Proof of Uniqueness for Bounded Data

Lecture 8.5

Theorem: The bounded solution to the diffusion equation

$$\begin{cases} u_t = k u_{xx} + f(x,t), & -\infty < x < +\infty,\ t>0 \\ u(x,0) = \phi(x) \end{cases}$$

is unique.

Proof: For uniqueness, we may assume that $f \equiv 0$, $\phi \equiv 0$.

We want to prove that $u \equiv 0$.

Let $T>0$. Consider the domain $\Omega = (x,t) \in (-\infty, +\infty) \times (0,T)$.

Let $\varepsilon > 0$ be small. Consider

$$u_\varepsilon(x) = u(x) - \varepsilon \frac{1}{k}\sqrt{1+x^2}\ \varepsilon t$$

Then u_ε satisfies

$$u_{\varepsilon,t} - k\, u_{\varepsilon,xx} = -\varepsilon\ \varepsilon \frac{1}{(1+x^2)^{\frac{3}{2}}}\ \leq 0$$

Since $u(x)$ is bounded, $t<T$, $u_\varepsilon(x)$ can not attain its maximum at $+\infty$. $u_\varepsilon(x)$ must attain its maximum in (x_0, t_0).

Case 1. $x_0 < +\infty$, $0 < t_0 < T$.

In this case $u_t = 0$, $u_{xx} \leq 0$

So $u_t - k u_{xx} \geq 0$. Contradiction

Case 2. $t_0 = T$. In this case, $u_{xx} \leq 0$, $u_t = \lim\limits_{\substack{t \to T \\ t<T}} \frac{u(T) - u(t)}{T-t} \geq 0$.

So $u_t - k u_{xx} \geq 0$. Contradiction again

Therefore $\quad t_0 = 0$

So $\quad \max u_\varepsilon(x) \le \max_{t_0=0} u_\varepsilon(x) = \max\left(u_\varepsilon(x,0) - \frac{\varepsilon}{k}\sqrt{1+x^2}\right)$

≤ 0

$u_\varepsilon(x) \le 0$

$\Rightarrow \quad u(x) \le \frac{\varepsilon}{k}\sqrt{1+x^2} - \varepsilon t$

Letting $\varepsilon \to 0+ \quad \Rightarrow \quad u(x) \le 0.$

Similarly we can show $\quad -u(x) \ge 0.$

Hence $\quad u(x) \equiv 0. \qquad \#$

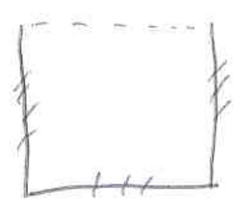
Maximum Principle for Heat equation

Theorem: Let u satisfy

$$u_t = k\,u_{xx}, \qquad 0 < x < L, \quad 0 < t < T$$

Then

$$\max_{\substack{0 \le x \le L \\ 0 \le t \le T}} u = \max_{\substack{0 \le x \le L,\ t=0 \\ x=0 \text{ or } x=L,\ 0 < t < T}} u$$



$U = (0,L) \times (0,T)$

$$\partial' U = \left\{ \begin{array}{l} 0 \le x \le L,\ t=0 \\ x=0,\ 0 \le t < T \\ x=L,\ 0 \le t < T \end{array} \right\}$$

i.e. $\quad \max_{\overline{U}} u = \max_{\partial' U} u$