SIGN-CHANGING SOLUTIONS FOR SUPERCRITICAL ELLIPTIC PROBLEMS IN DOMAINS WITH SMALL HOLES

E. N. DANCER AND JUNCHENG WEI

ABSTRACT. Let $\mathcal{D}$ be a bounded, smooth domain in $\mathbb{R}^N$, $N \geq 3$, $P \in \mathcal{D}$. We consider the boundary value problem in $\Omega = \mathcal{D} \backslash B_\delta(P)$,

$$\Delta u + |u|^{p-1}u = 0 \quad \text{ in } \Omega,$$

$$u = 0 \quad \text{on } \partial\Omega$$

with p supercritical, namely $p > \frac{N+2}{N-2}$. Given any positive integer m, we find a sequence

$$p_1 < p_2 < p_3 < \cdots, \quad \text{with } \lim_{k \to +\infty} p_k = +\infty,$$

such that if p is given, with $p \neq p_j$ for all j, then for all $\delta > 0$ sufficiently small, this problem has a sign-changing solution which has exactly $m+1$ nodal domains.

1. Introduction and statement of the main results

In this paper, we consider *sign-changing* solutions for the following nonlinear elliptic equation

$$\Delta u + |u|^{p-1}u = 0 \quad \text{ in } \Omega\,,\ u = 0 \quad \text{on } \partial\Omega\,, \tag{1.1}$$

where Ω is a domain with smooth boundary in $\mathbb{R}^N$ $(N \geq 3)$ and $p > 1$.

In the last twenty years there has been a great amount of activity in the study of positive solutions of (1.1). A main characteristic of (1.1) is the role played by the critical exponent $p = \frac{N+2}{N-2}$ in the solvability question. When $1 < p < \frac{N+2}{N-2}$, a positive solution can be found as an extremal for the best constant in the compact embedding of $H_0^1(\Omega)$ into $L^{p+1}(\Omega)$, namely a minimizer of the variational problem

$$\inf_{u \in H_0^1(\Omega)\backslash\{0\}} \frac{\int_\Omega |\nabla u|^2}{\left(\int_\Omega |u|^{p+1}\right)^{\frac{2}{p+1}}}\,.$$

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When $p \geq \frac{N+2}{N-2}$, this minimization procedure fails, so does existence in general: Pohozaev [28] discovered that no solution exists in this case if the domain is strictly star-shaped. In the classical paper [5], Brezis and Nirenberg considered the critical case $p = \frac{N+2}{N-2}$ and proved that compactness, and hence solvability, is restored by the addition of a suitable linear term. Coron [7] used a variational approach to prove that (1.1) is solvable for $p = \frac{N+2}{N-2}$ if Ω exhibits *a small hole*. Rey [32] established existence of multiple positive solutions if Ω exhibits several small holes. Bahri and Coron [1] established that solvability holds for $p = \frac{N+2}{N-2}$ whenever Ω has a non-trivial topology. Passaseo showed [26] that non-trivial topology of the domain is insufficient for the solvability in the supercritical case $p > \frac{N+2}{N-2}$. If p is supercritical but close to critical, bubbling positive solutions are found in domains with small holes, see [9, 10, 14, 16, 18]. In the purely supercritical case, there are very few results. We mention a recent result of del Pino and the second author [12] in which they considered (1.1) on Coron's domain where Ω has a small hole. They proved that there exist resonant sequences $\frac{N+2}{N-2} < p_1 < p_2 < ... < p_j < ...$ such that problem (1.1) has a positive solution as long as $p \neq p_j$.

In contrast to the achievements on positive solutions, very little progress has been made concerning the existence of *sign changing* solutions. When $p < \frac{N+2}{N-2}$, the existence of a least-energy *sign-changing* solution can be shown by using a variational method, see [2]. In the critical exponent case, i.e., $p = \frac{N+2}{N-2}$, Pohozaev's identity also gives nonexistence in star-shaped domains. Clapp and Weth [6] showed that least energy solutions still exist if the domain has some symmetries and nontrivial topology. On the other hand, in the case $p = \frac{N+2}{N-2} - \epsilon$, the existence of N pairs of sign changing solutions has been proved in Bartsch-Micheletti-Pistoia [4], and the existence of a *bubble-tower* sign-changing solution is considered in Pistoia-Weth [27]. When Ω has a small hole and $p = \frac{N+2}{N-2}$, the existence of many sign-changing solutions has been proved in Musso-Pistoia [19]. However, as far as the authors know, there are no results on the existence of nodal solutions in the supercritical case.

In this paper we consider Problem (1.1) for exponents p above the critical one in a Coron's type domain: one exhibiting a small hole. Thus we assume in what follows that the domain Ω has the form

$$\Omega = \mathcal{D} \setminus B_\delta(Q) \tag{1.2}$$

where $\mathcal{D}$ is a bounded domain with smooth boundary, $B_\delta(Q) \subset \mathcal{D}$ and $\delta > 0$ is to be taken small. Then we consider the problem of finding classical solutions of

$$\Delta u + |u|^{p-1}u = 0 \quad \text{in } \mathcal{D} \setminus B_\delta(Q)\,, \quad u = 0 \quad \text{on } \partial\mathcal{D} \cup \partial B_\delta(Q)\,. \tag{1.3}$$

In this paper, we establish results analogous to [12] for the existence of *nodal* solutions. Our main result states that there is a sequence of *resonant exponents*,

$$\frac{N+2}{N-2} < p_1 < p_2 < p_3 < \cdots\,, \quad \text{with } \lim_{k\to+\infty} p_k = +\infty \tag{1.4}$$

such that if p is supercritical and differs from all elements of this sequence then Problem (1.3) is solvable whenever δ is sufficiently small.

Theorem 1. *Suppose that m is a positive integer. There exists a sequence of the form (1.4) such that if $p > \frac{N+2}{N-2}$ and $p \neq p_j$ for all j, then there is a $\delta_0 > 0$ such that for any $\delta < \delta_0$, Problem (1.3) possesses at least one nodal solution which changes sign exactly $m-$times.*

In the background of our result is the problem

$$\Delta w + |w|^{p-1}w = 0 \quad \text{in } \mathbb{R}^N \setminus \bar{B}_1(0)\,, \tag{1.5}$$

$$w = 0 \ \text{ on } \partial B_1(0)\,, \quad \limsup_{|x|\to+\infty} |x|^{2-N} w(x) < +\infty\,. \tag{1.6}$$

The existence of a positive solution for (1.5)-(1.6) was given in [22]. In Section 2, we extend Ni's positive solution to a sign-changing solution. Namely, we show that given any positive integer m, problem (1.5)-(1.6) admits a radially symmetric solution $w = w(r)$ which changes sign exactly m times. We also show that this solution is unique and non-degenerate. The solutions we find have a profile similar to w suitably rescaled. More precisely, let us observe that

$$w_\delta(x) = \delta^{-\frac{2}{p-1}} w(\delta^{-1}|x-Q|) \tag{1.7}$$

solves uniquely the same problem with $B_1(0)$ replaced with $B_\delta(Q)$.

The idea is to consider w_δ as a first approximation for a solution of Problem (1.1), provided that $\delta > 0$ is chosen small enough. What we shall prove is that an actual solution of the problem, which differs little from w_δ does exist. To this end, it is necessary to show that the linearized operator around w is invertible. Here our approach departs from [12]: for positive solutions to (1.5)-(1.6), it is easy to show that $p \to w$ is analytic and hence the associated first eigenvalue is analytic. Here the map $p \to w$ is not analytic since w has interior zeroes (which change as p varies). We develop some new ideas to deal with this difficulty.

