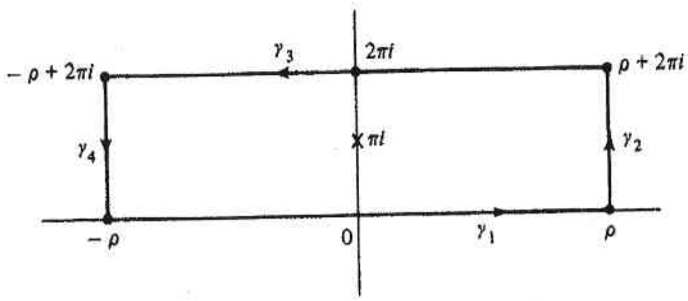

Figure 6.6 Closed contour for Example 3.

again approaching zero as $\rho \to \infty$ since $a > 0$.

As a result, on taking the limit as $\rho \to \infty$ we have

$$\lim_{\rho\to\infty} \int_{\Gamma_\rho} \frac{e^{az}}{1+e^z}\, dz = \left(1 - e^{a2\pi i}\right) \text{ p.v.} \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\, dx. \tag{8}$$

Now we use residue theory to evaluate the contour integral in Eq. (8). For each $\rho > 0$, the function $e^{az}/(1+e^z)$ is analytic inside and on Γ_ρ except for a simple pole at $z = \pi i$, the residue there being given by

$$\text{Res}(\pi i) = \left.\frac{e^{az}}{\dfrac{d}{dz}(1+e^z)}\right|_{z=\pi i} = \frac{e^{a\pi i}}{e^{\pi i}} = -e^{a\pi i} \tag{9}$$

(recall Example 2, Sec. 6.1). Consequently, putting Eqs. (8) and (9) together we obtain

$$\begin{aligned}\text{p.v.} \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\, dx &= \frac{1}{1-e^{a2\pi i}} \cdot (2\pi i)\left(-e^{a\pi i}\right) \\ &= \frac{-2\pi i}{e^{-a\pi i} - e^{a\pi i}} \\ &= \frac{\pi}{\sin a\pi}. \quad \blacksquare\end{aligned}$$

EXERCISES 6.3

Verify the integral formulas in Problems 1–7 with the aid of residues.

1. $\text{p.v.} \displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2} = \pi$

2. $\text{p.v.} \displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+9)^2}\, dx = \frac{\pi}{6}$

3. $\displaystyle\int_0^\infty \frac{x^2+1}{x^4+1}\,dx = \frac{\pi}{\sqrt{2}}$

4. $\displaystyle \text{p.v.}\int_{-\infty}^\infty \frac{dx}{\left(x^2+1\right)\left(x^2+4\right)} = \frac{\pi}{6}$

5. $\displaystyle \text{p.v.}\int_{-\infty}^\infty \frac{x}{\left(x^2+4x+13\right)^2}\,dx = -\frac{\pi}{27}$

6. $\displaystyle\int_0^\infty \frac{x^2}{\left(x^2+1\right)\left(x^2+4\right)}\,dx = \frac{\pi}{6}$

7. $\displaystyle\int_0^\infty \frac{x^6}{\left(x^4+1\right)^2}\,dx = \frac{3\pi\sqrt{2}}{16}$

8. Show that if $f(z) = P(z)/Q(z)$ is the quotient of two polynomials such that $\deg Q \geq 2 + \deg P$, where Q has no real zeros, then p.v. $\int_{-\infty}^\infty f(x)\,dx$ equals

$$-2\pi i \cdot \sum[\text{residues of } f(z) \text{ at the poles in the } \textit{lower} \text{ half-plane}].$$

9. Show that

$$\text{p.v.}\int_{-\infty}^\infty \frac{e^{2x}}{\cosh(\pi x)}\,dx = \sec 1$$

by integrating $e^{2z}/\cosh(\pi z)$ around rectangles with vertices at $z = \pm\rho,\ \rho + i,\ -\rho + i$.

