Solutions to Midterm Test One

1. $x^3(x-2\pi)^2 y'' + \sin x \, y' - y = 0$

$P(x) = x^3(x-2\pi)^2, \quad Q(x) = \sin x, \quad R(x) = -1$

1 (a) $P(x_0) = 0 \Rightarrow x_0 = 0$ is a singular point

Compute $(x-x_0)\dfrac{Q(x)}{P(x)} = x \cdot \dfrac{\sin x}{x^3(x-2\pi)^2} = \dfrac{\sin x}{x^2(x-2\pi)^2} \to \pm\infty$

as $x \to \pm 0$

So $x_0 = 0$ is an irregular singular point

(b) $P(\pi) \neq 0 \Rightarrow x_0 = \pi$ is an ordinary point

(c) $P(2\pi) = 0 \Rightarrow x_0 = 2\pi$ is a singular point

Compute $(x-x_0)\dfrac{Q(x)}{P(x)} = (x-2\pi)\dfrac{\sin x}{x^3(x-2\pi)^2}$

$= \dfrac{\sin x}{x^3(x-2\pi)} \to \dfrac{0}{0}$ type as $x \to 2\pi$

L'Hospital's rule $\to \dfrac{\cos x}{x^3 + 3x^2(x-2\pi)} \to \dfrac{\cos 2\pi}{(2\pi)^3}$

$\to \dfrac{1}{(2\pi)^3}$

$(x-x_0)^2 \dfrac{R(x)}{P(x)} = (x-2\pi)^2 \dfrac{(-1)}{x^3(x-2\pi)^2} = -\dfrac{1}{x^3} \to -\dfrac{1}{(2\pi)^3}$ as $x \to 2\pi$

So 2π is a regular singular point

(b). $x_0 = \pi$ is an ordinary point

$P(z) = z^3 (z-2\pi)^2 = 0 \Rightarrow z=0$ or $z=2\pi$

distance between π and $z=0$ or $z=2\pi$ is π

so the lower bound for the radius of convergence is π

$x_0 = 2\pi$ is a regular singular point with

$p_0 = \frac{1}{(2\pi)^3}, \quad q_0 = -\frac{1}{(2\pi)^3}$

So the equation for indicial roots is

$$r(r-1) + p_0 r + q_0 = r(r-1) + \frac{1}{2\pi^3} r - \frac{1}{(2\pi)^3} = 0$$

$$\Rightarrow (r-1)\left(r + \frac{1}{(2\pi)^3}\right) = 0$$

$$\Rightarrow r_1 = 1, \quad r_2 = -\frac{1}{(2\pi)^3}$$

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2. $3xy'' + 4y' + y = 0$

(a) The base point is $x_0 = 1$ so we set

$t = x - x_0 = x - 1 \Longleftrightarrow x = 1 + t$

The ODE becomes

$$3(1+t)y'' + 4y' + y = 3ty'' + 3y'' + 4y' + y = 0$$

Now let

$$y = \sum_{n=0}^{+\infty} a_n (x-x_0)^n = \sum_{n=0}^{+\infty} a_n t^n$$

$$y' = \sum_{n=0}^{+\infty} n a_n t^{n-1} = \sum_{n=1}^{+\infty} n a_n t^{n-1}$$

$$y'' = \sum_{n=0}^{+\infty} n(n-1) a_n t^{n-2} = \sum_{n=2}^{+\infty} n(n-1) a_n t^{n-2}$$

$$3t\, y'' = \sum_{n=2}^{+\infty} 3n(n-1) a_n t^{n-1} = \sum_{m=1}^{+\infty} 3(m+1) m\, a_{m+1} t^m$$

$$3y'' = \sum_{n=2}^{+\infty} 3n(n-1) a_n t^{n-2} = \sum_{m=0}^{+\infty} 3(m+2)(m+1) a_{m+2} t^m$$

$$4y' = \sum_{n=1}^{+\infty} 4n a_n t^{n-1} = \sum_{m=0}^{+\infty} 4(m+1) a_{m+1} t^m$$

$$y = \sum_{n=0}^{+\infty} a_n t^n = \sum_{m=0}^{+\infty} a_m t^m$$

so

$$\sum_{m=1}^{+\infty} 3(m+1) m\, a_{m+1} t^m + \sum_{m=0}^{+\infty} 3(m+2)(m+1) a_{m+2} t^m$$
$$+ \sum_{m=0}^{+\infty} 4(m+1) a_{m+1} t^m + \sum_{m=0}^{+\infty} a_m t^m = 0$$