It is easy to see that the result of Theorem 1 remains valid, with only minor modifications in the proof, for a problem of the form

$$\Delta u + f(x,u) = 0 \quad \text{in } \mathcal{D} \setminus B_\delta(Q)\,,$$

$$u = 0 \quad \text{on } \partial\mathcal{D} \cup \partial B_\delta(Q)\,.$$

where

$$\lim_{u\to 0} \frac{f(x,u)}{u} = \lambda < \lambda_1(\mathcal{D}),\ \lim_{u\to 0} \frac{f(x,u)}{|u|^{p-1}u} = C \tag{1.8}$$

where $\lambda_1(\mathcal{D})$ is the first eigenvalue of the Laplacian in $\mathcal{D}$ and $C > 0$. We can also get existence of multiple solutions in a domain of the form

$$\mathcal{D} \setminus \bigcup_{j=1}^{m} B_\delta(Q_i)\,.$$

An interesting question remains considering a non-spherical hole or, more generally, finding conditions which ensure solvability of rather general supercritical problems. A method beyond variational arguments or singular perturbations would be needed. (Note that by using the Kelvin transform and domain variation techniques, we can obtain some slightly weaker results for slightly non-spherical holes.)

2. The study of (1.5)-(1.6)

In this section, we study the existence of nodal solutions for (1.5)-(1.6). Our main result is the following theorem

Theorem 2. *Let m be a positive integer. Problem (1.5)-(1.6) admits a unique radially symmetric sign-changing solution $w = w(r)$ (with $w(r) > 0$ for r large).*

We first prove *existence*: by Kelvin's transformation $\mathtt{w}(r) = r^{2-N} w(\frac{1}{r})$, the equations (1.5)-(1.6) are equivalent to the following problem in a ball B_1

$$\begin{cases} \Delta \mathtt{w} + r^\alpha |\mathtt{w}|^{p-1}\mathtt{w} = 0 & \text{in } B_1, \\ \mathtt{w} = 0 & \text{on } \partial B_1 \end{cases} \tag{2.1}$$

where $\alpha = p(N-2) - (N+2) > 0$ and hence $p < \frac{N+2+2\alpha}{N-2}$. The existence of sign-changing solutions to (2.1) with exactly m zeroes has been proved by Naito [21]. Here we present a proof for the sake of completeness.

First we need the following radial lemma proved in [22].

Lemma 2.1. *Assume that* $u \in H^1_{0,r} = \{H^1_0(B_1), u = u(r)\}$. *Then we have*

$$|u(r)| \leq \frac{C}{r^{(N-2)/2}} \|\nabla u\|_{L^2(B_1)} \tag{2.2}$$

As a consequence, for $p < \frac{N+2+2\alpha}{N-2}$, *the map* $u \to r^\alpha |u|^{p+1}$ *is a compact map from* $H^1_0(B_1)$ *to* $L^1(B_1)$.

Let

$$E(u) = \int_{B_1} \frac{1}{2}|\nabla u|^2 - \int_{B_1} \frac{1}{p+1} r^\alpha |u|^{p+1},$$

$$\mathcal{N} = \{u \in H^1_{0,r}(B_1) | \int_{B_1} |\nabla u|^2 = \int_{B_1} r^\alpha |u|^{p+1}\}. \tag{2.3}$$

Let $\Gamma_m \subset H^1_{0,r}$ be the set of all functions $u \in H^1_{0,r}$ such that there exists radii $0 = r_0 < r_1 < ... < r_m < r_{m+1} = 1$ such that $u(r_1) = ... = u(r_m)$ and $u \cdot 1_{\{r_j \leq |x| \leq r_{j+1}\}} \in \mathcal{N}$ for $j = 0, ..., m$. By the compactness lemma 2.1, we have the following existence result which goes back to Nehari [25] in the one-dimensional case. (Later it was generalized to radial functions in higher space dimensions, see [30, 31, 3].)

Theorem 3. *Let* c_m

$$c_m := \inf_{u \in \Gamma_m} E(u). \tag{2.4}$$

Then c_m *can be achieved. Moreover, if* $\mathtt{w} \in \Gamma_m$ *satisfies* $E(\mathtt{w}) = c_m$ *and*

$$(-1)^j \mathtt{w}(x) \geq 0 \;\; \textit{for } r_j \leq |x| \leq r_{j+1}, \;\; j = 0, ..., m \qquad \textit{or}$$

$$(-1)^j \mathtt{w}(x) \leq 0 \;\; \textit{for } r_j \leq |x| \leq r_{j+1}, \;\; j = 0, ..., m,$$

then $\mathtt{w}$ *is a radial solution of (2.1) with precisely* m *interior zeros.*

Proof: We first use the radial lemma 2.1 to show that if $u \in \Gamma_m$ and $\int_{B_1} r^\alpha |u|^{p+1} \leq C_1$, then $r_1 \geq C$. The rest of the proof will be standard and thus omitted.

Since $u \cdot 1_{r \leq r_1} \in \mathcal{N}$, we deduce that

$$\int_{B_{r_1}} |\nabla u|^2 = \int_{B_{r_1}} r^\alpha |u|^{p+1} \tag{2.5}$$

Rescaling by $r = r_1 t$, we obtain

$$r_1^{-2-\alpha} \int_{B_1} |\nabla u|^2 = \int_{B_1} t^\alpha |u|^{p+1}$$

and hence by Lemma 2.1, we obtain

$$r_1^{-2-\alpha} \leq C(\int_{B_1} t^\alpha |u|^{p+1})^{\frac{p-1}{p+1}} \tag{2.6}$$

which implies

$$r_1^{-2-\alpha+(N+\alpha)\frac{p-1}{p+1}} \le C\Big(\int_{B_{r_0}} r^\alpha |u|^{p+1}\Big)^{\frac{p-1}{p+1}} \le C \tag{2.7}$$

Since $p < \frac{N+2+2\alpha}{N-2}$, we obtain that $r_1 \ge C$. □

Next we prove the *uniqueness*: Suppose that $\mathtt{w}_1$ and $\mathtt{w}_2$ are two radial solutions to (2.1) such that $\mathtt{w}_1(0) > 0, \mathtt{w}_2(0) > 0$ and $\mathtt{w}_1, \mathtt{w}_2$ have exactly m interior zeroes. Without loss of generality, we may assume that $\mathtt{w}_1(0) > \mathtt{w}_2(0)$. Then by rescaling and uniqueness of ODE boundary value problems, we have $\mathtt{w}_2(r) = \gamma^{\frac{2-\alpha}{p-1}}\mathtt{w}_1(\gamma r)$, where $\gamma = (\frac{\mathtt{w}_2(0)}{\mathtt{w}_1(0)})^{\frac{p+1}{2-\alpha}} < 1$. Since $\mathtt{w}_2(1) = 0$, we have $\mathtt{w}_1(\gamma) = 0$. Note that $\gamma < 1$ and hence $\mathtt{w}_1$ has at most $m-1$ zeroes in $(0,1)$. This implies that $\mathtt{w}_2$ has at most $m-1$ zeroes in $(0,1)$, a contradiction to our assumption. Thus $\mathtt{w}_1(0) = \mathtt{w}_2(0)$ and $\mathtt{w}_1 = \mathtt{w}_2$. □

3. The operator $\Delta + pw^{p-1}$ on $\mathbb{R}^N \setminus B_1(0)$

The purpose of this section is to establish an invertibility theory for the linearized operator associated to w. We consider the problem

$$\Delta\phi + p|w|^{p-1}\phi = h \quad \text{ in } \mathbb{R}^N \setminus \bar{B}_1(0)\,, \tag{3.1}$$

$$\phi = 0 \ \text{ on } \partial B_1(0)\,, \quad \lim_{|x|\to+\infty} \phi(x) = 0\,, \tag{3.2}$$

