Solutions to Assignment 2

1. $c(u) = 1-u$.

$$\frac{dt}{1} = \frac{dx}{1-u} = \frac{du}{0}$$

$$x - \xi = c(\cancel{u}f_0(\xi))\,t, \qquad u = f(\xi) = 1-\xi$$

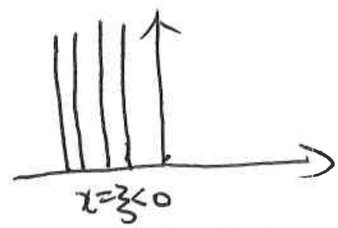
$$x - \xi = (1-(1-\xi))\,t = \xi t$$

$$\Rightarrow \quad \xi = \frac{x}{t+1}$$

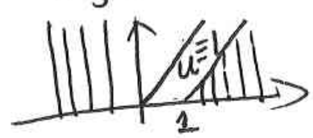
$$u = 1-\xi = 1 - \frac{x}{t+1}$$

2. $c(u) = u^2, \quad u(\xi,0) = f(\xi) = \begin{cases} 0, & \xi < 0 \\ 1, & 0<\xi<1 \\ 0, & \xi > 0 \end{cases}$

$$x - \xi = c(f(\xi))\,t, \quad u = f(\xi)$$



$\xi < 0 \Rightarrow f(\xi) = 0, \quad x = \xi, \ \xi < 0, \quad u = 0$



$0 < \xi < 1 \Rightarrow f(\xi) = 1, \quad x - \xi = t, \quad 0 < \xi = x - t < 1, \ u = 1$

$\xi > 1 \Rightarrow f(\xi) = 0, \quad x - \xi = 0, \quad x = \xi > 1, \ u = 0$

So there is a shock at $x = 1$, $\quad x = s(t)$

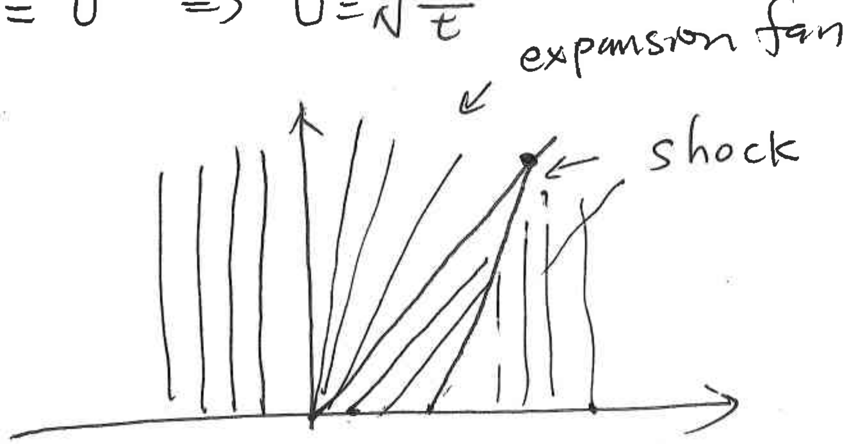
$$\begin{cases} \dfrac{ds}{dt} = \dfrac{Q^+ - Q^-}{u^+ - u^-} = \dfrac{\frac{1}{3}u^{+3} - \frac{1}{3}u^{-3}}{u^+ - u^-} = \dfrac{\frac{1}{3}\cdot 0^3 - \frac{1}{3}\cdot 1^3}{0-1} = \dfrac{1}{3} \\ s(0) = 1 \end{cases}$$

$$x = s(t) = \frac{t}{3} + 1$$

between $x=0$ and $x=t$, there is an expansion fan

$u=U(\frac{x}{t})$. Then

$$c(U)=\frac{x}{t}=U^2 \Rightarrow U=\sqrt{\frac{x}{t}}$$



The expansion fan and shock hit at

$$\begin{cases} x=t \\ x=\frac{t}{3}+1 \end{cases} \Rightarrow t=\frac{3}{2}, \quad x=\frac{3}{2}$$