10. Given that

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2},$$

integrate e^{-z^2} around a rectangle with vertices at $z = 0,\ \rho,\ \rho + \lambda i$, and λi (with $\lambda > 0$) and let $\rho \to \infty$ to derive

(a) $\displaystyle\int_0^\infty e^{-x^2}\cos(2\lambda x)\,dx = \frac{\sqrt{\pi}}{2}\,e^{-\lambda^2}$

(b) $\displaystyle\int_0^\infty e^{-x^2}\sin(2\lambda x)\,dx = e^{-\lambda^2}\int_0^\lambda e^{y^2}\,dy$

(The right-hand side of (b), as a function of λ, is known as the *Dawson integral* and is tabulated by Abramowitz and Stegun in Ref. [5].)

11. Show that

$$\int_0^\infty \frac{dx}{x^3+1} = \frac{2\pi\sqrt{3}}{9}$$

by integrating $1/(z^3+1)$ around the boundary of the circular sector $S_\rho : \{z = re^{i\theta} : 0 \leq \theta \leq 2\pi/3, 0 \leq r \leq \rho\}$ and letting $\rho \to \infty$.

12. Confirm the values of the integrals discussed in Prob. 18, Exercises 4.7.

13. Show that

$$\int_{-\infty}^{\infty} \frac{1}{\left(1+x^2\right)^{n+1}}\,dx = \frac{\pi(2n)!}{2^{2n}(n!)^2}, \qquad \text{for } n = 0, 1, 2, \ldots.$$

Summation of Series

14. Let $f(z)$ be a rational function of the form $P(z)/Q(z)$, where $\deg Q \geq 2 + \deg P$. Assume that no poles of $f(z)$ occur at the integer points $z = 0, \pm 1, \pm 2, \ldots$. Complete each of the following steps to establish the summation formula

$$\lim_{N\to+\infty} \sum_{k=-N}^{N} f(k) = -\{\text{sum of the residues of } \pi f(z)\cot(\pi z) \text{ at the poles of } f(z)\}. \tag{10}$$

(a) Show that for the function $g(z) := \pi f(z)\cot(\pi z)$, we have

$$\operatorname{Res}(g; k) = f(k), \quad k = 0, \pm 1, \pm 2, \ldots.$$

(b) Let Γ_N be the boundary of the square with vertices at $(N+\frac{1}{2})(1+i)$, $(N+\frac{1}{2})(-1+i)$, $(N+\frac{1}{2})(-1-i)$, $(N+\frac{1}{2})(1-i)$, taken in that order, where N is a positive integer. Show that there is a constant M independent of N such that $|\pi \cot(\pi z)| \leq M$ for all z on Γ_N.

(c) Prove that

$$\lim_{N\to+\infty} \int_{\Gamma_N} \pi f(z)\cot(\pi z)\,dz = 0,$$

where Γ_N is defined previously.

(d) Use the residue theorem and parts (a) and (c) to derive (10).

15. Using the summation formula in Prob. 14 verify that

(a) $\displaystyle\sum_{k=-\infty}^{\infty} \frac{1}{k^2+1} = \pi\coth(\pi)$ [HINT: Take $f(z) = 1/(z^2+1)$.]

(b) $\displaystyle\sum_{k=-\infty}^{\infty} \frac{1}{\left(k-\frac{1}{2}\right)^2} = \pi^2$

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ [HINT: The formula in Prob. 14 needs to be modified to compensate for the pole of $f(z) = 1/z^2$ at $z = 0$.]

16. Show that for n a positive integer,

$$\sum_{k=1}^{\infty} \frac{1}{k^{2n}} = (-1)^{n-1}\pi^{2n}\frac{2^{2n-1}}{(2n)!}B_{2n},$$

where the constants B_{2n} are the *Bernoulli numbers*, which are defined by the power series expansion

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} z^k.$$

[Compare Prob. 15(c).] [HINT: To determine the required residue at $z = 0$ when $f(z) = 1/z^{2n}$, show that

$$\pi z \cot(\pi z) = \sum_{k=0}^{\infty} (-1)^k \frac{B_{2k}}{(2k)!} (2\pi z)^{2k}.]$$

17. Show that if a is real and noninteger and $0 < r < 1$,

(a) $\displaystyle\sum_{k=-\infty}^{\infty} \frac{1}{(k+a)^2} = \pi^2 \csc^2 \pi a$

(b) $\displaystyle\sum_{k=-\infty}^{\infty} \frac{1}{k^2 + a^2} = \frac{\pi}{a} \coth \pi a$

