MATH305-201-2016/2017 Homework Assignment 3 (Due Date: Jan.30, 2017, by 5:30pm, in class or at my office)

1. Discuss the analyticity of the following complex functions
(a) $x^2 + y^2 + y - 2 + ix$; (b) $2y - ix$; (c) $(x + \frac{x}{x^2+y^2}) + i(y - \frac{y}{x^2+y^2})$

2. Use Cauchy-Riemann equation to find out the harmonic conjugate of the following functions
(a) $xy - x + y$; (b) $u = \log(x^2 + y^2)$ for $Re(z) > 0$; (c) $u = e^x \sin y$; (d) $u = \sin x \cosh(y)$

3. Show that if v is a harmonic conjugate of u in a domain D, then both $u^2 - v^2$ and uv are harmonic in D. Can you generalize this?

4. Suppose that functions u and v are harmonic in D. Are the following functions harmonic?
(a) $u + v$; (b) uv; (c) $\frac{\partial u}{\partial x}$; (d) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$
(Assume that harmonic functions are smooth functions with all derivatives.)

5. Find a harmonic function $\phi(x)$ in the infinite strip

$$\{z : -2 \leq Re(z) - Im(z) \leq 3\}$$

such that $\phi = 0$ on the left edge and $\phi = 4$ on the right edge. Hint: consider linear functions.

6. Find a harmonic function $\phi(x, y)$ in the region of the first duadrant between the curves $xy = 2$ and $xy = 4$ and take value 1 on the lower edge and the value 3 on the upper edge.

7. Suppose that f is analytic and nonzero in a domain D. Prove that $\log |f(z)|$ is harmonic in D. Use this to find a harmonic function $\phi(x, y)$ in the annulus $\{z : 1 \leq |z| \leq 2\}$ such that $\phi = 1$ on $\{|z| = 1\}$ and $\phi = 2$ on $\{|z| = 2\}$.

8. Let the polar coordinate be

$$x = r\cos\theta, y = r\sin\theta$$

(a) Suppose $u(x, y) = U(r, \theta)$. Find out $\frac{\partial U}{\partial r}, \frac{\partial U}{\partial \theta}$
(b) Let $f = u(x, y) + iv(x, y) = U(r, \theta) + iV(r, \theta)$ be analytic. (Here $u = U, v = V$.) Find out the Cauchy-Riemann equation in polar form.

9. Find the image of the $S = \{z : -1 \leq Re(z) \leq 1, -\frac{\pi}{2} \leq Im(z) \leq \pi\}$ under the map $f(z) = e^z$

10. Find all numbers z such that
(a) $z^3 = -1 - i$; (b) $e^z = -1 - i$; (c) $\sin(z) = -1 - i$