_{\lambda_j}(v_j-s_j)+O(1) \\
&= -\partial_x v_j+O(1).
\end{aligned} \tag{5.25}$$

Since $\partial_x v_j$ is an odd function,

$$\int_{B_{\delta_0}(p_j)}\rho h(p_j)e^{v_j}(v_j-s_j-1)(\partial_x v_j+O(1))=O(\lambda_j), \tag{5.26}$$

$$\int_{B_{\delta_0}(p_j)}\rho h(p_j)e^{v_j}(\partial_x v_j+O(1))=O(1), \tag{5.27}$$

and

$$\begin{aligned}
&\int_{B_{\delta_0}(p_j)} \rho h(p_j) e^{v_j} \nabla H_j(p_j)\cdot(x-p_j)\partial_x v_j \\
&= 384\nabla H_j(p_j)\int_{|y|\le\delta_0\sqrt{\frac{\rho h(p_j)}{\alpha_4}}e^{\frac{\lambda_j}{4}}} \frac{2|y|^2}{(1+|y|^2)^5}dy \qquad (5.28)\\
&= 64\pi^2\nabla H_j(p_j)+O(\varepsilon_j^4).
\end{aligned}$$

Thus

$$\begin{aligned}
&\int_{B_{\delta_0}(p_j)} \rho h(p_j) e^{v_j} \nabla H_j(p_j)\cdot(x-p_j)\partial_{p_j} u_j \\
&= (64\pi^2+O(\varepsilon_j))\nabla H_j(p_j)+O(\varepsilon_j^4) \qquad (5.29)\\
&= 64\pi^2\nabla H_j(p_j)+O(\varepsilon_j^3)
\end{aligned}$$

since $\nabla H_j(p_j)=\nabla H_j(q_j)+O(|p_j-q_j|)=O(|p_j-q_j|)=O(\varepsilon_j^2)$.

Using (3.24), by the similar computation as (5.8) and (5.9), we have

$$\begin{aligned}
\int_{B_{\delta_0}(p_j)} \rho h(p_j)e^{v_j} w\partial_{p_j}u_j &= \int_{B_{\delta_0}(p_j)} \rho h(p_j)e^{v_j} w\Big(\partial_{p_j}v_j+2\frac{\partial_{p_j}h(p_j)}{h(p_j)}+128\pi^2\partial_x R_4(x,p_j)|_{x=p_j}\Big)\\
&= O(\varepsilon_j^2\|w\|_{H_0^2(\Omega)})=O(\varepsilon_j^4).
\end{aligned} \qquad (5.30)$$

Also we have

$$|\int_{B_{\delta_0}(p_j)} E\partial_{p_j}u_j| = O(\varepsilon_j^3). \qquad (5.31)$$

On $\Omega\backslash\cup_{j=1}^m B_{\delta_0}(p_j)$, $\partial_{p_j}u_j=O(1)$. Hence,

$$\int_{\Omega\backslash B_{\delta_0}(p_j)} \Delta^2(u-64\pi^2G_4(x,p_j))\partial_{p_j}u_j=O(\varepsilon_j^2) \qquad (5.32)$$

and by $\int_\Omega he^u\sim e^{\lambda_j}$,

$$\int_{\Omega\backslash B_{\delta_0}(p_j)} \frac{\rho he^u}{\int_\Omega he^u}\partial_{p_j}u_j = O\Big(e^{-\lambda_j}\int_\Omega e^w\Big)=O(\varepsilon_j^4). \qquad (5.33)$$

Combining (5.24)-(5.33), we get part (ii).

Hence, we finish the proof of Lemma 3.2.

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