(c) $\displaystyle\sum_{k=-\infty}^{\infty} \frac{k^2 - a^2}{(k^2 + a^2)^2} = -\pi^2 \operatorname{csch}^2 \pi a$

(d) $\displaystyle\sum_{k=-\infty}^{\infty} \frac{1}{(k-r)^2 + a^2} = \frac{\pi}{2a} \frac{\sinh 2a\pi}{\sin^2 \pi r + \sinh^2 \pi a}$

(e) $\displaystyle\sum_{k=-\infty}^{\infty} \frac{(k-r)^2 - a^2}{\left[(k-r)^2 + a^2\right]^2} = \frac{\pi^2}{2} \frac{1 - \cos 2\pi r \cosh 2\pi a}{\left(\sin^2 \pi r + \sinh^2 \pi a\right)^2}$

(f) For which *complex* values of a are the preceding identities valid?

18. To evaluate sums of the form $\sum_{k=-\infty}^{\infty} (-1)^k f(k)$ involving a sign alternation, we modify the approach of Prob. 14 by replacing $\pi f(z) \cot(\pi z)$ by $\pi f(z) \csc(\pi z)$. Again assuming that $f(z)$ is a rational function of the form P/Q, with $\deg Q \geq 2 + \deg P$ and that f has no poles at the integer points, derive the formula

$$\sum_{k=-\infty}^{\infty} (-1)^k f(k) = -\{\text{sum of residues of } \pi f(z) \csc(\pi z) \text{ at the poles of } f\}.$$

19. Use the formula of Prob. 18 to verify that

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} = -\frac{\pi^2}{12}.$$

6.4 Improper Integrals Involving Trigonometric Functions

Our purpose in this section is to use residue theory to evaluate integrals of the general forms

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos mx \, dx, \qquad \text{p.v.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin mx \, dx,$$

respectively. Then we only have to perform *one* evaluation of residues, in the upper half-plane, and take the real or imaginary parts at the end. However, this shortcut is not valid for Example 3 since

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{\cos x}{x+i}\, dx \neq \text{Re p.v.} \int_{-\infty}^{\infty} \frac{e^{ix}}{x+i}\, dx\ .$$

In fact the left-hand member is pure imaginary, as we have seen.

EXERCISES 6.4

Using the method of residues, verify the integral formulas in Problems 1–3.

1. $\text{p.v.} \displaystyle\int_{-\infty}^{\infty} \frac{\cos(2x)}{x^2+1}\, dx = \frac{\pi}{e^2}$

2. $\text{p.v.} \displaystyle\int_{-\infty}^{\infty} \frac{x \sin x}{x^2-2x+10}\, dx = \frac{\pi}{3e^3}(3\cos 1 + \sin 1)$

3. $\displaystyle\int_{0}^{\infty} \frac{\cos x}{\left(x^2+1\right)^2}\, dx = \frac{\pi}{2e}$

Compute each of the integrals in Problems 4–9.

4. $\text{p.v.} \displaystyle\int_{-\infty}^{\infty} \frac{e^{3ix}}{x-2i}\, dx$

5. $\text{p.v.} \displaystyle\int_{-\infty}^{\infty} \frac{x \sin(3x)}{x^4+4}\, dx$

6. $\text{p.v.} \displaystyle\int_{-\infty}^{\infty} \frac{e^{-2ix}}{x^2+4}\, dx$

7. $\text{p.v.} \displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{\left(x^2+1\right)\left(x^2+4\right)}\, dx$

8. $\displaystyle\int_{0}^{\infty} \frac{x^3 \sin(2x)}{\left(x^2+1\right)^2}\, dx$

9. $\text{p.v.} \displaystyle\int_{-\infty}^{\infty} \frac{\cos(2x)}{x-3i}\, dx$

10. Derive the formula

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{\cos x}{x-w}\, dx = \begin{cases} \pi i e^{iw} & \text{if } \operatorname{Im} w > 0, \\ -\pi i e^{-iw} & \text{if } \operatorname{Im} w < 0. \end{cases}$$

11. Give conditions under which the following formula is valid:

$$\text{p.v.} \int_{-\infty}^{\infty} e^{imx} \frac{P(x)}{Q(x)}\, dx$$
$$= 2\pi i \cdot \sum \left[\text{residues of } e^{imz} P(z)/Q(z) \text{ at poles in the upper half-plane}\right].$$

