LOCATING THE BOUNDARY PEAKS OF LEAST-ENERGY SOLUTIONS TO A SINGULARLY PERTURBED DIRICHLET PROBLEM

TERESA D'APRILE AND JUNCHENG WEI

ABSTRACT. We consider the problem

$$\varepsilon^2\Delta v - v - \gamma_1 V v + f(v) = 0 \quad \Delta V + \gamma_2 |v|^2 = 0, \quad v = V = 0 \text{ on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^3$ is a smooth and bounded domain, $\varepsilon,\, \gamma_1,\, \gamma_2 > 0$, $v,\, V : \Omega \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$. We prove that this system has a *least-energy solution* v_ε which develops, as $\varepsilon \to 0^+$, a single spike layer located near the boundary, in striking contrast with the result in [35] for the single Schrödinger equation. Moreover the unique peak approaches the *most curved* part of $\partial\Omega$, i.e. where the boundary mean curvature assumes its maximum. Thus this elliptic system, even though it is a Dirichlet problem, acts more like Neumann problem for the single equation case. The technique employed is based on the so called energy method, which consists in the derivation of an asymptotic expansion for the energy of the solutions in power of ε up to the sixth order; from the analysis of the main terms of the energy expansion we derive the location of the peak in Ω.

1. INTRODUCTION

In this paper we study the following problem:

$$\begin{cases} \varepsilon^2\Delta v - v - \gamma_1 V v + f(v) = 0 \quad \text{in } \Omega, \\ \Delta V + \gamma_2 v^2 = 0 \quad \text{in } \Omega, \\ v,\, V > 0 \text{ in } \Omega, \quad v = V = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ is a smooth and bounded domain, $\varepsilon,\, \gamma_1,\, \gamma_2 > 0$, $v,\, V : \Omega \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$. Solutions of (1.1) correspond to the stationary waves for the following Schrödinger-Poisson system:

$$i\varepsilon\frac{\partial\psi}{\partial t} = -\varepsilon^2\Delta\psi + \psi + \gamma_1 V\psi - f(\psi), \quad \Delta V + \gamma_2|\psi|^2 = 0.$$

This system, first proposed by Benci-Fortunato ([2]) and later studied in [4], [10], can be used as a model in Quantum Mechanics to describe a charged particle interacting with its own electrostatic field. The purpose of this paper is to construct a single spike for the system (1.1) located near the boundary, where by single spike we intend a solution whose shape has the form of a unique peak which becomes highly concentrated when ε is sufficiently small.

When $\gamma_2 = 0$ (1.1) reduces to the single Schrödinger equation:

$$\varepsilon^2\Delta v - v + f(v) = 0, \tag{1.2}$$

1991 *Mathematics Subject Classification.* Primary 35B40, 35B45; Secondary 35J55, 92C15, 92C40.
Key words and phrases. Least energy solutions, Maxwell-Schrödinger Equation, mean curvature, energy expansion.

for which the existence of single and multiple spike solutions has been extensively studied. Concerning equation (1.2) in a bounded domain with Neumann boundary condition, Ni and Takagi in [32]-[33] first proved that for ε sufficiently small there is a least energy solution v_ε with the property that v_ε has exactly one maximum point P_ε in $\overline{\Omega}$, and P_ε must be located on $\partial\Omega$ and near the *most curved* part of the $\partial\Omega$, i.e., $\mathcal{H}(P_\varepsilon) \to \max_{P\in\partial\Omega} \mathcal{H}(P)$, where $\mathcal{H}(P)$ denotes the mean curvature of the boundary $\partial\Omega$. On the other hand, for equation (1.2) in a bounded domain with Dirichlet boundary conditions, Ni and Wei in [35] showed that the least energy solution develops a spike layer at the *most centered* part of the domain, i.e., $dist(P_\varepsilon, \partial\Omega) \to \max_{P\in\Omega} dist(P, \partial\Omega)$. Since then, there have been many papers looking for higher energy solutions. More specifically, solutions with multiple boundary peaks as well as multiple interior peaks have been established. It turns out that *a general guideline is that while multiple boundary spikes tend to cluster around the critical points of the boundary mean curvature $\mathcal{H}(P)$, the location of the interior spikes is governed by the distance between the peaks as well as from the boundary $\partial\Omega$* (see [1], [6], [7], [8], [16], [17], [23], [24], [25], [26], [27], [30], [31], [32], [33], [34], [35], [37], [38] and the references therein). In particular, it was established by Gui and Wei ([26]), that given two arbitrary integers l_1 and l_2 there exist solutions for the Neumann problem associated to (1.2) with l_1 peaks on the boundary and l_2 peaks in the interior.

The asymptotic analysis of (1.1) has been started very recently in the papers [12]-[15] and [36]. The radially symmetric case has been investigated in [11], [13], [14] and [36]. In [14] and [36] it is proved that for $1 < p < \frac{11}{7}$ there exists a family of positive radial solutions in $\mathbb{R}^3$ which concentrates at a sphere. In [11] and [13] the concentration on an internal sphere and on all the boundary $\partial\Omega$ respectively is produced for the problem (1.1) when Ω is the unit ball of $\mathbb{R}^3$. In the other recent papers [12] and [15] multiple interior spikes have been shown to exist for (1.1) in the case of general bounded domain $\Omega \subset \mathbb{R}^3$ (near the harmonic centers of Ω) or of all $\mathbb{R}^N$ respectively. However boundary peaked solutions have not yet been observed for (1.1) even in the case $\gamma_2 \neq 0$. This paper seems to be the first attempt in this line.

Before stating our main result we first enumerate the assumptions on the function f that will be steadily assumed:

(f1) $f \in C^2(\mathbb{R})$; $f(t) = 0$ for $t \leq 0$;
(f2) $\frac{f(t)}{t^3}$ is nondecresing in $t > 0$;
(f3) $f(t) = O(t^p)$ as $t \to +\infty$, where $3 < p < 5$;
(f4) there exists a constants $\theta > 4$ such that $0 < \theta F(t) \leq f(t)t$ for all $t > 0$, where $F(t) = \int_0^t f(s)ds$;

(f5) the problem in the whole space

$$\begin{cases} \Delta w - w + f(w) = 0, \;\; w > 0 \text{ in } \mathbb{R}^3, \\ w(0) = \max_{x \in \mathbb{R}^3} w(x), \quad \lim_{|x| \to +\infty} w(x) = 0, \end{cases}$$

has a unique solution w, which is nondegenerate, i.e., denoting by L the linearized operator

$$L : H^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3), \;\; L[u] := \Delta u - u + f'(w)u,$$

then

$$\text{Kernel}(L) = \text{span}\left\{ \frac{\partial w}{\partial x_1}, \frac{\partial w}{\partial x_1}, \frac{\partial w}{\partial x_3} \right\}.$$

By the well-known result of Gidas, Ni and Nirenberg ([21]) w is radially symmetric and strictly decreasing in $r = |x|$. Moreover, by classical regularity results, $w \in C^2(\mathbb{R}^3)$ and the following asymptotic behavior holds:

$$w(r), \; w''(r) = \frac{A}{r} e^{-r}\Big(1 + O\Big(\frac{1}{r}\Big)\Big), \quad w'(r) = -\frac{A}{r} e^{-r}\Big(1 + O\Big(\frac{1}{r}\Big)\Big), \tag{1.3}$$

where $A > 0$ is a suitable positive constant. Note that assumptions (f1)–(f3) imply

$$f(t) \le c_1 |t|^3 + c_2 |t|^p, \;\; F(t) \le C_1 |t|^4 + C_2 |t|^{p+1} \quad \forall t \ge 0. \tag{1.4}$$

Typical examples of f satisfying (f1)-(f5) include $f(t) = t_+^p$ where $3 < p < 5$. Other nonlinearities can be found in [5]. The uniqueness of w is proved in [29] for the case of power-like f; for a general nonlinearity, see [3]. The nondegeneracy condition can be derived from the uniqueness argument (see [32]).

In order to provide the exact formulation of the main result of this paper we recall the variational structure of the system (1.1): indeed for every $v \in H_0^1(\Omega)$ let $(-\Delta)^{-1}[v^2] \in H_0^1(\Omega)$ be the unique solution of the following problem

$$\Delta V + v^2 = 0 \text{ in } \Omega, \quad V = 0 \text{ on } \partial\Omega.$$

Then (1.1) is equivalent to

$$\begin{cases} \varepsilon^2 \Delta v - v - \gamma(-\Delta)^{-1}[v^2]v + f(v) = 0 \quad \text{in } \Omega, \\ v > 0 \text{ in } \Omega, \;\; v = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.5}$$

where $\gamma = \gamma_1 \gamma_2$, and associated to (1.5) is the following energy functional:

$$J_\varepsilon[v] := \frac{1}{2} \int_\Omega \big(\varepsilon^2 |\nabla v|^2 + |v|^2\big) dx + \frac{\gamma}{4} \int_\Omega (-\Delta)^{-1}[v^2] v^2 dx - \int_\Omega F(v) dx. \tag{1.6}$$

Our aim is to establish the existence of least energy solutions v_ε for (1.5) and to show that v_ε exhibits a *point-condensation phenomenon* as $\varepsilon \to 0^+$. More precisely, when ε is sufficiently small, v_ε has a single spike centered at a point P_ε which approaches the boundary as $\varepsilon \to 0^+$, while v_ε vanishes everywhere else. Hence a natural question arises immediately to ask which part of the boundary the points P_ε are situated near, and it is the purpose of this paper to answer this question and to give an accurate description of the profiles of the solutions v_ε. Indeed we shall prove that this unique peak must be situated near the *most curved* part of $\partial\Omega$, i.e. where the boundary mean curvature assumes its maximum; more precisely any limiting point P_0 of the family P_ε is such that $\mathcal{H}(P_0)$, the mean curvature of $\partial\Omega$ at P_0, is a maximum value of $\mathcal{H}(P)$ over $\partial\Omega$.

Now we proceed to state our main theorem.

Theorem 1.1. *Assume that $\Omega \subset \mathbb{R}^3$ is a smooth and bounded domain and that hypotheses (f1)-(f5) hold. Then for every $\varepsilon > 0$ there exists a least energy solution $v_\varepsilon \in H_0^1(\Omega)$ of (1.5). Furthermore, as $\varepsilon \to 0^+$, v_ε develops a spike near the maxima of the mean curvature; more precisely there exists $P_\varepsilon \in \Omega$ such that*

(1) $v_\varepsilon(x) = w\big(\frac{x-P_\varepsilon}{\varepsilon}\big) + o(1)$ *uniformly in* Ω*;*

(2) $dist(P_\varepsilon, \partial\Omega) = (1+o(1))\varepsilon \log \frac{1}{\varepsilon}$.

Finally, for every sequence $\varepsilon_n \to 0^+$, up to a subsequence,

(3) $P_{\varepsilon_n} \to P_0 \in \partial\Omega$ *where* $\mathcal{H}(P_0) = \mathcal{H}_0 := \max_{P\in\partial\Omega} \mathcal{H}(P)$.

Remark 1.1. *Notice that if, in addition, we assume the existence of a unique global maximum P_0 of $\mathcal{H}(P)$, Part (3) of Theorem 1.1 holds for all the families P_ε, without need to pass to sequences, and all the waves v_ε concentrate at that point P_0 as $\varepsilon \to 0^+$.*

It is interesting to see how the geometry of the domain determines exactly the location of the spike-layers as well as how this result is in striking contrast with the result in [35] for the single Schrödinger equation (1.2) with Dirichlet boundary condition, in which the least energy solutions are located at the *most centered* part of the domain. Furthermore, it is also known that there are *no* spike-layer solutions concentrating near the boundary for the Dirichlet problem associated to (1.2) (see [8], [39]). On the contrary least energy solutions with a single boundary peak closed to the maxima of the mean curvature are known for (1.2) with Neumann conditions. So we are in presence of a Dirichlet problem which acts more like a Neumann problem for the single equation case. To our knowledge, the only other result concerning boundary concentration occurring for a Dirichlet problem is known for the FitzHugh-Nagumo system by Dancer and Yan in [8] and [9]; however in [8] and [9] only the existence of such solutions is proved and the exact boundary limiting points are not determined. This paper seems to be the first one that succeeds in locating exactly the boundary spikes for a Dirichlet problem.

The proof of Theorem 1.1 is based on the *energy method*, i.e. on the derivation of an asymptotic formula for the smallest critical value $J_\varepsilon^* := J_\varepsilon[v_\varepsilon]$ as $\varepsilon \to 0^+$, in the spirit of [32], [33], [35]. However, here the technique is more complicated since we have to expand the energy up to the sixth order. The first object is to apply the Mountain-Pass Lemma to obtain a critical point v_ε of J_ε; furthermore we prove that v_ε is actually a least-energy solution of (1.5), by which it is meant that v_ε has the smallest energy J_ε^* among all the solutions to (1.5), and J_ε^* can be characterized as

$$J_\varepsilon^* = \inf_{v \in H_0^1(\Omega)} \max_{t \geq 0} J_\varepsilon[tv]. \tag{1.7}$$

Then we show that for ε sufficiently small v_ε is a single spike solution which is localized in a ε-neighborhood of a maximum point P_ε with $\frac{dist(P_\varepsilon, \partial\Omega)}{\varepsilon} \to +\infty$. Next, the critical step is to know the detailed structure of v_ε around P_ε. To do this we first use the solution w of the limiting problem (1.3) to construct a family of suitable functions $\tilde{w}_{\varepsilon,P}$ and then prove that the solution v_ε can be obtained as a suitable perturbation of $\tilde{w}_{\varepsilon,P_\varepsilon}$. To perform such approximation we make extensive use of the nondegeneracy condition (f5). Once we have obtained the shape of v_ε, we have to expand $J_\varepsilon[v_\varepsilon] = J_\varepsilon^*$ up to the order $O(\varepsilon^6)$. The first term in the expansion formula of J_ε^* is given by $I[w]\varepsilon^3$, where $I[w]$ is the energy of w:

$$I[w] = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla w|^2 + |w|^2)dx - \int_{\mathbb{R}^3} F(w)dx = \frac{1}{2}\int_{\mathbb{R}^3} f(w)wdx - \int_{\mathbb{R}^3} F(w)dx.$$

The first correction term in J_ε^* involves the distance function P_ε from the boundary, while the boundary mean curvature appears in the fourth term. The location of P_ε is determined by using $\tilde{w}_{\varepsilon,P}$ as comparison functions (for suitable $P \in \Omega$), i.e., according to the characterization (1.7), we compare J_ε^* with $\max_{t\geq 0} J_\varepsilon[t\tilde{w}_{\varepsilon,P}]$; such comparison gives information on the terms in the asymptotic expansion of J_ε^*, in particular on $dist(P_\varepsilon, \partial\Omega)$ as well as on which portion of the boundary P_ε approaches to.

The paper is organized as follows. Section 2 is devoted to introduce some notation and preliminaries. In Section 3 we construct the approximated solution $\tilde{w}_{\varepsilon,P}$ and we determine its shape. Section 4 contains the expansion of the functional J_ε on $\tilde{w}_{\varepsilon,P}$ as a function of ε and P. In Section 5 we construct the least energy solutions v_ε and prove that their shape can be approximated by $\tilde{w}_{\varepsilon,P_\varepsilon}$, for suitable $P_\varepsilon \in \Omega$, up to a certain order $\varepsilon^{\tau_\varepsilon}$; furthermore an upper bound for the critical values J_ε^* is derived by using $\tilde{w}_{\varepsilon,P}$ as comparison functions and computing $\max_{t\geq 0} J_\varepsilon[t\tilde{w}_{\varepsilon,P}]$. Finally the proof of Theorem 1.1 is completed in Section 6.

Acknowledgments. The research of the second author is supported by an Earmarked Grant from RGC of Hong Kong. This paper was begun while the first author was visiting The Chinese University of Hong Kong in April 2005. She gratefully acknowledges the Department of Mathematics at CUHK for their hospitality.

NOTATION

- Given $A \subset \mathbb{R}^3$ an open subset, $L^p(A)$ is the usual Lebesgue space endowed with the norm

$$\|u\|_{L^p}^p := \int_A |u|^p \, dx \text{ for } 1 \leq p < +\infty, \quad \|u\|_\infty = \sup_{x \in A} |u(x)|.$$

  Furthermore $H_0^1(A)$ is the usual Sobolev space endowed with the norm

$$\|u\|_{H^1}^2 = \int_A \left(|\nabla u|^2 + |u|^2 \right) dx.$$

- If $u : \mathbb{R}^N \to \mathbb{R}$ is a radially symmetric function, we will continue to denote by u the real function $r > 0 \mapsto u(x)$ with $|x| = r$.
- We will often use the symbol c or C for denoting different positive constants independent on ε. The value of c, C is allowed to vary from line to line (and also in the same formula).
- $o(1)$ denotes a vanishing quantity as $\varepsilon \to 0^+$.
- Given $\{a_\varepsilon\}_{\varepsilon > 0}$ and $\{b_\varepsilon\}_{\varepsilon > 0}$ two family of numbers, we write $a_\varepsilon = o(b_\varepsilon)$ (resp. $a_\varepsilon = O(b_\varepsilon)$) to mean that $\frac{a_\varepsilon}{b_\varepsilon} \to 0$ (resp. $|a_\varepsilon| \leq C|b_\varepsilon|$) as $\varepsilon \to 0^+$.

2. Preliminaries

In this section we collect some preliminary results concerning the variational structure of the system (1.5). In particular we recall some well-known facts on the representation formula for the Poisson equation: for a smooth domain there exists a unique Green's function $G(x, z)$ of the Laplace operator with Dirichlet boundary condition (see [28]). Furthermore G is symmetric in x and z and

$$0 < G(x, z) < \frac{1}{4\pi|x - z|} \quad \forall x, z \in \Omega \times \Omega, \ x \neq z. \tag{2.8}$$

Proposition 2.1. *Let Ω be a smooth and bounded domain of $\mathbb{R}^3$. For every $g \in L^2(\Omega)$ denote by $(-\Delta)^{-1}[g]$ the unique solution in $H_0^1(\Omega)$ of*

$$-\Delta\psi = g. \tag{2.9}$$

Then the following representation formula holds:

$$(-\Delta)^{-1}[g](x) = \int_\Omega G(x, z) g(z) dz. \tag{2.10}$$

Furthermore

a) $\int_\Omega (-\Delta)^{-1}[g] h dx = \int_\Omega (-\Delta)^{-1}[h] g dx$ *for every* g, $h \in L^2(\Omega)$;
b) $\|(-\Delta)^{-1}[g]\|_\infty \leq C\|g\|_{L^2}$ *for every* $g \in L^2(\Omega)$;
c) $\|(-\Delta)^{-1}[g]\|_\infty \leq \varepsilon^2 \|g\|_\infty + \frac{1}{\varepsilon}\|g\|_{L^1}$ *for every* $g \in L^\infty(\Omega)$;

d) *the functional* $\mathcal{J} : u \in H_0^1(\Omega) \mapsto \int_\Omega u^2(-\Delta)^{-1}[u^2]dx$ *is* C^1 *and*

$$\langle \mathcal{J}'[u], v\rangle = 4\int_\Omega uv(-\Delta)^{-1}[u^2]dx \quad \forall u, v \in H_0^1(\Omega).$$

Proof. By Lax-Milgram's Lemma we get the existence of a unique solution in $H_0^1(\Omega)$ of (2.9). The representation formula (2.10) holds for $u \in C_0^\infty(\Omega)$ (see, for example, [20, pg. 23, Theorem 1]); by density (2.10) can be extended to any $g \in L^2(\Omega)$. a) follows immediately from (2.10) and Fubini-Tonelli's theorem. By (2.8) for every $g \in L^2(\Omega)$, by using Hölder's inequality, we have

$$|(-\Delta)^{-1}[g](x)| \le \frac{1}{4\pi}\int_\Omega \frac{|g(z)|}{|z-x|}dz \le \frac{1}{4\pi}\|g\|_{L^2}\Big(\int_{|z|\le 2diam(\Omega)} \frac{1}{|z|^2}dz\Big)^{1/2} \le C\|g\|_{L^2},$$

while, for $g \in L^\infty(\Omega)$,

$$\begin{aligned}|(-\Delta)^{-1}[g](x)| &\le \frac{1}{4\pi}\int_{|z-x|\le\varepsilon} \frac{|g(z)|}{|z-x|}dz + \frac{1}{4\pi\varepsilon}\int_\Omega |g(z)|dz \\ &\le \frac{\|g\|_\infty}{4\pi}\int_{|y|\le\varepsilon} \frac{1}{|z|}dz + \frac{1}{4\pi\varepsilon}\|g\|_{L^1} = \frac{\varepsilon^2\|g\|_\infty}{2} + \frac{1}{4\pi\varepsilon}\|g\|_{L^1}\end{aligned}$$

and we obtain b)-c).

