Solution to Practice Problems

Ex. 1 : $\alpha^2 = 1$, source $= 0$, $f(x) = \sin(2\pi x)$, $L = \frac{1}{2}$

$BC_s = 0$

We are in Case A and so we use the formula

$$u(x,t) = \sum_{n=1}^{+\infty} b_n e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right)$$

$$= \sum_{n=1}^{+\infty} b_n e^{-(n\pi \cdot 2)^2 t} \sin(2n\pi x)$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx = 4\int_0^{\frac{1}{2}} \sin(2\pi x)\sin(2n\pi x)\,dx$$

$$= \begin{cases} 1, & n = 1 \\ 0, & n \neq 1 \end{cases}$$

So $u(x,t) = e^{-4\pi^2 t}\sin(2\pi x)$.

Ex. 2 $\alpha^2 = 1$, source $= 0$, $f(x) = 0$, $L = 1$

$BC_s \neq 0$

We are in Case B and so we use a steady-state problem

$0 = U_{xx}$, $U(0) = 1$, $U(1) = 0$

$U(x) = C_1 + C_2 x \Rightarrow$ $U(0) = 1 = C_1 + C_2 \cdot 0 \Rightarrow C_1 = 1$

$U(1) = 0 = C_1 + C_2 \Rightarrow C_2 = -1$

$U(x) = 1 - x$

Now $u(x,t) = U(x) + v(x,t) = 1 - x + v(x,t)$

$v_t = v_{xx}$

$v(x,0) = u(x,0) - U(x) = x - 1$

$V(0,t)=0, \quad v(1,t)=0$

So we can solve v as in Case A.

$$v(x,t)=\sum_{n=1}^{+\infty} b_n e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right)$$

$$= \sum_{n=1}^{+\infty} b_n e^{-(n\pi)^2 t} \sin(n\pi x)$$

where $b_n = \frac{2}{L}\int_0^L v(x,0)\sin(n\pi x)\,dx$

$$= 2\int_0^1 (x-1)\sin(n\pi x)\,dx$$

$$= 2\left(-\frac{x\cos n\pi x}{n\pi} + \frac{\sin(n\pi x)}{(n\pi)^2} + \frac{\cos n\pi x}{n\pi}\right)\Bigg|_0^1$$

$$= -\frac{2}{n\pi}$$

$$v(x,t) = -\sum_{n=1}^{+\infty} \frac{2}{n\pi} e^{-(n\pi)^2 t} \sin(n\pi x)$$

$$u(x,t) = 1-x - \sum_{n=1}^{+\infty} \frac{2}{n\pi} e^{-(n\pi)^2 t} \sin(n\pi x)$$

Ex. 3. Source $=0$, BCs $\neq 0$, but Neuman

$\alpha^2=3, \quad L=1,$

we are in case B.2.

Step 1: Set $u(x,t)=U(x,t)+v(x,t)$

where $U(x,t)=ax^2+bx+Ct$

$$U_t = 3U_{xx} \iff c = 6a$$

$$U'(0)=1 \Rightarrow b=1$$

$$U'(1)=2 \Rightarrow 2a+b=2 \Rightarrow a=\frac{1}{2}$$

so $U(x,t) = \frac{x^2}{2} + x + 3t$

Step 2 : $v(x,t)$ solves

$$\begin{cases} v_t = 3v_{xx}, & 0<x<1 \\ v(x,0) = u(x,0) - U(x,0) = 1 - \frac{x^2}{2} - x \\ v_x(0,t) = 0, \quad v_x(1,t) = 0 \end{cases}$$

