MATH 516-101 Homework TWO
Due Date: October 13, 2015

1. This problems concerns the Green's representation formula in a ball.

(a) using the Green's function in a ball to prove

$$r^{n-2}\frac{r-|x|}{(r+|x|)^{n-1}}u(0) \leq u(x) \leq r^{n-2}\frac{r+|x|}{(r-|x|)^{n-1}}u(0)$$

whenever u is positive and harmonic in $B_r(0)$.

(b) use (a) to prove the following result: let u be a harmonic function in R^n. Suppose that $u \geq 0$. Then $u \equiv Constan$

2. This problem concerns the heat equation

$$u_t = \Delta u$$

Let

$$\Phi(x-y,t) = (4\pi t)^{-n/2} e^{-\frac{|x-y|^2}{4t}}$$

(a) Show that $\int_{R^n} \Phi(x-y,t)dy = 1$ for all $t > 0$

(b) Show that there exists a generic constant C_n such that

$$\Phi(x-y,t) \leq C_n |x-y|^{-n}$$

Hint: maximize the function in t.

(c) Let $f(x)$ be a function such that $f(x_0-)$ and $f(x_0+)$ exists. Show that

$$\lim_{t\to 0}\int_R \Phi(x-x_0,t)f(y)dy = \frac{1}{2}(f(x_0-)+f(x_0+))$$

3. This problem concerns the one-dimensional wave equation

$$u_{tt} = c^2 u_{xx}$$

$$u(x,0) = f(x), u_t(x,0) = g(x)$$

(a) Show that all solutions to the following equation

$$u_{XY} = \frac{\partial^2 u}{\partial X \partial Y} = 0$$

are given by a combination of two functions

$$u = F(X) + G(Y)$$

(b) Show that all solutions to

$$u_{tt} = c^2 u_{xx}$$

are given by

$$u = F(x-ct) + G(x-ct)$$

Hint: let

$$X = x - t, Y = x + t$$

and then use (a)

(c) Prove the d'Alembert's formula: all solutions to

$$u_{tt} = c^2 u_{xx}$$

$$u(x,0) = f(x), u_t(x,0) = g(x)$$

are given by

$$u(x,t) = \frac{1}{2}(f(x+ct) + f(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$$

(d) use (c) to show that Maximum Principle does not hold for wave equation, i.e.,

$$\max_{\bar{U_T}} u(x,t) > \max_{\partial' U_T} u(x,t)$$

Hint: Let $f = 0$ and $g = 1$, $U = (-1,1)$ and choose T large.

4. This problem concerns Sobolev space

(a) Let $U = (-1,1)$ and

$$u(x) = |x|$$

What is its weak derivative $u^{'}$? Prove it rigorously.

(b) Does the second order weak derivative $u^{''}$ exist?

(c) For which integer k and positive $p > 1$, does u belong to $W^{k,p}(U)$?