We want to investigate under what conditions the homogeneous problem with $h = 0$ in (3.1)-(3.2) admits only the trivial solution. To this end, let us consider the eigenvalues of the problem

$$\psi'' + \frac{N-1}{r}\psi' + p|w|^{p-1}\psi + \nu\frac{\psi}{r^2} = 0, \psi = \psi(r) \tag{3.3}$$

$$\psi(1) = 0, \quad \psi(+\infty) = 0\ . \tag{3.4}$$

The $l-$th eigenvalue of (3.3)-(3.4) can be characterized variationally as

$$\nu_l(p) = \max_{\dim(V)\le l} \inf_{\psi\in V^\perp} \frac{\int_1^\infty |\psi'|^2 r^{N-1}dr - p\int_1^\infty w^{p-1}|\psi|^2 r^{N-1}dr}{\int_1^\infty \psi^2 r^{N-3}dr}, \tag{3.5}$$

where V runs through subspaces of $\mathcal{E}$ and $V^\perp$ is the set of $\psi \in \mathcal{E}$ satisfying $\int_1^\infty r^{N-3}\psi v = 0$ for all $v \in V$, and

$$\mathcal{E} = \{\psi(1) = 0, \int_1^\infty |\psi'(r)|^2 r^{N-1}dr < +\infty\}.$$

$\nu_1(p) \le \nu_2(p) \le ...$ are well defined thanks to Hardy's inequality,

$$\frac{(N-2)^2}{4}\int_1^\infty \psi^2 r^{N-3}dr \le \int_1^\infty |\psi'|^2 r^{N-1}dr.$$

Using Hardy's embedding and a simple compactness argument involving the fast decay of $w^{p-1} = o(r^{-4})$, yields the existence of an extremal for $\nu_l(p)$ which represents a solution to problem (3.3)-(3.4) for $\nu = \nu_l(p)$.

Let us consider now Problem (3.1)-(3.2) for $h = 0$, and assume that we have a solution ϕ. The symmetry of the domain $\mathbb{R}^N \setminus B_1(0)$ allows us to expand ϕ into spherical harmonics. We write ϕ as

$$\phi(x) = \sum_{k=0}^{\infty} \phi_k(r)\Theta_k(\theta), \quad r > 0,\, \theta \in S^{N-1}$$

where Θ_k, $k \geq 0$ are the eigenfunctions of the Laplace-Beltrami operator $-\Delta_{S^{N-1}}$ on the sphere S^{N-1}, normalized so that they constitute an orthonormal system in $L^2(S^{N-1})$. We take Θ_0 to be a positive constant, associated to the eigenvalue 0 and Θ_i, $1 \leq i \leq N$ is an appropriate multiple of $\frac{x_i}{|x|}$ which has eigenvalue $\lambda_i = N-1$, $1 \leq i \leq N$. In general, λ_k denotes the eigenvalue associated to Θ_k, we repeat eigenvalues according to their multiplicity and we arrange them in an non-decreasing sequence. We recall that the set of eigenvalues is given by $\{j(N-2+j) \,|\, j \geq 0\}$.

The components ϕ_k then satisfy the differential equations

$$\phi_k'' + \frac{N-1}{r}\phi_k' + \left(p|w|^{p-1} - \frac{\lambda_k}{r^2}\right)\phi_k = 0, \quad r \in (1,\infty), \tag{3.6}$$
$$\phi_k(1) = 0, \quad \phi_k(+\infty) = 0.$$

Let us consider first the radial mode $k = 0$, namely $\lambda_k = 0$. We observe that the function

$$Z_1(r) = rw'(r) + \frac{2}{p-1}w$$

satisfies

$$Z_1'' + \frac{N-1}{r}Z_1' + p|w|^{p-1}Z_1 = 0, \qquad \text{for all } r > 1,$$

but $Z_1(1) \neq 0$. Multiplying (3.6) by Z_1 and integrating by parts, we then obtain $\phi_k'(1) = 0$, which implies that $\phi_k = 0$ for the mode $k = 0$. (Note that this implies that w is non-degenerate in the space of radial functions.)

Let us consider now mode 1, namely $k = 1, \dots, N-1$, for which $\lambda_k = (N-1)$. In this case we also have an explicit solution which does not vanish at $r = 1$ but it does at $r = +\infty$. Simply $Z_1(r) = w'(r)$. But the same argument as above gives us $\phi_k'(1) = 0$ and hence $\phi_k = 0$, as desired.

Let us consider now modes 2 or higher. Here unfortunately life is harder. Not only we do not have an explicit solution to the ODE to rely on, but it could be the case that a non-trivial solution exists. Let us assume this is the case for an arbitrary mode $k \geq N$. Since $w^{'}(r)$ has exactly $m+1$ zeroes in $(1,+\infty)$ and $\lambda_k > \lambda_1$, by the standard Sturm-Liouville comparison theorem (Theorem 19 [29]), ϕ_k changes sign at most m times in $(1,\infty)$. On the other hand, by Sturm-Liouville theory, it is well-known that the eigenfunctions corresponding to ν_l must change sign in $(1,+\infty)$ at least $l-1$ times.Thus the only possibility for equation (3.6) to have a nontrivial solution for a given $k \geq N$ is that $\lambda_k = -\nu_l(p)$ for some $l = 1,...,m+1$.

In summary, we have proved the following result

Lemma 3.1. *Assume that p is such that*

$$\nu_l(p) \neq -j(N-2+j) \quad \text{for all } l = 1,...,m+1, j = 2,3,\ldots \tag{3.7}$$

where $\nu_l(p)$ is the $l-$th eigenvalue defined by (3.5). *Then Problem* (3.3)-(3.4) *with $h = 0$ admits only the solution $\phi = 0$.*

We will prove later that this non-resonance condition produces a good solvability theory for equation (3.1)-(3.2). Before doing so we will describe the set of exponents p for which condition (3.7) fails. We will prove

Proposition 3.1. *For each $l \leq m+1$ and $j \geq 2$ the set of numbers p for which $\nu_l(p) = -j(N-2+j)$ is non-empty and finite. In particular, there exists a sequence of the form*

$$\frac{N+2}{N-2} < p_1 < p_2 < p_3 < \cdots; \quad p_j \to +\infty \quad \text{as } j \to +\infty, \tag{3.8}$$

such that condition (3.7) *holds if and only if $p \neq p_j$ for all $j = 1,2,\ldots$.*

For the proof we need the following result, which contains elements of independent interest.

Lemma 3.2. (a) *As $p \downarrow \frac{N+2}{N-2}$, we have that $-\nu_l(p) \to \lambda_1 = N-1$ for $l = 1,...,m+1$.*

(b) *As $p \to +\infty$, it holds that*

$$-\nu_l(p) \to \infty, l = 1,...,m+1. \tag{3.9}$$

(c) *The set $\{\nu_l(p) = -j(N-2+j)\}$ consists of only isolated points.*

Proposition 3.1 is a direct consequence of Lemma 3.2. In fact, combining parts (a) and (b) we see that for each $l \leq m+1, j \geq 2$ the set of numbers p for which $\nu_l(p) = -j(N-2+j)$ is non-empty. Proposition 3.1 follows from this fact and (c) of Lemma 3.2.

Proof of Lemma 3.2 part (a). Let us set $p_* = \frac{N+2}{N-2}$. An alternative way of writing equation (1.5)-(1.6) and the eigenvalue problem (3.3)-(3.4) is by means of the so-called Emden-Fowler transformation,

$$\tilde{w}(s) = r^{\frac{2}{p-1}} w(r)\,, \quad \tilde{\psi}(s) = r^{\frac{2}{p-1}}\psi(r), \text{ where } r = e^s\,. \tag{3.10}$$

Then equation (1.5)-(1.6) is converted into

$$\tilde{w}'' + \alpha\tilde{w}' - \beta\tilde{w} + |\tilde{w}|^{p-1}\tilde{w} = 0\,, \quad \tilde{w}(0) = \tilde{w}(\infty) = 0,\ s \in [0,\infty) \tag{3.11}$$

where

$$\alpha = N - 2 - \frac{4}{p-1},\ \beta = \frac{2}{p-1}(N - 2 - \frac{2}{p-1}).$$

The eigenvalue problem (3.3)-(3.4) becomes

$$\tilde{\psi}'' + \alpha\tilde{\psi}' - \beta\tilde{\psi} + p|\tilde{w}|^{p-1}\tilde{\psi} + \nu\tilde{\psi} = 0, \quad \tilde{\psi}(0) = \tilde{\psi}(\infty) = 0,\ s \in [0,\infty)\,. \tag{3.12}$$

