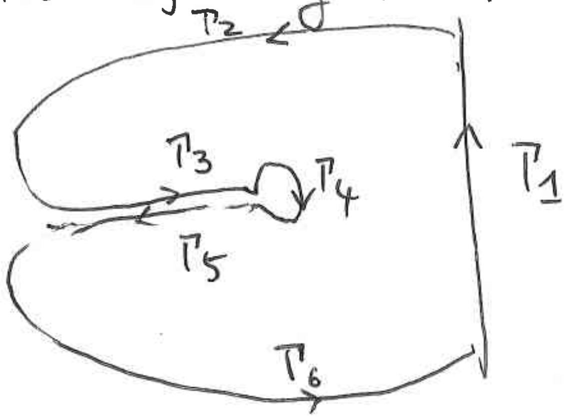
Solutions to Assignment 7, MATH301-201

1. We consider the following contour



for $f(z) = e^{zt} e^{-\sqrt{z}\cdot a}$

On Γ_3, $z = \rho e^{i\pi} = -\rho$, $\sqrt{z} = \rho^{\frac{1}{2}} e^{i\frac{\pi}{2}} = i\rho^{\frac{1}{2}}$

$$\int_{\Gamma_3} f(z)\,dz = \int_R^{\varepsilon} e^{-\rho t - i\rho^{\frac{1}{2}} a} (-d\rho) = \int_{\varepsilon}^{R} e^{-\rho t - i\rho^{\frac{1}{2}} a}\,d\rho$$

$\varepsilon \to 0, R \to +\infty$

$$\int_0^{+\infty} e^{-\rho t - i\rho^{\frac{1}{2}}\cdot a}\,d\rho \;\underset{=}{\overset{\rho = u^2}{}}\; 2\int_0^{+\infty} e^{-u^2 t - iua}\,u\,du$$

$$= 2\int_0^{+\infty} e^{-t\left(u + \frac{ia}{2t}\right)^2 - \frac{a^2}{4t}}\,u\,du$$

$$= 2e^{-\frac{a^2}{4t}} \int_0^{+\infty} e^{-tu^2} \left(u - \frac{ia}{2t}\right) du$$

$$= 2e^{-\frac{a^2}{4t}} \left[\frac{1}{2t} - \frac{ia}{2t}\cdot\frac{1}{t^{\frac{1}{2}}} \int_0^{+\infty} e^{-x^2}\,dx\right]$$

On Γ_5, $z = \rho e^{-i\pi} = -\rho$, $\sqrt{z} = \rho^{\frac{1}{2}} e^{-i\frac{\pi}{2}} = -i\rho^{\frac{1}{2}}$

Similarly, $\int_{\Gamma_5} f(z)\,dz = \int_{\varepsilon}^{R} e^{-\rho t + ia\rho^{\frac{1}{2}}} (-\rho)\,d\rho$

$$\to -\int_0^{+\infty} e^{-\rho t + i a \rho^{\frac{1}{2}}} \rho \, d\rho$$

$$= -\overline{\int_0^{+\infty} e^{-\rho t - i a \rho^{\frac{1}{2}}} \rho \, d\rho} = -2 e^{-\frac{a^2}{4t}} \left[\frac{1}{2t} + \frac{ia}{2t} \cdot \frac{1}{t^{\frac{1}{2}}} \frac{\sqrt{\pi}}{2} \right]$$

Thus $\frac{1}{2\pi i} \int_{\Gamma_1} f(z)\, dz = -\frac{1}{2\pi i} \int_{\Gamma_3} - \frac{1}{2\pi i} \int_{\Gamma_5}$

$$= -\frac{1}{2\pi i} \left[-\frac{4ia}{2t \, t^{\frac{1}{2}}} \cdot \frac{\sqrt{\pi}}{2} \right] e^{-\frac{a^2}{4t}} = \frac{a}{2\sqrt{\pi}\, t^{\frac{3}{2}}} e^{-\frac{a^2}{4t}}$$

So $L^{-1}\left[e^{-\sqrt{s}\, a} \right] = \frac{a}{2\sqrt{\pi}\, t^{\frac{3}{2}}} e^{-\frac{a^2}{4t}}$