Now we use the formula in Case A

$$v(x,t) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 t} \cos\left(\frac{n\pi}{L}x\right)$$

where $a_n = \frac{2}{L}\int_0^L v(x,0)\cos\left(\frac{n\pi}{L}x\right)dx$

$$= 2\int_0^1 \left(1-\frac{x^2}{2}-x\right)\cos(n\pi x)\,dx$$

$$= 2\int_0^1 \left(1-\frac{x^2}{2}-x\right) d\,\frac{\sin n\pi x}{n\pi}$$

$$= -2\int_0^1 (-1-x)\frac{\sin(n\pi x)}{n\pi}dx$$

$$= 2\int_0^1 (x+1)\frac{\sin(n\pi x)}{n\pi}dx$$

$$= 2\,(x+1)\frac{(-\cos n\pi x)}{(n\pi)^2}\Big|_0^1 + 2\int_0^1 \frac{\cos n\pi x}{(n\pi)^2}dx$$

$$= 2\,\frac{-2\cos n\pi + 1}{(n\pi)^2}, \quad n \neq 0$$

$$a_0 = 2\int_0^1 \left(1-\frac{x^2}{2}-x\right)dx = \frac{2}{3}$$

$$u(x,t) = \frac{x^2}{2} + x + 3t + \frac{1}{3} + \sum_{n=1}^{+\infty} \frac{2(1-2\cos n\pi)}{(n\pi)^2} e^{-3(n\pi)^2 t} \cos(n\pi x)$$ ④

Ex. 4 Source $\neq 0$, $\alpha^2 = 1$, $L=1$, $BC = 0$

Case C

We use eigenfunction expansion

The boundary conditions is mixed.

Eigenvalues: $\left(\frac{2n-1}{2L}\pi\right)^2 = \lambda_n = \left(\frac{2n-1}{2}\pi\right)^2$

$$X_n = \sin\left(\frac{2n-1}{2L}\pi\right) = \sin\left(\frac{2n-1}{2}\pi x\right)$$

$$u(x,t) = \sum_{n=1}^{+\infty} u_n(t) \sin\left(\frac{2n-1}{2}\pi x\right)^2$$

$$u_t = \sum_{n=1}^{+\infty} u_n'(t) \sin\left(\frac{2n-1}{2}\pi x\right)^2$$

$$u_{xx} = \sum_{n=1}^{+\infty} \left(-\left(\frac{2n-1}{2}\right)^2 \pi^2 u_n(t)\right) \sin\left(\frac{2n-1}{2}\pi x\right)$$

$$e^{-t}\sin(2\pi x) = \sum_{n=1}^{+\infty} S_n(t) \sin\left(\frac{2n-1}{2}\pi x\right)$$

$$S_n(t) = \frac{2}{1}\int_0^1 e^{-t} \sin(2\pi x) \sin\left(\frac{2n-1}{2}\pi x\right)$$

$$= 2\int_0^1 \sin(\alpha x)\sin(\beta x)\,dx\; e^{-t} \qquad \alpha = 2\pi,\ \beta = (n-\tfrac{1}{2})\pi$$

$$= \int_0^1 \left(\cos(\alpha-\beta)x - \cos(\alpha+\beta)x\right)dx\; e^{-t}$$

$$= \left[\frac{\sin(\alpha-\beta)}{\alpha-\beta} - \frac{\sin(\alpha+\beta)}{\alpha+\beta}\right] e^{-t}$$

$$= \left[\frac{+\sin(n-\frac{1}{2})\pi}{(n-\frac{5}{2})\pi} - \frac{\sin(n+\frac{1}{2})\pi}{(n+\frac{3}{2})\pi}\right] e^{-t}$$

$u(x,0)=0 \Rightarrow u_n(0)=0$

So $u_t = u_{xx} + e^{-t}\sin(2\pi x)$

$$\Leftrightarrow \begin{cases} u_n'(t) = -\left(\frac{2n-1}{2}\right)^2\pi^2 u_n + S_n(t) \\ u_n(0)=0 \end{cases}$$

$$\left(e^{\left(\frac{2n-1}{2}\right)^2\pi^2 t} u_n(t)\right)' = e^{\left(\frac{2n-1}{2}\right)^2\pi^2 t} S_n(t)$$

$$u_n(t) = e^{-\left(\frac{2n-1}{2}\right)^2\pi^2 t}\int_0^t e^{\left(\frac{2n-1}{2}\right)^2\pi^2 t} S_n(t)\,dt$$