Part d) is a direct computation. □

In view of d) of Proposition 2.1 the energy functional J_ε defined in (1.6) is of class $C^1(H_0^1(\Omega), \mathbb{R})$ and its critical points correspond to the solutions of (1.5). Furthermore J_ε can be rewritten as

$$J_\varepsilon[v] = \frac{\varepsilon^2}{2}\int_\Omega \big(|\nabla u|^2 + u^2\big)dx - \int_\Omega F(u)dx + \frac{\gamma}{4}\int_\Omega\int_\Omega G(x,z)u^2(x)u^2(z)dxdz.$$

3. Computation of $\tilde{w}_{\varepsilon,P}$

In this section we introduce some suitable approximated solutions and derive some crucial estimates: first set

$$w_{\varepsilon,P}(x) = w\Big(\frac{x-P}{\varepsilon}\Big), \quad x,\, P \in \mathbb{R}^3.$$

Next for every $P \in \mathbb{R}^3$ define $\tilde{w}_{\varepsilon,P}$ to be the unique solution of the problem

$$\varepsilon^2\Delta\tilde{w}_{\varepsilon,P} - \tilde{w}_{\varepsilon,P} + f(w_{\varepsilon,P}) = 0 \text{ in } \Omega, \quad \tilde{w}_{\varepsilon,P} = 0 \text{ on } \partial\Omega. \tag{3.11}$$

From the comparison principle it is immediate that

$$0 < \tilde{w}_{\varepsilon,P}(x) < w_{\varepsilon,P}(x) \quad \forall x \in \Omega, \forall P \in \mathbb{R}^3. \tag{3.12}$$

The goal is to obtain an asymptotic expansion of the approximations $\tilde{w}_{\varepsilon,P}$. To this aim some preparations are needed. First define the distance function d_P from the boundary $\partial\Omega$ by

$$d_P = dist(P, \partial\Omega), \quad P \in \mathbb{R}^3.$$

The regularity of Ω implies that $\partial\Omega$ satisfies the uniform interior and exterior sphere condition; that is, at each point $Q\in\partial\Omega$ there exist two balls B_1, B_2 such that $B_1\cap\bar\Omega=\{Q\}$, $B_2\cap(\mathbb{R}^3\setminus\Omega)=\{Q\}$, and the radii of the balls B_1 and B_2 are bounded from below by a positive constant; taking such constant as μ, we obtain that, set

$$\Gamma_\mu:=\{P\in\Omega\,|\,d_P\le\mu\},$$

for every $P\in\Gamma_\mu$ there exists a unique $\Sigma_P\in\partial\Omega$ such that $|\Sigma_P-P|=|\Sigma_P-P^*|=d_P=d_{P^*}$, (see, for example, [22], pg. 355), where $P^*=2\Sigma_P-P$ (i.e. P^* is the symmetric of P with respect to the tangent plane at $\partial\Omega$ in Σ_P). Notice that by construction, using (1.3),

$$w_{\varepsilon,P^*}(x)\le C\frac{\varepsilon}{d_P}e^{-\frac{|x-P^*|}{\varepsilon}}\le C\frac{\varepsilon}{d_P}e^{-\frac{d_P}{\varepsilon}}\quad\forall x\in\Omega,\ \forall P\in\Gamma_\mu. \tag{3.13}$$

$$w_{\varepsilon,P}(x)\le C\frac{\varepsilon}{d_P}e^{-\frac{|x-P|}{\varepsilon}}\le C\frac{\varepsilon}{d_P}e^{-\frac{d_P}{\varepsilon}}\quad\forall x\in\mathbb{R}^3\setminus\Omega,\ \forall P\in\mathbb{R}^3. \tag{3.14}$$

For every $P\in\Gamma_\mu$ let $\mathcal{H}_1(\Sigma_P)$, $\mathcal{H}_2(\Sigma_P)$ be the principal curvatures of $\partial\Omega$ at Σ_P, so that the mean curvature $\mathcal{H}(\Sigma_P)$ of $\partial\Omega$ at Σ_P is given by the average:

$$\mathcal{H}(\Sigma_P)=\frac{\mathcal{H}_1(\Sigma_P)+\mathcal{H}_2(\Sigma_P)}{2}.$$

We introduce a diffeomorfism which straightens a boundary portion near Σ_P: consider $T_P(x)$ the rotation and translation of coordinates which map Σ_P in 0, the inner normal to $\partial\Omega$ at Σ_P in the positive ℓ_3 coordinate axis and the principal directions corresponding to $\mathcal{H}_1(\Sigma_P),\mathcal{H}_2(\Sigma_P)$ in the ℓ_1, ℓ_2 axes. Then $T_P(P)=(0,0,d_P)$, $T_P(P^*)=(0,0,-d_P)$ and in some neighborhood of 0 the boundary $\partial(T_P\Omega)$ can be represented by

$$y_3=\frac12\sum_{i=1,2}\mathcal{H}_j(\Sigma_P)y_i^2+\omega_P(y'),\quad \lim_{y'\to0}\frac{\omega_P(y')}{|y'|^2}=0,\ \text{where } y'=(y_1,y_2). \tag{3.15}$$

Before providing in Proposition 3.1 the asymptotic expansion of the approximated solutions $\tilde w_{\varepsilon,P}$ we state first the following useful result.

Lemma 3.1. *Fix $a>0$, $b\ge0$. For $P\in\Gamma_\mu$ such that $\frac{d_P}{\varepsilon}$ is sufficiently large the following holds*

$$\Big\|w^a_{\varepsilon,P^*}\frac{|y'|^b}{\varepsilon^b}\Big\|_{L^\infty(\Omega)}+\varepsilon^{-3}\int_\Omega w^a_{\varepsilon,P^*}\frac{|y'|^b}{\varepsilon^b}dx\le Ce^{-\frac{2ad_P}{3\varepsilon}}\qquad (y=T_P(x)).$$

Proof. According to (3.13), for $\frac{d_P}{\varepsilon}$ sufficiently large we get

$$\begin{aligned}\|w^a_{\varepsilon,P^*}|y'|^b\|_{L^\infty(\Omega)}+\varepsilon^{-3}\int_\Omega w^a_{\varepsilon,P^*}|y'|^bdx&\le Ce^{-\frac{2ad_P}{3\varepsilon}}\Big(\|w^{a/3}_{\varepsilon,P^*}|y'|^b\|_{L^\infty(\Omega)}+\varepsilon^{-3}\int_\Omega w^{a/3}_{\varepsilon,P^*}|y'|^bdx\Big)\\&\le C\varepsilon^be^{-\frac{2ad_P}{3\varepsilon}}\Big(\Big\|w^{a/3}\Big(\frac{y+d_P\ell_3}{\varepsilon}\Big)\frac{|y'|^b}{\varepsilon^b}\Big\|_{L^\infty(\mathbb{R}^3)}+\varepsilon^{-3}\int_{\mathbb{R}^3}w^{a/3}\Big(\frac{y+d_P\ell_3}{\varepsilon}\Big)\frac{|y'|^b}{\varepsilon^b}dy\Big)\\&\le C\varepsilon^be^{-\frac{2d_P}{3\varepsilon}}\Big(\|w^{a/3}(y)|y'|^b\|_{L^\infty(\mathbb{R}^3)}+\int_{\mathbb{R}^3}w^{a/3}(y)|y'|^bdy\Big)\le C\varepsilon^be^{-\frac{2ad_P}{3\varepsilon}}.\end{aligned}$$

Proposition 3.1. *For $P \in \Omega$ such that $\frac{d_P}{\varepsilon}$ is sufficiently large the following estimates hold:*

i) $\tilde{w}_{\varepsilon,P}(x) = w_{\varepsilon,P}(x) + O(\varepsilon^4)$ *uniformly for* $x \in \Omega$ *and* $P \in \Omega \setminus \Gamma_\mu$;

ii) $\tilde{w}_{\varepsilon,P}(x) = w_{\varepsilon,P}(x) - w_{\varepsilon,P^*}(x) + O(\varepsilon)h_{\varepsilon,P}(x) + k_{\varepsilon,P}(x)$ *uniformly for* $x \in \Omega$ *and* $P \in \Gamma_\mu$ *where* $h_{\varepsilon,P}$ *and* $k_{\varepsilon,P}$ *solve*

$$\varepsilon^2 \Delta h_{\varepsilon,P} - h_{\varepsilon,P} = 0 \text{ in } \Omega, \quad h_{\varepsilon,P} = w_{\varepsilon,P^*}\frac{|y'|^2}{\varepsilon^2} + \varepsilon^4 \text{ on } \partial\Omega, \qquad y = T_P(x), \tag{3.16}$$

$$\varepsilon^2 \Delta k_{\varepsilon,P} - k_{\varepsilon,P} = -f(w_{\varepsilon,P^*}), \quad k_{\varepsilon,P} = 0 \text{ on } \partial\Omega. \tag{3.17}$$

Furthermore

$$\|h_{\varepsilon,P}\|_\infty^2 + \varepsilon^{-3}\int_\Omega h_{\varepsilon,P}^2 dx = O(\varepsilon^8 + e^{-\frac{4d_P}{3\varepsilon}}), \qquad \|k_{\varepsilon,P}\|_\infty^2 + \varepsilon^{-3}\int_\Omega k_{\varepsilon,P}^2 dx = O(e^{-\frac{4d_P}{\varepsilon}}) \tag{3.18}$$

uniformly for $P \in \Gamma_\mu$.

iii) $\varepsilon \nabla \tilde{w}_{\varepsilon,P}(P) = O(\varepsilon^4 + \varepsilon e^{-\frac{2d_P}{3\varepsilon}} + e^{-\frac{2d_P}{\varepsilon}})$.

Proof. The proof of Part i) is immediate: indeed $w_{\varepsilon,P} - \tilde{w}_{\varepsilon,P}$ satisfies

$$\varepsilon^2\Delta(w_{\varepsilon,P} - \tilde{w}_{\varepsilon,P}) - (w_{\varepsilon,P} - \tilde{w}_{\varepsilon,P}) = 0 \text{ in } \Omega, \quad w_{\varepsilon,P} - \tilde{w}_{\varepsilon,P} = w_{\varepsilon,P} \text{ on } \partial\Omega.$$

On the other hand by the definition of Γ_μ $w_{\varepsilon,P} \le Ce^{-\frac{\mu}{\varepsilon}} = O(\varepsilon^4)$ uniformly for $x \in \partial\Omega$ and $P \in \Omega \setminus \Gamma_\mu$. The maximum principle implies $w_{\varepsilon,P} - \tilde{w}_{\varepsilon,P} = O(\varepsilon^4)$ uniformly for $x \in \Omega$ and $P \in \Omega \setminus \Gamma_\mu$.

We go on with the proof of Part ii), which is more technical. During its proof it is understood, even though not stated plainly, that all the estimates hold uniformly for $P \in \Gamma_\mu$. First decompose

$$\tilde{w}_{\varepsilon,P} = w_{\varepsilon,P} - w_{\varepsilon,P^*} - \varepsilon\hat{h}_\varepsilon + k_\varepsilon$$

where $\hat{h}_\varepsilon$ solves the following problem

$$\varepsilon^2\Delta\hat{h}_\varepsilon - \hat{h}_\varepsilon = 0 \text{ in } \Omega, \quad \hat{h}_\varepsilon = \frac{w_{\varepsilon,P} - w_{\varepsilon,P^*}}{\varepsilon} \text{ on } \partial\Omega,$$

and $k_\varepsilon = k_{\varepsilon,P}$ solves (3.17).

The first object is to prove the following estimate for the boundary points:

$$\frac{|w_{\varepsilon,P}(x) - w_{\varepsilon,P^*}(x)|}{\varepsilon} \le Cw_{\varepsilon,P^*}(x)\frac{|y'|^2}{\varepsilon^2} + \varepsilon^4 \tag{3.19}$$

uniformly for $x \in \partial\Omega$. Indeed

$$\begin{aligned} d_P \le |x - P| = |y - d_P\ell_3| = \sqrt{d_P^2 + |y|^2 - 2d_P y_3} \ge |y'|, \\ d_P \le |x - P^*| = |y + d_P\ell_3| = \sqrt{d_P^2 + |y|^2 + 2d_P y_3} \ge |y'| \end{aligned} \tag{3.20}$$

uniformly for $x \in \partial\Omega$. Using (3.15) we have $y_3 = O(|y'|^2)$ on $\partial\Omega$; consequently

$$||x - P| - |x - P^*|| = \frac{||x-P|^2 - |x - P^*|^2|}{|x - P| + |x - P^*|} = \frac{4d_P|y_3|}{|x-P| + |x - P^*|} \le 2|y_3| \le C|y'|^2. \tag{3.21}$$

uniformly for $x \in \partial\Omega$. Take $x \in \partial\Omega$ and distinguish two cases: first assume $|y'| \geq \sqrt{\varepsilon}$; then, by (3.20), $|x-P|$, $|x-P^*| \geq \sqrt{\varepsilon}$, and, by (1.3),

$$|w_{\varepsilon,P}(x)|,\ |w_{\varepsilon,P^*}(x)| \leq C\sqrt{\varepsilon}e^{-\frac{1}{\sqrt{\varepsilon}}} \leq \varepsilon^5.$$

Next assume $|y'| \leq \sqrt{\varepsilon}$; then for every $r \in [\min\{|x-P|,|x-P^*|\}, \max\{|x-P|,|x-P^*|\}]$ by (3.21) we have $\frac{|r-|x-P^*||}{\varepsilon} \leq C$, by which, using again (1.3),

$$\Big|w'\Big(\frac{r}{\varepsilon}\Big)\Big| \leq C\frac{\varepsilon}{r}e^{-\frac{|r|}{\varepsilon}} \leq C\frac{\varepsilon}{|x-P^*|}e^{-\frac{|x-P^*|}{\varepsilon}} \leq Cw_{\varepsilon,P^*}(x);$$

hence, by applying the mean value theorem, we get

$$\big|w_{\varepsilon,P}(x) - w_{\varepsilon,P^*}(x)\big| \leq Cw_{\varepsilon,P^*}(x)\frac{||x-P|-|x-P^*||}{\varepsilon} \leq Cw_{\varepsilon,P^*}(x)\frac{|y'|^2}{\varepsilon}$$

uniformly for $x \in \partial\Omega$ with $|y'| \leq \sqrt{\varepsilon}$. Hence (3.19) holds. The maximum principle applies and gives $|\hat{h}_\varepsilon| \leq Ch_\varepsilon$ where $h_\varepsilon := h_{\varepsilon,P}$.

By multiplying both members of (3.16) by $h_\varepsilon(x) - w_{\varepsilon,P^*}(x)\frac{|y'|^2}{\varepsilon^2} - \varepsilon^4$ and since, using (1.3),

$$\Big|\nabla\Big(w_{\varepsilon,P^*}(x)\frac{|y'|^2}{\varepsilon^2}\Big)\Big| \leq \frac{C}{\varepsilon}w_{\varepsilon,P^*}(x)\Big(1+\frac{|y'|^2}{\varepsilon^2}\Big)$$

integrating by parts we get

$$\begin{aligned}
\int_\Omega(\varepsilon^2|\nabla h_\varepsilon|^2 + h_\varepsilon^2)dx &\leq C\int_\Omega w_{\varepsilon,P^*}\Big(1+\frac{|y'|^2}{\varepsilon^2}\Big)(\varepsilon|\nabla h_\varepsilon| + h_\varepsilon)dx + \varepsilon^4\int_\Omega h_\varepsilon dx\\
&\leq C\Big(\int_\Omega w_{\varepsilon,P^*}^2\Big(1+\frac{|y'|^2}{\varepsilon^2}\Big)^2dx\Big)^{1/2}\Big(\int_\Omega(\varepsilon^2|\nabla h_\varepsilon|^2 + h_\varepsilon^2)\Big)^{1/2} + C\varepsilon^4\Big(\int_\Omega h_\varepsilon^2\Big)^{1/2}\\
&\leq C\varepsilon^{3/2}e^{-\frac{2d_P}{3\varepsilon}}\Big(\int_\Omega(\varepsilon^2|\nabla h_\varepsilon|^2 + h_\varepsilon^2)dx\Big)^{1/2} + C\varepsilon^4\Big(\int_\Omega h_\varepsilon^2\Big)^{1/2}
\end{aligned}$$

where, in the last inequality, we have used Lemma 3.1. In the same way, by multiplying both members of (3.17) by k_ε, since by (1.4) $f(w) \leq Cw^3$, we have

$$\int_\Omega(\varepsilon^2|\nabla k_\varepsilon|^2 + k_\varepsilon^2)dx \leq C\int_\Omega w_{\varepsilon,P^*}^3k_\varepsilon dx \leq C\varepsilon^{3/2}e^{-\frac{2d_P}{\varepsilon}}\Big(\int_\Omega k_\varepsilon^2dx\Big)^{1/2}$$

and a first part of (3.18) follows. In order to complete the proof, first notice that by the maximum principle we derive h_ε, $k_\varepsilon \geq 0$ in Ω. Furthermore according to Lemma 3.1 $w_{\varepsilon,P^*}\frac{|y'|^2}{\varepsilon^2} \leq Ce^{-\frac{2d_P}{3\varepsilon}}$ on $\partial\Omega$, hence from the maximum principle, $\|h_{\varepsilon,P}\|_\infty = O(e^{-\frac{2d_P}{3\varepsilon}} + \varepsilon^4)$. In the same way $f(w_{\varepsilon,P^*}) \leq Cw_{\varepsilon,P^*}^3 \leq Ce^{-\frac{2d_P}{\varepsilon}}$ on Ω, then the comparison principle gives $\|k_{\varepsilon,P}\|_\infty = O(e^{-\frac{2d_P}{\varepsilon}})$.

Finally to prove Part iii), observe that $z_{\varepsilon,P} = (\varepsilon^4 + \varepsilon e^{-\frac{2d_P}{3\varepsilon}} + e^{-\frac{2d_P}{\varepsilon}})^{-1}\big(w - \tilde{w}_{\varepsilon,P}(\varepsilon x + P)\big)$ solves

$$\Delta z_{\varepsilon,P} = z_{\varepsilon,P} \text{ in } B(0,1)$$

(note that for $\frac{d_P}{\varepsilon}$ is sufficiently large we have $B(0,1) \subset \frac{\Omega - P}{\varepsilon}$). Furthermore by Parts i)-ii) it follows that $z_{\varepsilon,P}$ is uniformly bounded on $B(0,1)$ (note that $w_{\varepsilon,P^*}(\varepsilon x + P) \leq Ce^{-\frac{|P-P^*|}{\varepsilon}} \leq Ce^{-2\frac{d_P}{\varepsilon}}$ on

$B(0,1)$). Then from the well-known Schauder interior estimate $z_{\varepsilon,P}$ and its first and second derivatives are uniformly bounded on the compact sets of $B(0,1)$. Then iii) follows. $\square$

An easy consequence of Proposition 3.1 is the following corollary.

