Solutions to Assignment 3

1. (a). Well-posedness means

(1) Existence: Given $\phi(x)$, $\exists$ a solution $u_{\phi(x)}$

(2) Uniqueness: Given $\phi(x)$, $\exists$ a uniqueness u_ϕ

(3) stability: Let $|\tilde{\phi}(x) - \phi(x)| < \varepsilon$. Then

$$\|u_{\tilde{\phi}} - u_\phi\| < \varepsilon \qquad (\text{in some norm})$$

(b). First we check that $u(x,t) = \frac{1}{n} e^{n^2 t} \sin(nx)$ satisfies

$$u_t = \frac{1}{n} n^2 e^{n^2 t} \sin nx$$

$$u_{xx} = \frac{1}{n} e^{n^2 t} (-n^2) \sin nx$$

$$u_t + u_{xx} = 0$$

$$u(x,0) = \frac{1}{n} \sin nx$$

Let $\phi(x) = 0$. Then $u_\phi = 0$.

$$\tilde{\phi}(x) = \frac{1}{n} \sin nx, \quad u_{\tilde{\phi}} = \frac{1}{n} e^{n^2 t} \sin nx$$

Now $|\tilde{\phi} - \phi| \le \frac{1}{n} \to 0$ as $n \to +\infty$, while

$$\max_{-\infty < x < +\infty} |u_{\tilde{\phi}} - u_\phi| = \frac{1}{n} e^{n^2 t} \to +\infty \text{ as } n \to +\infty$$

Hence stability is violated.

2. (a) $\partial_x^2 + 2\partial_x \partial_y - \partial_y^2$

$$= (\partial_x + \partial_y)^2 - 3\partial_y^2$$

$$= (\partial_x + \partial_y)^2 - (\sqrt{3}\,\partial_y)$$

Let $\begin{cases} \partial_\xi = \partial_x + \partial_y \\ \partial_\eta = \sqrt{3}\,\partial_y \end{cases} \Rightarrow \begin{cases} \partial_x = \partial_\xi - \frac{1}{\sqrt{3}}\partial_\eta \\ \partial_y = \frac{1}{\sqrt{3}}\partial_\eta \end{cases}$

$$\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \partial_\xi \\ \partial_\eta \end{pmatrix} \Rightarrow \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Under the following change of variable

$$\xi = x, \quad \eta = -\frac{x}{\sqrt{3}} + \frac{y}{\sqrt{3}}$$

the equation becomes

$$\partial_\xi^2 - \partial_\eta^2$$

(b) $\partial_x^2 - 6\partial_x \partial_y + 9\partial_y^2 - 2\partial_x$

$$= (\partial_x - 3\partial_y)^2 - 2\partial_x$$

Let $\partial_\xi = \partial_x - 3\partial_y$, $\partial_\eta = 2\partial_x$ $\Rightarrow$ $\partial_x = \frac{1}{2}\partial_\eta$, $\partial_y = \frac{1}{6}\partial_\eta - \frac{1}{3}\partial_\xi$

$$\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{3} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} \partial_\xi \\ \partial_\eta \end{pmatrix} \Rightarrow \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{3} \\ \frac{1}{2} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Under the following change of variable

$$\xi = -\frac{x}{3}, \qquad \eta = \frac{1}{2}x + \frac{1}{6}y$$

the equation becomes

$$\partial_\xi^2 - \partial_\eta$$

(c) $4\partial_x^2 - 4\partial_x\partial_y + 5\partial_y^2$

$$= (2\partial_x - \partial_y)^2 + 4\partial_y^2$$

$$\left.\begin{array}{l}\partial_\xi = 2\partial_x - \partial_y \\ \partial_\eta = 2\partial_y\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\partial_x = \frac{1}{2}\partial_\xi + \frac{1}{4}\partial_\eta \\ \partial_y = \frac{1}{2}\partial_\eta\end{array}\right.$$

$$\begin{pmatrix}\partial_x \\ \partial_y\end{pmatrix} = \begin{pmatrix}\frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2}\end{pmatrix}\begin{pmatrix}\partial_\xi \\ \partial_\eta\end{pmatrix} \Rightarrow \begin{pmatrix}\xi \\ \eta\end{pmatrix} = \begin{pmatrix}\frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}x \\ y\end{pmatrix}$$

