The University of British Columbia
Midterm Examinations - November 2010

Mathematics 305

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Closed book examination. No notes, texts, or calculators allowed. Time: 50 minutes

Marks

[10] **1.** Define $f(z) = (z^2 + 1)^{1/3}$.

(i) Determine all of the branch points of $f(z)$ in the extended complex plane.

(ii) Construct a branch of $f(z)$ that is analytic in the unit disk and that satisfies $f(0) = 1$. For this branch, calculate $f(1+i)$.

[10] **2.** Let C be the curve $z = e^{i\theta}$ with $\pi/4 \leq \theta \leq 7\pi/4$ oriented counterclockwise. Calculate the following integrals:

$$\text{(i)} \quad \int_C \frac{1}{z}\, dz\,; \qquad\qquad \text{(ii)} \quad \int_C z\cos(z^2)\, dz$$

[20] **3.** Let C be the simple closed curve $|z| = 2$, oriented counterclockwise. Calculate the following integrals, providing justification for your results:

(i) $\int_C \frac{1}{(z-1)(z-3)}\, dz$

(ii) $\int_C z^{-2} e^{2z}\, dz$

(iii) $\int_C \frac{f(z)}{(z-1)(z-1-i)}\, dz$ where $f(z)$ is analytic in $|z| \geq 2$ and bounded by $|f(z)| < M$ in the region $|z| \geq 2$, where M is some positive constant.

(iv) $\int_{C_1} \frac{1}{4i+\sqrt{z}}\, dz$, where $\sqrt{z}$ denotes the branch of the square root with a branch cut on the positive real axis in the z-plane for which $\sqrt{-1} = i$. Here C_1 is the simple closed curve $|z+16| = 1$ oriented counterclockwise.

[10] **4.** Suppose that $f(z)$ is analytic inside and on a simple closed curve C. Assume also that $|f(z) - 1| < 1$ for z on C. Prove that there is no point z_0 with z_0 inside C for which $f(z_0) = 0$.

(i) Use this result to prove that there are no solutions to $z = -3e^{-z^2}$ in the disk $|z| \leq 1$. (Hint: Determine a convenient choice for $f(z)$ to use in the result above).

[50] Total Marks

The End

MATH 305 MIDTERM 2

PROBLEM 1 LET $f(z) = (z^2+1)^{1/3}$.

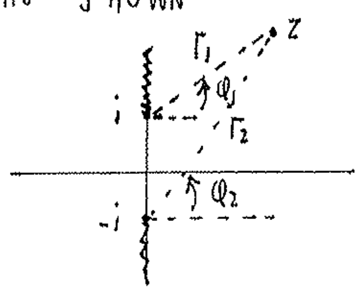
(i) WE FACTOR $f(z) = (z+i)^{1/3}(z-i)^{1/3} = (r_1 r_2)^{1/3} e^{i(\varphi_1+\varphi_2)/3}$.

CLEARLY $z = i,\ z = -i$ ARE BRANCH POINTS SINCE IF WE INCREASE φ_j BY 2π, i.e. $\varphi_j \to \varphi_j + 2\pi$, f DOES NOT RETURN TO THE SAME VALUE.

NOW CHECK $z = \infty$. LET $z = 1/\zeta$ SO $f(1/\zeta) = (1/\zeta + i)^{1/3}(1/\zeta - i)^{1/3}$

SO THAT $f(1/\zeta) = \zeta^{-2/3}(1+i\zeta)^{1/3}(1-i\zeta)^{1/3} \approx \zeta^{-2/3}$ FOR $|\zeta| \to 0$. THUS $\zeta = 0$ IS A BRANCH POINT OF $f(1/\zeta) \longrightarrow$ $z = \infty$ IS ALSO A BRANCH POINT.

(ii) WE WANT $f(z)$ ANALYTIC IN THE UNIT DISK. SO TRY BRANCH CUTS AS SHOWN



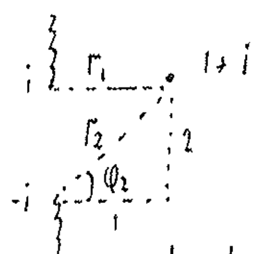
THUS

$$\left\{\begin{aligned} &f(z) = (r_1 r_2)^{1/3} e^{i(\varphi_1+\varphi_2)/3} \qquad (*) \\ &-3\pi/2 < \varphi_1 \le \pi/2 \\ &-\pi/2 \le \varphi_2 < 3\pi/2 . \end{aligned}\right.$$

NOW CALCULATE $f(0)$: FOR $z = 0$, $\varphi_1 = -\pi/2$, $\varphi_2 = \pi/2$, $r_1 = r_2 = 1$.