A further rescaling $\bar{w} = e^{\alpha s}\tilde{w}$ reduces (3.11) to

$$\bar{w}'' - (\beta+\alpha^2)\bar{w} + e^{-(p-1)\alpha s}|\bar{w}|^{p-1}\bar{w} = 0\,, \quad \bar{w}(0) = \bar{w}(\infty) = 0,\ s \in [0,\infty). \tag{3.13}$$

In the appendix, we shall prove that $\alpha \to 0, \beta \to \frac{(N-2)^2}{4}$ as $p \to p_*$, and

$$\bar{w} = \sum_{j=1}^{m+1} (-1)^{j+1} w_0(s - R_{\alpha,j}) \ +\ \textit{lower order terms}\,, \tag{3.14}$$

where w_0 is the unique homoclinic solution of the limiting equation,

$$w_0'' - \frac{(N-2)^2}{4} w_0 + w_0^{p_*} = 0, \quad w_0(0) = \max_{t\in R} w_0(t), \quad w_0(\infty) = 0 \tag{3.15}$$

and $R_{\alpha,1} \sim \log\frac{1}{|\alpha|} \to +\infty, |R_{\alpha,j} - R_{\alpha,j+1}| \sim \log\frac{1}{|\alpha|} \to +\infty$ as $\alpha \to 0$.

Next, we claim that for $l \le m+1$, $\nu_l(p) \to -(N-1)$ as $p \to p_*$. Note that by the rescaling (3.10), the eigenvalue has a variational characterization

$$\nu_l(p) = \max_{\dim(V)<l} \inf_{\psi\in V^\perp} \frac{\int_0^\infty e^{\alpha s}(|\psi'|^2 + \beta\psi^2 - pe^{-(p-1)\alpha s}|\bar{w}|^{p-1}|\psi|^2)ds}{\int_0^\infty e^{\alpha s}\psi^2 ds} \tag{3.16}$$

where V runs through the subspaces of $\mathcal{E}'$ and $V^\perp$ is the set of $\psi \in \mathcal{E}'$ satisfying $\int_0^\infty e^{\alpha s}\psi v = 0$ for all $v \in V$, and $\mathcal{E}' = \{\psi(0) = 0, \int_0^\infty e^{\alpha s}(|\psi'|^2 + \psi^2)ds < +\infty\}$. Note that the term involving the weight is relatively compact and it follows from a previous argument that the eigenvalues exist.

We observe that the limiting eigenvalue problem

$$\psi'' - \frac{(N-2)^2}{4}\psi + p_* w_0^{p_*-1}\psi = \mu\psi, \ \psi(\pm\infty) = 0 \tag{3.17}$$

admits eigenvalues

$$\mu_1 = N-1, \mu_2 = 0, \mu_3 < 0 \tag{3.18}$$

where the corresponding eigenfunction for the principal eigenvalue μ_1 is positive and denoted by Ψ_1. (In fact, it is known that problem (3.17) has eigenvalues such that $\mu_1 > 0, \mu_2 = 0, \mu_3 < 0$. A simple computation shows that we can take $\Psi_1 = w_0^{\frac{N}{N-2}}$ with $\mu_1 = N-1$ and $\Psi_2 = w_0^{'}, \mu_2 = 0$. See [13].) Now we take $\psi_j = \Psi_1(s - R_{\alpha,j}), j = 1, ..., m+1$. Let V be a given $m-$dimensional subspace and $v \in V$. Then there exists $c_1, ..., c_{m+1}$ (not all equal to 0) such that $\int_0^\infty e^{\alpha s} v(\sum_{j=1}^{m+1} c_j\psi_j) = 0$. We then compute that

$$\int_0^\infty e^{\alpha s}(|\psi^{'}|^2 + \psi^2 - pe^{-(p-1)\alpha s}|\bar{w}|^{p-1}\psi^2) \leq \sum_{j=1}^{m+1} c_j^2(-\mu_1 + o(1))\int_0^\infty e^{\alpha s}\psi_j^2 \tag{3.19}$$

and hence by the variational characterization of ν_{m+1} we deduce that

$$\nu_l(p) \leq \nu_{m+1}(p) \leq -(N-1) + o(1), l = 1, ..., m+1. \tag{3.20}$$

On the other hand, according to (3.18), $\nu_l(p) \to \mu_k \geq -(N-1)$ for some k. Thus by (3.20), we have $\nu_l(p) \to -(N-1)$ as $p \to p_*$, for $l \leq m+1$. □

Proof of Lemma 3.2 part (b).

We prove that $\nu_{m+1}(p) \to -\infty$ as $p \to +\infty$ (which then implies that $\nu_l(p) \to -\infty$ as $p \to +\infty$ for $l = 1, ..., m+1$). In fact, by the variational characterization of $\nu_{m+1}(p)$, we obtain that

$$\nu_{m+1}(p) = \max_{\dim(V)=m} \inf_{\int_{B_1^c}\frac{1}{r^2}\phi v = 0, \ \forall\ v \in V} \frac{\int_{B_1^c}(|\nabla\phi|^2 - p|w|^{p-1}\phi^2)}{\int_{B_1^c}\frac{1}{r^2}\phi^2} \tag{3.21}$$

Now let V be given. Let $w_j = w1_{r_{j-1}<r<r_j}, j = 1, ..., m+1$ where $r_0 = 0, r_{m+1} = \infty$. Then there exists $c_1, ..., c_{m+1}$ such that

$$\sum_{j=1}^{m+1} c_j \int_{B_1^c}\frac{1}{r^2} v w_j = 0, \ \forall v \in V. \tag{3.22}$$

Let $\phi = \sum_{j=1}^{m+1} c_j w_j$. We then compute

$$\begin{aligned} \int_{B_1^c} (|\nabla \phi|^2 - p|w|^{p-1}\phi^2) &= \sum_{j=1}^{m+1} c_j^2 \int_{B_1^c} (|\nabla w_j|^2 - p|w|^{p-1} w_j^2) \quad (3.23) \\ &= (1-p) \sum_{j=1}^{m+1} c_j^2 \int_{B_1^c} |\nabla w_j|^2 \end{aligned}$$

and

$$\int_{B_1^c} \frac{1}{r^2}\phi^2 = \sum_{j=1}^{m+1} c_j^2 \int_{B_1^c} \frac{1}{r^2} w_j^2 \leq C \sum_{j=1}^{m+1} c_j^2 \int_{B_1^c} |\nabla w_j|^2 \tag{3.24}$$

where the second inequality follows from Hardy's inequality.

From (3.24) and (3.24), we obtain that

$$\frac{\int_{B_1^c} (|\nabla \phi|^2 - p|w|^{p-1}\phi^2)}{\int_{B_1^c} \frac{1}{r^2}\phi^2} \leq C(1-p)$$

Thus from (3.21), we derive that

$$\nu_l(p) \leq \nu_{m+1}(p) \leq C(1-p), \forall l \leq m+1 \tag{3.25}$$

where C is independent of p. This shows that $-\nu_{m+1}(p) \to +\infty$ and hence $-\nu_l(p) \to +\infty$ for $l \leq m+1$, as $p \to +\infty$.

□

Proof of Lemma 3.2 part (c).

Let us sketch the main ideas first. Recall that in the positive solutions case [12], this is proved by first showing that the map

$$p \to \varphi_1^{-1} \mathtt{w}_p \tag{3.26}$$

is analytic, where $\mathtt{w}_p$ is the unique positive solution to (2.1) and φ_1 is the first eigenfunction of $-\Delta$ in B_1. Then the map $p \to \nu_l(p)$ is also analytic in p. Here this method is no longer applicable, since certainly the map $p \to \varphi_1^{-1}\mathtt{w}_p$ is not analytic. Furthermore, the location of zeroes of $\mathtt{w}_p$ changes as p varies. Our basic idea is to *freeze the zeroes* of $\mathtt{w}_p$ at $p = p_0$ and consider the analytical dependence of zeroes of $\mathtt{w}_p$ for p near p_0. Then we show that the eigenvalues $\nu_l(p)$ is also analytic in p for p near p_0.

