Notation $O((z-z_0)^n)$

The notation $O((z-z_0)^n)$ is a very useful notation.
It denotes any term that is

$$a_n(z-z_0)^n + a_{n+1}(z-z_0)^{n+1} + \cdots = O((z-z_0)^n)$$

Example:
$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$
$$= z - \frac{z^3}{3!} + O(z^5)$$
$$= z + O(z^3)$$

Properties of $O((z-z_0)^n)$

1. $(z-z_0)\cdot O((z-z_0)^n) = O((z-z_0)^{n+1})$

2. $\dfrac{O((z-z_0)^n)}{z-z_0} = O((z-z_0)^{n-1})$

3. $(1+O((z-z_0)^n))^m = 1+O((z-z_0)^n)$

4. $(1+O((z-z_0)^n))^{-m} = 1+O((z-z_0)^n)$

This notation is used whenever we are no longer needed to compute the remaining terms.

Ex. Classify the singularities and Compute the residues

(a) $\frac{e^z}{z^3} = \frac{1}{z^3}\left(1 + z + \frac{z^2}{2!} + O(z^3)\right)$

$= \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2}\frac{1}{z} + O(1)$

pole of order 3, residue $= \frac{1}{2}$

(b) $\frac{1}{\sin^3 z} = \frac{1}{\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} \cdots\right)^3} = \frac{1}{z^3}\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} \cdots\right)^{-3}$

$= \frac{1}{z^3}\left(1 - \frac{z^2}{3!} + O(z^4)\right)^{-3}$

$= \frac{1}{z^3}\left(1 + (-3)\cdot\left(-\frac{z^2}{3!}\right) + O(z^4)\right)$

$= \frac{1}{z^3}\left(1 + \frac{z^2}{2} + O(z^4)\right)$

$= \frac{1}{z^3} + \frac{1}{2}\frac{1}{z} + O(z)$

pole of order 3, residue $= \frac{1}{2}$

(c) $\frac{\sin(\pi z)}{(\sqrt{z} - 1)^3}$, $z = 1$, $\sqrt{z}$ = principal branch

$f(z) = \sqrt{z} - 1$, $z_0 = 1$, $g(z) = \sin(\pi z)$

$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots$

$= f'(1)(z-1) + \frac{f''(1)}{2}(z-1)^2 + O((z-1)^3)$

$g(z) = g(z_0) + g'(z_0)(z - z_0) + \frac{g''(z_0)}{2}(z - z_0)^2 + \cdots$

$= g'(1)(z-1) + \frac{g''(1)}{2}(z-1)^2 + O((z-1)^3)$

$$\frac{g(z)}{f^3(z)} = \frac{g'(1)(z-1) + \frac{g''(1)}{2}(z-1)^2 + O\left((z-1)^3\right)}{(z-1)^3\left[f'(1) + \frac{f''(1)}{2}(z-1) + O\left((z-1)^2\right)\right]^3}$$

$$= \frac{1}{(z-1)^2}\left[g'(1) + \frac{g''(1)}{2}(z-1) + O\left((z-1)^2\right)\right] \cdot \left[f'(1) + \frac{f''(1)}{2}(z-1) + O\left((z-1)^2\right)\right]^{-3}$$

$$= \frac{1}{(z-1)^2}\left[g'(1) + \frac{g''(1)}{2}(z-1) + O\left((z-1)^2\right)\right]\cdot\left[f'(1)\right]^{-3}\left[1 - \frac{3}{2}\frac{f''(1)}{f'(1)}(z-1) + O\left((z-1)^2\right)\right]$$

$$= \frac{1}{(z-1)^2}\left[g'(1)\right]\left[f'(1)\right]^{-3} + \frac{1}{z-1}\left(\left[\frac{g''(1)}{2}\right]\left[f'(1)\right]^{-3} - \frac{3}{2}g'(1)\frac{f''(1)}{f'(1)}\right) + O(1)$$

whee

$$f'(z) = \frac{1}{2}z^{-\frac{1}{2}}, \quad f'' = -\frac{1}{4}z^{-\frac{3}{2}}$$

$$g' = \pi\cos(\pi z), \quad g'' = -\pi^2 \sin \pi z$$

So

$$\text{Residue} = -\frac{3}{2}g'(1)\frac{f''(1)}{f'(1)} = -\frac{3}{2}(-\pi)\cdot\frac{\left(-\frac{1}{4}\right)}{\frac{1}{2}} = -\frac{3}{4}\pi$$