A Complete List of Formulas (and Theorems) in MATH400-101

Part I: First Order Equations

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)$$

1. Method of Characteristics:

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}$$

Suppose that the initial conditions are given by $(x_0(\xi), y_0(\xi), u_0(\xi))$. Then we need to solve

$$\begin{cases} \frac{dx}{ds} = a(x,y,u), x(0) = x_0(\xi) \\ \frac{dy}{ds} = b(x,y,u), y(0) = y_0(\xi) \\ \frac{du}{ds} = c(x,y,u), u(0) = u_0(\xi) \end{cases}$$

2. General Solutions for

$$a(x,y)u_x + b(x,y)u_y = c(x,y)$$

Method: 1) Solve the characteristics:

$$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}$$

to get $F(x,y;\xi) = 0$ and then solve $\xi = f(x,y)$. 2) Change variable

$$x' = x, y' = f(x,y), u(x,y) = U(x', y')$$

New equation for U:

$$aU_{x'} = c$$

and integrate.

3. Traffic Flow Problem:

$$\rho_t + c(\rho)\rho_x = 0, \rho(x,0) = \rho_0(x)$$

where $Q'(\rho) = c(\rho)$.

General solution:

$$x = c(\phi(\xi))t + \xi$$

Shock:

$$\frac{ds}{dt} = \frac{Q(\rho_+) - Q(\rho_-)}{\rho_+ - \rho_-}, s(t_0) = x_0$$

Expanding fan:

$$u = H(\Lambda) = H(\frac{x}{t}), \text{ where } c(H) = \Lambda$$

4. Fully nonlinear problem:

$$F(x,y,u,u_x,u_y) = 0$$

Charpit's equation:

$$\begin{cases} \frac{dx}{ds} = F_p, x(0) = x_0(\xi) \\ \frac{dy}{ds} = F_q, \ y(0) = y_0(\xi) \\ \frac{dp}{ds} = -F_x - pF_u, p(0) = p_0(\xi) \\ \frac{dq}{ds} = -F_y - qF_u, \ q(0) = q_0(\xi) \\ \frac{du}{ds} = pF_p + qF_q, \ u(0) = u_0(\xi) \end{cases}$$

where (p_0, q_0) is computed via

$$F(x_0, y_0, p_0, q_0) = 0, u_0' = p_0 x_0' + q_0 y_0'$$

Part I: Second order PDEs: general Formula

1. Wave Equation on the whole line:

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), -\infty < x < +\infty, \\ u(x,0) = \phi(x),\ u_t(x,0) = \psi(x), -\infty < x < +\infty \end{cases} \tag{1}$$

D'Alembert's formula

$$u(x,t) = \frac{1}{2}(\phi(x-ct) + \phi(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds + \frac{1}{2c}\int_0^t (\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dy)ds$$

2. Diffusion Equation on the whole line:

$$\begin{cases} u_t - ku_{xx} = f(x,t), -\infty < x < +\infty, t > 0 \\ u(x,0) = \phi(x), -\infty < x < +\infty \end{cases} \tag{2}$$

Solution formula

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy + \int_0^t \int_{-\infty}^{+\infty} S(x-y,t-s)f(y,s)dyds$$

where

$$S(x,t) = \frac{1}{\sqrt{4k\pi t}} e^{-\frac{x^2}{4kt}}$$

3. Laplace equation on a disk

$$\begin{cases} u_{xx} + u_{yy} = 0 \text{ in } \{x^2 + y^2 < a^2\} \\ u(a,\theta) = g(\theta) \end{cases} \tag{3}$$

Poisson's formula:

$$u(r,\theta) = \frac{a^2 - r^2}{2\pi 2} \int_0^{2\pi} \frac{g(\phi)}{a^2 + r^2 - 2ar\cos(\theta - \phi)} d\phi$$

4. Wave Equation on the half line:

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), 0 < x < +\infty, t > 0 \\ u(x,0) = \phi,\ u_t(x,0) = \psi(x), 0 < x < +\infty \\ u(0,t) = 0 \end{cases} \tag{4}$$

Method of Reflection: extend f, ϕ, ψ oddly to $(-\infty, +\infty)$.
There is a similar formula for Neumann boundary condition.
Inhomogeneous BC: $u(0,t) = h(t)$. Use $V(x,t) = u(x,t) - xh(t)$.