12. Given that $\int_0^\infty e^{-x^2}\, dx = \sqrt{\pi}/2$, integrate e^{iz^2} around the boundary of the circular sector $S_\rho : \{z = re^{i\theta} : 0 \le \theta \le \pi/4, 0 \le r \le \rho\}$, and let $\rho \to +\infty$ to prove that

$$\int_0^\infty e^{ix^2}\, dx = \frac{\sqrt{2\pi}}{4}(1+i).$$

6.5 Indented Contours

In the preceding sections the integrands f were assumed to be defined and continuous over the whole interval of integration. We turn now to the problem of evaluating special integrals where $|f(x)| \to \infty$ as x approaches certain finite points. Our first step is to give precise meaning to the integrals of f.

Let $f(x)$ be continuous on $[a, b]$ except at the point c, $a < c < b$. Then the *improper integrals* of f over the intervals $[a, c]$, $[c, b]$, and $[a, b]$ are defined by

$$\int_a^c f(x)\, dx := \lim_{r\to 0^+} \int_a^{c-r} f(x)\, dx,$$
$$\int_c^b f(x)\, dx := \lim_{s\to 0^+} \int_{c+s}^{b} f(x)\, dx,$$

and

$$\int_a^b f(x)\, dx := \lim_{r\to 0^+} \int_a^{c-r} f(x)\, dx + \lim_{s\to 0^+} \int_{c+s}^{b} f(x)\, dx, \tag{1}$$

provided the appropriate limit(s) exists. For example,

$$\int_0^1 \frac{1}{\sqrt{x}}\, dx = \lim_{s\to 0^+} \int_s^1 \frac{1}{\sqrt{x}} = \lim_{s\to 0^+} 2\sqrt{x}\Big|_s^1$$
$$= \lim_{s\to 0^+} \left[2 - 2\sqrt{s}\right] = 2,$$

and therefore one can say that the area under the graph in Fig. 6.9 is *finite*, despite the vertical asymptote.

On the other hand, the areas on either side of the vertical asymptote in the graph of $f(x) = 1/(x-2)$, depicted in Fig. 6.10, are both infinite, because

$$\int_{2+s}^{4} \frac{dx}{x-2} = \operatorname{Log}|x-2|\Big|_{x=2+s}^{x=4} = \operatorname{Log} 2 - \operatorname{Log} s \to \infty \quad \text{as} \quad s \to 0^+,$$

and indent around each of its simple poles, as indicated in Fig. 6.15. Then since $f(z)$ is analytic inside the closed contour, we obtain

$$\left(\int_{-\rho}^{-1-r_1} + \int_{-1+r_1}^{1-r_2} + \int_{1+r_2}^{\rho}\right) \frac{xe^{2ix}}{x^2-1}\,dx + J_{r_1} + J_{r_2} + J_\rho = 0, \tag{9}$$

where J_{r_1}, J_{r_2}, J_ρ are the integrals of $f(z)$ over S_{r_1}, S_{r_2}, C_ρ^+, respectively. Now by Jordan's lemma we have

$$\lim_{\rho\to\infty} J_\rho = 0,$$

and from Eq. (5) of Lemma 4,

$$\begin{aligned}\lim_{r_1\to 0^+} J_{r_1} &= -i\pi\,\mathrm{Res}(-1) = -i\pi \lim_{z\to -1}(z+1)f(z)\\ &= -i\pi \lim_{z\to -1}\frac{ze^{2iz}}{z-1} = \frac{-i\pi e^{-2i}}{2},\end{aligned}$$

and

$$\begin{aligned}\lim_{r_2\to 0^+} J_{r_2} &= -i\pi\,\mathrm{Res}(1) = -i\pi \lim_{z\to 1}(z-1)f(z)\\ &= -i\pi \lim_{z\to 1}\frac{ze^{2iz}}{z+1} = \frac{-i\pi e^{2i}}{2}.\end{aligned}$$

Hence on taking the limits in Eq. (9) we get

$$\text{p.v.}\int_{-\infty}^{\infty}\frac{xe^{2ix}}{x^2-1}\,dx = \frac{i\pi e^{-2i}}{2} + \frac{i\pi e^{2i}}{2} - 0 = i\pi\cos 2. \quad \blacksquare$$