Corollary 3.1. *Setting $\tilde{w}_{\varepsilon,P}=0$ for $x\notin\Omega$, we have $\tilde{w}_{\varepsilon,P}(\varepsilon x+P)\to w$ in $H^1(\mathbb{R}^3)$ and $L^\infty(\mathbb{R}^3)$ as $\frac{d_P}{\varepsilon}\to+\infty$ uniformly for $P\in\Omega$.*

Proof. First observe that $\|w_{\varepsilon,P}(\varepsilon x)\|_{H^1(\mathbb{R}^3\setminus\frac{\Omega}{\varepsilon})}$, $\|w_{\varepsilon,P}(\varepsilon x)\|_{L^\infty(\mathbb{R}^3\setminus\frac{\Omega}{\varepsilon})}\to 0^+$ as $\frac{d_P}{\varepsilon}\to+\infty$ uniformly for $P\in\Omega$. On the other hand for $P\in\Gamma_\mu$, since $\frac{d_{P^*}}{\varepsilon}=\frac{d_P}{\varepsilon}$, then we deduce $\|w_{\varepsilon,P^*}(\varepsilon x)\|_{H^1(\frac{\Omega}{\varepsilon})}$, $\|w_{\varepsilon,P^*}(\varepsilon x)\|_{L^\infty(\frac{\Omega}{\varepsilon})}\to 0^+$ as $\frac{d_P}{\varepsilon}\to+\infty$. Then by i) and ii) of Proposition 3.1 this implies that $\tilde{w}_{\varepsilon,P}(\varepsilon x)-w_{\varepsilon,P}(\varepsilon x)\to 0$ in $L^2(\mathbb{R}^3)$ and $L^\infty(\mathbb{R}^3)$ as $\frac{d_P}{\varepsilon}\to+\infty$ uniformly for $P\in\Omega$. By multiplying equation (3.11) by $\tilde{w}_{\varepsilon,P}$ and integrating by parts we get

$$\|\tilde{w}_{\varepsilon,P}(\varepsilon x)\|^2_{H^1(\mathbb{R}^3)}=\int_{\mathbb{R}^3}f(w)\tilde{w}_{\varepsilon,P}(\varepsilon x+P)dx\to\int_{\mathbb{R}^3}f(w)wdx=\|w\|^2_{H^1(\mathbb{R}^3)},$$

by which $\tilde{w}_{\varepsilon,P}(\varepsilon x+P)\to w$ in $H^1(\mathbb{R}^3)$ as $\frac{d_P}{\varepsilon}\to+\infty$ uniformly for $P\in\Omega$. $\square$

Our next lemma provides an estimate of the error up to $\tilde{w}_{\varepsilon,P}$ satisfies the system (1.5). To this aim set

$$\mathcal{S}_\varepsilon[v]=\varepsilon^2\Delta v-v-\gamma(-\Delta)^{-1}[v^2]v+f(v),\quad v\in H^2(\Omega).$$

Lemma 3.2. *For $P\in\Omega$ such that $\frac{d_P}{\varepsilon}$ is sufficiently large the following holds:*

$$\big|\mathcal{S}_\varepsilon[\tilde{w}_{\varepsilon,P}]\big|\le C\big(e^{-\frac{7d_P}{4\varepsilon}}+\varepsilon e^{-\frac{2d_P}{3\varepsilon}}+\varepsilon^2\big)w_{\varepsilon,P}^{1/4}.$$

Proof. According to c) of Proposition 2.1 and (3.12)

$$|(-\Delta)^{-1}[\tilde{w}^2_{\varepsilon,P}]\tilde{w}_{\varepsilon,P}|\le\varepsilon^2\big(\|w^2_{\varepsilon,P}\|_{L^\infty(\Omega)}+\varepsilon^{-3}\|w^2_{\varepsilon,P}\|_{L^1(\Omega)}\big)w_{\varepsilon,P}\le C\varepsilon^2 w_{\varepsilon,P}.$$

We just need to estimate the local term: by (3.11) and assumption (f1) we deduce

$$\big|\varepsilon^2\Delta\tilde{w}_{\varepsilon,P}-\tilde{w}_{\varepsilon,P}+f(\tilde{w}_{\varepsilon,P})\big|=\big|f(\tilde{w}_{\varepsilon,P})-f(w_{\varepsilon,P})\big|\le Cw_{\varepsilon,P}\big(w_{\varepsilon,P}-\tilde{w}_{\varepsilon,P}\big).$$

If $P\in\Omega\setminus\Gamma_\mu$ the thesis follows from Part i) of Proposition 3.1. Now assume $P\in\Gamma_\mu$; By using Part ii) of Proposition 3.1 we get

$$\big|\varepsilon^2\Delta\tilde{w}_{\varepsilon,P}-\tilde{w}_{\varepsilon,P}+f(\tilde{w}_{\varepsilon,P})\big|\le Cw_{\varepsilon,P}\big(w_{\varepsilon,P^*}+\varepsilon h_{\varepsilon,P}+k_{\varepsilon,P}\big)\le Cw_{\varepsilon,P}\big(w_{\varepsilon,P^*}+\varepsilon e^{-\frac{2d_P}{3\varepsilon}}+\varepsilon^4+e^{-\frac{2d_P}{\varepsilon}}\big).$$

In order to conclude by (1.3) and (3.13) we compute

$$w_{\varepsilon,P}w_{\varepsilon,P^*}\le Cw_{\varepsilon,P}^{1/4}e^{-3\frac{|x-P|}{4\varepsilon}-3\frac{|x-P^*|}{4\varepsilon}}e^{-\frac{d_P}{4\varepsilon}}\le Cw_{\varepsilon,P}^{1/4}e^{-3\frac{|P-P^*|}{4\varepsilon}}e^{-\frac{d_P}{4\varepsilon}}=Cw_{\varepsilon,P}^{1/4}e^{-\frac{7d_P}{4\varepsilon}}$$

uniformly for $x\in\Omega$ and $P\in\Gamma_\mu$. $\square$

4. Expansion of $J_\varepsilon[\tilde{w}_{\varepsilon,p}]$

This section is devoted to compute the energy of the approximated solutions $\tilde{w}_{\varepsilon,P}$. Since the computations are quite long and technical, for the sake of simplicity we provide the expansion for the internal energy and the nonlocal term $\int_\Omega(-\Delta)^{-1}[\tilde{w}^2_{\varepsilon,P}]\tilde{w}^2_{\varepsilon,P}$ separately in the next Propositions 4.1 and 4.2 respectively. First we need the auxiliary results provided by the following two lemmas.

Lemma 4.1. *For $P\in\Gamma_\mu$ such that $\frac{d_P}{\varepsilon}$ is sufficiently large the following holds*

$$\int_\Omega w^a_{\varepsilon,P}w^b_{\varepsilon,P^*}\frac{|y'|^c}{\varepsilon^c}dx\begin{cases}\le C\varepsilon^3\Big(\frac{d_P}{\varepsilon}\Big)^{\frac{c-a}{2}-b+1}e^{-2b\frac{d_P}{\varepsilon}} & \text{if } a>b>0,\ c\ge 0,\\ \le C\varepsilon^3\Big(\frac{d_P}{\varepsilon}\Big)^{\frac{c-3a+3}{2}}e^{-2a\frac{d_P}{\varepsilon}} & \text{if } a=b>0,\ c\ge 0,\\ \le C\varepsilon^3\Big(\frac{d_P}{\varepsilon}\Big)^{\frac{c-a+3}{2}-b}e^{-(a+b)\frac{d_P}{\varepsilon}} & \text{if } 0<a<b,\ c\ge 0,\end{cases}$$

where $y=T_P(x)$.

Proof. By using (1.3) and (3.13) we get

$$\begin{aligned}\int_\Omega w^a_{\varepsilon,P}w^b_{\varepsilon,P^*}\frac{|y'|^c}{\varepsilon^c}dx&\le C\frac{\varepsilon^b}{d_P^b}\int_{\mathbb{R}^3}w^a\Big(\frac{y-d_P\ell_3}{\varepsilon}\Big)e^{-b\frac{|y+d_P\ell_3|}{\varepsilon}}\frac{|y'|^c}{\varepsilon^c}dy\\ &\le C\frac{\varepsilon^{3+b}}{d_P^b}\int_{\mathbb{R}^3}w^a(y)e^{-b|y+\frac{2d_P\ell_3}{\varepsilon}|}|y'|^cdy\le C\frac{\varepsilon^{3+b}}{d_P^b}\int_{\mathbb{R}^3}e^{-a|y|}e^{-b|y+2\frac{d_P}{\varepsilon}\ell_3|}|y'|^{(c-a)_+}dy\end{aligned}$$

where $(c-a)_+=\max\{0,c-a\}$. Now observe that

$$\Big|y+2\frac{d_P}{\varepsilon}\ell_3\Big|=\sqrt{|y|^2+4\frac{d_P^2}{\varepsilon^2}+4y_3\frac{d_P}{\varepsilon}}\ge\frac{2d_P}{\varepsilon}+\frac{|y|^2}{4d_P}\varepsilon+y_3\ge\frac{2d_P}{\varepsilon}+\frac{|y|^2}{4d_P}\varepsilon-|y_3|,$$

by which

$$\int_{\mathbb{R}^3}e^{-a|y|}e^{-b|y+2\frac{d_P}{\varepsilon}\ell_3|}|y'|^{(c-a)_+}dy\le e^{-2b\frac{d_P}{\varepsilon}}\int_{\mathbb{R}^3}e^{-a|y|+b|y_3|}e^{-b\frac{|y|^2}{4d_P}\varepsilon}|y'|^{(c-a)_+}dy.$$

Now we distinguish the three cases: first assume $a>b$; then

$$\begin{aligned}\int_{\mathbb{R}^3}e^{-a|y|+b|y_3|}e^{-b\frac{|y|^2}{4d_P}\varepsilon}|y'|^{(c-a)_+}dy&\le\int_{\mathbb{R}}e^{-(a-b)|y_3|}dy_3\int_{\mathbb{R}^2}e^{-b\frac{|y'|^2}{4d_P}\varepsilon}|y'|^{(c-a)_+}dy'\\ &\le\Big(\frac{d_P}{\varepsilon}\Big)^{\frac{(c-a)_+}{2}+1}\int_{\mathbb{R}^3}e^{-(a-b)|y_3|}dy_3\int_{\mathbb{R}^2}e^{-b\frac{|y'|^2}{4}}|y'|^{(c-a)_+}dy'.\end{aligned}$$

Now assume $a=b$:

$$\int_{\mathbb{R}^3}e^{-a|y|+a|y_3|}e^{-a\frac{|y|^2}{4d_P}\varepsilon}|y'|^{(c-a)_+}dy\le\int_{\mathbb{R}^3}e^{-a\frac{|y|^2}{4d_P}\varepsilon}|y'|^{(c-a)_+}dy=\Big(\frac{d_P}{\varepsilon}\Big)^{\frac{(c-a)_++3}{2}}\int_{\mathbb{R}^3}e^{-a\frac{|y|^2}{4}}|y'|^{(c-a)_+}dy.$$

Finally, if $b>a$, then by (3.13)

$$\int_\Omega w^a_{\varepsilon,P}w^b_{\varepsilon,P^*}\frac{|y'|^c}{\varepsilon^c}dx\le\Big(\frac{\varepsilon}{d_P}\Big)^{b-a}e^{-(b-a)\frac{d_P}{\varepsilon}}\int_\Omega w^a_{\varepsilon,P}w^a_{\varepsilon,P^*}\frac{|y'|^c}{\varepsilon^c}dx$$

and the thesis follows from the previous cases. $\square$

Lemma 4.2. *The following limit holds*

$$\int_{\mathbb{R}^3} f(w)w(y+\rho\ell_3)dy = \frac{B}{\rho}(1+o(1))e^{-\rho} \text{ as } \rho\to+\infty, \quad B = A\int_{\mathbb{R}^3} f(w)e^{-y_3}dy.$$

Proof. The proof is an easy consequence of Lebesgue's Dominated Convergence Theorem. According to (1.3) for every $y\in\mathbb{R}^3$ we have

$$\lim_{\rho\to+\infty}\frac{w(y+\rho\ell_3)}{\frac{A}{\rho}e^{-\rho}} - e^{-y_3} = \lim_{\rho\to+\infty} e^{-|y+\rho\ell_3|+\rho} - e^{-y_3} = 0. \tag{4.22}$$

Observe that, if $|y|\le\frac{\rho}{2}$, then $|y+\rho\ell_3|\ge\frac{\rho}{2}$; hence, by using (1.3) we get

$$f(w)\frac{w(y+\rho\ell_3)}{w(\rho)} \le 2f(w)\frac{\rho}{|y+\rho\ell_3|}e^{|y|} \le 4f(w)e^{|y|}.$$

On the other hand, for $|y|\ge\frac{\rho}{2}$, by (1.3) and (1.4) we obtain

$$f(w)\frac{w(y+\rho\ell_3)}{w(\rho)} \le C\|w\|_\infty\frac{\rho}{|y|^3}e^{-3|y|+\rho} \le C\|w\|_\infty e^{-|y|}.$$

Since $f(w)e^{|y|}\in L^1(\mathbb{R}^3)$, the convergence (4.22) is dominated. □

Now we are ready to provide in the next two propositions the asymptotic formula for the energy $J_\varepsilon[v_\varepsilon]$.

Proposition 4.1. *The following asymptotic expansion holds:*

$$\begin{aligned}\frac{1}{2}\int_\Omega\left(\varepsilon^2|\nabla\tilde w_{\varepsilon,P}|^2+\tilde w_{\varepsilon,P}^2\right)dx - \int_\Omega F(\tilde w_{\varepsilon,P})dx =& I[w]\varepsilon^3+\varepsilon^3\alpha\Big(\frac{d_P}{\varepsilon}\Big)+O\Big(\varepsilon^4\sqrt{\frac{d_P}{\varepsilon}}e^{-2\frac{d_P}{\varepsilon}}\Big)\\ &+o(\varepsilon^6+\varepsilon^5e^{-\frac{d_P}{\varepsilon}}+\varepsilon^3e^{-3\frac{d_P}{\varepsilon}}),\end{aligned} \tag{4.23}$$

as $\frac{d_P}{\varepsilon}\to+\infty$ uniformly for $P\in\Omega$, where $\alpha:\mathbb{R}^+\to\mathbb{R}$ is defined by

$$\alpha(\rho) = \frac{1}{2}\int_{\mathbb{R}^3} f(w)w\big(y+2\rho\ell_3\big)dy = (1+o(1))\frac{B}{4\rho}e^{-2\rho} \text{ as } \rho\to+\infty. \tag{4.24}$$

Proof. We begin by observing that by assumption (f1) and (3.12) we get

$$F(\tilde w_{\varepsilon,P}) = F(w_{\varepsilon,P}) + f(w_{\varepsilon,P})(\tilde w_{\varepsilon,P}-w_{\varepsilon,P}) + O(w_{\varepsilon,P}(\tilde w_{\varepsilon,P}-w_{\varepsilon,P})^2)$$

uniformly for x, $P\in\Omega$, by which, using equation (3.11), it is easy to check that

$$\begin{aligned}I_\varepsilon[\tilde w_{\varepsilon,P}] :=& \frac{1}{2}\int_\Omega\left(\varepsilon^2|\nabla\tilde w_{\varepsilon,P}|^2+\tilde w_{\varepsilon,P}^2\right)dx - \int_\Omega F(\tilde w_{\varepsilon,P})dx = \frac{1}{2}\int_\Omega f(w_{\varepsilon,P})\tilde w_{\varepsilon,P}dx - \int_\Omega F(\tilde w_{\varepsilon,P})dx\\ =& -\int_\Omega F(w_{\varepsilon,P})dx - \frac{1}{2}\int_\Omega f(w_{\varepsilon,P})\tilde w_{\varepsilon,P}dx + \int_\Omega f(w_{\varepsilon,P})w_{\varepsilon,P}dx + O\Big(\int_\Omega w_{\varepsilon,P}(\tilde w_{\varepsilon,P}-w_{\varepsilon,P})^2dx\Big)\end{aligned}$$

uniformly for $P\in\Omega$. Notice that by (1.4) we have $f(w)\le Cw^3$, $F(w)\le Cw^4$; then (3.14) implies

$$\begin{aligned}&\frac{1}{2}\int_{\frac{\Omega}{\varepsilon}} f(w_{\varepsilon,P}(\varepsilon x))w_{\varepsilon,P}(\varepsilon x)dx - \int_{\frac{\Omega}{\varepsilon}} F(w_{\varepsilon,P}(\varepsilon x))dx\\ &= \int_{\mathbb{R}^3}\Big(\frac{1}{2}f(w)w - F(w)\Big)dx + o(\varepsilon^{-3\frac{d_P}{\varepsilon}})\int_{\mathbb{R}^3} wdx = I[w] + o(\varepsilon^{-3\frac{d_P}{\varepsilon}})\end{aligned}$$

as $\frac{d_P}{\varepsilon} \to +\infty$ uniformly for $P \in \Omega$. Hence we arrive at

$$\begin{aligned} I_\varepsilon[\tilde{w}_{\varepsilon,P}] =& I[w]\varepsilon^3 + \frac{1}{2}\int_\Omega f(w_{\varepsilon,P})(w_{\varepsilon,P} - \tilde{w}_{\varepsilon,P})dx + O\Big(\int_\Omega w_{\varepsilon,P}(\tilde{w}_{\varepsilon,P} - w_{\varepsilon,P})^2 dx\Big) \\ &+ o(\varepsilon^3 e^{-3\frac{d_P}{\varepsilon}}) \end{aligned} \tag{4.25}$$

as $\frac{d_P}{\varepsilon} \to +\infty$ uniformly for $P \in \Omega$.

Next we insert the expansion provided by Proposition 3.1 in (4.25) and distinguish the two cases. First assume $P \in \Omega \setminus \Gamma_\mu$. Then by Part i)

$$I_\varepsilon[\tilde{w}_{\varepsilon,P}] = I[w]\varepsilon^3 + o(\varepsilon^6 + \varepsilon^3 e^{-3\frac{d_P}{\varepsilon}})$$

and (4.23) holds uniformly for $P \in \Omega \setminus \Gamma_\mu$. Next assume $P \in \Gamma_\mu$. Then insert the estimate provided by Part ii) of Proposition 3.1 to obtain

$$\begin{aligned} I_\varepsilon[\tilde{w}_{\varepsilon,P}] &= I[w]\varepsilon^3 + \frac{1}{2}\int_\Omega f(w_{\varepsilon,P}) w_{\varepsilon,P^*} dx - \frac{1}{2}\int_\Omega f(w_{\varepsilon,P}) k_{\varepsilon,P} dx + O(\varepsilon)\int_\Omega f(w_{\varepsilon,P}) h_{\varepsilon,P} dx \\ &\quad + O\Big(\int_\Omega w_{\varepsilon,P} w_{\varepsilon,P^*}^2 dx\Big) + O(\varepsilon^2)\int_\Omega h_{\varepsilon,P}^2 dx + O\Big(\int_\Omega k_{\varepsilon,P}^2 dx\Big) + o(\varepsilon^3 e^{-3\frac{d_P}{\varepsilon}}) \\ &= I[w]\varepsilon^3 + \frac{\varepsilon^3}{2}\int_{\frac{\Omega}{\varepsilon}} f(w_{\varepsilon,P}(\varepsilon x)) w_{\varepsilon,P^*}(\varepsilon x) dx - \frac{1}{2}\int_\Omega f(w_{\varepsilon,P}) k_{\varepsilon,P} dx \\ &\quad + O(\varepsilon)\int_\Omega f(w_{\varepsilon,P}) h_{\varepsilon,P} dx + o\big(\varepsilon^3 e^{-3\frac{d_P}{\varepsilon}} + \varepsilon^5 e^{-\frac{d_P}{\varepsilon}} + \varepsilon^6\big) \end{aligned} \tag{4.26}$$

as $\frac{d_P}{\varepsilon} \to +\infty$ uniformly for $P \in \Gamma_\mu$, where we have used (3.18) and Lemma 4.1. Furthermore

$$\begin{aligned} \int_{\frac{\Omega}{\varepsilon}} f(w_{\varepsilon,P}(\varepsilon x)) w_{\varepsilon,P^*}(\varepsilon x) dx &= \int_{\mathbb{R}^3} f(w_{\varepsilon,P}(\varepsilon x)) w_{\varepsilon,P^*}(\varepsilon x) dx + o(\varepsilon^{-3\frac{d_P}{\varepsilon}}) \\ &= \int_{\mathbb{R}^3} f(w) w\Big(y + \frac{2d_P}{\varepsilon}\ell_3\Big) dy + o(\varepsilon^{-3\frac{d_P}{\varepsilon}}) = 2\alpha\Big(\frac{d_P}{\varepsilon}\Big) + o(\varepsilon^{-3\frac{d_P}{\varepsilon}}) \end{aligned} \tag{4.27}$$

as $\frac{d_P}{\varepsilon} \to +\infty$ uniformly for $P \in \Gamma_\mu$. The asymptotic formula (4.24) follows from Lemma 4.2.