Let w_p be the solution of (3.11) with m interior zeros. Let $\nu_l(p)$ be the $l-$th eigenvalue of (3.12) associated with w_p. We prove that if $\nu_l(p_0) = -j(N-2+j)$, then either $\nu_l(p) \neq -j(N-2+j)$ for $p \neq p_0$ and close to p_0 or $\nu_l(p) = -j(N-2+j)$ for all p close to p_0. Note that the latter case can not happen as it would imply $\nu_l(p) = -j(N-2+j)$ for all $p \in (\frac{N+2}{N-2}, \infty)$, which is impossible by (a) and (b) of Lemma 3.2.

Suppose that for $p = p_0$, the solution w_{p_0} has zeroes at $0 < s_1 < ... < s_m < +\infty$ where $s_0 = 0, s_{m+1} = \infty$. For $i = 0, ..., m$, we consider the rescaled Emden-Fowler equation on $[s_i, s_{i+1}]$:

$$w'' + \alpha z w' - \beta z^2 w + z^2 |w|^{p-1} w = 0, \tag{3.27}$$

where $\alpha = N - 2 - \frac{4}{p-1}, \beta = \frac{2}{p-1}(N - 2 - \frac{2}{p-1})$ and z is a real parameter and close to 1. Note that $u(t)$ is a solution of (3.11) on the interval $[z^{-1}s_i, z^{-1}s_{i+1}]$ if and only if $u(z^{-1}s)$ is a solution of (3.27) on $[s_i, s_{i+1}]$.

Given z and p, we consider the following Dirichlet problem

$$w'' + \alpha z w' - \beta z^2 w + z^2 |w|^{p-1} w = 0, s_i < s < s_{i+1}, w(s_i) = w(s_{i+1}) = 0. \tag{3.28}$$

By (3.10), equation (3.28) is equivalent to the following nonlinear problem on an anulus:

$$\begin{cases} \Delta u + u^p = 0 \text{ in } B_{r_{i+1}}(0) \backslash B_{r_i}(0), \\ u = u(r) > 0 \text{ and } u = 0 \text{ on } \partial(B_{r_{i+1}}(0) \backslash B_{r_i}(0)) \end{cases} \tag{3.29}$$

where $r_i = e^{z^{-1}s_i}, r_{i+1} = e^{z^{-1}s_{i+1}}$. Thus by standard PDE technique, there exists a positive (or negative) solution to (3.28), which is unique and non-degenerate by the theory of Ni and Nussbaum [24]. This implies that it also depends continuously on z and p. By the implicit function theorem and the real analyticity properties of Dancer [8], this positive (or negative) solution, viewed as an element of $C^1[s_i, s_{i+1}]$, depends analytically on z and p. (Note that s_i is fixed and doesn't depend on z and p.) We denote this solution by $w^i_{z,p}$. (In the interval $[s_m, +\infty)$, the solution $w^m_{1,p}$ is nothing but $\pm w_p(s - s_m)$, where w_p is the unique positive solution of (3.11). By [8], $w^m_{1,p}$ is analytic in p when considered as an element of a weighted Sobolev space $\|e^{\tau s} u\|_{W^{2,q}([s_m, +\infty)}$, where $\tau > 0$.)

Next, we want to choose $z = z^i_p$ (close to 1 and analytic in p) so that

$$w_p(t) = w^i_{z,p}(zt), \quad t \in [z^{-1}s_i, z^{-2}s_{i+1}]. \tag{3.30}$$

For the interval $[s_m, +\infty)$, we may set $z = 1$. We now find z^i_p near 1 and real analytic in p so that the solution in the original variables is C^1 across s_{i+1} (and hence C^2 across s_{i+1}). We do this successively starting from the infinite interval $[s_m, +\infty)$ (where it is trivial). At s_{i+1}, we have

$$z^i_p (w^i_{z^i_p, p})'(s_{i+1}) = z^{i+1}_p (w^{i+1}_{z^{i+1}_p, p})'(s_{i+1}). \tag{3.31}$$

If we can prove that the map $z \to z(w^i_{z,p_0})'(s_{i+1})$ has non-zero derivative at $z = 1$, we can use the implicit function theorem to prove that z^i_p is analytic in p near $p = p_0$. (Note that the right hand side of (3.31) is analytic in p and $z^i_{p_0} = 1$.)

Now $v(t) = w^i_{z,\ p_0}(zt)$ satisfies

$$v'' + \alpha v' - \beta v + v^{p_0} = 0 \tag{3.32}$$

and $v(z^{-1}s_i) = v(z^{-1}s_{i+1}) = 0$. Thus $z(w^i_{z,p_0})'(zt)$ is a solution of the linearized equation of (3.32) at $w^i_{z,p_0}(zt)$. By a translation, we can assume without loss of generality that $s_{i+1} = 0$. Thus $v(0) = 0$. Note that $w^i_{z,p_0}(zt) = 0$ for $t = z^{-1}s_i$. Let us consider how the solution $\hat{u}_\gamma$ of (3.32) satisfying $\hat{u}_\gamma(0) = 0, \hat{u}'_\gamma(0) = \gamma$ varies with γ when γ is close to $z^i_{p_0}(w^i_{z^i_{p_0},p_0})'(0) = Y$. It will be of the form $w^i_{1,p_0}(t) + (\gamma - Y)h(t) + o(\gamma - Y)$ and the corresponding result for the time derivative. Here $h(t)$ is the solution of the linearized equation satisfying $h(0) = 0, h'(0) = -1$. By the non-degeneracy, $h(s_i) \neq 0$. It follows that $\hat{u}_\gamma$ crosses zero at a point near s_i of the form $s_i + a(\gamma - Y) + o(\gamma - Y)$ where

$$a = -\frac{h(s_i)}{(w^i_{1,p_0})'(s_i)} \neq 0.$$

We used here the fact that $(w^i_{1,p_0})'(s_i) \neq 0$ by the uniqueness of the initial value problem (3.28). Hence the map $\gamma \to$ the largest negative zero of $\hat{u}$ is a local diffeomorphism. Now the largest negative zero of $\hat{u}_\gamma$ is $z^{-1}s_i$ which has non-zero derivative in z at $z = 1$. Thus γ has non-zero derivative in z at $z = 1$, which gives the claim (since $\gamma = z(w^i_{z,p_0})'(0)$).

We now consider the linearized problem (3.12) for a possible eigenvalue $\nu_l(p_0)$. We once again work on fixed intervals $[s_i, s_{i+1}]$ by our rescaling. Once again, we work from the right. Since α, β and $|w^i_{z,p}|^{p-1}$ are real analytic and z^i_p are real analytic in p, our linearized equation on $[s_i, s_{i+1}]$

$$\phi'' + \alpha z^i_p \phi' - \beta (z^i_p)^2 \phi + p(z^i_p)^2 (w^i_{z,p})^{p-1}\phi + (z^i_p)^2 \nu_l(p_0)\phi = 0, \tag{3.33}$$

has coefficients real analytic in p and hence by a Liapunov-Schmidt reduction argument applied to the maximal operator of the linearized equation (3.33) on $[s_i, s_{i+1}]$, there is a basis $\{\phi^i_1, \phi^i_2\}$ for the kernel depending real analytically on p. Note that the maximal operator is onto and Fredholm for fixed p and thus this is a well-known result. See [15]. Here as in [15], the maximal operator has no boundary conditions and we work on the space L^q for q large. The unbounded interval is slightly different in that we only have a single solution decaying at ∞ and this depends analytically on p, by similar arguments.