$m=0$ term:

$$3 \cdot 2 \cdot 1\, a_2 + 4a_1 + a_0 = 0$$

$$a_2 = -\frac{2}{3} a_1 - \frac{1}{6} a_0$$

$m \geq 1$ term:

$$\left(3(m+1)m + 4(m+1)\right) a_{m+1} + 3(m+2)(m+1) a_{m+2} + a_m = 0$$

$$a_{m+2} = -\frac{3m+4}{3(m+2)} a_{m+1} - \frac{a_m}{3(m+2)(m+1)}$$

So

$$a_3 = -\frac{7}{9}a_2 - \frac{1}{18}a_1$$

$$= -\frac{7}{9}\left(-\frac{2}{3}a_1 - \frac{1}{6}a_0\right) - \frac{1}{18}a_1$$

$$= \left(\frac{14}{27} - \frac{1}{18}\right)a_1 + \frac{7}{54}a_0$$

$$= \frac{25}{54}a_1 + \frac{7}{54}a_0$$

Thus

$$y = a_0 + a_1 t - \frac{2}{3}a_1 t^2 - \frac{1}{6}a_0 t^2 + \frac{25}{54}a_1 t^3 + \frac{7}{54}a_0 t^3 + \cdots$$

$$= a_0\left(1 - \frac{1}{6}(x-1)^2 + \frac{7}{54}(x-1)^3\right)$$

$$+ a_1\left((x-1) - \frac{2}{3}(x-1)^2 + \frac{25}{54}(x-1)^3\right)$$

$$+ \cdots$$

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2 (b) base point $x_0 = 0$,

$$p_0 = \lim_{x\to 0} x \cdot \frac{4}{3x} = \frac{4}{3}, \quad q_0 = \lim_{x\to 0} x^2 \frac{1}{3x} = 0$$

Equation for indicial root

$$r(r-1) + \frac{4}{3}r + 0 = 0 \Rightarrow r\left(r + \frac{1}{3}\right) = 0$$

$$r_1 = -\frac{1}{3}, \quad r_2 = 0.$$

Since $x^{\frac{1}{3}} y(x) \to 1 \Rightarrow$ we choose $r = -\frac{1}{3}$

$$y = \sum_{n=0}^{+\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{+\infty} (n+r) a_n x^{n+r}$$

$$y'' = \sum_{n=0}^{+\infty} (n+r)(n+r-1) a_n x^{n+r-1}$$

$3x y'' + 4y' + y$

$$= \sum_{n=0}^{+\infty} \Big((n+r)\big(3(n+r-1)+4\big)\Big) a_n x^{n+r-1} + \sum_{n=0}^{+\infty} (n+r) a_n x^{n+r}$$

$$= \Big(r\big(3(r-1)+4\big)\Big) a_0 + \sum_{n=1}^{+\infty} (n+r)\big(3(n+\tfrac{1}{3}-1)+4\big) a_n x^{n+r-1} + \sum_{n=0}^{+\infty} (n+r) a_n x^{n+r}$$

$$= r(3r+1) a_0 x^{r-1} + \sum_{n=0}^{+\infty} (n+1+r)\big(3(n+1)\big) a_{n+1} x^{n+r} + \sum_{n=0}^{+\infty} (n+r) a_n x^{n+r}$$

r-term: $r(3r+1)a_0 = 0 \Rightarrow a_0$ free

n+r-term: $(n+1+r)3(n+1)a_{n+1} + (n+r)a_n = 0$

$$a_{n+1} = -\frac{n+r}{3(n+1+r)(n+1)} a_n$$

$$a_1 = -\frac{r}{3(1+r)} a_0 = -\frac{-\frac{1}{3}}{3(1-\frac{1}{3})} a_0 = \frac{1}{2} a_0$$

so $y = x^{-\frac{1}{3}}(a_0 + a_1 x + \cdots) = x^{-\frac{1}{3}}(a_0 + \frac{1}{2} a_0 x + \cdots)$