2. Use Laplace Transform $\Rightarrow$

$$s^2 \hat{u} - s\, u(0) - u'(0) - 2(s\hat{u} - u(0)) + 5\hat{u} = \frac{1}{s} + \frac{5}{s^2+1}$$

$$(s^2 - 2s + 5)\hat{u} = as + 1 - 2a + \frac{1}{s} + \frac{5}{s^2+1}$$

Thus $$\hat{u}(s) = \frac{as + 1 - 2a + \frac{1}{s} + \frac{5}{s^2+1}}{(s-1)^2 + 4} = \frac{as^2(s^2+1) + (1-2a)s(s^2+1) + s^2 + 1 + 5s}{s(s^2+1)\left((s-1)^2+4\right)}$$

By the Inversion Formula,

$$u(t) = \sum_{j=1}^{m} \operatorname{Res}(e^{st}\hat{u}(s); s_j)$$

$$= \frac{1}{5} e^{0\cdot t} + \left. \frac{as^2(s^2+1) + (1-2a)s(s^2+1) + s^2+1 + 5s}{s(2s)(s^2-2s+5)} \right|_{s=i} e^{it}$$

$$+ (\cdots) \Big|_{s=-i} e^{-it}$$

$$+ \frac{as^2(s^2+1)+(1-2a)s(s^2+1)+s^2+1+5s}{s(s^2+1)(2s-2)}\Bigg|_{s=1+2i} e^{(1+2i)t} + (\cdot\cdot)\Bigg|_{s=1-2i} e^{(1-2i)t}$$

$$= \frac{1}{5} + \frac{1}{5}\cos t + \sin t + \left(a-\frac{7}{10}\right)e^t\cos 2t - \frac{1}{2}\left(a-\frac{7}{10}\right)e^t\sin 2t$$

When $a=\frac{7}{10}$, the solution is bounded.

3. We solve

$$y''' + y = f, \quad y(0)=y_0,\ y'(0)=y_1,\ y''(0)=y_2$$

$$s^3\hat{y} - s^2y_0 - sy_1 - y_2 + \hat{y} = \hat{f}$$

$$\hat{y} = \frac{s^2y_0+sy_1+y_2}{s^3+1} + \frac{\hat{f}}{s^3+1}$$

The inverse Laplace transform of $\frac{1}{s^3+1}$ is

$$\sum_{j=1}^{3} \mathrm{Res}\left(e^{st}\frac{1}{s^3+1}; s_j\right) = \sum_{j=1}^{3}\frac{e^{s_jt}}{3s_j^2} \qquad s_j^3=-1$$

$$s_1=-1,\ s_2=e^{\frac{\pi}{3}i},\ s_3=e^{\frac{5}{3}\pi i}$$

$$= -\sum_{j=1}^{3}\frac{s_je^{s_jt}}{3}$$

$$= -\frac{(-1)}{3}e^{-t} + \frac{1}{3}\left(-\cos\frac{\sqrt{3}}{2}t + \sqrt{3}\sin\frac{\sqrt{3}}{2}t\right)e^{+\frac{1}{2}t}$$

So
$$y = \sum_{j=1}^{3}\left(e^{st}\frac{s^2y_0+sy_1+y_2}{3s^2}; s_j\right) + \int_0^t g(t-\tau)f(\tau)\,d\tau$$

$$= e^{-t}\frac{y_0-y_1+y_2}{3} + e^{\frac{1}{2}t}\left[\left(\frac{2}{3}y_0+y_1-\frac{1}{3}y_2\right)\cos\frac{\sqrt{3}}{2}t + \left(\sqrt{3}y_1+\frac{2}{3}y_2\right)\sin\frac{\sqrt{3}}{2}t\right]$$
$$+\int_0^t g(t-\tau)f(\tau)\,d\tau$$

where

$$g(t) = \frac{1}{3}e^{-t} + \frac{1}{3}\left(-\cos\frac{\sqrt{3}}{2}t + \sqrt{3}\sin\frac{\sqrt{3}}{2}t\right)e^{\frac{1}{2}t}.$$