EXERCISES 6.5

1. Compute each of the following limits along the given circular arcs.

(a) $\lim_{r\to 0^+}\int_{T_r}\frac{2z^2+1}{z}\,dz$, where $T_r : z = re^{i\theta}, 0 \le \theta \le \frac{\pi}{2}$

(b) $\lim_{r\to 0^+}\int_{\Gamma_r}\frac{e^{3iz}}{z^2-1}\,dz$, where $\Gamma_r : z = 1 + re^{i\theta}, \frac{\pi}{4} \le \theta \le \pi$

(c) $\lim_{r\to 0^+}\int_{\gamma_r}\frac{\mathrm{Log}\,z}{z-1}\,dz$, where $\gamma_r : z = 1 + re^{-i\theta}, \pi \le \theta \le 2\pi$

(d) $\lim_{r\to 0^+}\int_{S_r}\frac{e^z-1}{z^2}\,dz$, where $S_r : z = re^{-i\theta}, \pi \le \theta \le 2\pi$

Using the technique of residues, verify each of the integral formulas in Problems 2–8.

2. $\text{p.v.}\int_{-\infty}^{\infty}\frac{e^{2ix}}{x+1}\,dx = \pi i e^{-2i}$

3. p.v. $\int_{-\infty}^{\infty} \frac{e^{ix}}{(x-1)(x-2)}\,dx = \pi i\left(e^{2i} - e^{i}\right)$

4. $\int_{0}^{\infty} \frac{\sin(2x)}{x\left(x^2+1\right)^2}\,dx = \pi\left(\frac{1}{2} - \frac{1}{e^2}\right)$

5. $\int_{0}^{\infty} \frac{\cos x - 1}{x^2}\,dx = -\frac{\pi}{2}$

6. p.v. $\int_{-\infty}^{\infty} \frac{\sin x}{\left(x^2+4\right)(x-1)}\,dx = \frac{\pi}{5}\left[\cos(1) - e^{-2}\right]$

7. p.v. $\int_{-\infty}^{\infty} \frac{x\cos x}{x^2-3x+2}\,dx = \pi[\sin(1) - 2\sin(2)]$

8. p.v. $\int_{-\infty}^{\infty} \frac{\cos(2x)}{x^3+1}\,dx = \frac{\pi}{3}e^{-\sqrt{3}}\left[\sin(1) + \sqrt{3}\cos(1)\right] + \frac{\pi\sin(2)}{3}$

9. Compute p.v. $\int_{-\infty}^{\infty} \frac{\sin^3 x}{x^3}\,dx$. $\left[\text{HINT: } \sin^3 x = \operatorname{Im}\left(\frac{3e^{ix}}{4} - \frac{e^{3ix}}{4} - \frac{1}{2}\right).\right]$

10. Verify that

$$\int_{0}^{\infty} \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}.$$

[HINT: $\sin^2 x = \frac{1}{2}(1-\cos 2x) = \frac{1}{2}\operatorname{Re}\left(1 - e^{2ix}\right)$.]

11. Compute p.v. $\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x - 1}\,dx$ for $0 < a < 1$. [HINT: Indent the contour of Fig. 6.6 around the points $z = 0$ and $z = 2\pi i$.]

12. Verify that for $a > 0$ and $b > 0$

$$\int_{0}^{\infty} \frac{\sin(ax)}{x\left(x^2+b^2\right)}\,dx = \frac{\pi}{2b^2}\left(1 - e^{-ab}\right).$$

6.6 Integrals Involving Multiple-Valued Functions

In attempting to apply residue theory to compute an integral of $f(x)$, it may turn out that the complex function $f(z)$ is multiple-valued. If this happens, we need to modify our procedure by taking into account not only isolated singularities but also branch points and branch cuts. In fact we may find it necessary to integrate along a branch cut, so we turn first to a discussion of this technique.

To be specific, let α denote a real number, but not an integer, and let $f(z)$ be the branch of z^α obtained by restricting the argument of z to lie between 0 and 2π; that is,

$$f(z) = e^{\alpha(\operatorname{Log} r + i\theta)}, \qquad \text{where } z = re^{i\theta},\ 0 < \theta < 2\pi. \tag{1}$$