After integration by parts, using (3.17),

$$\int_\Omega f(w_{\varepsilon,P}) k_{\varepsilon,P} dx = \int_\Omega \big(-\varepsilon^2 \Delta \tilde{w}_{\varepsilon,P} + \tilde{w}_{\varepsilon,P}\big) k_{\varepsilon,P} dx = \int_\Omega \tilde{w}_{\varepsilon,P} f(w_{\varepsilon,P^*}) dx = o(\varepsilon^3 e^{-3\frac{d_P}{\varepsilon}}). \tag{4.28}$$

as $\frac{d_P}{\varepsilon} \to +\infty$ uniformly for $P \in \Gamma_\mu$, where the last estimate follows from (3.13).

By (1.3) a direct computation shows that

$$\Big|\varepsilon^2 \Delta\Big(w_{\varepsilon,P^*}(x)\frac{|y'|^2}{\varepsilon^2}\Big) - w_{\varepsilon,P^*}(x)\frac{|y'|^2}{\varepsilon^2}\Big| \le C w_{\varepsilon,P^*}^3(x)\frac{|y'|^2}{\varepsilon^2} + C w_{\varepsilon,P^*}(x)\Big(1 + \frac{|y'|}{\varepsilon}\Big),$$

by which, using equation (3.16) and integrating by parts,

$$
\begin{aligned}
&\Big|\int_\Omega f(w_{\varepsilon,P})\Big(h_{\varepsilon,P}-w_{\varepsilon,P^*}\frac{|y'|^2}{\varepsilon^2}-\varepsilon^4\Big)dx\Big|\\
&\qquad=\Big|\int_\Omega\big(-\varepsilon^2\Delta\tilde w_{\varepsilon,P}+\tilde w_{\varepsilon,P}\big)\Big(h_{\varepsilon,P}-w_{\varepsilon,P^*}\frac{|y'|^2}{\varepsilon^2}-\varepsilon^4\Big)dx\Big|\\
&\qquad\le C\int_\Omega \tilde w_{\varepsilon,P}w^3_{\varepsilon,P^*}\frac{|y'|^2}{\varepsilon^2}dx+C\int_\Omega\tilde w_{\varepsilon,P}w_{\varepsilon,P^*}\Big(1+\frac{|y'|}{\varepsilon}\Big)dx+\varepsilon^4\int_\Omega\tilde w_{\varepsilon,P}dx\\
&\qquad\le C\varepsilon^3\sqrt{\frac{d_P}{\varepsilon}}e^{-2\frac{d_P}{\varepsilon}}+C\varepsilon^7
\end{aligned}
\tag{4.29}
$$

by Lemma 4.1. Finally Lemma 4.1 also gives

$$
\int_\Omega f(w_{\varepsilon,P})w_{\varepsilon,P^*}\frac{|y'|^2}{\varepsilon^2}dx+\varepsilon^4\int_\Omega f(w_{\varepsilon,P})dx\le C\varepsilon^3e^{-2\frac{d_P}{\varepsilon}}+C\varepsilon^7.
\tag{4.30}
$$

The conclusion follows by inserting (4.27), (4.28), (4.29), (4.30) into (4.26). □

Proposition 4.2. *The following asymptotic expansion holds:*

$$
\frac{\gamma}{4}\int_\Omega \tilde w^2_{\varepsilon,P}(-\Delta)^{-1}[\tilde w^2_{\varepsilon,P}]dx=I_2\varepsilon^5-I_3\frac{\varepsilon^6}{d_P}+o(\varepsilon^{11/2}e^{-\frac{d_P}{2\varepsilon}}+\varepsilon^5e^{-\frac{d_P}{\varepsilon}}+\varepsilon^4e^{-2\frac{d_P}{\varepsilon}})+O(\varepsilon^6),
\tag{4.31}
$$

as $\frac{d_P}{\varepsilon}\to+\infty$ *uniformly for* $P\in\Omega$*, where* I_2*,* I_3 *are positive constants; furthermore*

$$
\begin{aligned}
\frac{\gamma}{4}\int_\Omega \tilde w^2_{\varepsilon,P}(-\Delta)^{-1}[\tilde w^2_{\varepsilon,P}]dx=&I_2\varepsilon^5-I_3\frac{\varepsilon^6}{d_P}-I_4\mathcal{H}(\Sigma_P)\varepsilon^6\\
&+o(\varepsilon^{11/2}e^{-\frac{d_P}{2\varepsilon}}+\varepsilon^5e^{-\frac{d_P}{\varepsilon}}+\varepsilon^4e^{-2\frac{d_P}{\varepsilon}}+\varepsilon^6)
\end{aligned}
\tag{4.32}
$$

as $\frac{d_P}{\varepsilon}\to+\infty$ *and* $d_P\to0$ *uniformly for* $P\in\Gamma_\mu$*, where* I_4 *is a positive constant.*

Proof. By (3.12) and a) of Proposition 2.1 we can write

$$
\int_\Omega \tilde w^2_{\varepsilon,P}(-\Delta)^{-1}[\tilde w^2_{\varepsilon,P}]dx=\int_\Omega w^2_{\varepsilon,P}(-\Delta)^{-1}[w^2_{\varepsilon,P}]dx+O\Big(\int_\Omega w^2_{\varepsilon,P}(-\Delta)^{-1}\big[w_{\varepsilon,P}(w_{\varepsilon,P}-\tilde w_{\varepsilon,P})\big]dx\Big).
$$

First assume $P\in\Omega\setminus\Gamma_\mu$: b) of Proposition 2.1 and Part i) of Proposition 3.1 imply

$$
\begin{aligned}
\int_\Omega \tilde w^2_{\varepsilon,P}(-\Delta)^{-1}[\tilde w^2_{\varepsilon,P}]dx&=\int_\Omega w^2_{\varepsilon,P}(-\Delta)^{-1}[w^2_{\varepsilon,P}]dx+O\Big(\varepsilon^4\int_\Omega w^2_{\varepsilon,P}dx\Big)\\
&=\int_\Omega w^2_{\varepsilon,P}(-\Delta)^{-1}[w^2_{\varepsilon,P}]dx+O(\varepsilon^7)
\end{aligned}
\tag{4.33}
$$

uniformly for $P\in\Omega\setminus\Gamma_\mu$. Now assume $P\in\Gamma_\mu$; according to Part ii) of Proposition 3.1,

$$
\begin{aligned}
\int_\Omega \tilde w^2_{\varepsilon,P}(-\Delta)^{-1}[\tilde w^2_{\varepsilon,P}]dx=&\int_\Omega w^2_{\varepsilon,P}(-\Delta)^{-1}[w^2_{\varepsilon,P}]dx+O\Big(\int_\Omega w^2_{\varepsilon,P}(-\Delta)^{-1}\big[w_{\varepsilon,P}w_{\varepsilon,P^*}\big]dx\Big)\\
&+O\Big(\varepsilon\int_\Omega w^2_{\varepsilon,P}(-\Delta)^{-1}[h_{\varepsilon,P}]dx\Big)+O\Big(\int_\Omega w^2_{\varepsilon,P}(-\Delta)^{-1}[k_{\varepsilon,P}]dx\Big)
\end{aligned}
\tag{4.34}
$$

uniformly for $P\in\Gamma_\mu$. Let us analyze the error terms: by using b) of Proposition 2.1 and Lemma 4.1 we obtain

$$
\int_\Omega w^2_{\varepsilon,P}(-\Delta)^{-1}\big[w_{\varepsilon,P}w_{\varepsilon,P^*}\big]dx\le C\varepsilon^{3/2}e^{-2\frac{d_P}{\varepsilon}}\int_\Omega w^2_{\varepsilon,P}dx\le C\varepsilon^{9/2}e^{-2\frac{d_P}{\varepsilon}}.
\tag{4.35}
$$

uniformly for $P \in \Gamma_\mu$. Again by b) of Proposition 2.1 and (3.18) we get

$$\int_\Omega w_{\varepsilon,P}^2(-\Delta)^{-1}[h_{\varepsilon,P}]dx \le C\varepsilon^{3/2}(\varepsilon^2 + e^{-\frac{2d_P}{3\varepsilon}})\int_{\mathbb{R}^3} w_{\varepsilon,P}^2 dx \le C(\varepsilon^{13/2} + \varepsilon^{9/2}e^{-\frac{2d_P}{3\varepsilon}}). \tag{4.36}$$

and

$$\int_\Omega w_{\varepsilon,P}^2(-\Delta)^{-1}[k_{\varepsilon,P}]dx \le C\varepsilon^{3/2}e^{-2\frac{d_P}{\varepsilon}}\int_{\mathbb{R}^3} w_{\varepsilon,P}^2 dx \le C\varepsilon^{9/2}e^{-2\frac{d_P}{\varepsilon}} \tag{4.37}$$

uniformly for $P \in \Gamma_\mu$. Combining (4.33), (4.34), (4.35), (4.36) and (4.37) we arrive at

$$\int_\Omega \tilde{w}_{\varepsilon,P}^2(-\Delta)^{-1}[\tilde{w}_{\varepsilon,P}^2]dx = \int_\Omega w_{\varepsilon,P}^2(-\Delta)^{-1}[w_{\varepsilon,P}^2]dx + o(\varepsilon^{11/2}e^{-\frac{d_P}{2\varepsilon}} + \varepsilon^4 e^{-2\frac{d_P}{\varepsilon}} + \varepsilon^6)$$

as $\frac{d_P}{\varepsilon} \to +\infty$ uniformly for $P \in \Omega$. Thus it is sufficient to estimate

$$\int_\Omega w_{\varepsilon,P}^2(-\Delta)^{-1}[w_{\varepsilon,P}^2]dx.$$

To this aim denote by V_0 the unique solution of

$$\Delta V_0 + w^2 = 0 \text{ in } \mathbb{R}^3, \quad V_0 \to 0 \text{ as } |x| \to +\infty$$

i.e. $V_0(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{1}{|x-z|}w^2(z)dz$. Then V_0 is radial and its equation in radial coordinates becomes $(r^2V_0')' + r^2w^2 = 0$ by which, after integration,

$$V_0'(r) = -\frac{1}{r^2}\int_0^r s^2w^2ds = -\frac{1}{r^2}\Big(\int_0^{+\infty} s^2w^2ds + O(e^{-2r})\Big) = -\frac{c_0}{r^2}(1 + O(e^{-2r})) \tag{4.38}$$

for some $c_0 > 0$ and, by integrating again,

$$V_0(r) = -\int_r^{+\infty} V_0'(s)ds = \frac{c_0}{r} + O(e^{-2r}). \tag{4.39}$$

Now set

$$V_{\varepsilon,P}(x) = V_0\Big(\frac{x-P}{\varepsilon}\Big), \quad x \in \Omega,\ P \in \mathbb{R}^3.$$

First assume $P \in \Omega \setminus \Gamma_\mu$. Note that $(-\Delta)^{-1}[w_{\varepsilon,P}^2] - \varepsilon^2 V_{\varepsilon,P}$ solve

$$\Delta\big((-\Delta)^{-1}[w_{\varepsilon,P}^2] - \varepsilon^2V_{\varepsilon,P}\big) = 0 \text{ in } \Omega, \quad (-\Delta)^{-1}[w_{\varepsilon,P}^2] - \varepsilon^2V_{\varepsilon,P} = -\varepsilon^2V_{\varepsilon,P} \text{ on } \partial\Omega$$

and by (4.39) $V_{\varepsilon,P} = O(\varepsilon)$ uniformly for $x \in \partial\Omega$ and $P \in \Omega \setminus \Gamma_\mu$. From the maximum principle it follows that $(-\Delta)^{-1}[w_{\varepsilon,P}^2] - \varepsilon^2V_{\varepsilon,P} = O(\varepsilon^3)$ uniformly for $x \in \partial\Omega$ and $P \in \Omega\setminus\Gamma_\mu$. Hence using (3.14) we obtain

$$\int_\Omega w_{\varepsilon,P}^2(-\Delta)^{-1}[w_{\varepsilon,P}^2] = \varepsilon^2\int_\Omega w_{\varepsilon,P}^2V_{\varepsilon,P}dx + O(\varepsilon^3)\int_\Omega w_{\varepsilon,P}^2 = \varepsilon^5\int_{\mathbb{R}^3} w^2V_0dx + o(\varepsilon^5e^{-\frac{d_P}{\varepsilon}}) + O(\varepsilon^6)$$

and (4.31) holds uniformly for $P \in \Omega \setminus \Gamma_\mu$. Now we assume $P \in \Gamma_\mu$ and decompose

$$(-\Delta)^{-1}[w_{\varepsilon,P}^2] = \varepsilon^2\big(V_{\varepsilon,P} - V_{\varepsilon,P^*} + Z_{\varepsilon,P} - W_{\varepsilon,P}\big) \tag{4.40}$$

where $Z_{\varepsilon,P}$ and $W_{\varepsilon,P}$ solve the following problems:

$$\Delta Z_{\varepsilon,P} + \frac{1}{\varepsilon^2}w_{\varepsilon,P^*}^2 = 0 \text{ in } \Omega, \quad Z_{\varepsilon,P} = 0 \text{ on } \partial\Omega, \tag{4.41}$$

$$\Delta W_{\varepsilon,P} = 0 \text{ in } \Omega, \quad W_{\varepsilon,P} = V_{\varepsilon,P} - V_{\varepsilon,P^*} \text{ on } \partial\Omega. \tag{4.42}$$

For the sake of clarity we divide the remaining part of the proof into 4 steps. In order to simplify the notation, during the steps we will write d and $\mathcal{H}_j$ in the place of d_P and $\mathcal{H}_j(\Sigma_P)$, since this can be done without causing confusion. We let it be understood that all the asymptotic estimates hold uniformly with respect to the choice of $P \in \Gamma_\mu$.

Step 1. The following holds:

$$V_{\varepsilon,P}(x) - V_{\varepsilon,P^*}(x) = c_0\varepsilon\frac{\sum_{j=1,2}\mathcal{H}_j\hat{y}_j'^2}{(1+|\hat{y}|^2)^{3/2}} + o(\varepsilon) \ \text{as} \ \frac{d}{\varepsilon}\to+\infty,\ d\to 0^+,$$

uniformly for $x \in \partial\Omega$, where $\hat{y} = \frac{y}{d} = \frac{T_P x}{d}$.

By (3.15), we compute

$$|x-P| = \sqrt{d^2+|y|^2-2dy_3} = \sqrt{d^2+|y|^2+O(d|y'|^2)} = d\sqrt{1+|\hat{y}|^2+o(|\hat{y}'|^2)}$$

as $d\to 0^+$ uniformly for $x\in\partial\Omega$. In the same way

$$|x-P^*| = \sqrt{d^2+|y|^2+2dy_3} = d\sqrt{1+|\hat{y}|^2+o(|\hat{y}'|^2)}, \quad r_{x,P} = d(\sqrt{1+|\hat{y}|^2+o(|\hat{y}'|^2)})$$

as $d\to 0^+$ uniformly for $x\in\partial\Omega$ and $r_{x,P}\in\mathcal{I}_{x,P} := (\min\{|x-P|,|x-P^*|\},\max\{|x-P|,|x-P^*|\})$. Hence by using again (3.15) we get

$$\begin{aligned}\frac{|x-P^*|-|x-P|}{r_{x,P}^2} &= \frac{|x-P^*|^2-|x-P|^2}{r_{x,P}^2(|x-P|+|x-P^*|)} = \frac{2dy_3}{d^3(1+|\hat{y}|^2+o(|\hat{y}'|^2))^{3/2}}\\ &= \frac{d^3\sum_j\mathcal{H}_j\hat{y}_j'^2+2d\omega_P(d\hat{y}')}{d^3(1+|\hat{y}|^2+o(|\hat{y}'|^2))^{3/2}} = \frac{\sum_j\mathcal{H}_j\hat{y}_j'^2}{(1+|\hat{y}|^2+o(|\hat{y}'|^2))^{3/2}} + \frac{O(\omega_P(d\hat{y}'))}{d^2(1+|\hat{y}|^2)^{3/2}}\\ &= \frac{\sum_j\mathcal{H}_j\hat{y}_j'^2}{(1+|\hat{y}|^2)^{3/2}} + o(1) + \frac{O(\omega_P(d\hat{y}'))}{d^2(1+|\hat{y}|^2)^{3/2}}\end{aligned} \tag{4.43}$$

as $d\to 0$ uniformly for $x\in\partial\Omega$ and $r_{x,P}\in\mathcal{I}_{x,P}$. Observe that

$$\frac{\omega_P(d\hat{y}')}{d^2(1+|\hat{y}|^2)^{3/2}} = o(1) \text{ as } d\to 0 \text{ unif. for } x\in\partial\Omega. \tag{4.44}$$

Indeed by (3.15)

$$\frac{\omega_P(d\hat{y}')}{d^2(1+|\hat{y}|^2)^{3/2}} = \frac{o(d^2|\hat{y}'|^2)}{d^2(1+|\hat{y}|^2)^{3/2}} = o(1) \text{ unif. for } |\hat{y}'| \le \frac{1}{\sqrt{d}}.$$

While, if $|\hat{y}'| \ge \frac{1}{\sqrt{d}}$, then $\frac{\omega_P(d\hat{y}')}{d^2(1+|\hat{y}|^2)^{3/2}} \le \frac{C|y'|^2}{(1+|\hat{y}|^2)^{3/2}} = o(1)$. By inserting (4.44) in (4.43) we deduce

$$\frac{|x-P^*|-|x-P|}{r_{x,P}^2} = \sum_j\mathcal{H}_j\frac{\hat{y}_j'^2}{(1+|\hat{y}|^2)^{3/2}} + o(1) \text{ as } d\to 0^+$$

uniformly for $x\in\partial\Omega$ and $r_{x,P}\in\mathcal{I}_{x,P}$.

Then by the mean value theorem, using (4.38), we derive

$$V_{\varepsilon,P}(x)-V_{\varepsilon,P^*}(x)=-c_0\varepsilon(1+O(e^{-2\frac{d}{\varepsilon}}))\frac{|x-P|-|x-P^*|}{r_{x,P}^2}$$
$$=c_0\varepsilon(1+O(e^{-2\frac{d}{\varepsilon}}))\frac{\sum_j\mathcal{H}_j\hat{y}_j'^2}{(1+|\hat{y}|^2)^{3/2}}+o(\varepsilon)=\frac{c_0\varepsilon\sum_j\mathcal{H}_j\hat{y}_j'^2}{(1+|\hat{y}|^2)^{3/2}}+o(\varepsilon)$$

as $d\to 0^+$, $\frac{d}{\varepsilon}\to+\infty$ uniformly for $x\in\partial\Omega$. Hence Step 1 holds.