4.(a) $f(t+T) = -f(t)$ so

$$\int_0^{+\infty} e^{-st} f(t) = \int_0^T e^{-st} f(t)\,dt - e^{-sT}\int_0^T e^{-st} f(t)\,dt + e^{-2sT}\int_0^T e^{-st} f(t)\,dt - \cdots$$

$$= \left(1 - e^{-sT} + e^{-2sT} - \cdots\right)\int_0^T e^{-st} f(t)\,dt$$

$$= \frac{\int_0^T e^{-st} f(t)\,dt}{1+e^{-sT}}$$

(b). $$\hat{f}(s) = \frac{\int_0^T e^{-st} f(t)\,dt}{1+e^{-sT}} = \frac{\int_0^T e^{-st}\,dt}{1+e^{-sT}} = \frac{\frac{1}{s}\left(1-e^{-sT}\right)}{1+e^{-sT}}$$

5. $s\hat{y} - y_0 + \hat{y} = \hat{f}$

$$\hat{f} = \frac{\int_0^1 e^{-st} f(t)\,dt}{1-e^{-s}} = \frac{\int_{\frac{1}{2}}^1 e^{-st}\,dt}{1-e^{-s}}$$

$$= \frac{\frac{1}{s}\left(e^{-\frac{s}{2}} - e^{-s}\right)}{1-e^{-s}}$$

$$\hat{y} = \frac{y_0}{(s+1)} + \frac{\frac{1}{s}\left(e^{-\frac{s}{2}} - e^{-s}\right)}{(s+1)\left(1-e^{-s}\right)}$$

$$= \frac{\frac{1}{s+1}\,\frac{1}{s}\left(e^{-\frac{s}{2}} - e^{-s}\right)}{1-e^{-s}} - \frac{a}{s+1} + \frac{a}{s+1} + \frac{y_0}{s+1}$$

where $a = \left.\dfrac{\frac{1}{s}\left(e^{-\frac{s}{2}} - e^{-s}\right)}{1-e^{-s}}\right|_{s=-1} = \dfrac{-\left(e^{\frac{1}{2}} - e\right)}{1-e} = \dfrac{e-e^{\frac{1}{2}}}{1-e}$

Hence when $y_0 = \dfrac{e^{\frac{1}{2}} - e}{1-e} = -a$, y becomes periodic.

To find $y(t)$, we have

$$\hat{y}(s) = \frac{1}{1-e^{-s}}\, p(s), \quad \text{where } p(s) = \frac{1}{s+1}\,\frac{1}{s}\left(e^{-\frac{s}{2}} - e^{-s}\right) - \frac{a}{(s+1)}\left(1-e^{-s}\right)$$

$$= \left(\frac{1}{s} - \frac{1}{s+1}\right)e^{-\frac{s}{2}} - \frac{a}{s+1} + e^{-s}(\cdots).$$

Its inverse is

$$y(t) = \begin{cases} \left(1-e^{-t}\right)u_{\frac{1}{2}}(t) - a e^{-t}, & 0 < t < 1 \\ y(t+1), & t > 1 \end{cases}$$

6. $s\hat{u} - u(x,0) = \hat{u}_{xx}$

$$\begin{cases} \hat{u}_{xx} - s\hat{u} = -e^{-x} \\ \hat{u}(0,s) = \frac{1}{s} \end{cases}$$

So $\hat{u} = A e^{-\sqrt{s}\,x} + \frac{1}{s-1} e^{-x}$

where $\hat{u}(0,s) = \frac{1}{s} \Rightarrow A + \frac{1}{s-1} = \frac{1}{s}$ $\quad A = \frac{1}{s} - \frac{1}{s-1}$

So $\hat{u} = \frac{1}{s} e^{-\sqrt{s}\,x} - \frac{1}{s-1} e^{-\sqrt{s}\,x} + \frac{1}{s-1} e^{-x}$

Now the inverse of $\frac{1}{s} e^{-\sqrt{s}\,x}$ is $1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{t}}} e^{-p^2}\,dp$.

the inverse of $\frac{1}{s-1}$ is e^t, the inverse of $e^{-\sqrt{s}\,x}$ is $\frac{x}{2\sqrt{\pi}\, t^{\frac{3}{2}}} e^{-\frac{x^2}{4t}}$

Note that the inverse of $\hat{f}(s)\,\hat{g}(s)$ is $\int_0^t f(t-\tau)\, g(\tau)\, d\tau$.

So

$$u(x,t) = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{t}}} e^{-p^2}\,dp - \int_0^t e^{t-\tau} \frac{x}{2\sqrt{\pi}\,\tau^{\frac{3}{2}}} e^{-\frac{x^2}{4\tau}}\, d\tau + e^{-x+t}$$