$$= e^{-\left(\frac{2n-1}{2}\right)^2\pi^2 t}\left[\frac{\sin(n-\frac{1}{2})\pi}{(n-\frac{5}{2})\pi} - \frac{\sin(n-\frac{1}{2})\pi}{(n+\frac{3}{2})\pi}\right]\int_0^t e^{\left((n-\frac{1}{2})^2\pi^2-1\right)t}\,dt$$

$$= e^{-\left(\frac{2n-1}{2}\right)^2\pi^2 t}\left[\frac{\sin(n-\frac{1}{2})\pi}{(n-\frac{5}{2})\pi} - \frac{\sin(n-\frac{1}{2})\pi}{(n+\frac{3}{2})\pi}\right]\frac{1}{(n-\frac{1}{2})^2\pi^2-1}\left[e^{\left((n-\frac{1}{2})^2\pi^2-1\right)t}-1\right]$$

Ex. 5. Source $=0$, $\alpha^2=1$, $L=1$, $BC\neq 0$. $\boxed{\text{Case B.1}}$

steady-state-problem:

$$\begin{cases} U_{xx} - U = 0 \\ U'(0)=1,\ U(1)=0 \end{cases}$$

$U = c_1 e^{x} + c_2 e^{-x}$. Since $U(1)=0 \Rightarrow U(x) = c\sinh(x-1)$

$U'(0)=1 \Rightarrow U(x) = \dfrac{\sinh(x-1)}{\cosh 1}$

set $u(x,t) = U(x) + v(x,t)$.

Then $v(x,t)$ satisfies

$$\begin{cases} v_t = v_{xx} - v \\ v(x,0) = u(x,0) - U(x) = -\dfrac{\sinh(x-1)}{\cosh 1} \\ v_x(0,t) = 0,\ v(1,t) = 0 \end{cases}$$

Eigenfunction expansion

$$v(x) = \sum_{n=1}^{+\infty} v_n(t) \cos\left(\left(\frac{2n-1}{2L}\right)\pi x\right)$$

$$= \sum_{n=1}^{+\infty} v_n(t) \cos\left(\left(n-\tfrac{1}{2}\right)\pi x\right)$$

$$v_t = \sum_{n=1}^{+\infty} v_n'(t) \cos\left(\left(n-\tfrac{1}{2}\right)\pi x\right)$$

$$v_{xx} = \sum_{n=1}^{+\infty} \left(-\left(n-\tfrac{1}{2}\right)^2\pi^2\right) v_n(t) \cos\left(\left(n-\tfrac{1}{2}\right)\pi x\right)$$

$$-v = \sum_{n=1}^{+\infty} -v_n(t) \cos\left(\left(n-\tfrac{1}{2}\right)\pi x\right)$$

So

$$v_t = v_{xx} - v$$

$$\Leftrightarrow \quad v_n'(t) = -\left(\left(n-\tfrac{1}{2}\right)^2\pi^2 + 1\right) v_n(t) \quad \Rightarrow \quad v_n(t) = v_n(0)\, e^{-\left(\left(n-\frac{1}{2}\right)^2\pi^2+1\right)t}$$

$$v_n(0) = \frac{2}{L}\int_0^L v(x,0) \cos\left(\left(n-\tfrac{1}{2}\right)\pi x\right) dx$$

$$= 2\int_0^1 \left(-\frac{1}{\cosh 1}\sinh(x-1)\right) \cos\left(\left(n-\tfrac{1}{2}\right)\pi x\right) dx$$