THUS $(*)$ GIVES $f(0) = 1$ AS DESIRED

NOW LET $z = 1+i$. THEN



$r_1 = 1 \quad \varphi_1 = 0$

$r_2 = \sqrt{5}$,

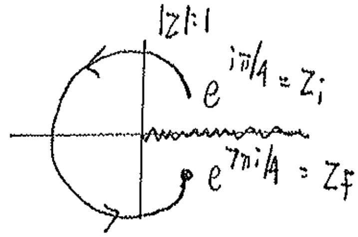
$\varphi_2 = \tan^{-1}(2)$.

THUS $f(1+i) = (\sqrt{5})^{1/3} e^{i\tan^{-1}(2)/3} = 5^{1/6} e^{i\tan^{-1}(2)/3}$.

REMARK ALSO COULD HAVE CHOSEN BRANCH BY WRITING

$f(z) = \exp\left(\frac{1}{3}\,\mathrm{LOG}(z^2+1)\right)$ LOG = PRINCIPAL BRANCH

PROBLEM 2 LET C BE AS SHOWN



(i) DEFINE $\log_1 z$ TO BE BRANCH OF $\log z$ WITH CUT AS SHOWN ON positive Real AXIS. Then SINCE $\log_1 z$ IS analytic IN Region containing C,

$$\int_C \frac{1}{z}\,dz = \log_1 z_f - \log_1 z_i = (\ln|z_f| + i7\pi/4) - (\ln|z_i| + i\pi/4)$$

$$\int_C \frac{1}{z}\,dz = 3\pi i/2 \quad \text{SINCE } |z_f| = |z_i| = 1.$$

THE OTHER METHOD IS DIRECT integration. Let $z = e^{it}$

SO $\dfrac{dz}{z} = \dfrac{ie^{it}}{e^{it}}\,dt = i\,dt. \longrightarrow \displaystyle\int_C \frac{dz}{z} = \int_{\pi/4}^{7\pi/4} i\,dt = 3\pi i/2.$

(ii) $\displaystyle\int_C z\cos(z^2)\,dz$. THE ANTI-derivative IS $\frac{1}{2}\sin(z^2)$.

$$\int_C z\cos(z^2)\,dz = \frac{1}{2}\sin(z_f^2) - \frac{1}{2}\sin(z_i^2) = \frac{1}{2}\sin\left(e^{7\pi i/2}\right) - \frac{1}{2}\sin\left(e^{\pi i/2}\right)$$

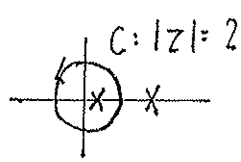
$$\int_C z\cos(z^2)\,dz = \frac{1}{2}\left[\sin(-i) - \sin(i)\right] = -\sin(i).$$

BUT $\sin(i) = +i\sinh(1)$ $\quad\sin(-i) = -i\sinh(1)$.

SO $\displaystyle\int_C z\cos(z^2)\,dz = -i\sinh(1).$

PROBLEM 3

(i) $\displaystyle\int_C \frac{dz}{(z-1)(z-3)}$



ONLY $z=1$ IS inside contour.

SO $\dfrac{1}{(z-1)(z-3)} = \dfrac{A}{z-1} + \dfrac{B}{z-3} \longrightarrow A(z-3) + B(z-1) = 1$

let $z=3 \rightarrow B = 1/2$

$z=1 \rightarrow A = -1/2$.

THUS $\displaystyle\int_C \frac{dz}{(z-1)(z-3)} = -\frac{1}{2}\int_C \frac{1}{z-1}\,dz + \frac{1}{2}\int_C \frac{1}{z-3}\,dz = -\frac{1}{2}(2\pi i) + 0$ (SINCE $z=3$ NOT INSIDE C).

THUS $\displaystyle\int_C \frac{dz}{(z-1)(z-3)} = -\pi i.$

(ii) $\displaystyle I = \int_C \frac{e^{2z}}{z^2}\,dz$: $C: |z|=2$ counter-clockwise.

RECALL $\displaystyle f'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\,dz$

WHEN z_0 inside C AND f analytic inside AND ON C.

