MATH 516-101 Homework One
Due Date: September 29th, 2015

1. This problem concerns the Newtonian potential

$$u(x)=\int_{R^3}\frac{1}{|x-y|}f(y)dy \tag{0.1}$$

For the following three parts, pick up only **one part** to finish

a) Show that if $|f(y)|\leq\frac{C}{|y|^\alpha}$ for $\alpha\in(2,3)$. Then $|u(x)|\leq\frac{C}{|x|^{\alpha-2}}$ for $|x|>1$

Proof. For $|x|=R>>1$ we divide the integral into three parts:

$$u(x)=\int_{|x-y|\leq\frac{|x|}{2}}\frac{f(y)}{|x-y|}dy+\int_{\frac{|x|}{2}\leq|x-y|\leq 2|x|}\frac{f(y)}{|x-y|}dy+\int_{|x-y|\geq 2|x|}\frac{f(y)}{|x-y|}dy$$
$$=I_1+I_2+I_3$$

For I_1, there holds $|y|\geq|x|-\underset{\geq}{|x-y|}\frac{|x|}{2}$ and hence

$$I_1\leq\int_{|x-y|\leq\frac{|x|}{2},|y|\geq\frac{|x|}{2}}\frac{C}{|x-y|}\frac{C}{|y|^\alpha}dy\leq\frac{C}{|x|^\alpha}\int_{|x-y|\leq\frac{|x|}{2}}\frac{1}{|x-y|}dy$$
$$\leq\frac{C}{|x|^\alpha}\int_0^{\frac{|x|}{2}}\frac{r^2dr}{r}\leq\frac{C}{|x|^{\alpha-2}}$$

For I_3, we can perform similar analysis: $|y-x|\leq|y|+|x|\leq|y|+\frac{|y-x|}{2}$. Thus $|y|\geq\frac{|x-y|}{2}$

$$I_3\leq\int_{|x-y|\geq 2|x|}\frac{C}{|x-y|}\frac{C}{|y|^\alpha}dy\leq\int_{|x-y|\geq 2|x|}\frac{1}{|x-y|^{1+\alpha}}dy$$
$$\leq C\int_{2|x|}^\infty\frac{r^2dr}{r^{1+\alpha}}dr\leq\frac{C}{|x|^{\alpha-2}}$$

since $\alpha>2$.

It remains to estimate I_2:

$$I_2\leq\int_{\frac{|x|}{2}\leq|x-y|\leq 2|x|}\frac{C}{|x|}\frac{C}{1+|y|^\alpha}dy\leq\frac{C}{|x|}(\int_{|y|\leq 1}\frac{1}{1+|y|^\alpha}+\int_{1\leq|y|\leq 3|x|}\frac{C}{|y|^\alpha}dy)$$
$$\leq\frac{C}{|x|}(C+\int_1^{3|x|}\frac{r^2dr}{r^\alpha})$$
$$\leq\frac{C}{|x|}(C+|x|^{3-alpha})\leq\frac{C}{|x|^{\alpha-2}}$$

since $\alpha<3$

□

b) Show that if $|f(y)|\leq\frac{C}{|y|^3}$, then $|u(x)|\leq\frac{C}{|x|}\log|x|$ for $|x|>1$

Proof. The proof is similar to (a) except in the last part, $\int_1^{3|x|}\frac{r^2dr}{r^3}\sim C\log|x|$

□

c) Show that if $|f(y)|\leq\frac{C}{|y|^\alpha}$ for $\alpha>3$, then $|u(x)|\leq\frac{C}{|x|}$ for $|x|>1$

Proof. The proof is similar to (a) except in the last part, $\int_1^{3|x|} \frac{r^2 dr}{r^3} \leq C$

□

2. This problem concerns the Mean-Value-Property (MVP). We say $v \in C^2(\bar{U})$ is *subharmonic*, if

$$-\Delta v \leq 0 \quad \text{in } U$$

a) Prove that for subharmonic functions

$$v(x) \leq \frac{1}{|B(x,r)|} \int_{B(x,r)} v dy, \quad \forall B(x,r) \subset U$$

Hint: use the formula for $\psi^{'}(r)$.

Proof. Let $\psi(r) = \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} v dy$. By the computation done at class,

$$\psi^{'}(r) = r \int_{B_r(x)} \Delta v \geq 0$$

and hence

$$\psi(0) \leq \psi(r)$$

hence

$$v(x) \leq \leq \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} v dy$$

Integrating from 0 to r we obtain

$$v(x) \leq \frac{1}{|B(x,r)|} \int_{B(x,r)} v dy$$

□

b) Prove that the Maximum Principle holds for subharmonic functions on bounded domains

$$\max_{\bar{U}} v = \max_{\partial U} v$$

Proof. Repeat the proof done in the class for the harmonic case.

□

c) Let u be harmonic functions in U. Show that u^2 and $|\nabla u|^2$ are subharmonic functions.

Proof.

$$\Delta u^2 = 2u\Delta u + 2|\nabla u|^2 = 2|\nabla u|^2$$
$$\Delta |\nabla u|^2 = \sum_{i,j} u_{ij}^2 + \nabla u \cdot \nabla \Delta u = \sum_{i,j} u_{ij}^2$$

□

d) Let u satisfy

$$-\Delta u = f \text{ in } U, \quad u = g \quad \text{on } \partial U$$

Show that there exists a generic constant $C = C(n, U)$ such that

$$\max_U u \leq C(\max_U |f| + \max_{\partial U} |g|)$$

Proof. Consider $v(x) = u(x) + \frac{|x|^2}{2n}\max_U |f| - \max_{\partial U}|g| - \frac{\max_U |x|^2}{2n}\max_U |f|$ and show that v is subharmonic and then apply b).

Then $-\Delta v = -\Delta u + \max_U |f| = f + \max_U |f| \geq 0$ so v is subharmonic. By (b),

$$\max_{\bar{U}} v = \max_{\partial U} v \leq 0$$

and hence

$$\max_{\bar{U}} u \leq C(\max_U |f| + \max_{\partial U}|g|)$$

□

3. This problem concerns Green's function and Green's representation formula.

a) Write the Green's function for the unit ball $B(0,1)$.

Proof. For $n \geq 3$,

$$G_{B_1(0)}(x,y) = \frac{1}{(n-2)|\partial B_1|}\left(\frac{1}{|x-y|^{n-2}} - \frac{|x|^{n-2}}{|x-|x|^2 y|^{n-2}}\right)$$

For $n = 2$,

$$G_{B_1(0)}(x,y) = \frac{1}{2\pi}\left(\log\frac{1}{|x-y|} - \log\frac{|x|}{|x-|x|^2 y|}\right)$$

□

b) Use a) and reflection to find the Green's function in half ball $B^+(0,1) = B(0,1) \cap \{x_n > 0\}$.

Proof. Let $G_{B_1(0)}(x,y)$ be defined at (a). Fix $x \in B^+(0,1)$ we define $x^* = (x', -x_n)$. Then

$$G(x,y) = G_{B_1}(x,y) - G_{B_1}(x^*,y)$$

□