Homework Assignment 7 (Due Date: April 17, 2014)

Please hand in either to my office or to my mailbox at the Department of Mathematics by noon of April 17, 2014. The solutions will be put on my website after noon of April 17.

1. (20pts) Solve the following exterior domain problem
(a) $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, r > 1, 0 \leq \theta < 2\pi,\ \ u(1,\theta) = \cos^2\theta$ and u is bounded.
(b) $u_{rr} + \frac{2}{r}u_r = 0, r > 1, u(1) = 1, \lim_{r\to+\infty} u(r) = -2$

2. (20pts) Use the energy method to show that the solution, if exists, to the following diffusion equation is unique

$$\begin{cases} u_t = k\Delta u,\ \ x \in D, t > 0 \\ u(x,0) = \phi(x), x \in D \\ \frac{\partial u}{\partial n} + a(x,t)u = g(x,t), x \in \partial D \end{cases}$$

where D is a bounded domain in R^2 and $a(x,t) > 0$.
Hint: consider $E(t) = \frac{1}{2}\int_D v^2(x,t)dx$. Use the divergence theorem to compute $\frac{dE}{dt}$.

3. (20pts) Use the method of separation of variables to solve the following PDE:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0\ \text{ in } D = \{(r,\theta) \mid 1 < r < 2, 0 < \theta, \frac{\pi}{4}\}$$

$$u(1,\theta) = -\cos 4\theta,\ \ u(2,\theta) = 1$$

$$u_\theta(r,0) = 0, u_\theta(r,\frac{\pi}{4}) = 0$$

4. (20pts) Find an infinite series representation (in terms of the Bessel function) for the wave equation problem

$$\begin{cases} u_{tt} = c^2(u_{rr} + \frac{1}{r}u_r),\ \ 0 \leq r < a, t > 0 \\ u(a,t) = 0, t \geq 0 \\ u(r,0) = \phi(r), u_t(r,0) = \psi(r) \end{cases}$$

5. (20pts) Find an infinite series representation (in terms of the Bessel function) for the diffusion equation problem

$$\begin{cases} u_t = k(u_{rr} + \frac{1}{r}u_r + u_{zz}),\ \ 0 \leq r < a, 0 < z < b,\ \ t > 0 \\ u(a,z,t) = 0, u_z(r,0,t) = 0, u_z(r,b,t) = 0 \\ u(r,z,0) = \phi(r,z) \end{cases}$$