Step 2. For $P\in\Gamma_\mu$ such that $\frac{d}{\varepsilon}$ is sufficiently large we have

$$V_{\varepsilon,P}(x)-V_{\varepsilon,P^*}(x)=O(\varepsilon)\ \textit{unif. for}\ x\in\partial\Omega.$$

By (4.39), fixed $\eta>0$, for $P\in\Gamma_\mu$ with $d\geq\eta$ we have

$$V_{\varepsilon,P}(x),\ V_{\varepsilon,P^*}(x)\leq\frac{c_0\varepsilon}{\eta}+O(e^{-\frac{2\eta}{\varepsilon}})=O(\varepsilon)\ \text{unif. for}\ x\in\partial\Omega.$$

Then Step 2 follows by Step 1.

Step 3. Set $\hat{W}_{\varepsilon,P}(\hat{y})=W_{\varepsilon,P}(x)$ where $\hat{y}=\frac{T_Px}{d}$. Then for $P\in\Gamma_\mu$ such that $\frac{d}{\varepsilon}$ is sufficiently large

$$\hat{W}_{\varepsilon,P}(\hat{y})=O(\varepsilon)\ \textit{unif. for}\ \hat{y}\in\frac{T_P\Omega}{d}. \tag{4.45}$$

Furthermore $\frac{1}{\varepsilon}\hat{W}_{\varepsilon,P}\to\hat{W}_1+\hat{W}_2$ as $\frac{d}{\varepsilon}\to+\infty$ and $d\to 0^+$ uniformly on compact sets of $\mathbb{R}^3_+$, where $\hat{W}_j$ satisfies

$$\Delta\hat{W}_j=0\ \textit{in}\ \mathbb{R}^3_+,\quad \hat{W}_j=c_0\mathcal{H}_j\frac{\hat{y}_j'^2}{(1+|\hat{y}'|^2)^{3/2}}\ \textit{on}\ \partial\mathbb{R}^3_+. \tag{4.46}$$

By Step 2, from the comparison principle it follows immediately that if $\frac{d}{\varepsilon}$ is large enough $\hat{W}_{\varepsilon,P}(\hat{y})=O(\varepsilon)$ uniformly for $\hat{y}\in\frac{T_P\Omega}{d}$.

By (4.42) and Step 1, from the comparison principle it follows that as $d\to 0^+$ and $\frac{d}{\varepsilon}\to+\infty$

$$\hat{W}_{\varepsilon,P}(\hat{y})=\varepsilon\sum_{j=1,2}\hat{W}^j_{\varepsilon,P}(\hat{y})+o(\varepsilon)$$

uniformly for $\hat{y}\in\frac{T_P\Omega}{d}$, where $\hat{W}^j_{\varepsilon,P}$, solve

$$\Delta\hat{W}^j_{\varepsilon,P}=0\ \text{in}\ \frac{T_P\Omega}{d},\quad \hat{W}^j_{\varepsilon,P}=g_j:=c_0\mathcal{H}_j\frac{\hat{y}_j'^2}{(1+|\hat{y}|^2)^{3/2}}\ \text{on}\ \partial\Big(\frac{T_P\Omega}{d}\Big),\quad j=1,2. \tag{4.47}$$

Denote by $D^{1,2}$ the closure of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|u\|^2_{D^{1,2}}=\int_{\mathbb{R}^3}|\nabla u|^2dx$. An easy computation shows that $g_j\in D^{1,2}$. By multiplying both members of (4.47) by $\hat{W}^j_{\varepsilon,P}-g_j$ and integrating on $\frac{T_P\Omega}{d}$ we obtain

$$\int_{\frac{T_P\Omega}{d}}|\nabla(\hat{W}^j_{\varepsilon,P}-g_j)|^2d\hat{y}=-\int_{\frac{T_P\Omega}{d}}\nabla(\hat{W}^j_{\varepsilon,P}-g_j)\nabla g_jd\hat{y}\leq\|\hat{W}^j_{\varepsilon,P}-g_j\|_{D^{1,2}}\|g_j\|_{D^{1,2}}$$

where we have set $\hat{W}^j_{\varepsilon,P}=g_j$ in $\mathbb{R}^3\setminus\frac{T_P\Omega}{d}$. Hence we deduce that $\{\hat{W}^j_{\varepsilon,P}-g_j\}_\varepsilon$ and, consequently, $\{\hat{W}^j_{\varepsilon,P}\}_\varepsilon$ are bounded in $D^{1,2}$. Then, if we consider a generic sequence $\varepsilon_n\to 0^+$ and $P_n\in\Omega$ such that

$d_{P_n} \to 0$ and $\frac{d_{P_n}}{\varepsilon_n} \to +\infty$, up to a subsequence we may assume $\hat{W}^j_{\varepsilon_n, P_n} \rightharpoonup \hat{W}_j$ as $n \to +\infty$ weakly in $D^{1,2}$ and a.e. in $\mathbb{R}^3$. Note that, since by construction $\chi_{\frac{1}{d_{P_n}} T_{P_n}\Omega} \to \chi_{\mathbb{R}^3_+}$ (denoting χ the characteristic function), then $\hat{W}^j_{\varepsilon_n, P_n} \to g_j$ a.e. in $\mathbb{R}^3_-$ and $\hat{W}_j|_{\mathbb{R}^3_+}$ solves (4.46). The uniqueness of the solution of (4.46) implies that all the family $\hat{W}^j_{\varepsilon,P}$ converges to $\hat{W}_j$ in $D^{1,2}$ and a.e. in $\mathbb{R}^3_+$ as $d \to 0^+$ and $\frac{d}{\varepsilon} \to +\infty$.

For every fixed compact set $K \subset \mathbb{R}^3_+$, for small d we have $K \subset \frac{T_P\Omega}{d}$. Since $\{\hat{W}^j_{\varepsilon,P}\}_\varepsilon$ are harmonic functions uniformly bounded on $\frac{T_P\Omega}{d}$, then the classical Schauder's internal estimates imply that $\hat{W}^j_{\varepsilon,P} \to \hat{W}_j$ uniformly in K.

Step 4. End of the proof:

$$\frac{\gamma}{4}\int_\Omega w^2_{\varepsilon,P}(-\Delta^{-1})[w^2_{\varepsilon,P}]dx = \varepsilon^5\Big(I_2 - I_3\frac{\varepsilon}{d} + O(\varepsilon) + o(e^{-\frac{d}{\varepsilon}})\Big) \quad as\ \frac{d}{\varepsilon} \to +\infty,$$

$$\frac{\gamma}{4}\int_\Omega w^2_{\varepsilon,P}(-\Delta^{-1})[w^2_{\varepsilon,P}]dx = \varepsilon^5\Big(I_2 - I_3\frac{\varepsilon}{d} - \varepsilon I_4\mathcal{H}(\Sigma_P) + o(\varepsilon) + o(e^{-\frac{d}{\varepsilon}})\Big)\ as\ \frac{d}{\varepsilon} \to +\infty,\ d \to 0^+.$$

From (4.40) we obtain

$$\begin{aligned}\int_\Omega w^2_{\varepsilon,P}(-\Delta^{-1})[w^2_{\varepsilon,P}]dx =& \varepsilon^5\int_{\frac{\Omega}{\varepsilon}} w^2_{\varepsilon,P}(\varepsilon x)\big(V_{\varepsilon,P}(\varepsilon x) - V_{\varepsilon,P^*}(\varepsilon x)\big)dx \\ &+ \varepsilon^5\int_{\frac{\Omega}{\varepsilon}} w^2_{\varepsilon,P}(\varepsilon x)Z_{\varepsilon,P}(\varepsilon x)dx - \varepsilon^5\int_{\frac{\Omega}{\varepsilon}} w^2_{\varepsilon,P}(\varepsilon x)W_{\varepsilon,P}(\varepsilon x)dx.\end{aligned} \tag{4.48}$$

(3.14) and (4.39) imply

$$\begin{aligned}\frac{\gamma}{4}\int_{\frac{\Omega}{\varepsilon}} w^2_{\varepsilon,P}(\varepsilon x)\big(V_{\varepsilon,P}(\varepsilon x) - V_{\varepsilon,P^*}(\varepsilon x)\big)dx &= \frac{\gamma}{4}\int_{\mathbb{R}^3} w^2 V_0 dx - \frac{\gamma}{4}\int_{\mathbb{R}^3} w^2 V_0\Big(x - \frac{P^*-P}{\varepsilon}\Big)dx + o(\varepsilon^{-\frac{d}{\varepsilon}}) \\ &= I_2 - \frac{\gamma}{4}\int_{\mathbb{R}^3} w^2\frac{c_0}{|x - \frac{P-P^*}{\varepsilon}|}dx + O\Big(\int_{\mathbb{R}^3} w^2 e^{-2|x-\frac{P-P^*}{\varepsilon}|}dx\Big) + o(\varepsilon^{-\frac{d}{\varepsilon}}) \\ &= I_2 - \gamma\pi c_0 V_0\Big(\frac{P-P^*}{\varepsilon}\Big) + O\Big(e^{-\frac{2d}{\varepsilon}}\int_{\mathbb{R}^3} w^2 e^{|x|}dx\Big) + O(e^{-\frac{d}{\varepsilon}}) = I_2 - I_3\frac{\varepsilon}{d} + o(e^{-\frac{d}{\varepsilon}}),\end{aligned}$$

where we have set $I_2 = \frac{\gamma}{4}\int_{\mathbb{R}^3} w^2 V_0$ and $I_3 = \frac{\gamma\pi c_0^2}{2}$.

Using c) of Proposition 2.1, Lemma 3.1 and (4.41), for $\frac{d}{\varepsilon}$ large enough we get

$$\int_{\frac{\Omega}{\varepsilon}} w^2_{\varepsilon,P}(\varepsilon x)Z_{\varepsilon,P}(\varepsilon x)dx \le C\|Z_{\varepsilon,P}\|_{L^\infty(\Omega)} \le C\|w^2_{\varepsilon,P^*}\|_{L^\infty(\Omega)} + C\varepsilon^{-3}\|w^2_{\varepsilon,P^*}\|_{L^1(\Omega)} \le Ce^{-\frac{4d}{3\varepsilon}},$$

and, using (4.45),

$$\int_{\frac{\Omega}{\varepsilon}} w^2_{\varepsilon,P}(\varepsilon x)W_{\varepsilon,P}(\varepsilon x)dx = O(\varepsilon);$$

then the first part of the thesis of Step 4 follows. Finally as $\frac{d}{\varepsilon}\to+\infty$

$$\begin{aligned}\int_{\frac{\Omega}{\varepsilon}} w^2_{\varepsilon,P}(\varepsilon x)W_{\varepsilon,P}(\varepsilon x)dx &= \int_{|x-\frac{P}{\varepsilon}|\le\sqrt{\frac{d}{\varepsilon}}} w^2_{\varepsilon,P}(\varepsilon x)W_{\varepsilon,P}(\varepsilon x)dz + o(\varepsilon)\\ &= \int_{|y-\frac{d}{\varepsilon}\ell_3|\le\sqrt{\frac{d}{\varepsilon}}} w^2\Big(y-\frac{d}{\varepsilon}\ell_3\Big)\hat W_{\varepsilon,P}\Big(\frac{\varepsilon}{d}y\Big)dy + o(\varepsilon) = \int_{|z|\le\sqrt{\frac{d}{\varepsilon}}} w^2(z)\hat W_{\varepsilon,P}\Big(\ell_3+\frac{\varepsilon}{d}z\Big)dz + o(\varepsilon).\end{aligned}$$

By Step 3 we immediately deduce $\frac{1}{\varepsilon}\hat W_{\varepsilon,P}\big(\ell_3+\frac{\varepsilon}{d}z\big)\to\hat W_1(\ell_3)+\hat W_2(\ell_3)$ uniformly on $|z|\le\sqrt{\frac{d}{\varepsilon}}$ as $\frac{d}{\varepsilon}\to+\infty$ and $d\to0^+$. On the other hand it is well known (see, for example [20], pg. 37) that the following representation formula holds for (4.46):

$$\hat W_j(\hat y) = c_0\mathcal{H}_j\frac{\hat y_3}{2\pi}\int_{\mathbb{R}^2}\frac{|\hat z_j'|^2}{(1+|\hat z'|^2)^{3/2}}\frac{d\hat z'}{[(\hat z_1-\hat y_1)^2+(\hat z_2-\hat y_2)^2+\hat y_3^2]^{3/2}},$$

by which

$$\hat W_j(\ell_3) = c_0\mathcal{H}_j\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{|\hat z_j'|^2}{(1+|\hat z'|^2)^3}d\hat z',\quad j=1,2.$$

Hence we obtain

$$\frac{\gamma}{4}\int_{\frac{\Omega}{\varepsilon}} w^2_{\varepsilon,P}(\varepsilon x)W_{\varepsilon,P}(\varepsilon x)dx = \varepsilon\frac{I_4}{2}\sum_{j=1,2}\mathcal{H}_j + o(\varepsilon) = \varepsilon I_4\mathcal{H}(\Sigma_P)+o(\varepsilon)$$

as $\frac{d}{\varepsilon}\to+\infty$ and $d\to0^+$, where $I_4=\frac{c_0\gamma}{2}\int_{\mathbb{R}^3}w^2dx\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{|\hat z_1'|^2}{(1+|\hat z'|^2)^3}d\hat z'$. □

5. Least Energy Solutions

The object of this section is to prove the first part of Theorem 1.1; in particular we are going to show the existence of a least energy solution v_ε for the problem (1.5), i.e. a solution with the important property of minimizing the energy J_ε among all solutions of (1.5). Furthermore we will provide a detailed description of its shape which has the form of a single spike near the boundary. The more delicate matter of the location of the spike at the boundary will be the subject of the next Section which concludes the proof.

We begin with the following existence theorem.

Theorem 5.1. *For every $\varepsilon>0$ the value J^*_ε defined by (1.7) is a positive critical value of J_ε with critical point v_ε. Furthermore J^*_ε is the least energy critical value of J_ε. Finally for every family $d_\varepsilon\to0^+$ with $\frac{d_\varepsilon}{\varepsilon}\to+\infty$*

$$\begin{aligned}J^*_\varepsilon \le& I[w]\varepsilon^3 + I_2\varepsilon^5 - I_3\frac{\varepsilon^6}{d_\varepsilon} + \varepsilon^3\alpha\Big(\frac{d_\varepsilon}{\varepsilon}\Big) - I_4\varepsilon^6\mathcal{H}_0 + O\Big(\varepsilon^4\sqrt{\frac{d_\varepsilon}{\varepsilon}}e^{-2\frac{d_\varepsilon}{\varepsilon}}\Big)\\ &+ o(\varepsilon^6+\varepsilon^{11/2}e^{-\frac{d_\varepsilon}{2\varepsilon}}+\varepsilon^5e^{-\frac{d_\varepsilon}{\varepsilon}}+\varepsilon^4e^{-2\frac{d_\varepsilon}{\varepsilon}}+\varepsilon^3e^{-3\frac{d_\varepsilon}{\varepsilon}})\ \text{as}\ \varepsilon\to0^+.\end{aligned}\tag{5.49}$$

where $\alpha:\mathbb{R}^+\to\mathbb{R}$ has been introduced in Proposition 4.1.

Proof. Fix $\varepsilon > 0$. The object is to apply the Mountain-Pass Lemma to the functional J_ε. To this aim we first prove that J_ε satisfies the Palais-Smale condition. Let $\{v_n\}_n \subset H_0^1(\Omega)$ be such that

$$|J_\varepsilon[v_n]| \le M, \quad J_\varepsilon'[v_n] \to 0$$

for some constant $M > 0$. Then, using assumption (f4),

$$\begin{aligned} 4M + o(1)\|v_n\|_{H^1} \ge 4J_\varepsilon[v_n] - \langle J_\varepsilon'[v_n], v_n\rangle &= 3\varepsilon^2 \int_\Omega |\nabla v_n|^2 dx + 3\int_\Omega |v_n|^2 dx + \int_\Omega (f(v_n)v_n - 4F(v_n))\, dx \\ &\ge 3\varepsilon^2 \int_\Omega |\nabla v_n|^2 dx + 3\int_\Omega |v_n|^2 dx. \end{aligned}$$

Hence $\{v_n\}_n$ turns out to be bounded in $H_0^1(\Omega)$; then, up to a subsequence, using Rellich's theorem, for some $v \in H_0^1(\Omega)$

$$v_n \rightharpoonup v \text{ in } H_0^1(\Omega), \qquad v_n \to v \text{ in } L^s(\Omega) \text{ for } 1 \le s < 6.$$

Using the continuity of $(-\Delta)^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega)$ we deduce

$$(-\Delta)^{-1}[v_n^2] \to (-\Delta)^{-1}[v^2] \text{ in } H_0^1(\Omega)$$

and, consequently, using assumption (f3),

$$-v_n + f(v_n) - \gamma(-\Delta)^{-1}[v_n^2]v_n \to -v + f(v) - \gamma(-\Delta)^{-1}[v^2]v \text{ in } H^{-1}(\Omega).$$

Hence we arrive at

$$-\varepsilon^2\Delta v_n = J_\varepsilon'[v_n] - v_n + f(v_n) - \gamma(-\Delta)^{-1}[v_n^2]v_n \to -v + f(v) - \gamma(-\Delta)^{-1}[v^2]v \text{ in } H^{-1}(\Omega).$$

Again the continuity of $(-\Delta)^{-1}$ allows us to conclude $\varepsilon^2 v_n \to (-\Delta)^{-1}[-v + f(v) - \gamma(-\Delta)^{-1}[v^2]v]$ in $H_0^1(\Omega)$. Hence the P.-S. condition is satisfied for J_ε.

Next observe that $J_\varepsilon[0] = 0$; moreover combining (1.4) and the Sobolev's embeddings

$$\begin{aligned} J_\varepsilon[v] &\ge \frac{\varepsilon^2}{2}\int_\Omega |\nabla v|^2 dx + \frac{1}{2}\int_\Omega |v|^2 dx - C\int_\Omega |v|^4 dx - C\int_\Omega |v|^{p+1} dx \\ &\ge \Big(\frac{\varepsilon^2}{2} - C\|v\|_{H^1}^2 - C\|v\|_{H^1}^{p-1}\Big)\|v\|_{H^1}^2 \ge \rho_\varepsilon > 0 \end{aligned} \tag{5.50}$$

provided that $\|v\|_{H^1}$ is sufficiently small. Condition (f4) can be restated as a differential equation for the function F of the form $\frac{d}{dt}\log F \ge \frac{\theta}{t}$ for $t > 0$, which implies

$$F(t) \ge C(|t|^\theta - 1) \quad \forall t \ge 0. \tag{5.51}$$

Hence, fixed $v \in H_0^1(\Omega) \setminus \{0\}$ such that $v_+ = \max\{v, 0\} \ne 0$, we get

$$J_\varepsilon[\lambda v] \le \frac{\lambda^2}{2}\int_\Omega \left(\varepsilon^2|\nabla v|^2 + |v|^2\right)dx + \gamma\frac{\lambda^4}{4}\int_\Omega v^2(-\Delta)^{-1}[v^2]dx - C\frac{\lambda^\theta}{4}\int_\Omega |v_+|^\theta dx + C \to -\infty \tag{5.52}$$

as $\lambda \to +\infty$. The well-known Mountain-Pass Lemma applies and gives that the following is actually a critical value for J_ε:

$$\tilde{J}_\varepsilon^* = \inf_{q\in\mathcal{Q}} \max_{t\in[0,1]} J_\varepsilon[q(t)]$$

where $\mathcal{Q} = \{q \in C([0,1], H_0^1(\Omega)) \,|\, q(0) = 0,\ J_\varepsilon[q(1)] < 0\}$. Denoted by v_ε the associated critical point, hence v_ε solves the equation

$$\varepsilon^2 \Delta v_\varepsilon - v_\varepsilon + f(v_\varepsilon) - \gamma v_\varepsilon (-\Delta)^{-1}[v_\varepsilon^2] = 0 \text{ in } \Omega. \tag{5.53}$$

It remains to show that $v_\varepsilon > 0$. Indeed, multiplying (5.53) by $v_\varepsilon^- = \max\{0, -v_\varepsilon\}$, and using (f1) we see that

$$\int_\Omega |\nabla v_\varepsilon^-|^2\, dx + \int_\Omega |v_\varepsilon^-|^2\, dx + \gamma \int_\Omega (v_\varepsilon^-)^2 (-\Delta)^{-1}[v_\varepsilon^2]\, dx = 0$$

which implies $v_\varepsilon^- = 0$. By the strong maximum principle $v_\varepsilon > 0$ in Ω. Then v_ε is a solution to (1.5).