Suppose we have found a solution ϕ^{j+1} as linear combination of ϕ_1^{j+1} and ϕ_2^{j+1} which depends analytically on p. Namely, $\phi^{j+1} = a^{j+1}(p)\phi_1^{j+1} + b^{j+1}(p)\phi_2^{j+1}$ and a^{j+1}, b^{j+1} are analytic in p. At r_j, we want $a^j(p), b^j(p)$ depending analytically on p so that

$$a^j(p)\phi_1^j(s_{j+1}) + b^j(p)\phi_2^j(s_{j+1}) = \phi^{j+1}(s_{j+1}), \tag{3.34}$$

$$a^j(p)(z_p^j)^{-1}(\phi_1^j)'(s_{j+1}) + b^j(p)(z_p^j)^{-1}(\phi_2^j)'(s_{j+1}) = (z_p^{j+1})^{-1}(\phi^{j+1})'(s_{j+1}) \tag{3.35}$$

where the z_p^j and z_p^{j+1} comes from the scalings since we want the solution to be C^1 in the original variables.

By Cramer's rule, $a^j(p)$ and $b^j(p)$ are analytic in p (since all the other terms are). Thus $\phi^j = a^j(p)\phi_1^j + b^j(p)\phi_2^j$ is analytic in p. We eventually find $\phi^1(0)$ is analytic in p and hence $\phi^1(0) = 0$ holds for all p near p_0 or $p = p_0$ is an isolated zero of $\phi^1(0) = 0$. In other words, for this choice of l, $\nu_l(p) = -j(N-2+j)$ for p near p_0 or $\nu_l(p) \neq -j(N-2+j)$ for all p near $p_0, p \neq p_0$. This is exactly what we claimed.

$\square$

Finally, we consider the full problem (3.1)-(3.2), namely

$$\Delta\phi + p|w|^{p-1}\phi = h \quad \text{ in } \mathbb{R}^N \setminus \bar{B}_1(0)\,,$$

$$\phi = 0 \;\text{ on } \partial B_1(0)\,, \qquad \lim_{|x|\to+\infty} \phi(x) = 0\,.$$

As in [12], we fix a small number $\sigma > 0$ and consider the norms

$$\|\phi\|_* = \sup_{|x|>1} |x|^{N-2-\sigma}|\phi(x)| + \sup_{|x|>1} |x|^{N-1-\sigma}|\nabla\phi(x)| \tag{3.36}$$

and

$$\|h\|_{**} = \sup_{|x|>1} |x|^{N-\sigma}|h(x)|. \tag{3.37}$$

Then we have the following proposition whose proof is exactly the same as that of Proposition 2.2 of [12].

Proposition 3.2. *Assume that p satisfies condition* (3.7)*. Then for any h with $\|h\|_{**} < +\infty$, Problem* (3.1)-(3.2) *has a unique solution $\phi = T(h)$ with $\|\phi\|_* < +\infty$. Besides, there exists a constant $C(p) > 0$ such that*

$$\|T(h)\|_* \leq C\|h\|_{**}.$$

Proposition 3.2 has a local version. Let $Q=0$ and $\mathcal{D}_\delta=\frac{1}{\delta}\mathcal{D}$.
For the following linear problem

$$\Delta\phi+pw^{p-1}\phi=h \qquad \text{in } \mathcal{D}_\delta\setminus\bar{B}_1(0)\,, \tag{3.38}$$

$$\phi=0 \quad \text{on } \partial B_1(0)\,\cup\,\partial\mathcal{D}_\delta \tag{3.39}$$

we have

Proposition 3.3. *Assume that p satisfies condition* (3.7)*. Then there is a number δ_0 such that for all $\delta<\delta_0$ and any h with $\|h\|_{**}<+\infty$, Problem* (3.38)-(3.39) *has a unique solution $\phi=T_\delta(h)$ with $\|\phi\|_*<+\infty$. Besides, there exists a constant $C(p,\mathcal{D})>0$ such that*

$$\|T_\delta(h)\|_*\le C\|h\|_{**}.$$

4. Proof of Theorem 1

Let $Q=0$, and $\mathcal{D}_\delta=\delta^{-1}\mathcal{D}$. Without loss of generality, we also assume that $B_3(0)\subset\mathcal{D}$ and $w(r)>0$ for $r>r_m$. Let us assume the validity of condition of condition (3.7) or, equivalently, that $p\neq p_j$ for all j, with p_j the sequence in (3.8). Problem (1.3) is, after setting $v(x)=\delta^{\frac{2}{p-1}}u(\delta x)$, equivalent to

$$\Delta v+|v|^{p-1}v=0 \ \text{ in } \mathcal{D}_\delta\setminus\bar{B}_1(0)\,,\quad v=0 \ \text{ on } \partial B_1(0)\cup\partial\mathcal{D}_\delta. \tag{4.1}$$

Instead of considering (4.1), we modify the nonlinearity in the following way: let $\chi(t)$ be a smooth cut-off function which equals one for $|t|<2r_m$ and zero for $|t|>4r_m$. Let

$$g(x,v)=|v|^{p-1}v\chi(|x|)+v_+^p(1-\chi(|x|)),\ \text{ where } v_+=\max(v,0) \tag{4.2}$$

We now consider the following problem

$$\Delta v+g(x,v)=0 \qquad \text{in } \mathcal{D}_\delta\setminus\bar{B}_1(0)\,,\ v=0 \ \text{ on } \partial B_1(0)\cup\partial\mathcal{D}_\delta. \tag{4.3}$$

The rest of the proof of Theorem 1 is exactly the same as in [12]. For the reader's convenience, we give a sketch here.

Let us consider the smooth cut-off function η_δ, which equals 1 in $B(0,2\delta^{-1})$ and 0 outside $B(0,3\delta^{-1})$. We search for a solution v to problem (4.3) of the form

$$v=\eta_\delta w+\phi,$$

which is equivalent to the following problem for ϕ:

$$\begin{cases}\Delta\phi+p|\eta_\delta w|^{p-1}\phi=N_1(\phi)+N_2(\phi)+E \quad \text{in } \mathcal{D}_\delta\setminus\bar{B}_1(0)\,,\\ \quad\phi=0 \ \text{ on } \partial B_1(0)\,\cup\,\partial\mathcal{D}_\delta\end{cases} \tag{4.4}$$

where

$$-N_1(\phi)=g(x,\eta_\delta w+\phi)-g(x,\eta_\delta w)-g'(x,\eta_\delta w)\phi,$$

$$N_2(\phi) = p(1-\eta_\delta^{p-1})|w|^{p-1}$$

and

$$E = -\Delta(\eta_\delta w) - g(x, \eta_\delta w) = -\Delta(\eta_\delta w) - \eta_\delta^p |w|^{p-1} w.$$

Note that $g^{'}(x, \eta_\delta w) = p|w|^{p-1}$.

According to Proposition 3.3 we thus have a solution to (4.3) if ϕ solves the fixed point problem

$$\phi = T_\delta(N_1(\phi) + N_2(\phi) + E)\,. \tag{4.5}$$

Let us estimate E. We have, explicitly,

$$-E = \eta(\eta_\delta^{p-1} - 1)|w|^{p-1}w + 2\nabla\eta_\delta \nabla w + w\Delta\eta_\delta$$

We clearly have, globally, $|E(x)| \le C\delta^N$ and hence

$$\|E\|_{**} \le C\delta^\sigma\,. \tag{4.6}$$

Let us now estimate $\|N_1(\phi) + N_2(\phi)\|_{**}$. We observe that

$$\begin{aligned}\|N_2(\phi)\|_{**} = \|p(1-\eta_\delta^{p-1})|w|^{p-1}\phi\|_{**} &\le C \sup_{|x|\ge\delta^{-1}} |x|^{N-\sigma}|w(x)|^{p-1}|\phi(x)| \\ &\le C\delta^2\|\phi\|_*. \end{aligned} \tag{4.7}$$