Under the following change of variable

$$\xi = \frac{x}{2}, \quad \eta = \frac{x}{4} + \frac{y}{2}$$

the equation becomes

$$\partial_\xi^2 + \partial_\eta^2$$

3. By d'Alembert's formula: $c=1$, $\phi(x)=x$, $\psi(x)=e^x$

$$u(x,t) = \frac{1}{2}\left[\phi(x-t)+\phi(x+t)\right] + \frac{1}{2}\int_{x-t}^{x+t} \psi(s)\, ds$$

$$= \frac{1}{2}\left[x-t+x+t\right] + \frac{1}{2}\int_{x-t}^{x+t} e^s\, ds$$

$$= x + \frac{1}{2}e^x\left(e^t - e^{-t}\right)$$

$$= x + e^x \sinh x.$$

4. The equation is

$$\partial_t^2 - \partial_t\partial_x - \partial\partial_x^2 = (\partial_t - 2\partial_x)(\partial_t + \partial_x)$$

Let $\partial_\xi = \partial_t - 2\partial_x$ $\Rightarrow$ $\partial_t = \frac{1}{3}\partial_\xi + \frac{2}{3}\partial_\eta$

$\partial_\eta = \partial_t + \partial_x$ $\quad$ $\partial_x = -\frac{1}{3}\partial_\xi + \frac{1}{3}\partial_\eta$

$$\begin{pmatrix}\partial_t\\ \partial_x\end{pmatrix} = \begin{pmatrix}\frac{1}{3} & \frac{2}{3}\\ -\frac{1}{3} & \frac{1}{3}\end{pmatrix}\begin{pmatrix}\partial_\xi\\ \partial_\eta\end{pmatrix} \Rightarrow \begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}\frac{1}{3} & -\frac{1}{3}\\ \frac{2}{3} & \frac{1}{3}\end{pmatrix}\begin{pmatrix}t\\ x\end{pmatrix}$$

$$\xi = \frac{t}{3} - \frac{x}{3}, \quad \eta = \frac{2t}{3} + \frac{x}{3}$$

Now $u = F(\xi) + G(\eta) = F\left(-\frac{1}{3}(x-t)\right) + G\left(\frac{1}{3}(x+2t)\right)$

$= f(x-t) + g(x+2t).$

$$\left.\begin{aligned} &f(x) + g(x) = \sin x\\ &-f'(x) + 2g'(x) = 0 \Rightarrow -f(x) + 2g(x) = 0 \end{aligned}\right\} \begin{aligned} g(x) &= \tfrac{1}{3}\sin x\\ f(x) &= \tfrac{2}{3}\sin x \end{aligned}$$

So the sol'n is

$$u = \frac{2}{3}\sin(x-t) + \frac{1}{3}\sin(x+2t)$$

5. Use d'Alembert's Formula

$$f(x,t) = \cos x, \quad \phi(x) = \sin x, \quad \psi(x) = 1+x$$

$$u(x,t) = \frac{1}{2}\left[\sin(x-ct) + \sin(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}(1+s)\,ds + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} \cos y\,dy\,ds$$

$$= \sin x\cos ct + \frac{1}{2c}\left[s + \frac{s^2}{2}\right]\Bigg|_{x-ct}^{x+ct} + \frac{1}{2c}\int_0^t \sin y\Big|_{x-c(t-s)}^{x+c(t-s)} ds$$

$$= \sin x\cos ct + \frac{1}{2c}\left(2ct + \frac{1}{2}\cdot 4ct\cdot x\right) + \frac{2}{2c}\int_0^t \cos x\sin c(t-s)\,ds$$

$$= \sin x\cos ct + t + t\cdot x + \frac{1}{c}\cos x\left(\frac{1}{c}\cos c(t-s)\Big|_0^t\right).$$

$$= \sin x\cos ct + t(1+x) + \frac{1}{c^2}\cos x(1-\cos ct)$$