Practice Problems for Laplace equation

Formula: We use the method of separation of Variables to solve

$$\Delta u = \begin{cases} u_{xx} + u_{yy}, & \text{in rectangular domains} \\ u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} & \text{in circular domains} \end{cases}$$

<u>step 1</u>: Use separated functions

$$u = X(x)Y(y) \quad (\text{for rectangular domains})$$
$$= R(r)\Theta(\theta) \quad (\text{for circular domains})$$

$\Rightarrow$ Find two ODEs

<u>Step 2</u>: Use homogeneous BCs to find (EVP) and (ODE)

<u>Step 3</u>: Solve (EVP) first, then (ODE)

<u>Step 4</u>: Sum up. use inhomogeneous BC to obtain the coefficients

Problems

(1). Solve

$$
\begin{cases}
u_{xx} + u_{yy} = 0, & 0 < x < \pi, \quad 0 < y < \pi \\
u(x,0) = u(x,\pi) = 0 & \\
u(0,y) = 1, & u(\pi, y) = y
\end{cases}
$$

(2) Solve

$$
\begin{cases}
u_{xx} + u_{yy} = 0, & 0 < x < \pi, \quad 0 < y < \pi \\
u_y = 0 \text{ for } y = 0 \text{ and for } y = \pi & \\
u = 0 \quad \text{for } x = 0 & \\
u = \frac{1}{2}(1 + \cos 2y) \quad \text{for } x = \pi &
\end{cases}
$$

(3) Solve

$$
\begin{cases}
u_{xx} + u_{yy} = 0, & 0 < x < a, \quad 0 < y < b \\
u_y = 0 \text{ for } y = 0 \text{ and for } y = \pi & \\
u_x = 0 \quad \text{for } x = 0 & \\
u_x = 1 + a \cos y \quad \text{for } x = \pi &
\end{cases}
$$

For which a, does the problem admit a sol'n?

(4) Solve

$$\begin{cases} u_{xx}+u_{yy}=0, & 0<x<a, \quad 0<y<+\infty \\ u(x,0)=x \\ u(0,y)=u_x(a,y)=0 \end{cases}$$

(5). Solve

$$\begin{cases} u_{rr}+\frac{1}{r}u_r+\frac{u_{\theta\theta}}{r^2}=0, & 0\le r<2, \quad 0\le\theta<2\pi \\ u(2,\theta)=1+3\sin 2\theta \end{cases}$$

(6) Solve

$$\begin{cases} u_{rr}+\frac{1}{r}u_r+u_{\theta\theta}=0, & 0\le r<1, \quad 0\le\theta<\pi \\ u(1,\theta)=\sin\theta \\ u(r,0)=0, \quad u_\theta(r,\pi)=0 \end{cases}$$

(7) Solve

$$\begin{cases} u_{rr}+\frac{1}{r}u_r+\frac{u_{\theta\theta}}{r^2}=0, & 1<r<2, \quad 0\le\theta<2\pi \\ u(1,\theta)=\sin\theta, \quad u(2,\theta)=\cos\theta \end{cases}$$

(8) Solve

$$\begin{cases} u_{rr} + \frac{1}{r} u_r + \frac{u_{\theta\theta}}{r^2} = 0, & 1 < r < +\infty, \ 0 \le \theta < 2\pi \\ u(1, \theta) = \cos^2\theta \\ u \ \text{is bounded} \end{cases}$$

(9) Solve

$$\begin{cases} u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0, & 1 < r < 2, \quad 0 \le \theta < \frac{\pi}{2} \\ u(1, \theta) = 1, \quad u(2, \theta) = \sin\theta \\ u(r, 0) = 0, \quad u(r, \frac{\pi}{2}) = 0 \end{cases}$$