THUS SET $z_0 = 0$ AND $f(z) = e^{2z}$.

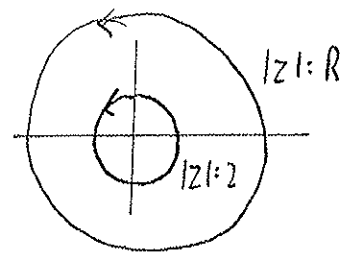
THEN $\displaystyle I = 2\pi i\, f'(0) = 2\pi i\,[2e^{z}]\big|_{z=0} = 4\pi i \longrightarrow \int_C \frac{e^{2z}}{z^2}\,dz = 4\pi i.$

(iii) DEFINE $\displaystyle g(z) = \frac{f(z)}{(z-1)(z-1-i)}$. NOTE THAT $g(z)$ IS ANALYTIC IN $|z| \geq 2$

SINCE f IS ANALYTIC IN $|z| \geq 2$ AND $z_1 = 1$, $z_2 = 1+i$ SATISFY $|z_1| = 1 < 2$, $|z_2| = \sqrt{2} < 2$.

THUS, DEFORM AS SHOWN

$\displaystyle I = \int_C g(z)\,dz = J(R) = \int_{C_R} g(z)\,dz$



$R > 2$.

BUT NOW LET $R \to \infty$ AND ESTIMATE

$$|J(R)| \leq \max_{C_R} \frac{|f(z)|}{|z-1|\,|z-(1+i)|}(2\pi R) \leq \frac{M(2\pi R)}{(R-1)(R-\sqrt{2})} \to 0 \text{ AS } R \to \infty.$$

THUS $\displaystyle J(R) = I = 0 \longrightarrow \int_C \frac{f(z)}{(z-1)(z-1-i)}\,dz = 0.$

(iv) $\displaystyle I = \int_{C_1} \frac{dz}{4i + \sqrt{z}}$. THE CONTOUR C_1 AND BRANCH CUT ARE AS SHOWN

THUS THE ONLY POSSIBLE SINGULARITY IS IF $4i + \sqrt{z} = 0$ FOR z inside C_1.

BUT THIS IS IMPOSSIBLE SINCE FOR BRANCH CUT WITH $0 \leq \varphi < 2\pi$

WE HAVE $\sqrt{z} = r^{1/2} e^{i\varphi/2} \longrightarrow \mathrm{IM}(\sqrt{z}) = r^{1/2}\sin(\varphi/2) \geq 0$ ON $0 \leq \varphi < 2\pi$.

NOTICE: $\sqrt{-1} = 1e^{i\pi/2} = i$ THUS $\sqrt{-16} = 4i$ NOT $\sqrt{-16} = -4i$. $\Longrightarrow I = 0$.

PROBLEM 4

LET $g(z) = f(z) - 1$. THEN BY ASSUMPTION

$|g(z)| < 1$ FOR z ON C

$g(z)$ ANALYTIC INSIDE AND ON C.

BY MAX-MODULUS PRINCIPLE, IT FOLLOWS THAT

(*) $|g(z)| < 1$ FOR z INSIDE AND ON C.

SUPPOSE THAT THERE WAS A z_0 FOR WHICH $f(z_0) = 0$ WITH z_0 INSIDE C. THEN $|g(z_0)| = 1$. THIS CONTRADICTS *).

(i) WRITE $z = -3e^{-z^2}$ AS $ze^{z^2} = -3 \rightarrow \frac{1}{3}ze^{z^2} + 1 = 0$.

SO DEFINE $f(z) = \frac{1}{3}ze^{z^2} + 1$.

THEN $f(z)$ IS analytic AND $f(z) - 1 = \frac{1}{3}ze^{z^2}$.

NOW ON $|z| = 1$, $|f(z) - 1| = \frac{1}{3}|z||e^{z^2}| \leq \frac{1}{3}e^{|z^2|} \leq \frac{e^1}{3} = \frac{2.71..}{3} < 1$.

(HERE WE USED $|e^w| \leq e^{|w|}$).

THUS $|f(z) - 1| < 1$ FOR $|z| = 1$

$\Longrightarrow$ $f(z)$ HAS NO ZEROES inside $|z| \leq 1$.

(i.e. $z \neq -3e^{-z^2}$ FOR ALL z IN $|z| \leq 1$)