Sturm–Liouville–Problem (S-L-P).

(1) Any eigenvalue problem

$$P(x)\, y'' + Q(x)\, y' + \lambda R(x)\, y = 0$$

can be reduced to

$$(p(x)\, y')' - q(x)\, y + \lambda\, r(x)\, y = 0$$

(2). $$\begin{cases} (p y')' - q y + \lambda r y = 0, & 0 < x < L \\ \alpha_1 y(0) + \alpha_2 y'(0) = 0, & \beta_1 y(L) + \beta_2 y'(L) = 0 \end{cases}$$

has infinitely many eigenvalues and eigenfunctions

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots, \qquad \lambda_n \longrightarrow +\infty$$

$$y_1, y_2, \ldots, y_n$$

and any function (smooth) can be expanded

$$f(x) = \sum_{n=1}^{+\infty} A_n\, y_n(x), \quad \text{where} \quad A_n = \frac{\int_0^L f(x)\, y_n}{\int_0^L r(x)\, y_n^2}$$

(3) If $\alpha_1 \alpha_2 \leq 0$, $\beta_1 \beta_2 \geq 0$, then $\lambda > 0$.

(4) For heat, wave or Laplace equation, we can use the method of separation of variables to solve (S-L-P) and find the solution

Problems

(1) Find the equations for the eigenvalue problem

$$\begin{cases} y'' + \lambda y = 0, & 0 < x < L \\ y'(0) - y(0) = 0, & y(L) = 0 \end{cases}$$

(2) Find the algebraic equations for the S-L-P

$$\begin{cases} y'' + \lambda y = 0, & 0 < x < L \\ y'(0) - y(0) = 0, & y'(L) + y(L) = 0 \end{cases}$$

(3). Transform the following eigenvalue problem into (S-L-P)

$$x y'' + 2y + \lambda x y = 0$$

(4) Transform the following eigenvalue problem to (S-L-P)

$$y'' + 2xy + \lambda y = 0$$

(5) Solve

$$\begin{cases} u_t = \alpha^2 u_{xx}, \quad 0 < x < L \\ u(x,0) = f(x) \\ u'(0,t) - u(0,t) = 0, \quad u(L,t) = 0 \end{cases}$$

(6) Solve

$$\begin{cases} u_{tt} = c^2 u_{xx}, & 0 < x < L \\ u(x,0) = f(x), & u_t(x,0) = g(x) \\ u'(0,t) - u(0,t) = 0, & u(L,t) = 0 \end{cases}$$

(7) Solve

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < a, \quad 0 < y < b \\ u_x(0,y) - u(0,y) = 0, & u(a,y) = 0 \\ u(x,0) = 0, & u(x,b) = g(x) \end{cases}$$

(8) Solve (S-L-P)

$$\begin{cases} (x^2 y')' + \lambda y = 0, & 0 < x < L \\ y'(0) - y(0) = 0, & y(L) = 0 \end{cases}$$

(9) Solve
$$\begin{cases} u_t = \alpha^2 u_{xx}, & 0 < x < L \\ u(x,0) = f(x) \\ u_x(0,t) - u(0,t) = A, & u(L,t) = B \end{cases}$$

(10) Solve
$$\begin{cases} u_{tt} = c^2 (x^2 u_x)_x, & 0 < x < L \\ u(x,0) = f(x), & u_t(x,0) = g(x) \\ u_x(0,t) - u(0,t) = 0, & u(L,t) = 0 \end{cases}$$