5. Diffusion Equation on the half line:

$$\begin{cases} u_t - ku_{xx} = f(x,t), 0 < x < +\infty, t > 0 \\ u(x,0) = \phi, 0 < x < +\infty \\ u(0,t) = 0 \end{cases} \tag{5}$$

Method of Extension: extend f and ϕ oddly. Solution formula:
$u(x,t) = \int_0^{\infty} (S(x-y,t) - s(x+y,t)\phi(y))dy + \int_0^t \int_0^{\infty} (S(x-y,t-s) - S(x+y,t-s))f(y,s)dyds$

Part III: Boundary Value Problems and Method of Separation of Variables

1. Method of Separation of Variations

Step 1: Find the right separated functions. Plug into PDE and obtain two ODEs.
Step 2: Plug BC (homogeneous or natural BC). Distinguish EVP and ODE.
Step 3: Solve (EVP) and (ODE)

Step 4: Sum-up.Plug in the inhomogeneous BC.

2. Standard Eigenvalue Problems

$$X'' + \lambda X = 0, 0 < x < l$$

2.1) Dirichlet BC: $X(0) = X(l) = 0$

$$\lambda_n = \frac{(n\pi)^2}{l^2}, X_n = \sin(\frac{n\pi}{l}x), n = 1, 2, ...$$

2.2) Neumann BC: $X'(0) = X'(l) = 0$

$$\lambda_n = \frac{(n\pi)^2}{l^2}, X_n = \cos(\frac{n\pi}{l}x), n = 0, 1, 2, ...$$

2.3) Periodic BC: $X(0) = X(l), X'(0) = X'(l)$

$$\lambda_n = \frac{(2n\pi)^2}{l^2}, X_n = a\cos(\frac{2n\pi}{l}x) + b\sin(\frac{2n\pi}{l}x), n = 0, 1, 2, ...$$

2.4) Summary of Robin boundary condition eigenvalue problems

$$\begin{cases} X'' + \lambda X = 0, 0 < x < l, \\ X'(0) - a_0 X(0), X'(l) + a_l X(l) = 0 \end{cases}$$

Hyperbola:

$$a_0 + a_l + a_0 a_l l = (a_0 + \frac{1}{l})(a_l + \frac{1}{l}) - \frac{1}{l^2} = 0$$

divides the parameter space (a_0, a_l) into Five Regions. Depending on the regions, the number of negative or zero eigenvalues can be determined.

Equation for negative eigenvalues:

$$\lambda = -\gamma^2,\ \tanh(\gamma l) = -\frac{(a_0 + a_l)\gamma}{\gamma^2 + a_0 a_l},\ X = \cosh(\gamma x) + \frac{a_0}{\gamma}\sinh(\gamma x)$$

Equation for zero eigenvalue

$$a_0 + a_l + a_0 a_l l = 0, \lambda = 0, X = 1 - a_0 x$$

Equation for positive eigenvalue

$$\lambda = \beta^2,\ \tan(\beta l) = \frac{(a_0 + a_l)\beta}{\beta^2 - a_0 a_l},\ \ X = \cosh(\beta x) + \frac{a_0}{\beta}\sinh(\beta x)$$

3. Sturm-Liouville Eigenvalue Problem

$$(p(x)X)' + \lambda w(x) X = 0, 0 < x < l,$$

$$X'(0) - h_0 X(0) = 0, X'(l) + h_1 X(l) = 0$$

Lagrange's identity:

$$\int_0^l [f(pg')' - g(pf')'] = (pfg' - pf'g)|_0^l$$

1) all eigenvalues are real

2) $\lambda_1 > 0$ if $h_0 > 0, h_1 > 0$

3) Different eigenfunctions are orthogonal with respect to the weight function w:

$$\int_0^l w(x) X_n X_m dx = 0$$

4) eigenvalues are discrete and approach to infinity

$$\lambda_1 < \lambda_2 < ... < \lambda_n < ..., \lambda_n \to +\infty$$

5) Expansion with respect to the eigenfunctions:

$$f(x) = \sum_n A_n X_n(x)$$

$$A_n = \frac{\int_0^l X_n f w(x) dx}{\int_0^l X_n^2 w(x) dx}$$

4. Bessel functions:

$$R'' + \frac{1}{r}R' - \frac{n^2}{r^2}R + \lambda R = 0, 0 \le r < a, \ R(a) = 0$$