We are going to prove that, as a consequence of condition (f2), we can characterize the values $\tilde{J}_\varepsilon^*$ in a simpler way, i.e. $\tilde{J}_\varepsilon^* = J_\varepsilon^*$. First notice that, since $J_\varepsilon[tv] \to +\infty$ for $v \in H_0^1(\Omega) \setminus \{0\}$ with $v_+ = 0$, then J_ε^* can also be defined as

$$J_\varepsilon^* = \inf_{v\in H_0^1(\Omega),\, v_+\neq 0} \sup_{t\geq 0} J_\varepsilon[tv].$$

For $v \in H_0^1(\Omega)$, $v_+ \neq 0$, we have $J_\varepsilon[\lambda v] < 0$ for large λ, by which we immediately obtain $\tilde{J}_\varepsilon^* \leq J_\varepsilon^*$. In order to prove the opposite inequality, we first note that fixed $v \in H_0^1(\Omega)$ with $v_+ \neq 0$, the function $\lambda > 0 \mapsto J_\varepsilon[\lambda v]$ has a unique nontrivial critical point which is a maximum point. Indeed

$$\frac{d}{d\lambda} J_\varepsilon[\lambda v] = \langle J_\varepsilon'[\lambda v], v\rangle = \lambda^3 \Big(\frac{1}{\lambda^2} \int_\Omega (\varepsilon^2 |\nabla v|^2 + |v|^2) dx + \gamma \int_\Omega v^2 (-\Delta)^{-1}[v^2] dx - \int_\Omega \frac{f(\lambda v)}{\lambda^3 v^3} v^4 dx \Big);$$

hypothesis (f2) implies that the bracket on the right hand side is a decreasing function of λ. Noting that, by (5.50), $J_\varepsilon[\lambda v] > 0$ for $\lambda > 0$ small and $J_\varepsilon[\lambda v] \to -\infty$ as $\lambda \to +\infty$, we obtain the assertion. Define $\lambda_v > 0$ as the unique nontrivial critical point of $J_\varepsilon[\lambda v]$. Since $J_\varepsilon'[v_\varepsilon] = 0$, it is clear that $\frac{d}{d\lambda} J_\varepsilon[\lambda v_\varepsilon]_{\lambda=1} = 0$, which implies $\lambda_{v_\varepsilon} = 1$ and, consequently

$$J_\varepsilon^* \leq J_\varepsilon[v_\varepsilon] = \tilde{J}_\varepsilon^*.$$

Observe that

$$J_\varepsilon^* = \inf_{v\in M_\varepsilon} J_\varepsilon[v]$$

where

$$\begin{aligned} M_\varepsilon :&= \{\lambda_v v \,|\, v \in H_0^1(\Omega),\, v_+ \neq 0\} \\ &= \Big\{ v \in H_0^1(\Omega) \,\Big|\, \langle J_\varepsilon'[v], v\rangle := \int_\Omega (\varepsilon^2 |\nabla v|^2 + |v|^2) dx + \gamma \int_\Omega v^2 (-\Delta)^{-1}[v^2] dx - \int_\Omega f(v) v dx = 0 \Big\}. \end{aligned}$$

Since any nontrivial critical point of J_ε lies on M_ε, then J_ε^* is the smallest critical value of J_ε and, consequently, v_ε is a least energy solution for (1.5).

To prove (5.49) choose $Q_\varepsilon \in \Omega$ such that $d_\varepsilon = d_{Q_\varepsilon}$ and $Q_\varepsilon \to P_0$ where $\mathcal{H}(P_0) = \mathcal{H}_0$. Then set $\lambda_\varepsilon = \lambda_{\tilde{w}_{\varepsilon,Q_\varepsilon}} > 0$, i.e.

$$\lambda_\varepsilon \int_\Omega (\varepsilon^2|\nabla \tilde{w}_{\varepsilon,Q_\varepsilon}|^2 + |\tilde{w}_{\varepsilon,Q_\varepsilon}|^2)dx + \lambda_\varepsilon^3\gamma \int_\Omega \tilde{w}^2_{\varepsilon,Q_\varepsilon}(-\Delta)^{-1}[\tilde{w}^2_{\varepsilon,Q_\varepsilon}]dx - \int_\Omega f(\lambda_\varepsilon \tilde{w}_{\varepsilon,Q_\varepsilon})\tilde{w}_{\varepsilon,Q_\varepsilon}dx = 0,$$

which can be written, using Proposition 4.2, as

$$\lambda_\varepsilon \int_\Omega (\varepsilon^2|\nabla \tilde{w}_{\varepsilon,Q_\varepsilon}|^2 + |\tilde{w}_{\varepsilon,Q_\varepsilon}|^2)dx + O(\lambda_\varepsilon^3\varepsilon^5) - \int_\Omega f(\lambda_\varepsilon \tilde{w}_{\varepsilon,Q_\varepsilon})\tilde{w}_{\varepsilon,Q_\varepsilon}dx = 0. \tag{5.54}$$

According to assumption (f2) we have $f(t) \geq ct^3$ for all $t \geq 0$; hence using (1.4) and Corollary 3.1 we get

$$\frac{\varepsilon^3\int_{\mathbb{R}^3}(|\nabla w|^2 + |w|^2)dx + O(\lambda_\varepsilon^2\varepsilon^5)}{C\varepsilon^3(\lambda_\varepsilon^2\int_{\mathbb{R}^3}|w|^4dx + \lambda_\varepsilon^{p-1}\int_{\mathbb{R}^3}|w|^{p+1}dx)} \leq \lambda_\varepsilon \leq \frac{\varepsilon^3\int_{\mathbb{R}^3}(|\nabla w|^2 + |w|^2)dx + O(\lambda_\varepsilon^2\varepsilon^5)}{c\varepsilon^3\lambda_\varepsilon^2\int_{\mathbb{R}^3}|w|^4dx},$$

by which $\{\lambda_\varepsilon\}$ turns out to be bounded from above and below away from 0. We state that $\lambda_\varepsilon \to 1$ as $\varepsilon \to 0^+$. Indeed, assume by absurd that $\lambda_{\varepsilon_n} \to \bar\lambda > 0$ and $\bar\lambda \neq 1$ for some sequence $\varepsilon_n \to 0^+$. Using Corollary 3.1 in (5.54) we get

$$\bar\lambda \int_{\mathbb{R}^3}(|\nabla w|^2 + |w|^2)dx - \int_{\mathbb{R}^3} f(\bar\lambda w)wdx = 0,$$

i.e. $\bar\lambda$ is a critical point of $\lambda > 0 \mapsto I[\lambda w]$. On the other hand $\lambda = 1$ is obviously a critical point too, then the uniqueness of the nontrivial critical point (which can be proved as for $J_\varepsilon[\lambda w]$) gives the contradiction. Hence $\lambda_\varepsilon \to 1$ as $\varepsilon \to 0^+$. Using equation (3.11), (5.54) leads to

$$\int_\Omega \lambda_\varepsilon f(w_{\varepsilon,Q_\varepsilon})\tilde{w}_{\varepsilon,Q_\varepsilon}dx - \int_\Omega f(\lambda_\varepsilon \tilde{w}_{\varepsilon,Q_\varepsilon})\tilde{w}_{\varepsilon,Q_\varepsilon}dx = O(\varepsilon^5).$$

Observe that by assumption (f1) and (3.12) we get

$$\lambda_\varepsilon f(w_{\varepsilon,Q_\varepsilon}) - f(\lambda_\varepsilon\tilde{w}_{\varepsilon,Q_\varepsilon}) = (\lambda_\varepsilon - 1)(f(\tilde{w}_{\varepsilon,Q_\varepsilon}) - f'(\tilde{w}_{\varepsilon,Q_\varepsilon})\tilde{w}_{\varepsilon,Q_\varepsilon} + o(1)) + O(w_{\varepsilon,Q_\varepsilon}(\tilde{w}_{\varepsilon,Q_\varepsilon} - w_{\varepsilon,Q_\varepsilon}))$$

uniformly in Ω. Hence Proposition 3.1 leads to

$$\begin{aligned} O(\varepsilon^5) = (\lambda_\varepsilon - 1)\Big(&\int_\Omega (f(\tilde{w}_{\varepsilon,Q_\varepsilon}) - f'(\tilde{w}_{\varepsilon,Q_\varepsilon})\tilde{w}_{\varepsilon,Q_\varepsilon})\tilde{w}_{\varepsilon,Q_\varepsilon}dx + o(1)\int_\Omega \tilde{w}_{\varepsilon,Q_\varepsilon}dx\Big) \\ &+ O\Big(\int_\Omega w_{\varepsilon,Q_\varepsilon}w_{\varepsilon,Q_\varepsilon^*}dx\Big) + O\Big(\int_\Omega w_{\varepsilon,Q_\varepsilon}(\varepsilon h_{\varepsilon,Q_\varepsilon} + k_{\varepsilon,Q_\varepsilon})dx\Big) \end{aligned}$$

and then, combining (3.18), Corollary 3.1 and Lemma 4.1 we deduce

$$\begin{aligned} O(\varepsilon^2) &= (\lambda_\varepsilon - 1)\Big(\int_{\mathbb{R}^3}(f(w) - f'(w)w)wdx + o(1)\Big) + O(e^{-2\frac{d_\varepsilon}{\varepsilon}}) + \varepsilon^{-3}O\big(\|w_{\varepsilon,Q_\varepsilon}\|_{L^2}\|\varepsilon h_{\varepsilon,Q_\varepsilon} + k_{\varepsilon,Q_\varepsilon}\|_2\big) \\ &= (\lambda_\varepsilon - 1)\Big(\int_{\mathbb{R}^3}(f(w) - f'(w)w)wdx + o(1)\Big) + O(\varepsilon^5 + \varepsilon e^{-\frac{2d_\varepsilon}{3\varepsilon}} + e^{-2\frac{d_\varepsilon}{\varepsilon}}). \end{aligned}$$

In view of (f2) $\frac{f(t)}{t}$ is strictly increasing for $t > 0$, hence $f(w) - f'(w)w < 0$. Thus we arrive at

$$1 - \lambda_\varepsilon = O(\varepsilon^2 + \varepsilon e^{-\frac{2d_\varepsilon}{3\varepsilon}} + e^{-2\frac{d_\varepsilon}{\varepsilon}}).$$

Setting $\Lambda_\varepsilon(\lambda)=J_\varepsilon[\lambda\tilde w_{\varepsilon,Q_\varepsilon}]$, it is immediate that $\Lambda_\varepsilon\in C^2$ and $\Lambda'_\varepsilon(\lambda_\varepsilon)=0$; then, for some $t_\varepsilon\to 1$

$$J^*_\varepsilon\le J_\varepsilon[\lambda_\varepsilon\tilde w_{\varepsilon,Q_\varepsilon}]=\Lambda_\varepsilon(\lambda_\varepsilon)=\Lambda_\varepsilon(1)-(\lambda_\varepsilon-1)^2\Lambda''_\varepsilon(t_\varepsilon)=J_\varepsilon[\tilde w_{\varepsilon,Q_\varepsilon}]+O(\varepsilon^4+\varepsilon^2e^{-\frac{4d_\varepsilon}{3\varepsilon}}+e^{-4\frac{d_\varepsilon}{\varepsilon}})\Lambda''_\varepsilon(t_\varepsilon)$$

and the thesis follows from Propositions 4.1-4.2 once we have observed that

$$\varepsilon^{-3}\Lambda''_\varepsilon(t_\varepsilon)\to\int_{\mathbb{R}^3}(|\nabla w|^2+|w|^2)dx-\int_{\mathbb{R}^3}f'(w)w^2dx$$

□

Corollary 5.1. *As a corollary of Theorem 5.1, choosing $d_\varepsilon=\varepsilon\log\frac{1}{\varepsilon}+\varepsilon\log\log\frac{1}{\varepsilon}$ and using* (4.24)*, it is easy to verify that the main term in the envelope on the right hand side of* (5.49) *is represented by the first four; hence we obtain*

$$\begin{aligned}J^*_\varepsilon&\le I[w]\varepsilon^3+I_2\varepsilon^5-I_3\frac{\varepsilon^5}{|\log\varepsilon|+\log|\log\varepsilon|}+\varepsilon^5\frac{B(1+o(1))}{4|\log\varepsilon|^2(|\log\varepsilon|+\log|\log\varepsilon|)}\\&=I[w]\varepsilon^3+I_2\varepsilon^5-I_3\frac{\varepsilon^5}{|\log\varepsilon|}+o\Big(\frac{\varepsilon^5}{|\log\varepsilon|^{5/3}}\Big)\ \text{as}\ \varepsilon\to0^+,\end{aligned}$$

since $\frac{1}{|\log\varepsilon|+\log|\log\varepsilon|}-\frac{1}{|\log\varepsilon|}=O\big(\frac{\log|\log\varepsilon|}{|\log\varepsilon|^2}\big)=o\big(\frac{1}{|\log\varepsilon|^{5/3}}\big)$.

The goal of the next two propositions is to obtain more precise information on the shape of the least-energy solution v_ε; in particular we will prove that v_ε develops a spike near the boundary whose profile resembles the approximation $\tilde w_{\varepsilon,P_\varepsilon}$ (for suitable $P_\varepsilon\in\Omega$) constructed in Section 3 up to the order $O(\tau_\varepsilon)$ (for suitable $\tau_\varepsilon\to0^+$).

Proposition 5.1. *For $\varepsilon>0$ sufficiently small v_ε is a single spike solution; more precisely there exists $P_\varepsilon\in\Omega$ such that P_ε is a the maximum point of v_ε on Ω and*

$$\frac{d_{P_\varepsilon}}{\varepsilon}\to+\infty\ \text{as}\ \varepsilon\to0^+;\tag{5.55}$$

furthermore v_ε satisfies

$$v_\varepsilon=\tilde w_{\varepsilon,P_\varepsilon}+\psi_\varepsilon$$

where

$$\|\psi_\varepsilon\|_\infty+\varepsilon^{-3}\int_\Omega(\varepsilon^2|\nabla\psi_\varepsilon|^2+|\psi_\varepsilon|^2)dx\to0\ \text{as}\ \varepsilon\to0^+.$$

Proof. First we observe that, setting $u_\varepsilon(x)=v_\varepsilon(\varepsilon x)$ for $x\in\frac{\Omega}{\varepsilon}$ and $u_\varepsilon=0$ in $\mathbb{R}^3\setminus\frac{\Omega}{\varepsilon}$, by assumption (f4) we have

$$\begin{aligned}\Big(\frac12-\frac1\theta\Big)\int_{\mathbb{R}^3}(|\nabla u_\varepsilon|^2+|u_\varepsilon|^2)dx&=\varepsilon^{-3}\Big(\frac12-\frac1\theta\Big)\int_\Omega(\varepsilon^2|\nabla v_\varepsilon|^2+|v_\varepsilon|^2)dx\le\varepsilon^{-3}J_\varepsilon[v_\varepsilon]-\frac{\varepsilon^{-3}}{\theta}\langle J'_\varepsilon[v_\varepsilon],v_\varepsilon\rangle\\&=\varepsilon^{-3}J_\varepsilon[v_\varepsilon]=\varepsilon^{-3}J^*_\varepsilon=I[w]+O(\varepsilon^2)\ \text{as}\ \varepsilon\to0^+,\end{aligned}$$

where in the last inequality we have used Corollary 5.1. Then $\{u_\varepsilon\}$ is bounded in $H^1(\mathbb{R}^3)$ for small ε. For the sake of simplicity we divide the remaining part of the proof into four Claims.

Claim 1. For every $R>0$ and $1\le q<6$:

$$\lim_{\varepsilon\to 0^+}\sup_{d_P\le\varepsilon R}\int_{B(\frac{P}{\varepsilon},R)} u_\varepsilon^q dx = 0.$$

For otherwise, there exist $q\in[1,6)$, $R>0$, a sequence $u_n := u_{\varepsilon_n}$ and $P_n\in\mathbb{R}^3$ with $d_{P_n}\le\varepsilon_n R$ such that

$$\lim_{n\to+\infty}\int_{B(\frac{P_n}{\varepsilon_n},R)} u_n^q dx > 0.$$

Then set $\tilde u_n(y) = u_n(x+\frac{P_n}{\varepsilon_n})$ where $y = T_{P_n}x$ and, without loss of generality, we may assume $dist(\frac{P_n}{\varepsilon_n},\frac{\partial\Omega}{\varepsilon_n}) = \frac{d_{P_n}}{\varepsilon_n}\to\delta\in[0,R]$ and $\tilde u_n\rightharpoonup\tilde u$ weakly in $H^1(\mathbb{R}^3)$. Since by construction $\chi_{\frac{1}{\varepsilon_n}T_{P_n}\Omega}\to\chi_{\mathbb{R}^3_+}$ (denoting χ the characteristic function), then, setting $\mathbb{R}^3_{\delta,+} = \{y\in\mathbb{R}^3\,|\,y_n > -\delta\}$, we deduce $\tilde u\in H^1_0(\mathbb{R}^3_{\delta,+})$ and (since by b) of Proposition 2.1 $(-\Delta)^{-1}[v_\varepsilon^2]\le C\|v_\varepsilon\|^2_{L^4} = O(\varepsilon^{3/2})$) $\tilde u$ solves

$$\Delta\tilde u-\tilde u+f(\tilde u)=0 \text{ in } \mathbb{R}^3_{\delta,+},\quad \tilde u\ge 0 \text{ in } \mathbb{R}^3_{\delta,+},\quad \tilde u = 0 \text{ on } \partial\mathbb{R}^3_{\delta,+}. \tag{5.56}$$

Theorem 1.1 in [18] implies $\tilde u = 0$, which is a contradiction since

$$\int_{B(0,R)}\tilde u^q dx = \lim_{n\to+\infty}\int_{B(0,R)}\tilde u_n^q dx = \lim_{n\to+\infty}\int_{B(\frac{P_n}{\varepsilon_n},R)} u_n^2 dx > 0.$$

Hence Claim 1 holds.

Claim 2. There exists $\eta>0$ such that if $\varepsilon\in(0,1)$ and Q_ε is a local maximum point for v_ε, then

$$v_\varepsilon(Q_\varepsilon)\ge\eta.$$

Indeed

$$v_\varepsilon(Q_\varepsilon)-f(v_\varepsilon(Q_\varepsilon))\le v_\varepsilon(Q_\varepsilon)+v_\varepsilon(Q_\varepsilon)(-\Delta)^{-1}[v_\varepsilon^2](Q_\varepsilon)-f(v_\varepsilon(Q_\varepsilon)) = \varepsilon^2\Delta v_\varepsilon(Q_\varepsilon)\le 0.$$

Using (1.4) we obtain the thesis.

