Be sure this exam has 6 pages including the cover

The University of British Columbia

MATH 305, Sections 201

Midterm Examination 1, Feb. 10, 2017, 50minutes

Name ______________________ Signature ______________________

Student Number ______________

This exam consists of 5 questions. No notes. Write your answer in the blank page provided.

Problem	max score	score
1.	20	
2.	20	
3.	20	
4.	20	
5.	20	
total	100	

(20 points) 1. Find all solutions in the complex plane to the following:

$$\sin(z) = 2i$$

Hint: $\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$

Solutions:

Method 1. $\sin z = \sin x \cosh y + i \cos x \sinh y = 2i$

$\Rightarrow \sin x \cosh y = 0 \Rightarrow \sin x = 0 \Rightarrow x = k\pi$

$\cos x \sinh y = 2 \Rightarrow \cos k\pi \sinh y = 2$

k - even $\Rightarrow \sinh y = 2 \Rightarrow e^{2y} - 4e^{y} - 1 = 0$

$e^y = \frac{4 \pm \sqrt{20}}{2} = 2 \pm \sqrt{5} > 0 \Rightarrow$

$e^y = 2 + \sqrt{5}$

$x = 2m\pi$, $y = \ln(2+\sqrt{5})$

k - odd $\Rightarrow \sinh y = -2 \Rightarrow e^{2y} + 4e^{y} - 1 = 0$

$e^y = -2 + \sqrt{5}$, $y = \ln(\sqrt{5}-2)$

$x = (2m+1)\pi$, $y = \ln(\sqrt{5}-2)$

Method 2. $e^{iz} - e^{-iz} = 4i^2 = -4 \Rightarrow e^{2iz} + 4e^{iz} - 1 = 0$

$e^{iz} = -2 \pm \sqrt{5}$

"+" sign: $e^{iz} = -2 + \sqrt{5}$, $iz = \log(\sqrt{5}-2) = \ln|\sqrt{5}-2| + i(2k\pi)$

$z = -i \ln|\sqrt{5}-2| + 2k\pi$

"-" sign: $e^{iz} = -2 - \sqrt{5}$, $iz = \log(-2-\sqrt{5}) = \ln|2+\sqrt{5}| + i(\pi + 2k$

$z = -i \ln|2+\sqrt{5}| + (2k+1)\pi$

(20 points) 2. Find the image of the following domain

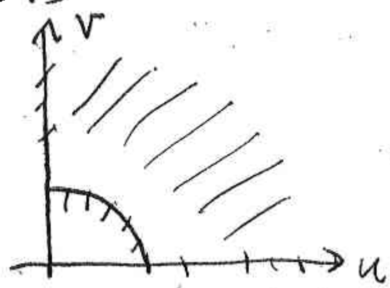
$$\{x > 0,\ 0 < y < \frac{\pi}{2}\}$$

under the mapping $w = e^z$.

Solution: $w = e^z = e^x \cos y + i e^x \sin y$

$$\left.\begin{array}{l} u = e^x \cos y \\ v = e^x \sin y \end{array}\right\} \Rightarrow \begin{array}{l} u^2+v^2 = e^{2x} > 1 \\ y = \arctan \frac{v}{u} \in (0, \frac{\pi}{2}) \end{array}$$

The image is



$\{(u,v) \mid u^2+v^2>1, \quad u>0,\ v>0\}$

(20 points) 3. Find a branch cut for $f(z) = (z(z+2)(z-3))^{\frac{1}{2}}$ so that it is analytic in $C\backslash((-\infty,-2]\cup[0,3])$ and $f(-1)=2$.

Solution: Branch points are $z = 0, -2, 3$

$$(z(z+2)(z-3))^{\frac{1}{2}} = r_1^{\frac{1}{2}} r_2^{\frac{1}{2}} r_3^{\frac{1}{2}} e^{i\left(\frac{\varphi_1}{2}+\frac{\varphi_2}{2}+\frac{\varphi_3}{2}\right)}$$

$r_1 = |z-3|$, $r_2 = |z|$, $r_3 = |z+2|$

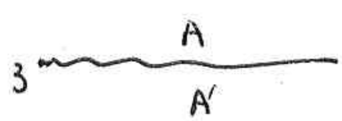
$\varphi_1 = \arg(z-3)$, $\varphi_2 = \arg z$, $\varphi_3 = \arg(z+2)$

To make it analytic at -1, we take a cut

$$0 < \varphi_1 < 2\pi, \quad 0 < \varphi_2 < 2\pi$$

$$\boxed{\pi < \varphi_3 < 3\pi}$$

First we show that this cut make it $f(z)$ analytic for $z = x > 3$.



	φ_1	φ_2	φ_3	$\frac{\varphi_1+\varphi_2+\varphi_3}{2}$
A	0	0	2π	π
A'	2π	2π	2π	3π

differ 2π

Second, at $z = -1$

$$\varphi_1 = \pi, \quad \varphi_2 = \pi, \quad \varphi_3 = 2\pi$$

So

$$f(z) = r_1^{\frac{1}{2}} r_2^{\frac{1}{2}} r_3^{\frac{1}{2}} e^{i\left(\frac{\pi}{2}+\frac{\pi}{2}+\frac{2\pi}{2}\right)} = 2$$

RmK: If $-\pi < \varphi_3 < \pi$, then $f(-1) = -2$

(20 points) 4. If $u(x, y) = 2xy + x$, find an analytic function $f(z)$ with $Re(f(z)) = u(x, y)$ and $f(0) = i$.

Solution: Let $f = u + iv$. By Cauchy-Riemann

$$u_x = v_y \Rightarrow v_y = 2y + 1$$

$$\Rightarrow v = y^2 + y + \varphi(x)$$

$$u_y = -v_x \Rightarrow v_x = -u_y = -2x = \varphi'(x)$$

$$\varphi'(x) = -2x, \quad \varphi = -x^2 + a$$

$$f = 2xy + x + i(y^2 + y - x^2 + a)$$

$$f(0) = i \Rightarrow a \cdot i = i \Rightarrow a = 1$$

So $f = 2xy + x + i(y^2 + y - x^2 + 1)$

$$= z - iz^2 + i$$

(20 points) 5. Show that in general $log(z^2) \neq 2\ log(z)$.

Solution: By definition

$$\log(z^2) = \ln|z^2| + i(\mathrm{Arg}(z^2) + 2k\pi), \quad \text{where } -\pi < \mathrm{Arg}(z^2) \le \pi, \ k = 0, \pm 1, \cdots$$

$$\log z = \ln|z| + i(\mathrm{Arg}(z) + 2m\pi)$$

$$2\log z = 2\ln|z| + 2i(\mathrm{Arg}(z) + 2m\pi)$$

$$= \ln|z^2| + i(2\mathrm{Arg}(z) + 4m\pi), \quad m = 0, \pm 1, \cdots$$

Now we choose

$z=1$, $\mathrm{Arg}(z^2) = 0$

$k=1$,

then $\exists$ no m such that

$$2\log z = \ln|z^2| + i(2\mathrm{Arg}(z) + 4m\pi) = i(4m\pi) = i \cdot 2\pi$$

So $\log(z^2) \neq 2\log z$ in general