$x^{\frac{1}{3}} y(x) \to 1 \Rightarrow a_0 = 1$, $\quad y = x^{-\frac{1}{3}}(1 + \frac{1}{2}x + \cdots)$

3. $L=\frac{3}{2}$, $f(x)=\cos(\pi x)$, $u_x(0,t)=0$, $u(L,t)=0$ (6)

mixed BC.

Step 1 $u=X(x)T(t) \Rightarrow$

$$X(x)T'(t)=\left(X''(x)+X(x)\right)T(t)$$

$$\Rightarrow \frac{T'(t)}{T(t)}=\frac{X''(x)+X(x)}{X(r)}=-\lambda$$

$$\Rightarrow \quad T'+\lambda T=0$$
$$X''+(\lambda+1)X=0$$

Step 2. BCs $\Rightarrow X'(0)=0,\ X(\frac{3}{2})=0$

$$(EVP)\begin{cases} X''+(\lambda+1)X=0 \\ X'(0)=0,\ X(\frac{3}{2})=0 \end{cases} \qquad T'+\lambda T=0 \Rightarrow (ODE)$$

Step 3. Solve (EVP): let $\rho=\lambda+1$

$$\begin{cases} X''+\rho X=0 \\ X'(0)=0,\ X(\frac{3}{2})=0 \end{cases}$$

Case 1 $\rho<0$, $\rho=-\gamma^2<0$

$X=c_1e^{-\gamma x}+c_2e^{\gamma x}$, $X'(0)=0 \Rightarrow -c_1+c_2=0$

$X(\frac{3}{2})=0 \Rightarrow c_1e^{-\frac{3}{2}\gamma}+c_2e^{\frac{3}{2}\gamma}=0$

$\Rightarrow c_1=c_2=0$

Case 2. $p=0$.

$$X = c_1 + c_2 x$$

$$X'(0)=0 \Rightarrow c_2=0$$

$$X(\tfrac{3}{2})=0 \Rightarrow c_1=0$$

Case 3 $\rho=\beta^2>0$

$$X = c_1\cos\beta x + c_2\sin\beta x$$

$$X'(0)=0 \Rightarrow c_2=0$$

$$X(\tfrac{3}{2})=0 \Rightarrow c_1\cos\beta\cdot\tfrac{3}{2}=0 \Rightarrow \cos(\tfrac{3}{2}\beta)=0$$

$$\frac{3}{2}\beta = \frac{2n-1}{2}\pi, \quad n=1,2,\cdots$$

$$\beta = \frac{(2n-1)}{3}\pi, \quad n=1,2,\cdots$$

$$\rho = \frac{(2n-1)^2}{9}\pi^2, \qquad X = c_1\cos\beta x = c_1\cos\frac{2n-1}{3}\pi x$$

$$\lambda = \rho-1 = \frac{(2n-1)^2}{9}\pi^2-1$$

$$T' \cancel{+} \lambda T=0 \Rightarrow T = e^{-\lambda t} = e^{-\left(\frac{(2n-1)^2}{9}\pi^2-1\right)t}$$

Step 4. Sum up

$$u(x,t) = \sum_{n=1}^{+\infty} b_n X_n(x)T_n(t) = \sum_{n=1}^{+\infty} b_n\cos\left(\frac{2n-1}{3}\pi x\right)e^{-\left(\frac{(2n-1)^2}{9}\pi^2-1\right)t}$$

$$u(x,0)=\cos\pi x$$