for $t<\frac{3}{2}$, the solution is given by

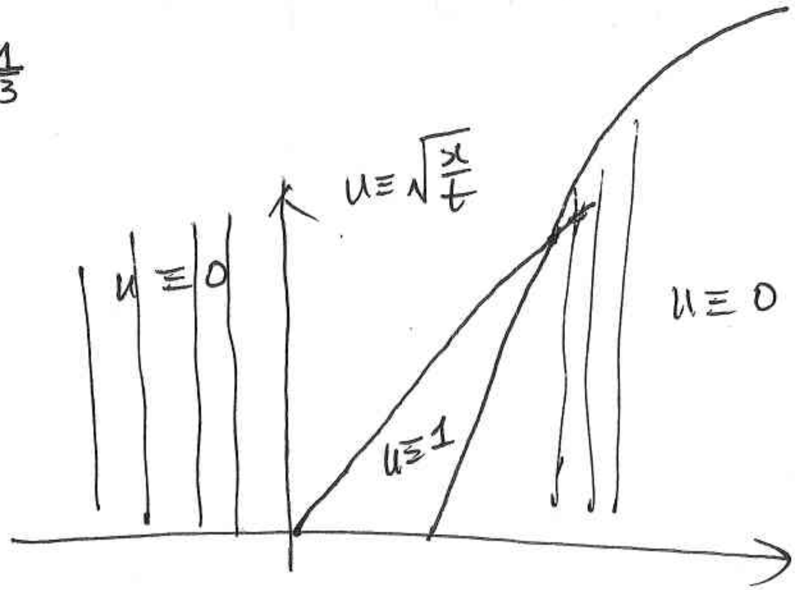
$$u(x,t)=\begin{cases} 0, & x<0 \\ \sqrt{\frac{x}{t}}, & 0<x<t \\ 1, & t<x<\frac{t}{3}+1 \\ 0, & \frac{t}{3}+1<x \end{cases}$$

After $t>\frac{3}{2}$, there exists a shock curve

$$u^+\equiv 0, \quad u^-=\sqrt{\frac{x}{t}}=\sqrt{\frac{s}{t}}$$

$$\begin{cases} \dfrac{ds}{dt} = \dfrac{Q^+ - Q^-}{u^+ - u^-} = \dfrac{0 - \frac{1}{3}\left(\sqrt{\frac{s}{t}}\right)^3}{0 - \sqrt{\frac{s}{t}}} = \dfrac{s}{3t} \\ s(0) = \dfrac{3}{2} \end{cases}$$

$$s = \left(\frac{3}{2}\right)^{\frac{2}{3}} t^{\frac{1}{3}}$$



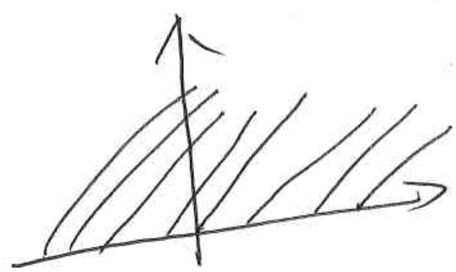
Problem 3. There are three initial conditions

For $\rho(x,0) = \frac{3}{8}$ $\Rightarrow$ initial curve is $(\xi, 0)$, $u_0(\xi) = \frac{3}{8}$

$\Rightarrow$ $x - \xi = c(u_0(\xi))\, t$

$x - \xi = c\left(\frac{3}{8}\right) t$

$c(\rho) = Q'(\rho) = 1 - \frac{2}{3}\rho$, $\quad c\left(\frac{3}{8}\right) = \frac{3}{4}$

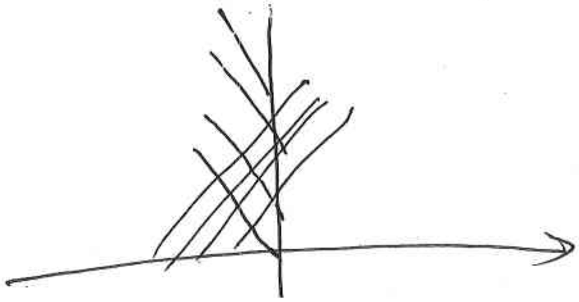


$x - \frac{3}{4} t = \xi$, $\quad -\infty < \xi < +\infty$

For $\rho(0-, t) = 3$, $\Rightarrow$ $\begin{cases} \dfrac{dt}{ds} = 1, & t(0) = \xi \\ \dfrac{dx}{ds} = c(\rho), & x(0) = 0 \\ \dfrac{d\rho}{ds} = 0, & \rho(0) = 3 \end{cases}$ $\Rightarrow$ $t - \dfrac{x}{c(\rho)} = \xi > 0$

