Final Review

There are Four Block materials for this course

I. Series expansions of 2nd order ODE

II. Fourier Series (full, sine, cosine)

III. Method of Separation of Variables for heat equation, wave equation and Laplace equation

IV. Finite difference method for heat equation.

In the following I give a summary on each Block

Block I Series expansions of 2nd order ODE

$$P(x)\, y'' + Q(x)\, y' + R(x)\, y = 0$$

- $P(x_0) \neq 0 \Rightarrow x_0$ is an ordinary point
- $P(x_0) = 0 \Rightarrow x_0$ is a singular point
- $P(x_0)=0$, $(x-x_0)\dfrac{Q(x)}{P(x)} \to p$, $(x-x_0)^2 \dfrac{R(x)}{P(x)} \to q \Rightarrow x_0$ is a regular singular point

Case 1. Ordinary point

Let $t = x - x_0 \Rightarrow x = x_0 + t$ and

$$y = \sum_{n=0}^{+\infty} a_n (x - x_0)^n = \sum_{n=1}^{+\infty} a_n t^n$$

Expand $P(x)$, $Q(x)$, $R(x)$ at x_0

Case 2. Regular singular point

$$y = \sum_{n=0}^{+\infty} a_n (x - x_0)^{n+r}$$

where r is one of the indicial roots of

$$r(r-1) + pr + q = 0$$

provided $0 < r_2 - r_1 \neq$ integer.

For $r = r_2$, it always works

Block II. Fourier series

Case 1. Full Fourier series of periodic functions with period $2L$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x$$

where
$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x \, dx$$
$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi}{L} x \, dx$$

Case 2: Fourier Sine series

$$f(x) = \sum_{n=1}^{+\infty} a_n \sin \frac{n\pi}{L} x$$

$$a_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi}{L} x \, dx$$

Case 3: Fourier cosine series

$$f(x) = \sum_{n=1}^{+\infty} a_n \cos \frac{n\pi}{L} x + \frac{a_0}{2}$$

$$a_n = \frac{2}{L}\int_0^L f(x) \cos \frac{n\pi}{L} x \, dx, \quad n = 0, 1, 2, \cdots$$

Convergence of Fourier series:

Fourier series $\longrightarrow \frac{1}{2}\left[f(x+) + f(x-) \right]$

where f is the periodic extension of $f(x)$

Block III. Method of Separation of Variables

III.1. Basic steps

Step 1: Look for separate sol'ns $\Rightarrow$ obtain two (ODE)s

- heat eqn, wave eqn, $u = X(x)T(t)$
- Laplace eqn
  - rectangular domain, $u = X(x)Y(y)$
  - circular domain, $u = R(r)\Theta(\theta)$

Step 2: homogeneous BCs $\Rightarrow$ identify (EVP) and (ODE)

Step 3: Solve (EVP) first. Then (ODE)

Step 4. Sum up $\Rightarrow$ use inhomogeneous ICs or BCs to obtain Fourier coefficients

III.2 Inhomogeneous BCs and Sources

A. 0 BCs, 0 Source $\Rightarrow$ III.1

B. $\neq 0$ BCs, 0 source.

B.1 Use steady-state-solution $U(x)$ and let
$$u = U(x) + v(x,t)$$

B.2 If it is NBC, then let $U(x,t)$ be
$$U(x,t) = ax^2 + bx + ct$$
and U satisfy $U_t = \alpha^2 U_{xx}$ + BC

C. $= 0$ BCs, $\neq 0$ Source

$$\left.\begin{aligned} u(x,t) &= \sum u_n(t)\, X_n(x) \\ s(x,t) &= \sum s_n(t)\, X_n(x) \\ f(x) &= \sum a_n X_n(x) \end{aligned}\right\} \xrightarrow[\text{PDE}]{\text{throw into}} \text{infinitely many ODEs}$$

D. $\neq 0$ BCs, $\neq 0$ Source

Use B to get rid of BCs and use C.

III.3 Wave equation

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = f(x), \quad u_t(x,0) = g(x) \end{cases}$$

d'Alembert's Formula

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

III.4 (S-L-P):

$$P(x)y'' + Q(x)y' + \lambda R(x)y = 0$$

can be reduced to

$$(py')' - qy + \lambda r y = 0.$$

Block

IV : Finite difference method:

Δt, Δx

$$u_n^k = u(n\Delta x, k\Delta t)$$

For heat equation

$$u_n^{k+1} = u_n^k + \frac{\alpha^2 \Delta t}{\Delta x^2}\left(u_{n+1}^k - 2u_n^k + u_{n-1}^k\right)$$

For wave equation

$$\begin{cases} u_n^{k+1} = 2u_n^k - u_{n-1}^k + \dfrac{c^2\Delta t^2}{\Delta x^2}\left(u_{n+1}^k - 2u_n^k + u_{n-1}^k\right) \\ u_n^{-1} = u_n^1 - 2(\Delta t)\, g(n\Delta x) \end{cases}$$

ICs: $u_n^0 = f(n\Delta x)$

For BCs; if it is NBC, add a ghost column next to it

Final Remark: Several trick things

(1) The solution to

$$ay'' + by' + cy = 0$$

is $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$, $ar^2 + br + c = 0$

(2) The solution to

$$ax^2 y'' + bxy' + cy = 0$$

is $y = x^r$, $ar(r-1) + br + c = 0$

$r_1 \neq r_2$, $y = c_1 x^{r_1} + c_2 x^{r_2}$

$r_1 = r_2$, $y = c_1 x^r + c_2 x^r \ln x$

$r_1 = \lambda + i\mu$, $y = c_1 x^\lambda \cos(\mu \ln x) + c_2 x^\lambda \sin(\mu \ln x)$

(3) The sol'n to

$$y'' - \beta^2 y = 0$$

Can be written as: $y = c_1 e^{\beta x} + c_2 e^{-\beta x}$

or $y = c_1 \cosh \beta x + c_2 \sinh \beta x$

or $y = c_1 \cosh \beta(x - x_0) + c_2 \sinh \beta(x - x_0)$