Claim 3. For every $R>0$

$$\limsup_{\varepsilon\to 0^+}\sup_{d_P\le\varepsilon R} v_\varepsilon(P) = 0.$$

Fix $R>0$ sufficiently large such that

$$\frac{1}{2\pi R}\int_{\mathbb{R}^3} f(u_n)dx\le\frac{\eta}{2}$$

and let $P_\varepsilon\in\Omega$ be the maximum point for v_ε in $\{x\in\Omega\,|\,d_P\le\varepsilon R\}$. Assume by absurd that there is a sequence $\varepsilon_n\to 0$ such that $v_n(P_n) := v_{\varepsilon_n}(P_{\varepsilon_n})\not\to 0$. First observe that $\frac{d_{P_n}}{\varepsilon_n}\to 0$. Otherwise, up to a subsequence, $\frac{d_{P_n}}{\varepsilon_n}\to\delta>0$, and, proceeding as in Claim 1, setting $\tilde u_n(y) = u_n(x+\frac{P_n}{\varepsilon_n})$ with

$y = T_{P_n}(x)$, we deduce $\tilde{u}_n \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^3)$. On the other hand by standard elliptic regularity we have $\tilde{u}_n \to 0$ uniformly on compact sets of $\mathbb{R}^3_{\delta,+}$, then $v_n(P_n) = \tilde{u}_n(0) \to 0$ which is a contradiction. Hence $\frac{d_{P_n}}{\varepsilon_n} \to 0$, which implies that, at least for large n, P_n is a local maximum point for v_n in $B(P_n, \varepsilon_n \frac{R}{2})$. Consequently, by Claim 2, $v_n(P_n) \geq \eta$ and, by (1.4), $\frac{f(u_n)}{v_n(P_n)} \leq C(|u_n|^2 + |u_n|^{p-1})$ for $|z - \frac{P_n}{\varepsilon_n}| \leq \frac{R}{2}$.

By using the comparison principle it is immediate that $0 \leq u_\varepsilon \leq w_\varepsilon$ where w_ε solves

$$\Delta w_\varepsilon + f(u_\varepsilon) = 0 \text{ in } \frac{\Omega}{\varepsilon}, \quad w_\varepsilon = 0 \text{ on } \frac{\partial\Omega}{\varepsilon}.$$

Fix $\frac{3}{2} < a < \frac{6}{p-1}$ and let $a' < 3$ be such that $\frac{1}{a} + \frac{1}{a'} = 1$; by (2.8), using Hölder's inequality, we get

$$\begin{aligned}
\eta \leq v_n(P_n) \leq w_{\varepsilon_n}\Big(\frac{P_n}{\varepsilon_n}\Big) &\leq \frac{1}{4\pi}\int_{\frac{\Omega}{\varepsilon_n}} \frac{f(u_n)}{|\frac{P_n}{\varepsilon_n} - z|}dz \leq \frac{1}{4\pi}\int_{|z-\frac{P_n}{\varepsilon_n}|\leq\frac{R}{2}} \frac{f(u_n)}{|\frac{P_n}{\varepsilon_n} - z|}dz + \frac{1}{2\pi R}\int_{\mathbb{R}^3} f(u_n)dz \\
&\leq Cv_n(P_n)\int_{|z-\frac{P_n}{\varepsilon_n}|\leq\frac{R}{2}} \frac{|u_n|^2 + |u_n|^{p-1}}{|\frac{P_n}{\varepsilon_n} - z|}dz + \frac{\eta}{2} \\
&\leq Cv_n(P_n)\Big(\int_{|z|\leq\frac{R}{2}} \frac{1}{|z|^{a'}}dz\Big)^{1/a'}\Big(\int_{|z-\frac{P_n}{\varepsilon_n}|\leq\frac{R}{2}} (|u_n|^{2a} + |u_n|^{a(p-1)})dz\Big)^{1/a} + \frac{\eta}{2};
\end{aligned}$$

then Claim 1 implies

$$\int_{|z-\frac{P_n}{\varepsilon_n}|\leq\frac{R}{2}} |u_n|^{2a}dz, \quad \int_{|z-\frac{P_n}{\varepsilon_n}|\leq\frac{R}{2}} |u_n|^{a(p-1)}dz \to 0.$$

Hence we deduce

$$\eta \leq v_n(P_n) \leq o\big(v_n(P_n)\big) + \frac{\eta}{2}.$$

by which $\eta \leq v_n(P_n) \leq o(1) + \frac{\eta}{2}$ and the contradiction follows.

Claim 4. End of the proof.

Let P_ε be the maximum point for v_ε in Ω. According to Claims 2 and 3 $v_\varepsilon(P_\varepsilon) \geq \eta$ and $\frac{d_{P_\varepsilon}}{\varepsilon} \to +\infty$ as $\varepsilon \to 0^+$. Let $\varepsilon_n \to 0^+$ be an arbitrary sequence and set $P_n = P_{\varepsilon_n}$ and $\bar{u}_n(x) = u_{\varepsilon_n}(x + \frac{P_n}{\varepsilon_n})$. Since $dist(\frac{P_n}{\varepsilon_n}, \frac{\partial\Omega}{\varepsilon_n}) = \frac{d_{P_n}}{\varepsilon_n} \to +\infty$, we may assume $\bar{u}_n \rightharpoonup \bar{u}$ weakly in $H^1(\mathbb{R}^3)$ where $\bar{u}$ solves

$$\Delta\bar{u} - \bar{u} + f(\bar{u}) = 0 \text{ in } \mathbb{R}^3. \tag{5.57}$$

The elliptic regularity theory implies $\lim_{|x|\to+\infty} \bar{u} = 0$ (see [19], Theorem 5, and [40]) and $\bar{u}_n \to \bar{u}$ in $C^1_{loc}(\mathbb{R}^3)$; in particular $\bar{u}(0) = \max_{x\in\mathbb{R}^3} \bar{u}(x) \geq \eta$ and $\bar{u} \geq 0$, consequently, from the strong maximum principle $\bar{u} > 0$ in $\mathbb{R}^3$. Assumption (f5) implies $\bar{u} = w$. Then by Fatou's Lemma we get

$$\varepsilon_n^{-3}\int_\Omega \Big(\frac{1}{\theta}f(v_{\varepsilon_n})v_{\varepsilon_n} - F(v_{\varepsilon_n})\Big)dx = \int_{\mathbb{R}^3}\Big(\frac{1}{\theta}f(\bar{u}_n)\bar{u}_n - F(\bar{u}_n)\Big) \geq \int_{\mathbb{R}^3}\Big(\frac{1}{\theta}f(w)w - F(w)\Big)dx + o(1),$$

by which, since $J'_\varepsilon[v_\varepsilon] = I'[w] = 0$, using Corollary 5.1,

$$\begin{aligned}
\Big(\frac{1}{2}-\frac{1}{\theta}\Big)\int_{\mathbb{R}^3}(|\nabla \bar u_n|^2+|\bar u_n|^2)dx &= \varepsilon_n^{-3}\Big(\frac{1}{2}-\frac{1}{\theta}\Big)\int_\Omega(\varepsilon^2|\nabla v_{\varepsilon_n}|^2+|v_{\varepsilon_n}|^2)dx\\
&\le \varepsilon_n^{-3}J_{\varepsilon_n}[v_{\varepsilon_n}]-\frac{\varepsilon_n^{-3}}{\theta}\langle J'_{\varepsilon_n}[v_{\varepsilon_n}],v_{\varepsilon_n}\rangle-\varepsilon_n^{-3}\int_\Omega\Big(\frac{1}{\theta}f(v_{\varepsilon_n})v_{\varepsilon_n}-F(v_{\varepsilon_n})\Big)dx\\
&\le I[w]-\int_{\mathbb{R}^3}\Big(\frac{1}{\theta}f(w)w-F(w)\Big)dx+o(1)\\
&= I[w]-\frac{1}{\theta}\langle I'[w],w\rangle-\int_{\mathbb{R}^3}\Big(\frac{1}{\theta}f(w)w-F(w)\Big)dx+o(1)=\Big(\frac{1}{2}-\frac{1}{\theta}\Big)\int_{\mathbb{R}^3}(|\nabla w|^2+|w|^2)dx+o(1).
\end{aligned}$$

The weakly lower semicontinuity of the norm implies the opposite inequality, hence $\int_{\mathbb{R}^3}(|\nabla\bar u_n|^2+|\bar u_n|^2)dx\to\int_{\mathbb{R}^3}(|\nabla w|^2+|w|^2)dx$, which implies $\bar u_n\to w$ in $H^1(\mathbb{R}^3)$. The arbitrariness of ε_n leads to

$$u_\varepsilon\Big(\cdot+\frac{P_\varepsilon}{\varepsilon}\Big)\to w \text{ in } H^1(\mathbb{R}^3) \text{ and } C^1_{loc}(\mathbb{R}^3). \tag{5.58}$$

It remains to prove that

$$\lim_{|x|\to+\infty}u_\varepsilon\Big(\cdot+\frac{P_\varepsilon}{\varepsilon}\Big)=0 \text{ uniformly with respect to } \varepsilon.$$

For otherwise there would be a sequence $\varepsilon_n\to0$, $Q_n\in\Omega$ with $\frac{|Q_n-P_n|}{\varepsilon_n}\to+\infty$ and $u_{\varepsilon_n}(\frac{Q_n}{\varepsilon_n})\ge c>0$. According to Claim 3 $\frac{d_{Q_n}}{\varepsilon_n}\to+\infty$ as $n\to+\infty$, hence $u_{\varepsilon_n}(x+\frac{Q_n}{\varepsilon_n})\rightharpoonup u$ in $H^1(\mathbb{R}^3)$ and in $C^1_{loc}(\mathbb{R}^3)$ (with u solving (5.57)), by which $u(0)=\lim u_{\varepsilon_n}(\frac{Q_n}{\varepsilon_n})\ge c$; in particular $u\ne0$, which is in contradiction with (5.58). Hence we have proved that $u_\varepsilon(\cdot+\frac{P_\varepsilon}{\varepsilon})\to w$ in $H^1(\mathbb{R}^3)$ and $L^\infty(\mathbb{R}^3)$, which implies, using Corollary 3.1, that $\psi_\varepsilon(\varepsilon x):=u_\varepsilon(x)-\tilde w_{\varepsilon,P_\varepsilon}(\varepsilon x)\to0$ in $H^1(\mathbb{R}^3)$ and $L^\infty(\mathbb{R}^3)$. □

In the last proposition of this section we go further in the analysis began in the previous one and provide an estimate on the error order up to the approximation $\tilde w_{\varepsilon,P_\varepsilon}$ works in the expansion of v_ε.

Proposition 5.2. *For $\varepsilon>0$ sufficiently small v_ε satisfies*

$$v_\varepsilon=\tilde w_{\varepsilon,P_\varepsilon}+\tau_\varepsilon\phi_\varepsilon,$$

where $\tau_\varepsilon=\max\{e^{-\frac{5d_\varepsilon}{3\varepsilon}},\varepsilon e^{-\frac{3d_\varepsilon}{5\varepsilon}},\varepsilon^{5/3}\}$ and, for ε sufficiently small, ϕ_ε verifies

$$\|\phi_\varepsilon\|_\infty+\varepsilon^{-3}\int_\Omega(\varepsilon^2|\nabla\phi_\varepsilon|^2+|\phi_\varepsilon|^2)dx\le C. \tag{5.59}$$

Proof. Substituting $v_\varepsilon=\tilde w_{\varepsilon,P_\varepsilon}+\tau_\varepsilon\phi_\varepsilon$ into the equation $\mathcal{S}_\varepsilon[v_\varepsilon]=0$ we obtain

$$\varepsilon^2\Delta\phi_\varepsilon-\phi_\varepsilon+f'(\tilde w_{\varepsilon,P_\varepsilon})\phi_\varepsilon=-\tau_\varepsilon^{-1}\mathcal{S}_\varepsilon[\tilde w_{\varepsilon,P_\varepsilon}]+N_\varepsilon[\phi_\varepsilon]+M_\varepsilon[\phi_\varepsilon], \tag{5.60}$$

where

$$\begin{aligned}
N_\varepsilon[\phi]&=-\tau_\varepsilon^{-1}\big(f(\tilde w_{\varepsilon,P_\varepsilon}+\tau_\varepsilon\phi)-f(\tilde w_{\varepsilon,P_\varepsilon})-\tau_\varepsilon f'(\tilde w_{\varepsilon,P_\varepsilon})\phi\big),\\
M_\varepsilon[\phi]&=\gamma\tau_\varepsilon(\tilde w_{\varepsilon,P_\varepsilon}+\tau_\varepsilon\phi)(-\Delta)^{-1}[\phi^2]+2\gamma(\tilde w_{\varepsilon,P_\varepsilon}+\tau_\varepsilon\phi)(-\Delta)^{-1}[\tilde w_{\varepsilon,P_\varepsilon}\phi]+\gamma(-\Delta)^{-1}[\tilde w^2_{\varepsilon,P_\varepsilon}]\phi.
\end{aligned}$$

By Lemma 3.2 for ε sufficiently small we have

$$|\tau_\varepsilon^{-1}\mathcal{S}_\varepsilon[\tilde{w}_{\varepsilon,P_\varepsilon}]| \le C\max\{e^{-\frac{d_\varepsilon}{12\varepsilon}}, e^{-\frac{d_\varepsilon}{15\varepsilon}}, \varepsilon^{1/3}\}w_{\varepsilon,P_\varepsilon}^{1/4} \quad \text{unif. in } \Omega \tag{5.61}$$

while, since by Proposition 5.1 $\|\tau_\varepsilon\phi_\varepsilon\|_\infty = \|\psi_\varepsilon\|_\infty = o(1)$, $\|\tau_\varepsilon\phi_\varepsilon\|_{L^2}^2 = \|\psi_\varepsilon\|_{L^2}^2 = o(\varepsilon^3)$, by the mean value theorem we get

$$|N_\varepsilon[\phi_\varepsilon]| \le C\tau_\varepsilon\phi_\varepsilon^2 = o(1)|\phi_\varepsilon| \quad \text{unif. in } \Omega \tag{5.62}$$

and, using c) of Proposition 2.1, we have $\tau_\varepsilon(-\Delta)^{-1}[\phi_\varepsilon^2]$, $\tau_\varepsilon|(-\Delta)^{-1}[\tilde{w}_{\varepsilon,P_\varepsilon}\phi_\varepsilon]| = o(\varepsilon^2\tau_\varepsilon^{-1}) = o(1)$ and $|(-\Delta)^{-1}[\tilde{w}_{\varepsilon,P_\varepsilon}^2]| = O(\varepsilon^2)$ by which

$$|M_\varepsilon[\phi_\varepsilon]| = o(1)w_{\varepsilon,P_\varepsilon} + o(1)|\phi_\varepsilon|, \tag{5.63}$$

uniformly in Ω. Suppose that there exists a sequence $\varepsilon_n \to 0^+$ such that, setting $\phi_n := \phi_{\varepsilon_n}$, $\|\phi_n\|_\infty \to +\infty$ and let $x_n \in \Omega$ be such that $|\phi_n(x_n)| = \|\phi_n\|_\infty$. We may assume, without loss of generality, that x_n is a maximum. Then we claim that

$$\frac{|x_n - P_{\varepsilon_n}|}{\varepsilon_n} \le C. \tag{5.64}$$

Otherwise, up to a subsequence, $\frac{|x_n - P_{\varepsilon_n}|}{\varepsilon_n} \to +\infty$. Then by (3.12) we have $\tilde{w}_{\varepsilon_n,P_{\varepsilon_n}}(x_n) \to 0$, and, consequently, $f'(\tilde{w}_{\varepsilon_n,P_{\varepsilon_n}}(x_n)) \to 0$, while $\Delta\phi_n(x_n) \le 0$. Combining (5.60)-(5.63) we arrive at

$$0 = \varepsilon_n^2\Delta\phi_n(x_n) - (1+o(1))\|\phi_n\|_\infty + o(1) \le -(1+o(1))\|\phi_n\|_\infty + o(1)$$

and hence $\|\phi_n\|_\infty = o(1)$, in contrast to the hypothesis. Then (5.64) holds, and we may assume $\frac{x_n - P_{\varepsilon_n}}{\varepsilon_n} \to \bar{x}$. Now set $\hat{\phi}_n(x) = \frac{\phi_n}{\|\phi_n\|_\infty}(\varepsilon_n x + P_{\varepsilon_n})$ for $x \in \frac{\Omega - P_{\varepsilon_n}}{\varepsilon_n}$. By dividing both members of (5.60) by $\|\phi_n\|_\infty$ and using (5.61)-(5.63), we deduce that $\hat{\phi}_n$ solves

$$\Delta\hat{\phi}_n - \hat{\phi}_n + f'(\tilde{w}_{\varepsilon,P_{\varepsilon_n}}(\varepsilon_n x + P_{\varepsilon_n}))\hat{\phi}_n = o(1)|\hat{\phi}_n| + o(1)w^{1/4}, \quad |\hat{\phi}_n| \le 1 \text{ in } \frac{\Omega - P_{\varepsilon_n}}{\varepsilon_n}. \tag{5.65}$$

By multiplying both members of (5.65) by $\hat{\phi}_n$ we get

$$\int_{\mathbb{R}^3}(|\nabla\hat{\phi}_n|^2 + |\hat{\phi}_n|^2)dx \le \int_{\mathbb{R}^3}(|f'(\tilde{w}_{\varepsilon_n,P_{\varepsilon_n}})(\varepsilon_n x)| + w^{1/4})dx + o(1)\int_{\mathbb{R}^3}|\hat{\phi}_n|^2dx \le C + o(1)\int_{\mathbb{R}^3}|\hat{\phi}_n|^2dx,$$

i.e. $\hat{\phi}_n$ is bounded in $H^1(\mathbb{R}^3)$. Hence we may assume $\hat{\phi}_n \rightharpoonup \hat{\phi}$ weakly in $H^1(\mathbb{R}^3)$. Since by Proposition 5.1 $\frac{d_{P_\varepsilon}}{\varepsilon} \to +\infty$ we get $\chi_{\frac{\Omega - P_{\varepsilon_n}}{\varepsilon_n}} \to \chi_{\mathbb{R}^3} = 1$; then, by Corollary 3.1, $\hat{\phi}$ solves

$$\Delta\hat{\phi} - \hat{\phi} + f'(w)\hat{\phi} = 0, \quad |\hat{\phi}| \le 1 \text{ in } \mathbb{R}^3.$$

Furthermore elliptic estimates lead to $\hat{\phi} \in H^2(\mathbb{R}^3)$ and $\hat{\phi}_n \rightharpoonup \hat{\phi}$ in $C_{loc}^1(\mathbb{R}^3)$, which implies $\hat{\phi}(\bar{x}) = \lim_n \hat{\phi}_n\left(\frac{x_n - P_{\varepsilon_n}}{\varepsilon_n}\right) = \lim_n \frac{\phi_n(x_n)}{\|\phi_n\|_\infty} = 1$. In particular $\hat{\phi} \ne 0$. Assumption (f5) leads to

$$\hat{\phi} = \sum_{j=1}^3 a_j\frac{\partial w}{\partial x_j}$$

for some constants a_1, a_2, a_3. On the other hand by Part iii) of Proposition 3.1 and the choice of τ_ε we have $\varepsilon\nabla\tilde{w}_{\varepsilon,P_\varepsilon}(P_\varepsilon)=o(\tau_\varepsilon)$, by which

$$0=\nabla v_{\varepsilon_n}(P_{\varepsilon_n})=\nabla(\tilde{w}_{\varepsilon_n,P_{\varepsilon_n}}+\tau_{\varepsilon_n}\phi_n)(P_{\varepsilon_n})=o(\varepsilon_n^{-1}\tau_{\varepsilon_n})+\tau_{\varepsilon_n}\varepsilon_n^{-1}\|\phi_n\|_\infty\nabla\hat{\phi}_n(0).$$

This implies $\nabla\hat{\phi}(0)=0$, i.e. (since $\frac{\partial^2 w}{\partial x_k\partial x_j}(0)=0$ if $j\neq k$ and $\frac{\partial^2 w}{\partial x_j^2}(0)=w''(0)$) $a_jw''(0)=0$. The contradiction will follow if we prove that $w''(0)\neq 0$. Otherwise, from assumption (f5) $w(0)=f(w(0))$. By (f2) $\frac{f(t)}{t}$ is strictly increasing for $t>0$, while $w(0)=\max_{\mathbb{R}^3}w$; hence $w-f(w)\geq 0$ in $\mathbb{R}^3$. This is a contradiction since $\int_{\mathbb{R}^3}|\nabla w|^2+\int_{\mathbb{R}^3}(w-f(w))w=0$. Thus $|\phi_\varepsilon|\leq C$ for small ε. Finally, multiplying the equation (5.60) by ϕ_ε, integrating over Ω and using (5.61)-(5.63), we obtain

$$\begin{aligned}\varepsilon^2\int_\Omega|\nabla\phi_\varepsilon|^2dx+\int_\Omega|\phi_\varepsilon|^2dx&\leq\int_\Omega f'(\tilde{w}_{\varepsilon,P_\varepsilon})\phi_\varepsilon^2dx+o(1)\int_\Omega\phi_\varepsilon w_{\varepsilon,P_\varepsilon}^{1/4}dx+o(1)\int_\Omega|\phi_\varepsilon|^2dx\\&\leq C\int_\Omega f'(\tilde{w}_{\varepsilon,P_\varepsilon})dx+o(1)\int_\Omega w_{\varepsilon,P_\varepsilon}^{1/4}dx+o(1)\int_\Omega|\phi_\varepsilon|^2dx\leq C\varepsilon^{-3}+o(1)\int_\Omega|\phi_\varepsilon|^2dx.\end{aligned}$$

□

6. Proof of Theorem 1.1

Now we have all the ingredients to prove Theorem 1.1. We have just to combine the results of Proposition 5.1 and 5.2. In what follows we write $\hat{d}_\varepsilon$ in place of d_{P_ε}.