To estimate $N_1(\phi)$, let us assume first $p < 2$. Then we estimate

$$|x|^{N-\sigma}|N_1(\phi)| \;\le\; C|x|^{N-\sigma}|\phi(x)|^p \;\le\; |x|^{N-\sigma}|x|^{-N-2}\|\phi\|_*^p \;\le\; C\|\phi\|_*^p\,,$$

so that

$$\|N_1(\phi)\|_{**} \le C\|\phi\|_*^p\,.$$

Let us assume now $p \ge 2$. In this case we have

$$|N(\phi)| \;\le\; C\,(\,|w|^{p-2}\phi^2 + |\phi|^p\,).$$

Now, we directly check that

$$|x|^{N-\sigma}w^{p-2}\phi^2 \;\le\; C|x|^{(p-2)(2-N)-N+4+\sigma}\|\phi\|_*^2.$$

The exponent of $|x|$ in the last expression is always negative. In fact, this is obvious if $N \ge 5$, while if $N = 3, 4$ supercriticality implies $p \ge 3$. On the other hand,

$$|x|^{N-\sigma}|\phi|^p \;\le\; C\,|x|^{N-\sigma-p(N-2-\sigma)}\,\|\phi\|_*^p \;\le\; |x|^{-2+(p-1)\sigma}\,\|\phi\|_*^p\,.$$

We conclude from these estimates that, for any $p > \frac{N+2}{N-2}$,

$$\|N_1(\phi)\|_{**} \le C\,(\,\|\phi\|_*^p + \|\phi\|_*^2\,)\,. \tag{4.8}$$

Let us consider now the operator

$$\mathcal{T}(\phi) = T_\delta(N_1(\phi) + N_2(\phi) + E)$$

defined in the region

$$\mathcal{B} = \{\phi \in C^1(\bar{\mathcal{D}}_\delta \setminus B_1(0)) \ / \ \|\phi\|_* \le \delta^{\frac{\sigma}{2}} \}\,.$$

Using estimates (4.6), (4.8), we immediately get that $\mathcal{T}(\mathcal{B}) \subset \mathcal{B}$, provided that δ is sufficiently small. We observe that, in the bounded domain $\mathcal{D}_\delta \setminus B_1(0)$,

$$T_\delta = (\Delta + p|w|^{p-1})^{-1}$$

maps boundedly C^0 into $C^{1,\alpha}$, hence compactly into C^1. It follows that the map $\mathcal{T}$ is actually compact on the closed, bounded subset of C^1 given by $\mathcal{B}$. The existence of a fixed point of $\mathcal{T}$ on $\mathcal{B}$ thus follows from Schauder's theorem.

This yields a solution to (4.3). Note that for our solution $v = \eta_\delta w + \phi$, we have $v > 0$ for $1.5 r_m < |x| < 5 r_m$ since $v = w + \phi$ and $\|\phi\|_* \le C\delta^{\frac{\sigma}{2}}$. Thus, v actually satisfies $\Delta v + v_+^p = 0$ for $|x| > 2r_m$ and $\Delta v + |v|^{p-1} v = 0$ for $|x| < 2r_m$. By Maximum Principle applied to v on $\mathcal{D}_\delta \setminus B_{2r_m}$, we have that $v > 0$ in $\mathcal{D}_\delta \setminus B_{2r_m}$. In summary, v satisfies (4.1). Furthermore, since $v > 0$ for $|x| > 2r_m$and $v = w + \phi$ for $|x| < 2r_m$, we deduce that v has exactly $m+1$ nodal domains. This concludes the proof of the theorem. □

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Appendix: Proof of (3.14)

In this appendix, we use the so-called localized energy method to prove (3.14). For background on the localized energy method, we refer to the survey article [23]. In particular, we follow the steps in [20].

By a rescaling, (3.13) becomes the following nonlinear problem:

$$u'' - u + e^{-\epsilon y}|u|^{p-1}u = 0, u > 0 \text{ for } y > 0, u(0) = u(+\infty) = 0 \quad (4.9)$$

where $\epsilon = c_1(p-1)\alpha, p = p_* + c_2\epsilon$ for some $c_1, c_2 > 0$. Note that $\alpha \to 0, \epsilon \to 0$ as $p \to p_*$.

Associated with (4.9) is the following energy functional:

$$J[u] = \frac{1}{2}\int_{\mathbb{R}_+} (|u'|^2 + u^2) - \frac{1}{p+1}\int_{\mathbb{R}_+} e^{-\epsilon y}|u|^{p+1}, u \in H_0^1(\mathbb{R}_+) \quad (4.10)$$

The localized energy method consists of four steps:

Step 1: Choose good approximate functions

Let w be the unique solution of

$$w'' - w + w^p = 0, w(0) = \max_{y\in\mathbb{R}} w(y), w(\pm\infty) = 0 \quad (4.11)$$

Let $t > 0$. We define

$$w_t := w(y-t) - \rho(t)\beta(y) \quad (4.12)$$

where $\beta(y) = e^{-y}, \rho(t) = w(t)$. Then $w_t \in H_0^1(\mathbb{R}_+)$.

Let $\mathbf{t} = (t_1, ..., t_m)$. We define a configuration space:

$$\Lambda = \left\{ \mathbf{t} = (t_1, ..., t_N) \left| \begin{array}{l} t_m < m(1+m^3\eta)\log\frac{1}{\epsilon}, t_1 \geq (1-\eta)\,\epsilon\log\frac{1}{\epsilon}, \\ t_j - t_{j-1} > (1-\eta)\,\epsilon\log\frac{1}{\epsilon}, j = 2, ..., m \end{array} \right. \right\}, \quad (4.13)$$

where $\eta \in (0, \frac{1}{100m^4})$ is a fixed number.

For $\mathbf{t} \in \Lambda$, we define

$$w_{\mathbf{t}} = \sum_{j=1}^m (-1)^{j+1} w_{t_j}. \quad (4.14)$$

Let

$$S[u] = u'' - u + e^{-\epsilon y}|u|^{p-1}u.$$

Note that

$$\sum_{j=1}^m e^{-t_j} + \sum_{j=2}^m e^{-(t_j - t_{j-1})} = O(\epsilon^{1-\eta}).$$

Then, similar to the computations in Lemma 2.2 and Lemma 2.3 of [20], we obtain

$$S[w_{\mathbf{t}}] = O(\epsilon^{\frac{1}{2}+2\sigma}) \quad (4.15)$$

and

$$J[w_{\mathbf{t}}] = A(\epsilon) + \sum_{j=1}^m (A_0+\epsilon^\sigma)e^{t_j - t_{j-1}} + B_0\epsilon\sum_{j=1}^N t_j + \sum_{j=1}^N (C_0+\epsilon^\sigma)e^{-t_j} + O(\epsilon) \quad (4.16)$$

where $A(\epsilon)$ is some function of ϵ only, $A_0, B_0, C_0 > 0$ and σ is a small positive number in $(0,1)$.

Step 2: Nonlinear Liapunov-Schmidt reduction

Similar to the proof of Proposition 4.2 of [20], there exists a unique $(\phi_{\epsilon,\mathbf{t}}, c_1, ..., c_m)$ such that

$$S[w_{\mathbf{t}} + \phi_{\epsilon,\mathbf{t}}] = \sum_{j=1}^{m} c_j w^{'}(y - t_j) \tag{4.17}$$

and we also have

$$\|\phi_{\epsilon,\mathbf{t}}\|_{L^\infty(\mathbb{R}_+)} \leq C\epsilon^{\frac{1}{2}+\sigma}. \tag{4.18}$$

Step 3: Reduction to finite dimensional space.

We set

$$\mathcal{K}_\epsilon(\mathbf{t}) = J[w_{\mathbf{t}} + \phi_{\epsilon,\mathbf{t}}] : \Lambda \to \mathbb{R}.$$

Then if $\mathbf{t}^\epsilon = (t_1^\epsilon, ..., t_m^\epsilon)$ is a critical point of $\mathcal{K}_\epsilon$, the corresponding function $u_\epsilon = w_{\mathbf{t}^\epsilon} + \phi_{\epsilon,\mathbf{t}^\epsilon}$ is a critical point for $J[u]$ and hence a solution to (4.9). So it is enough to find a critical point of $\mathcal{K}_\epsilon$.