Ex. 6. source $\neq 0$, $BC \doteq 0$, $\alpha^2 = 1$, $L = 1$,

$\boxed{\text{Case C}}$

eigenfunction expansion

$$u(x,t) = \sum_{n=1}^{+\infty} u_n(t) \sin(n\pi x)$$

$$u_t = \sum_{n=1}^{+\infty} u_n'(t) \sin(n\pi x)$$

$$u_{xx} = \sum_{n=1}^{+\infty} (-n^2\pi^2)\, u_n(t) \sin(n\pi x)$$

$$S(x,t) = e^{-t} \sin(\pi x) = \sum_{n=1}^{+\infty} S_n(t) \sin(n\pi x)$$

$$\Rightarrow \quad S_n(t) = \begin{cases} 0, & \text{if } n \neq 1 \\ e^{-t}, & \text{if } n = 1 \end{cases}$$

$$u_t = u_{xx} + e^{-t} \sin \pi x$$

$$u_n'(t) = -(n^2\pi^2)\, u_n(t) + S_n(t)$$

$$\left(e^{n^2\pi^2 t} u_n\right)' = e^{n^2\pi^2 t} S_n(t)$$

$$e^{n^2\pi^2 t} u_n(t) - u_n(0) = \int_0^t e^{n^2\pi^2 \tau} S_n(\tau)\, d\tau$$

$$u_n(t) = u_n(0)\, e^{-n^2\pi^2 t} + \int_0^{\tau} e^{n^2\pi^2(\tau - t)} S_n(\tau)\, d\tau$$

$$u(x,0) = \sin(2\pi x)$$

$$\Rightarrow \quad u_n(0) = \frac{2}{L}\int_0^L u(x,0) \sin(n\pi x)\, dx = \begin{cases} 1, & n = 2 \\ 0, & n \neq 2 \end{cases}$$

Thus for $n \neq 1, 2$, we have $u_n(0) = 0$, $S_n(\tau) = 0 \Rightarrow$

$$u_n(t) = 0$$

For $n = 1$, $u_1(0) = 0$, so

$$u_1(t) = \int_0^t e^{\pi^2(\tau - t)} e^{-\tau} \, d\tau$$

$$= e^{-\pi^2 t} \frac{1}{\pi^2 - 1} \left(e^{(\pi^2 - 1)t} - 1 \right)$$

For $n = 2$, $u_2(0) = 1$, $S_2 = 0$, so

$$u_2(t) = e^{-4\pi^2 t}$$

Thus

$$u(x, t) = \frac{1}{\pi^2 - 1} \left(e^{-t} - e^{-\pi^2 t} \right) \sin(\pi x) + e^{-4\pi^2 t} \sin(2\pi x)$$

Ex. 7. source $\neq 0$, BC $\neq 0$, BC depends on t.

Case E.

$A(t) = t$, $B(t) = 2$

So $$U(x, t) = A(t) + \frac{B(t) - A(t)}{L} x$$

$$= t + \frac{2 - t}{1} x = t + (2 - t) x$$

$$U_t = 1 - x$$

set $u(x,t) = U(x,t) + v(x,t)$

Then

$$v_t = v_{xx} + x e^{-t} - U_t$$

$$= v_{xx} + x e^{-t} \, (1-x)$$

$$v(x,0) = u(x,0) - U(x,0) = 0 - 2x = -2x$$

$$v(0,t) = 0, \quad v(1,t) = 0.$$

$$\begin{cases} v_t = v_{xx} + x e^{-t} + x - 1 \\ v(x,0) = -2x \\ v(0,t) = 0, \; v(1,t) = 0 \end{cases}$$

Eigenfunction expansion

$$v(x,t) = \sum_{n=1}^{+\infty} v_n(t) \sin(n\pi x)$$

$$s(x,t) = x(e^{-t}+1) - 1 = \sum_{n=1}^{+\infty} S_n(t) \sin(n\pi x)$$

$$v(x,0) = -2x = \sum_{n=1}^{+\infty} v_n(0) \sin(n\pi x)$$

$$v_n'(t) = -n^2\pi^2 v_n(t) + S_n(t)$$

$$\Rightarrow \quad v_n(t) = e^{-n^2\pi^2 t} v_n(0) + \int_0^t e^{n^2\pi^2 \tau} S_n(\tau) \, d\tau \; e^{-n^2\pi^2 t}$$

$$v_n(0) = 2\int_0^1 (-2x)\sin(n\pi x)\,dx = (-4)\left[-x\frac{\cos n\pi x}{n\pi}\right]\Bigg|_0^1 = \frac{4\cos n\pi}{n\pi}$$