Then

$$\lambda = \frac{z_{m,n}^2}{a^2}, R(r) = J_n(\frac{z_{m,n}}{a}r), m = 1, 2, ...$$

where Bessel function of order zero of first kind:

$$J_0'' + \frac{1}{z}J_0 + J_0 = 0, J_0(0) = 1$$

Bessel function of order n:

$$J_n'' + \frac{1}{z}J_n + J_n - \frac{n^2}{z^2}J_n = 0, J_n(z) \sim z^n$$

The zeroes of J_n is denoted as

$$0 < z_{1,n} < z_{2,n} < ... < z_{m,n} < ...$$

5. Method of Separation of Variables for heat equation/wave equation

1) Diffusion equation without source:

$$u_t = k u_{xx}, 0 < x < l$$

$$u(x, 0) = \phi$$

$$u(0, t) = 0, u(l, t) = 0$$

$$u(x, t) = \sum_{n=1}^{\infty} a_n e^{-k\lambda_n t} \sin(\frac{n\pi}{l}x)$$

where

$$a_n = \frac{2}{l}\int_0^l \phi(x) \sin(\frac{n\pi}{l}x) dx$$

Similar formula for wave equation.

2) Diffusion equation with source:

$$u_t = k u_{xx} + f(x, t),$$

$$u(x, 0) = \phi$$

$$u(0, t) = h(t), u(l, t) = k(t)$$

Expansion:

$$u = \sum_{n=1}^{\infty} u_n(t) \sin(\frac{n\pi}{l}x)$$

$$f(x, t) = \sum_{n=1}^{\infty} f_n(t) \sin(\frac{n\pi}{l}x)$$

$$\phi(x) = \sum_{n=1}^{\infty} \phi_n \sin(\frac{n\pi}{l}x)$$

Then we need to solve

$$u_n^{'} + k\lambda_n u_n = \frac{2n\pi}{l^2}(h(t) - (-1)^n k(t)) + f_n(t)$$

$$u_n(0) = \phi_n$$

3) Wave equation with source:

$$u_{tt} = c^2 u_{xx} + f(x,t),$$

$$u(x,0) = \phi, u_t(x,0) = \psi$$

$$u(0,t) = h(t), u(l,t) = k(t)$$

Expansion:

$$u = \sum_{n=1}^{\infty} u_n(t) \sin(\frac{n\pi}{l}x)$$

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin(\frac{n\pi}{l}x)$$

$$\phi(x) = \sum_{n=1}^{\infty} \phi_n \sin(\frac{n\pi}{l}x)$$

$$\psi(x) = \sum_{n=1}^{\infty} \psi_n \sin(\frac{n\pi}{l}x)$$

Then we need to solve

$$u_n^{''} + c^2\lambda_n u_n = \frac{2n\pi}{l^2}(h(t) - (-1)^n k(t)) + f_n(t)$$

$$u_n(0) = \phi_n, u_n^{'}(0) = \psi_n$$

4) Higher-dimensional Diffusion equation without source:

$$u_t = k(u_{rr} + \frac{1}{r}u_r + \frac{u_{\theta\theta}}{r^2}), 0 \le r < a, 0 \le \theta < 2\pi$$

$$u(r,\theta,0) = \phi(r,\theta)$$

$$u(a,\theta,z) = 0$$

$$u = \sum_{m,n} e^{-k(\frac{z_{m,n}^2}{a^2})t} J_n(\frac{z_{m,n}}{a}r)(a_{m,n}\cos m\theta + b_{m,n}\sin\theta)$$

$$a_{m,n} = \frac{\int_0^a \int_0^{2\pi} r\phi(r,z) J_n(\frac{z_{m,n}}{a}r)\cos m\theta d\theta dr}{\int_0^a \int_0^{2\pi} r\phi(r,z) J_m^2(\frac{z_{m,n}}{a}r)\cos^2(m\theta) d\theta dr}$$