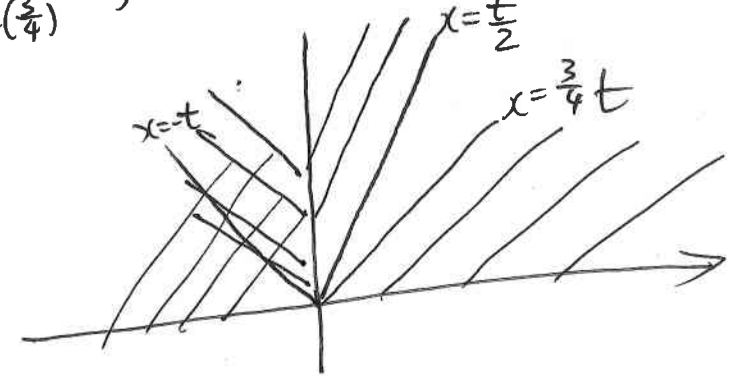
$t - \frac{x}{c(3)} = \xi > 0$, $c(3) = 1 - \frac{2}{3} \times 3 = -1$

$t + x = \xi > 0$.



For $\rho(0^+, x) = \frac{3}{4}$, $\Rightarrow c(\frac{3}{4}) = \frac{1}{2}$

$t - \frac{x}{c(\frac{3}{4})} = \xi > 0 \Rightarrow t - 2x = \xi > 0$



On the left plane, we need a shock curve:

$$\begin{cases} \frac{ds}{dt} = \frac{Q^+ - Q^-}{u^+ - u^-} = \frac{\rho^+(1-\frac{\rho^+}{3}) - \rho^-(1-\frac{\rho^-}{3})}{\rho^+ - \rho^-} \\ \quad = 1 - \frac{\rho^+ + \rho^-}{3} \\ \quad = 1 - \frac{\frac{3}{8} + 3}{3} = 1 - \frac{9}{8} = -\frac{1}{8} \\ s(0) = 0 \end{cases}$$

The shock curve is $x = -\frac{1}{8}t$

On the right plane, we need an expansion fan

$$c(U) = \frac{x}{t} \quad \Rightarrow 1 - \frac{2}{3}U = \frac{x}{t} \Rightarrow \quad U = \frac{3}{2}\left(1 - \frac{x}{t}\right)$$

So the final solution is

$$u(x,t) = \begin{cases} \frac{3}{8}, & x < -\frac{1}{8}t \\ 3, & -\frac{t}{8} < x < 0 \\ \frac{3}{4}, & 0 < x < \frac{t}{2} \\ \frac{3}{2}\left(1 - \frac{x}{t}\right), & \frac{t}{2} < x < \frac{3}{4}t \\ \frac{3}{8}, & \frac{3}{4}t < x \end{cases}$$

Problem 4: $\quad F = q - \frac{1}{2}p^2$

Initial conditions: $\quad (\xi, 0), \quad u_0(\xi) = 2\xi$

$$x_0(\xi) = \xi, \quad y_0(0) = 0, \quad u_0(\xi) = 2\xi$$

$$u_0' = x_0' p_0 + y_0' q_0 \quad \Rightarrow \quad 2 = 1 \cdot p_0 \Rightarrow p_0 = 2$$

$$q_0 - \frac{1}{2}p_0^2 = 0 \Rightarrow q_0 = 2.$$

$$\begin{cases} \frac{dx}{ds} = F_p = -p, \quad x(0) = \xi & \Rightarrow x = -2s + \xi \Rightarrow \xi = x + 2y. \\ \frac{dy}{ds} = F_q = 1, \quad y(0) = 0 & \Rightarrow y = s \\ \frac{dp}{ds} = -F_x - F_u p = 0, \quad p(0) = 2 & \Rightarrow p = 2 \\ \frac{dq}{ds} = -F_x - F_u q = 0, \quad q(0) = 2 & \Rightarrow q = 2 \\ \frac{du}{ds} = pF_p + qF_q = -p^2 + q, \quad u(0) = 2\xi & \Rightarrow u = -2s + 2\xi \end{cases}$$

$$= -2y + 2(x + 2y)$$

$$= 2x + 2y$$