$$S_n(t) = 2\int_0^1 S(x,t)\sin(n\pi x)\,dx$$

$$= 2\int_0^1 \left(x(e^{-t}+1)-1\right)\sin(n\pi x)\,dx$$

$$= \cancel{2} - \frac{2\cos n\pi}{n\pi}(e^{-t}+1) + \frac{2(\cos n\pi - 1)}{n\pi}$$

$$= -\frac{2\cos n\pi}{n\pi}e^{-t} - \frac{2}{n\pi}$$

So

$$V_n(t) = \frac{4\cos n\pi}{n\pi}e^{-n^2\pi^2 t} + \int_0^t e^{n^2\pi^2\tau}\left(-\frac{2\cos n\pi}{n\pi}e^{-\tau} - \frac{2}{n\pi}\right)d\tau\, e^{-n^2\pi^2 t}$$

$$= \frac{4\cos n\pi}{n\pi}e^{-n^2\pi^2 t} - \frac{2\cos n\pi}{n\pi}\cdot\frac{1}{n^2\pi^2-1}\left[e^{(n^2\pi^2-1)t}-1\right]e^{-n^2\pi^2 t}$$

$$- \frac{2}{n\pi}\cdot\frac{1}{n^2\pi^2}\left[1 - e^{-n^2\pi^2 t}\right]$$

Ex. 8 Source $\neq 0$, BC $\neq 0$ Case D

BC is Neumann.

~~steady~~ : Solve $U = ax^2 + bx + ct$ with $U_t = 5U_{xx}$

$$U_x(0,t) = 1, \quad U_x(1,t) = 0$$

So $c = 10a$, $b = 1$, $2a + b = 0 \Rightarrow a = -\frac{1}{2}$

$$c = -5$$

So $U(x,t) = -\frac{1}{2}x^2 + x - 5t$

Set $u(x,t) = U(x,t) + v(x,t)$

$$\begin{cases} v_t = 5 v_{xx} + 1 \\ v(x,0) = u(x,0) - U(x,0) = \frac{1}{2}x^2 - x \\ v_x(0,t) = 0, \quad v_x(1,t) = 0 \end{cases}$$

Eigenfunction expansion

$$v(x,t) = \frac{a_0(t)}{2} + \sum_{n=1}^{+\infty} a_n(t) \cos(n\pi x)$$

$$s(x,t) = \frac{S_0(t)}{2} + \sum S_n(t) \cos(n\pi x)$$

$$\Rightarrow S_0(t) = 2, \quad S_n(t) = 0, \ n = 1, 2, \cdots$$

$$a_n'(t) = -5n^2\pi^2 a_n(t) + S_n(t)$$

$$a_n = a_n(0) e^{-5n^2\pi^2 t} + e^{-5n^2\pi^2 t} \int_0^t e^{5n^2\pi^2 \tau} S_n(\tau) \, d\tau$$

where $a_n(0) = 2 \int_0^1 v(x,0) \cos(n\pi x) \, dx$

$$= 2 \int_0^1 \left(\tfrac{1}{2}x^2 - x\right) \cos(n\pi x) \, dx$$

$$= 2 \int_0^1 \left(\tfrac{1}{2}x^2 - 1\right) d \frac{\sin n\pi x}{n\pi} = -2 \int_0^1 \frac{\sin n\pi x}{n\pi} (x) \, dx$$

$$= \frac{2}{(n\pi)^2}\int_0^1 x\, d(\cos n\pi x) = \frac{2\cos n\pi}{(n\pi)^2}$$

~~Thus~~ for $n=0$, $a_0(0) = 2\int_0^1 (\frac{1}{2}x^2 - 1)\,dx = 2\cdot(\frac{1}{6} - 1) = -\frac{5}{3}$

$$a_0(t) = -\frac{5}{3} + e^{\cancel{-5\pi}} \int_0^t 2\, d\tau = 2 - \frac{5}{3} = \frac{1}{3}$$