6 Method of separation of variables applied to Laplace equation

6.1) Laplace equation in rectangle and cubes

$$u(x,y) = X(x)Y(y)$$

6.2) Radial Domains

$$u(r,\theta) = R(r)\Theta(\theta)$$

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{u_{\theta\theta}}{r^2}$$

1) solution to

$$\Delta u = 0, 0 \le r < a, 0 \le \theta < 2\pi,\ u(a, \phi) = h(\phi)$$

is given by

$$u(r, \theta) = a_0 + \sum_{n=1}^{n} r^n (a_n \cos(n\theta) + b_n \sin(n\theta))$$

where

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} h(\phi) d\phi, a_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(\phi \cos(n\phi) d\phi, b_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(\phi \sin(n\phi) d\phi,$$

6.4) Laplace equation on wedges, annulus, exterior of disk

7. Method of separation of variables for Diffusion equation in a disk: polar coordinate

$$u_t = k(u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta})$$

$$u(r, \theta, 0) = \phi(r, \theta)$$

$$u(a, \theta, t) = 0$$

The solution is given by

$$u(r, \theta, t) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} J_n(\frac{z_{m,n}}{a} r)(a_{m,n} \cos(n\theta) + b_{m,n} \sin(n\theta)) e^{-k \frac{z_{m,n}^2}{a^2} t}$$

where

$$a_{m,n} = \frac{\int_0^a \int_0^{2\pi} \phi(r, \theta) \cos(n\theta) d\theta J_n(\frac{z_{m,n}}{a} r) r dr}{\pi \int_0^a J_n^2(\frac{z_{m,n}}{a} r) r dr}$$

$$b_{m,n} = \frac{\int_0^a \int_0^{2\pi} \phi(r, \theta) \sin(n\theta) d\theta J_n(\frac{z_{m,n}}{a} r) r dr}{\pi \int_0^a J_n^2(\frac{z_{m,n}}{a} r) r dr}$$

Part IV: Properties of Second Order PDEs

1. Classification of second order equations

$$a_{11} u_{xx} + 2a_{12} u_{xy} + a_{22} u_{yy} + b_1 u_x + b_2 u_y = 0$$

Type of PDEs: Elliptic, Parabolic, Hyperbolic

Change of variables to standard form

$$\partial_x = a_{11} \partial_\xi + a_{12} \partial_\eta$$

$$\partial_y = a_{21} \partial_\xi + a_{22} \partial_\eta$$

Then

$$\xi = a_{11} x + a_{21} y$$

$$\eta = a_{12} x + a_{22} y$$

2. Well-posedness of PDE problems: (a) existence (b) uniqueness (c) stability

3. For wave equation

$$u_{tt} = c^2 u_{xx}$$

Domain of dependence, domain of influence

4. The energy of wave equation

$$E(t) = \frac{1}{2} \int_{-\infty}^{\infty} u_t^2(x, t) dt + \frac{c^2}{2} \int_{-\infty}^{+\infty} u_x^2(x, t) dx$$

$$\frac{dE}{dt} = 0$$

The energy of diffusion equation

$$E(t) = \frac{1}{2}\int_{-\infty}^{+\infty} u(x,t)dx$$

$$\frac{dE}{dt} \leq 0$$

Uniqueness of wave and diffusion equations.

5. For Laplace equation

$$\Delta u = f \text{ in } D$$

5.1) Uniqueness: The solution to Dirichlet BC is unique; the solution to Neumann BC is unique, up to a constant; the solution to Robin BC is unique provided $a \geq 0, a \not\equiv 0$.

Uniqueness proved by energy method (and divergence theorem):

$$\int_D u\Delta u = \int_{\partial D} u\frac{\partial u}{\partial \nu} - \int_D |\nabla u|^2$$

5.2) Laplace equation on a disk: Poisson formula

$$\Delta u = 0 \text{ in } 0 \leq r < a, 0 \leq \theta < 2\pi,\ u(a,\theta) = h(\theta)$$

$$u(r,\theta) = \frac{a^2 - r^2}{2\pi}\int_0^{2\pi} \frac{h(\phi)}{a^2 + r^2 - 2ar\cos(\theta - \phi)}d\phi$$

5.3) Mean Value Theorem: If $\Delta u = 0$ then

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(a,\phi)d\phi$$

5.4) Maximum Principle: If $\Delta u = 0$ in D then $\max_{\bar{D}} u = \max_{\partial D} u$ and equality holds if and only if $u \equiv Constant$