An easy computation shows that

$$\mathcal{K}_\epsilon(\mathbf{t}) = J[w_{\mathbf{t}}] + O(\epsilon). \tag{4.19}$$

Step 4: A minimization procedure

We consider the following minimization problem:

$$\min_{\mathbf{t}\in\Lambda} \mathcal{K}_\epsilon(\mathbf{t}). \tag{4.20}$$

By choosing $t_j = j \log \frac{1}{\epsilon}$, we obtain an upper bound for

$$\min_{\mathbf{t}\in\Lambda} \mathcal{K}_\epsilon(\mathbf{t}) \leq A(\epsilon) + \frac{m(m+1)}{2} B_0 \epsilon \log \frac{1}{\epsilon} + O(\epsilon). \tag{4.21}$$

On the other hand, if $\mathbf{t} \in \partial\Lambda$, then we have either $t_1 = (1-\eta)\log\frac{1}{\epsilon}$, or $t_j - t_{j-1} = (1-\eta)\log\frac{1}{\epsilon}$, or $t_N = m(1+m^3\eta)\log\frac{1}{\epsilon}$. In any case, we shall derive that

$$\mathcal{K}_\epsilon(\mathbf{t}) \geq A(\epsilon) + (\frac{m(m+1)}{2} + \eta) B_0 \epsilon \log \frac{1}{\epsilon} \tag{4.22}$$

which contradicts with (4.21). This shows that the above minimization problem (4.20) has a solution $\mathbf{t}^\epsilon = (t_1^\epsilon, ..., t_m^\epsilon) \in \Lambda$. Thus $u_\epsilon = \sum_{j=1}^{m} (-1)^{j+1} w(y - t_j^\epsilon) + \phi_{\epsilon,\mathbf{t}^\epsilon}$ is a solution to (4.9) with exactly m zeroes. By uniqueness, (3.14) is proved. □

References

[1] A. Bahri, J.M. Coron, On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain, *Comm. Pure Appl. Math.* 41(1988), 255-294.

[2] T. Bartsch and Z.Q. Wang, On the existence of sign changing solutions for semilinear Dirichlet problems, *Topol. Methods Nonlinear Anal.* 7(1996), no.1, 115-131.

[3] T. Bartsch and M. Willem. Infinitely many radial solutions of a semilinear elliptic problem on $\mathbf{R}^N$. *Arch. Rational Mech. Anal.*, 124(3):261–276, 1993.

[4] T. Bartsch, A.M. Micheletti and A. Pistoia, On the existence and the profile of nodal solutions of elliptic equations involving critical gwoth, *Cal. Var. PDE*, to appear.

[5] H. Brezis, L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. *Comm. Pure Appl. Math. 36* (1983), 437–47.

[6] M. Clapp and T. Weth, Minimal nodal solutions of the pure critical exponent problem on a symmetric domain, *Cal. Var. PDE* 21(2004), 1-14.

[7] J.M. Coron, Topologie et cas limite des injections de Sobolev, *C.R. Acad. Sc. Paris,* 299, Series I (1984), 209-212.

[8] E.N. Dancer, Real analyticity and non-degeneracy. *Math. Ann.* 325 (2003), no. 2, 369–392.

[9] M. del Pino, P. Felmer, M. Musso, Two-bubble solutions in the super-critical Bahri-Coron's problem *Cal. Var. PDE* 16 (2003), no. 2, 113–145.

[10] M. del Pino, P. Felmer, M. Musso, Multi-peak solutions for super-critical elliptic problems in domains with small holes. *J. Diff. Eqns.* 182 (2002), no. 2, 511–540.

[11] M. del Pino, P. Felmer and K. Tanaka, An elementary construction of complex patterns in nonlinear Schrödinger equations, *Nonlinearity* 15(2002), 1653-1671.

[12] M. del Pino and J. Wei, Supercritical elliptic problems in domains with small holes, *Ann. Inst. H. Poincare Anal. Non Lineaire*, to appear.

[13] V. Felli and M. Schneider, Perturbation results of critical elliptic equations of Caffarelli-Kohn-Nirenberg type, *J. Diff. Eqns.* 91(2003), 121-142.

[14] Y. Ge, R. Jing, F. Pacard, Bubble towers for supercritical semilinear elliptic equations. *J. Funct. Anal.* 221 (2005), no. 2, 251–302.

[15] S. Goldberg, Unbounded Linear Operators, Theory and Applications. Dover Publications, Inc., New York, 1985.

[16] S. Khenissy, O. Rey, An criterion for existence of solutions to the supercritical Bahri-Coron's problem. *Houston J. Math.* 30 (2004), no. 2, 587-613.

[17] A. Micheletti, A. Pistoia, On the effect of the domain geometry on the existence of sign changing solutions to elliptic problems with critical and supercritical growth. *Nonlinearity* 17 (2004), no. 3, 851–866

[18] R. Molle, D. Passaseo, Positive solutions of slightly supercritical elliptic equations in symmetric domains.*Ann. Inst. H. Poincare Anal. Non Lineaire* 21 (2004), no. 5, 639–656.

[19] M. Musso and A. Pistoia, Sign changing solutions to Bahri-Coron's problem in pierced domains, preprint.

[20] A. Malchiodi, W.-M. Ni and J. Wei, Multiple clustered layer solutions for semilinear Neumann problems on a ball, *Ann. Inst. H. Poincare Anal. Non Lineaire* 22(2005), 143-163.

[21] Y.Y. Naito, Bounded solutions with prescribed number of zeroes for the Emden-Fowler differential equation, *Hiroshima Math. J.* 24(1994), 177-220.

[22] W.-M. Ni, A nonlinear Dirichlet problem on the unit ball and its applications, *Indiana Univ. Math. J.* 31 (1982), no. 6, 801–807.

[23] W.-M. Ni, *Qualitative properties of solutions to elliptic problems.* Stationary partial differential equations. Vol. I, 157–233, Handb. Differ. Equ., North-Holland, Amsterdam, 2004.

[24] W.-M. Ni and R. Nussbaum, Uniqueness and nonuniqueness for positive solutions of $\Delta u + f(u,r) = 0$, *Comm. Pure Appl. Math.* 38(1985), 67-108.

[25] Z. Nehari. Characteristic values associated with a class of non-linear second-order differential equations. *Acta Math.*, 105:141–175, 1961.

[26] D. Passaseo, Nonexistence results for elliptic problems with supercritical nonlinearity in nontrivial domains. *J. Funct. Anal.* 114 (1993), no. 1, 97–105

[27] A. Pistoia and T. Weth, Sign changing bubble tower solutions in a slightly subcritical semilinear Dirichlet problem, *Ann. Inst. Poincare Anal. Non Lineaire*, to appear.

[28] S. Pohozaev, Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$, *Soviet. Math. Dokl.* 6(1965), 1408-1411.

[29] M. Protter and H. Weinberger, Maximum Principles in Differential Equations, Prentice-Hall Inc.

[30] M. Struwe. Infinitely many solutions of superlinear boundary value problems with rotational symmetry. *Arch. Math. (Basel)*, 36(4):360–369, 1981.

[31] M. Struwe. Superlinear elliptic boundary value problems with rotational symmetry. *Arch. Math. (Basel)*, 39(3):233–240, 1982.

[32] O. Rey, On a variational problem with lack of compactness: the effect of small holes in the domain, *C. R. Acad. Sci. Paris Ser. I Math.* 308 (1989), no. 12, 349–352.

School of Mathematics and Statistics, University of Sydney, Australia

E-mail address: normd@maths.usyd.edu.au

J. Wei - Department of Mathematics, Chinese University of Hong Kong, Shatin, Hong Kong

E-mail address: wei@math.cuhk.edu.hk