$n \neq 0$, $S_n(t) = 0$

$$a_n(t) = \frac{2\cos n\pi}{(n\pi)^2} e^{-5n^2\pi^2 t}$$

Ex. 9. Source $\neq 0$, BC $\neq 0$, Case D

step 1. Solve steady-state-problem

$$\begin{cases} U_{xx} = 0 \\ U(0) = 1, \quad U(1) = 2 \end{cases}$$

$U = c_1 + c_2 x \Rightarrow U(x) = 1 + x$.

step 2. $u(x,t) = U(x,t) + v(x,t)$

$$\begin{cases} v_t = v_{xx} + e^{-t} \\ v(x,0) = u(x,0) - U(x,0) = -1 - x \\ v(0,t) = 0, \quad v(1,t) = 0 \end{cases}$$

Eigenfunction expansion

$$v(x,t) = \sum v_n(t) \sin(n\pi x)$$

$$e^{-t} = \sum g_n(t) \sin(n\pi x)$$

$$v(x,0) = \sum v_n(0) \sin(n\pi x)$$

$$v_n' = -n^2\pi^2 v_n + g_n$$

$$v_n(t) = v_n(0)\, e^{-n^2\pi^2 t} + e^{-n^2\pi^2 t}\int_0^t e^{n^2\pi^2 \tau}\, g_n(\tau)\, d\tau$$

Now $\quad v_n(0) = 2\int_0^1 (-1-x)\sin(n\pi x)\,dx$

$$= \cancel{\frac{2(\cos n\pi - 1)}{n\pi}}$$

$$= 2\int_0^1 (x+1)\, d\left(\frac{\cos n\pi x}{n\pi}\right)$$

$$= \frac{4\cos n\pi - 2}{n\pi}$$

$$g_n(\tau) = 2\int_0^1 \sin n\pi x\, dx \; e^{-t} = 2\,\frac{1-\cos n\pi}{n\pi}\, e^{-t}$$

$$v_n(t) = \frac{4\cos n\pi - 2}{n\pi}\, e^{-n^2\pi^2 t} + e^{-n^2\pi^2 t}\,\frac{2(1-\cos n\pi)}{n\pi}\int_0^1 e^{(n^2\pi^2-1)\tau}\, d\tau$$

Ex. 10. Source $\neq 0$, but time-independent

BC $\neq 0$, Neuman

Case D.

Step 1. Solve steady-state-problem

$$\begin{cases} U_{xx} - U + x = 0 \\ U'(0) = 1, \quad U'(1) = 0 \end{cases}$$

$$U(x) = x + c_1 \cosh(x-1) + c_2 \sinh(x-1).$$

$U'(0) = 1 \Rightarrow \quad 1 + c_1 \sinh(-1) + c_2 \cosh 1 = 0$

$U'(1) = 0 \Rightarrow \quad 1 + c_2 = 0$

$$c_2 = -1, \quad c_1 = \frac{1 - \cosh 1}{\sinh 1}$$

$$U(x) = x + \frac{1 - \cosh 1}{\sinh 1} \cosh(x-1) - \sinh(x-1)$$

Step 2 let $u(x,t) = U(x) + v(x,t)$

$$\Rightarrow \begin{cases} v_t = v_{xx} - v \\ v(x,0) = u(x,0) - U(x) = -U(x) \\ v_x(0,t) = 0, \quad v_x(1,t) = 0 \end{cases}$$

Eigenfunction expansion

$$v(x,t) = \frac{\tilde{a}_0(t)}{2} + \sum_{n=1}^{+\infty} v_n(t) \cos(n\pi x)$$

$$v_t = \frac{v_0'}{2} + \sum v_n'(t) \cos(n\pi x)$$

$$v_{xx} = \sum (-n^2\pi^2) v_n(t) \cos n\pi x$$

So

$$v_n' = -n^2\pi^2 v_n - v_n$$

$$\Rightarrow v_n(t) = v_n(0)\, e^{-(n^2\pi^2+1)t}$$

where

$$v_n(0) = 2\int_0^1 v(x,0) \cos n\pi x \, dx$$

$$= 2\int_0^1 (-U(x)) \cos(n\pi x)\, dx.$$