Step 1. The following holds:

$$J_\varepsilon^*=J_\varepsilon[v_\varepsilon]=J_\varepsilon[\tilde{w}_{\varepsilon,P_\varepsilon}]+O(\varepsilon^3\tau_\varepsilon^2)=J_\varepsilon[\tilde{w}_{\varepsilon,P_\varepsilon}]+o(\varepsilon^5e^{-\frac{\hat{d}_\varepsilon}{\varepsilon}}+\varepsilon^3e^{-3\frac{\hat{d}_\varepsilon}{\varepsilon}}+\varepsilon^6)\tag{6.66}$$

as $\varepsilon\to 0^+$.

Using a)-b) of Proposition 2.1 and (5.59) for $\varepsilon>0$ sufficiently small we get

$$\int_\Omega\left|v_\varepsilon^2(-\Delta)^{-1}[v_\varepsilon^2]dx-\tilde{w}_{\varepsilon,P_\varepsilon}^2(-\Delta)^{-1}[\tilde{w}_{\varepsilon,P_\varepsilon}^2]-4\tau_\varepsilon\tilde{w}_{\varepsilon,P_\varepsilon}\phi_\varepsilon(-\Delta)^{-1}[\tilde{w}_{\varepsilon,P_\varepsilon}^2]\right|dx=O(\varepsilon^3\tau_\varepsilon^2),$$

by which

$$\begin{aligned}J_\varepsilon[v_\varepsilon]=&J_\varepsilon[\tilde{w}_{\varepsilon,P_\varepsilon}]-\tau_\varepsilon\int_\Omega\mathcal{S}_\varepsilon[\tilde{w}_{\varepsilon,P_\varepsilon}]\phi_\varepsilon dx-\int_\Omega\big(F(\tilde{w}_{\varepsilon,P_\varepsilon}+\tau_\varepsilon\phi_\varepsilon)-F(\tilde{w}_{\varepsilon,P_\varepsilon})-\tau_\varepsilon f(\tilde{w}_{\varepsilon,P_\varepsilon})\phi_\varepsilon\big)dx\\&+\tau_\varepsilon^2\frac{\varepsilon^2}{2}\int_\Omega|\nabla\phi_\varepsilon|^2dx+\frac{\tau_\varepsilon^2}{2}\int_\Omega|\phi_\varepsilon|^2dx+O(\varepsilon^3\tau_\varepsilon^2).\end{aligned}$$

Observe that $|F(\tilde{w}_{\varepsilon,P_\varepsilon}+\tau_\varepsilon\phi_\varepsilon)-F(\tilde{w}_{\varepsilon,P_\varepsilon})-\tau_\varepsilon f(\tilde{w}_{\varepsilon,P_\varepsilon})\phi_\varepsilon|\leq C\tau_\varepsilon^2|\phi_\varepsilon|^2$, while, by Lemma 3.2, $\mathcal{S}_\varepsilon[\tilde{w}_{\varepsilon,P_\varepsilon}]=o(\tau_\varepsilon w_{\varepsilon,P_\varepsilon}^{1/4})$; (5.59) leads to $J_\varepsilon[v_\varepsilon]=J_\varepsilon[\tilde{w}_{\varepsilon,P_\varepsilon}]+o(\varepsilon^3\tau_\varepsilon^2)$. The choice of τ_ε allows us to conclude.

Step 2. $\frac{\hat{d}_\varepsilon}{\varepsilon|\log\varepsilon|}\to 1$ *as* $\varepsilon\to 0^+$ *and* $\frac{\hat{d}_\varepsilon}{\varepsilon}\geq\log\frac{1}{\varepsilon}+\frac{1}{3}\log\log\frac{1}{\varepsilon}$ *for* ε *sufficiently small.*

By inserting in (6.66) the estimates (4.23), (4.24) and (4.31) of Propositions 4.1-4.2 (taking into account that $\varepsilon^3 e^{-3\frac{\hat d_\varepsilon}{\varepsilon}}=o(\frac{\varepsilon^4}{\hat d_\varepsilon}e^{-2\frac{\hat d_\varepsilon}{\varepsilon}})$), we obtain

$$J^*_\varepsilon = I[w]\varepsilon^3 + I_2\varepsilon^5 - I_3\frac{\varepsilon^6}{\hat d_\varepsilon} + \varepsilon^4(1+o(1))B\frac{e^{-2\frac{\hat d_\varepsilon}{\varepsilon}}}{4\hat d_\varepsilon} + O\Big(\varepsilon^4\sqrt{\frac{\hat d_\varepsilon}{\varepsilon}}e^{-2\frac{\hat d_\varepsilon}{\varepsilon}}\Big) + o(\varepsilon^{11/2}+\varepsilon^5 e^{-\frac{\hat d_\varepsilon}{\varepsilon}}) \tag{6.67}$$

as $\varepsilon\to 0^+$. First we will prove that $\limsup_{\varepsilon\to0^+}\frac{\hat d_\varepsilon}{\varepsilon|\log\varepsilon|}\le 1$. Assume by contradiction the existence of a sequence $\varepsilon_n\to 0^+$ such that $\frac{\hat d_{\varepsilon_n}}{\varepsilon_n}>(1+\eta)\log\frac{1}{\varepsilon_n}$ for some $\eta>0$. In this case it is easy to show that the main part in the expansion on the right hand side of (6.67) is given by the first three terms, and the others are negligible, i.e.

$$J^*_{\varepsilon_n} \ge I[w]\varepsilon_n^3 + I_2\varepsilon_n^5 - I_3(1+o(1))\frac{\varepsilon_n^5}{(1+\eta)|\log\varepsilon_n|},$$

in contradiction with Corollary 5.1. Hence $\limsup_{\varepsilon\to0^+}\frac{\hat d_\varepsilon}{\varepsilon|\log\varepsilon|}\le 1$. Next we will prove that $\frac{\hat d_\varepsilon}{\varepsilon}\ge \log\frac1\varepsilon+\frac13\log\log\frac1\varepsilon$ for small ε. Otherwise let $\varepsilon_n\to0^+$ be a sequence such that $\frac{\hat d_{\varepsilon_n}}{\varepsilon_n}<\log\frac{1}{\varepsilon_n}+\frac13\log\log\frac{1}{\varepsilon_n}$. Hence we easily get $\varepsilon_n^4\sqrt{\frac{\hat d_{\varepsilon_n}}{\varepsilon_n}}e^{-2\frac{\hat d_{\varepsilon_n}}{\varepsilon_n}}, \varepsilon_n^5 e^{-\frac{\hat d_{\varepsilon_n}}{\varepsilon_n}} = o(\frac{\varepsilon_n^4}{\hat d_{\varepsilon_n}}e^{-2\frac{\hat d_{\varepsilon_n}}{\varepsilon_n}})$ and (6.67) becomes

$$J^*_{\varepsilon_n} \ge I[w]\varepsilon_n^3 + I_2\varepsilon_n^5 - I_3\frac{\varepsilon_n^6}{\hat d_{\varepsilon_n}} + \varepsilon_n^4 B\frac{e^{-2\frac{\hat d_{\varepsilon_n}}{\varepsilon_n}}}{8\hat d_{\varepsilon_n}} + O(\varepsilon_n^{11/2}) \text{ as } \varepsilon\to0^+. \tag{6.68}$$

Set

$$\rho_n(x) = -I_3\frac{\varepsilon_n^2}{x} + B\frac{e^{-2x}}{8x},\quad x\ge\frac{\hat d_{\varepsilon_n}}{\varepsilon_n}$$

and compute

$$\rho_n'(x) = I_3\frac{\varepsilon_n^2}{x^2} - B\frac{e^{-2x}}{8x^2} - B\frac{e^{-2x}}{4x}.$$

By taking the logarithm, $\rho_n'(x)\ge0$ in $(\frac{\hat d_{\varepsilon_n}}{\varepsilon_n},+\infty)$ implies

$$x\ge \log\frac{1}{\varepsilon_n}+\frac12\log x+\frac12\log\Big(\frac{1}{2x}+1\Big)+\frac12\log\frac{B}{4I_3}\ge\log\frac{1}{\varepsilon_n}+\frac12\log\log\frac{1}{\varepsilon_n}.$$

In particular, for n sufficiently large, ρ_n is decreasing in $(\frac{\hat d_{\varepsilon_n}}{\varepsilon_n},\log\frac{1}{\varepsilon_n}+\frac13\log\log\frac{1}{\varepsilon_n})$, by which, proceeding as in Corollary 5.1,

$$\rho_n\Big(\frac{\hat d_{\varepsilon_n}}{\varepsilon_n}\Big)\ge\rho_n\Big(\log\frac{1}{\varepsilon_n}+\frac13\log\log\frac{1}{\varepsilon_n}\Big) = -I_3\frac{\varepsilon_n^2}{|\log\varepsilon_n|}+B(1+o(1))\frac{\varepsilon_n^2}{8|\log\varepsilon_n|^{5/3}}.$$

Inserting this inequality in (6.68) we obtain

$$J^*_{\varepsilon_n}\ge I[w]\varepsilon_n^3+I_2\varepsilon_n^5-I_3\frac{\varepsilon_n^5}{|\log\varepsilon_n|}+B(1+o(1))\frac{\varepsilon_n^5}{8|\log\varepsilon_n|^{5/3}}$$

again in contradiction with Corollary 5.1.

Step 3. $\mathcal H(\Sigma_{P_\varepsilon})\to\mathcal H_0=\max_{P\in\partial\Omega}\mathcal H(P)$.

In view of Step 2 all the error terms in the expansion (4.23), (4.32) and in (5.49) (with $d_\varepsilon = \hat{d}_\varepsilon$) are $o(\varepsilon^6)$; for example $\varepsilon^4\sqrt{\frac{\hat{d}_\varepsilon}{\varepsilon}}e^{-2\frac{\hat{d}_\varepsilon}{\varepsilon}} = O(\frac{\varepsilon^6}{|\log|^{1/6}}) = o(\varepsilon^6)$. Hence we can write

$$I[w]\varepsilon^3 + \varepsilon^3\alpha\Big(\frac{\hat{d}_\varepsilon}{\varepsilon}\Big) + I_2\varepsilon^5 - I_3\frac{\varepsilon^6}{\hat{d}_\varepsilon} - I_4\mathcal{H}(\Sigma_{P_\varepsilon})\varepsilon^6 + o(\varepsilon^6) \leq J_\varepsilon[v_\varepsilon]$$
$$\leq I[w]\varepsilon^3 + \varepsilon^3\alpha\Big(\frac{\hat{d}_\varepsilon}{\varepsilon}\Big) + I_2\varepsilon^5 - I_3\frac{\varepsilon^6}{\hat{d}_\varepsilon} - I_4\mathcal{H}_0\varepsilon^6 + o(\varepsilon^6)$$

and Step 3 follows immediately.

Step 4. End of the proof.

By Corollary 3.1 and Proposition 5.2 we deduce Part (1) of Theorem 1.1. Parts (2) and (3) have been proved in the Steps 2-3.

References

[1] P. Bates, E. N. Dancer, J. Shi. *Multi-spike stationary solutions of the Cahn-Hilliard equation in higher-dimension and instability,* Adv. Differential Equations **4** (1999), 1–69.

[2] V. Benci, D. Fortunato. *An eigenvalue problem for the Schrödinger-Maxwell equations,* Topol. Methods Nonlinear Anal. **11** (1998), no. 2, 283–293.

[3] C. C. Chen, C. S. Lin. *Uniqueness of the ground state solution of $\Delta u + f(u) = 0$ in $\mathbb{R}^N, N \geq 3$,* Comm. Partial Differential Equations **16** (1991), 1549–1572.

[4] G. M. Coclite, V. Georgiev. *Solitary waves for Maxwell-Schrödinger equations,* Electronic J. Differential Equations **94** (2004), 1–31.

[5] E. N. Dancer. *Stable and finite Morse index solutions on $\mathbb{R}^N$ or on bounded domains with small diffusion. II,* Indiana Univ. Math. J. **53** (2004), no. 1, 97–108.

[6] E. N. Dancer, S. Yan. *Effect of the domain geometry on the existence of multipeak solutions for an elliptic problem,* Topol. Methods Nonlinear Anal. **14** (1999), no. 1, 1–38.

[7] E. N. Dancer, S. Yan. *Multipeak solutions for a singularly perturbed Neumann problem,* Pacific J. Math. **189** (1999), no. 2, 241–262.

[8] E. N. Dancer, S. Yan. *Interior and boundary peak solutions for a mixed boundary value problem,* Indiana Univ. Math. J. **48** (1999), no. 4, 1177–1212.

[9] E. N. Dancer, S. Yan. *Peak solutions for an elliptic system of Fitzhugh-Nagumo type*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **2** (2003), no. 4, 679–709.

[10] T. D'Aprile, D. Mugnai. *Existence of solitary waves for the nonlinear Klein-Gordon Maxwell and Schrödinger-Maxwell equations*, Proc. Roy. Soc. Edinburgh Sect. A **134** (2004), no. 5, 893–906.

[11] T. D'Aprile, J. Wei. *Boundary layers for a coupled elliptic system*, preprint 2005.

[12] T. D'Aprile, J. Wei. *Clustered solutions around harmonic centers to a coupled elliptic system*, preprint 2004.

[13] T. D'Aprile, J. Wei. *Layered solutions for a semilinear elliptic system in a ball*, preprint 2005.

[14] T. D'Aprile, J. Wei. *On Bound states concentrating on spheres for the Maxwell-Schrödinger equation*, SIAM J. Math. Anal., to appear.

[15] T. D'Aprile, J. Wei. *Standing waves in the Maxwell-Schrödinger equation and an optimal configuration problem*, Calc. Var. Partial Differential Equations, to appear.

[16] M. Del Pino, P. Felmer. *Spike-layered solutions of singularly perturbed elliptic problems in a degenerate setting,* Indiana Univ. Math. J. **48** (1999), no. 3, 883–898.

[17] M. Del Pino, P. Felmer, J. Wei. *On the role of mean curvature in some singularly perturbed Neumann problems,* SIAM J. Math. Anal. **31** (1999), no. 1, 63–79.

[18] M. J. Esteban, P. L. Lions. *Existence and non-existence results for semilinear elliptic problems in unbounded domains*, Proc. Roy. Soc. Edinburgh Sect. A **93** (1982), 1–14.

[19] H. Egnell. *Asymptotic results for finite energy solutions of semilinear elliptic equations*, J. Differential Equations, **98** (1992), no. 1, 34–56.

[20] L. C. Evans. *Partial Differential Equations*, American Mathematical Society, Providence, Rhode Island (1998).

[21] B. Gidas, W. M. Ni, L. Nirenberg. Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^N$, *Adv. Math. Suppl Stud.* **7A** (1981), 369–402.
[22] D. Gilbarg, N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order*, Springer Verlag Berlin Heidelberg 2001.
[23] M. Grossi, A. Pistoia. *On the effect of critical points of distance function in superlinear elliptic problems,* Adv. Differential Equations **5** (2000), no. 10-12, 1397–1420.
[24] M. Grossi, A. Pistoia, J. Wei. *Existence of multipeak solutions for a semilinear Neumann problem via nonsmooth critical point theory,* Calc. Var. Partial Differential Equations **11** (2000), no. 2, 143–175.
[25] C. Gui, J. Wei. *Multiple interior spike solutions for some singular perturbed Neumann problems*, J. Differential Equations **158** (1999), 1–27.
[26] C. Gui, J. Wei. *On multiple mixed interior and boundary peak solutions for some singularly perturbed Neumann problems*, Canad. J. Math. **52** (2000), 522–538.
[27] C. Gui, J. Wei, M. Winter. *Multiple boundary peak solutions for some singularly perturbed Neumann problems*, Ann. Inst. H. Poincaré Anal. Non Linéaire **17** (2000), 249–289.
[28] L. L. Helms. *Introduction to Potential Theory*, John Wiley & sons Inc., New York, 1969.
[29] M. K. Kwong. *Uniqueness of positive solutions of* $\Delta u - u + u^p = 0$ *in* $\mathbb{R}^N$, Arch. Rational Mech. Anal. **105** (1991), 243–266.
[30] Y. Y. Li. *On a singularly perturbed equation with Neumann boundary conditions,* Commun. Partial Differential Equations **23** (1998), no. 3-4, 487–545.
[31] Y.-Y. Li, L. Nirenberg. *The Dirichlet problem for singularly perturbed elliptic equations*, Comm. Pure Appl. Math. **51** (1998), 1445–1490.
[32] W. M. Ni, I. Takagi. *Locating the peaks of least-energy solutions to a semilinear Neumann problem,* Duke Math. J. **70** (1993), no. 2, 247–281.
[33] W. M. Ni, I. Takagi. *On the shape of least-energy solutions to a semi-linear Neumann problem,* Comm. Pure Appl. Math. **44** (1991), no. 7, 819–851.
[34] W. M. Ni, I. Takagi, J. Wei. *On the location and profile of spike-layer solutions to singularly perturbed semilinear Dirichlet problems: intermediate solutions*, Duke Math. J. **94** (1998), no. 3, 597–618.
[35] W. M. Ni, J. Wei. *On the location and profile of spike-layer solutions to singularly perturbed semilinear Dirichlet problems*, Comm. Pure Appl. Math. **48** (1995), 723–761.
[36] D. Ruiz. *Semiclassical states for coupled Schrödinger-Maxwell equations: concentration around a sphere,* Math. Models Methods Appl. Sci. **15** (2005), no. 1, 141–164.
[37] J. Wei. *On the boundary spike layer solutions of a singularly perturbed semilinear Neumann problem*, J. Differential Equations **134** (1997), no. 1, 104–133.
[38] J. Wei. *On the construction of single-peaked solutions to a singularly perturbed semilinear Dirichlet problem,* J. Differential Equations **129** (1996), no. 2, 315–333.
[39] J. Wei. *Conditions for two-peaked solutions of singularly perturbed elliptic equations,* Manuscripta Math. **96** (1998), no. 1, 113–131.
[40] X. P. Zhu. *Multiple entire solutions of a semilinear elliptic equation*, Nonlinear Analysis TMA **12** (1998), 1297–1316.

Dipartimento di Matematica, via E. Orabona 4, 70125 Bari, Italy
E-mail address: `daprile@dm.uniba.it`

Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong
E-mail address: `wei@math.cuhk.edu.hk`